Chapter 4
Convergence with Nonhomogeneous Boundary Conditions

4.1 The Setting

In this chapter, we consider the continuous wave equation

$$\begin{cases} \partial_{tt} y - \partial_{xx} y = 0, & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = 0, \quad y(t,1) = v(t), & t \in (0,T), \\ (y(0,\cdot), \partial_t y(0,\cdot)) = (y^0, y^1), \end{cases} \tag{4.1}$$

with

$$(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1), \qquad v \in L^2(0,T). \tag{4.2}$$

Following [36] (see also [33, 35]), system (4.1) can be solved uniquely in the sense of transposition and the solution y belongs to

$$C([0,T]; L^2(0,1)) \times C^1([0,T]; H^{-1}(0,1)).$$

Let us briefly recall the main ingredients of this definition of solution in the sense of transposition and this result.

The key idea is the following. Given smooth functions f, the solutions φ of

$$\begin{cases} \partial_{tt} \varphi - \partial_{xx} \varphi = f, & (t,x) \in (0,T) \times (0,1), \\ \varphi(t,0) = \varphi(t,1) = 0, & t \in (0,T), \\ (\varphi(T,\cdot), \partial_t \varphi(T,\cdot)) = (0,0), \end{cases} \tag{4.3}$$

which are smooth for smooth f, should satisfy

$$\begin{aligned} \int_0^T \int_0^1 y f \, \mathrm{d}x \, \mathrm{d}t = & - \int_0^T v(t) \partial_x \varphi(t,1) \, \mathrm{d}t \\ & - \int_0^1 y^0(x) \partial_t \varphi(0,x) \, \mathrm{d}x + \langle y^1, \varphi(0,\cdot) \rangle_{H^{-1}, H_0^1}. \end{aligned} \tag{4.4}$$

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Thus one should first check that if $f \in L^1(0,T;L^2(0,1))$, then the solution φ of Eq. (4.3) belongs to the energy space $C([0,T];H_0^1(0,1)) \cap C^1([0,T];L^2(0,1))$ and is such that $\partial_x \varphi(t,1) \in L^2(0,T)$ with the following continuity estimate:

$$\|(\varphi, \partial_t \varphi)\|_{L^\infty(0,T;H_0^1(0,1)\times L^2(0,1))} + \|\partial_x \varphi(t,1)\|_{L^2(0,T)} \leq C\|f\|_{L^1(0,T;L^2(0,1))}. \tag{4.5}$$

Of course, there, the first term can be estimated easily through the energy identity, whereas the estimate on the normal derivative of φ at $x = 1$ is a hidden regularity result that can be easily proved using multiplier techniques.

Assuming Eq. (4.5), the map

$$\mathscr{L}(f) = -\int_0^T v(t)\partial_x \varphi(t,1)\,\mathrm{d}t - \int_0^1 y^0(x)\partial_t \varphi(0,x)\,\mathrm{d}x + \langle y^1, \varphi(0,\cdot)\rangle_{H^{-1},H_0^1}$$

is continuous on $L^1(0,T;L^2(0,1))$ and thus there is a unique function y in the space $L^\infty(0,T;L^2(0,1))$ that represents $\mathscr{L}$, which is by definition the solution y of Eq. (4.1) in the sense of transposition. The solution y actually belongs to the space $C([0,T];L^2(0,1))$ since it can be approximated in $L^\infty(0,T;L^2(0,1))$ by smooth functions by taking smooth approximations of v, y^0, and y^1.

A similar duality argument shows that $\partial_t y$ belongs to $C([0,T];H^{-1}(0,1))$.

Let us finally mention the following regularity result (see [34]): if $(y^0, y^1) \in H_0^1(0,1) \times L^2(0,1)$ and $v \in H^1(0,T)$ satisfies $v(0) = 0$, then the solution y of Eq. (4.1) satisfies

$$y \in C([0,T];H^1(0,1)) \cap C^1([0,T];L^2(0,1)) \text{ and } \Delta y \in C([0,T];H^{-1}(0,1)). \tag{4.6}$$

Now, the goal of this chapter is to study the convergence of the solutions of

$$\begin{cases} \partial_{tt} y_{j,h} - \dfrac{1}{h^2}(y_{j+1,h} - 2y_{j,h} + y_{j-1,h}) = 0, & (t,j) \in (0,T) \times \{1,\ldots,N\}, \\ y_{0,h} = 0, \quad y_{N+1,h}(t) = v_h(t), & t \in (0,T), \\ (y_h(0), \partial_t y_h(0)) = (y_h^0, y_h^1), \end{cases} \tag{4.7}$$

towards the solution y of Eq. (4.1), under suitable convergence assumptions on the data (y_h^0, y_h^1) and v_h to (y^0, y^1) and v.

As in Chap. 3, y_h will be identified with its Fourier extension $\mathbb{F}_h(y_h)$. This will allow us to identify the $H^{-1}(0,1)$-norm of f_h as

$$\|f_h\|_{H^{-1}(0,1)} = \|z_h\|_{H_0^1(0,1)}, \text{ where } z_h \text{ solves } -\partial_{xx} z_h = f_h \text{ on } (0,1), \quad z_h(0) = z_h(1).$$

Note that, expanding these discrete functions on the Fourier basis, one can check (see Proposition 4.1 below) that this norm is equivalent to $\|\tilde{z}_h\|_{H_0^1(0,1)}$, where $\tilde{z}_h$ solves

$$-\frac{1}{h^2}\left(\tilde{z}_{j+1,h} + \tilde{z}_{j-1,h} - 2\tilde{z}_{j,h}\right) = f_{j,h}, \quad j \in \{1,\ldots,N\}, \quad \tilde{z}_{0,h} = \tilde{z}_{N+1,h} = 0.$$

The outline of this Chap. 4 is as follows. Since we are working with the $H^{-1}(0,1)$-norm, it will be convenient to present some further convergence results for the discrete Laplace operator. In Sect. 4.3 we give some uniform bounds on the solutions y_h of Eq. (4.7). In Sect. 4.4 we derive explicit rates of convergence for smooth solutions. In Sect. 4.5 we explain how these results yield various convergence results. In Sect. 4.6, we illustrate our theoretical results by numerical experiments.

4.2 The Laplace Operator

In this section, we focus on the convergence of the discrete Laplace operator Δ_h, defined for discrete functions $z_h = (z_{j,h})_{j\in\{1,\ldots,N\}}$ by

$$(\Delta_h z_h)_j = \frac{1}{h^2}(z_{j+1,h} - 2z_{j,h} + z_{j-1,h}), \quad j \in \{1,\ldots,N\}, \text{ with } z_{0,h} = z_{N+1,h} = 0. \tag{4.8}$$

In particular, we give various results that will be used afterwards.

Let us first recall that the operator $-\Delta_h$ is self-adjoint positive definite on $\mathbb{R}^N$ according to the analysis done in Sect. 2.2. Besides, its eigenvectors w^k and eigenvalues $\lambda_k(h) = \mu_k(h)^2$ are explicit; the k-th eigenvector $w^k(x) = \sqrt{2}\sin(k\pi x)$ is independent of $h > 0$ and $\mu_k(h) = 2\sin(k\pi h/2)/h$.

4.2.1 Natural Functional Spaces

In this section, we focus on the case of "natural" functional spaces, i.e., in our case $H_0^1(0,1)$, $L^2(0,1)$, and $H^{-1}(0,1)$.

As already mentioned, we have the following:

Proposition 4.1. *If f_h is a discrete function, then there exists a constant C independent of $h \in (0,1)$ such that*

$$\frac{1}{C}\|f_h\|_{H^{-1}} \le \left\|(-\Delta_h)^{-1} f_h\right\|_{H_0^1} \le C\|f_h\|_{H^{-1}}. \tag{4.9}$$

To simplify notations, for $f \in H^{-1}(0,1)$, we shall often denote by $(-\partial_{xx})^{-1} f$ the solution $z \in H_0^1(0,1)$ of

$$-\partial_{xx} z = f \quad \text{on } (0,1), \qquad z(0) = z(1) = 0.$$

Proof. Since f_h is a discrete function, it can be expanded in Fourier series as follows:

$$f_h = \sum_{k=1}^{N} f_k w^k.$$

Then the expansions of $z = (-\partial_{xx})^{-1} f_h$ and $z_h = (-\Delta_h)^{-1} f_h$ are known:

$$z = \sum_{k=1}^{N} \frac{f_k}{\mu_k^2} w^k, \qquad z_h = \sum_{k=1}^{N} \frac{f_k}{\mu_k(h)^2} w^k.$$

Hence

$$\|z\|_{H_0^1}^2 = \sum_{k=1}^{N} \frac{|f_k|^2}{\mu_k^2}, \qquad \|z_h\|_{H_0^1}^2 = \sum_{k=1}^{N} \frac{|f_k|^2}{\mu_k^2} \frac{\mu_k^4}{\mu_k(h)^4}.$$

Since for all $k \in \{1, \ldots, N\}$,

$$1 \leq \frac{\mu_k^4}{\mu_k(h)^4} \leq \frac{\pi^4}{16},$$

we easily get Proposition 4.1. □

We now prove the following convergence result:

Theorem 4.1. *Let $f \in L^2(0,1)$ and expand it in Fourier series as*

$$f = \sum_{k=1}^{\infty} f_k w^k, \tag{4.10}$$

and set

$$f_h = \sum_{k=1}^{N} f_k w^k. \tag{4.11}$$

Let then z be the solution of

$$-\partial_{xx} z = f, \text{ on } (0,1), \qquad z(0) = z(1) = 0, \tag{4.12}$$

and z_h of

$$-(\Delta_h z_h)_j = f_{j,h}, \quad j \in \{1, \ldots, N\}. \tag{4.13}$$

Then

$$\|f - f_h\|_{H^{-1}} + \|z - z_h\|_{H_0^1} \leq Ch \|f\|_{L^2} \tag{4.14}$$

$$\|z - z_h\|_{L^2} \leq Ch^2 \|f\|_{L^2}. \tag{4.15}$$

Remark 4.1. Of course, Theorem 4.1 is very classical and can be found for many different discretization schemes and in particular for finite-element methods; see for instance the textbook [46].

Proof. Our proof is of course based on the fact that the functions w^k are eigenvectors of both the continuous and discrete Laplace operators. Note that it is straightforward to check that

$$\|f - f_h\|_{H^{-1}} \leq Ch \|f\|_{L^2}.$$

We thus focus on the comparison between z and z_h. Again, we use the fact that the expansions of z and z_h in Fourier are explicit:

$$z = \sum_{k=1}^{\infty} \frac{f_k}{\mu_k^2} w^k, \qquad z_h = \sum_{k=1}^{N} \frac{f_k}{\mu_k(h)^2} w^k. \tag{4.16}$$

Now, computing the H_0^1-norm of $z - z_h$ is easy:

$$\begin{aligned} \|z - z_h\|_{H_0^1}^2 &= \sum_{k=1}^{N} \frac{|f_k|^2}{\mu_k^2} \left(1 - \frac{\mu_k^2}{\mu_k(h)^2}\right)^2 + \sum_{k=N+1}^{\infty} \frac{|f_k|^2}{\mu_k^2} \\ &\le C \sum_{k=1}^{N} |f_k|^2 k^2 h^4 + \frac{1}{N^2} \sum_{k=N+1}^{\infty} |f_k|^2, \end{aligned}$$

where we have used that

$$\frac{1}{\mu_k^2} \left(1 - \frac{\mu_k^2}{\mu_k(h)^2}\right)^2 \le C k^2 h^4, \quad \forall k \in \{1, \ldots, N\}. \tag{4.17}$$

Hence

$$\|z - z_h\|_{H_0^1}^2 \le C \left(N^2 h^4 + \frac{1}{N^2}\right) \|f\|_{L^2}^2 .$$

Since $N + 1 = 1/h$, this concludes the proof of Eq. (4.14).

Similarly, one derives

$$\|z - z_h\|_{L^2}^2 \le C \left(h^4 + \frac{1}{N^4}\right) \|f\|_{L^2}^2 ,$$

which immediately implies Eq. (4.15). □

From Proposition 4.1 and Theorem 4.1 we deduce:

Theorem 4.2. *Let $f \in H^{-1}(0,1)$ and f_h be a sequence of discrete functions such that*

$$\lim_{h \to 0} \|f - f_h\|_{H^{-1}} = 0.$$

Then

$$\lim_{h \to 0} \left\|(-\partial_{xx})^{-1} f - (-\Delta_h)^{-1} f_h\right\|_{H_0^1} = 0. \tag{4.18}$$

Besides, if $f \in L^2(0,1)$ and f_h satisfies, for some $\theta > 0$,

$$\|f - f_h\|_{H^{-1}} \le C_0 h^{\theta},$$

then

$$\left\|(-\partial_{xx})^{-1} f - (-\Delta_h)^{-1} f_h\right\|_{H_0^1} \le C \left(h \|f\|_{L^2} + C_0 h^{\theta}\right). \tag{4.19}$$

Proof. The first part of Theorem 4.2 easily follows by the density of $L^2(0,1)$ functions in $H^{-1}(0,1)$, the uniform stability result of Proposition 4.1 and the convergence result of Theorem 4.1, similarly as in the proof of Proposition 3.5. The details are left to the reader.

The second part of Theorem 4.2 consists of taking $\tilde{f}_h$ as in Eq. (4.11), for which we have

$$\left\|f-\tilde{f}_h\right\|_{H^{-1}} \leq Ch\|f\|_{L^2} \quad \text{and} \quad \left\|(-\Delta_h)^{-1}\tilde{f}_h-(-\partial_{xx})^{-1}f\right\|_{H_0^1} \leq Ch\|f\|_{L^2}.$$

Then Proposition 4.1 implies that

$$\left\|(-\Delta_h)^{-1}f_h-(-\Delta_h)^{-1}\tilde{f}_h\right\|_{H_0^1} \leq C\left\|f_h-\tilde{f}_h\right\|_{H^{-1}}.$$

Of course, these three last estimates imply Eq. (4.19). □

Finally, we mention this last result:

Theorem 4.3. *Let $f \in L^2(0,1)$ and $z=(-\partial_{xx})^{-1}f$. Then there exists C such that*

$$|\partial_x z(1)|^2 \leq C\|f\|_{L^2}\|f\|_{H^{-1}}. \tag{4.20}$$

Similarly, there exists $C>0$ such that for all $h \in (0,1)$, if f_h is a discrete function and $z_h=(-\Delta_h)^{-1}f_h$, we have

$$\left|\frac{z_{N,h}}{h}\right|^2 \leq C\|f_h\|_{L^2}\|f_h\|_{H^{-1}}. \tag{4.21}$$

Besides, taking f_h as in Eq. (4.11)*, we have*

$$\left|\partial_x z(1)+\frac{z_{N,h}}{h}\right| \leq C\sqrt{h}\|f\|_{L^2}. \tag{4.22}$$

Proof. We prove this result using the multiplier technique. Since $-\partial_{xx}z=f$, multiplying the equation by $x\partial_x z$, easy integrations by parts show

$$|\partial_x z(1)|^2=-2\int_0^1 fx\partial_x z+\int_0^1|\partial_x z|^2.$$

Of course, this implies Eq. (4.20) from the fact that $\|z\|_{H_0^1}=\|f\|_{H^{-1}}$.

In order to prove estimate (4.21), we develop a similar multiplier argument. Namely, we multiply the equation

$$-(\Delta_h z_h)_j=f_{j,h}, \quad j \in \{1,\ldots,N\},$$

by $j(z_{j+1,h}-z_{j-1,h})$. We thus obtain

$$\left|\frac{z_{N,h}}{h}\right|^2=-2h\sum_{j=1}^N jh\left(\frac{z_{j+1,h}-z_{j-1,h}}{h}\right)f_{j,h}+h\sum_{j=0}^N\left(\frac{z_{j+1,h}-z_{j,h}}{h}\right)^2.$$

Hence

$$\left|\frac{z_{N,h}}{h}\right|^2 \le C\|f_h\|_{L^2}\|z_h\|_{H_0^1} + C\|z_h\|_{H_0^1}^2 \le C\|f_h\|_{L^2}\|f_h\|_{H^{-1}} + C\|f_h\|_{H^{-1}}^2,$$

which yields estimate (4.21).

We now aim at proving Eq. (4.22). First remark that z_h also solves

$$-\partial_{xx}z_h = \tilde{f}_h, \text{ on } (0,1), \quad z_h(0) = z_h(1) = 0,$$

where

$$\tilde{f}_h = \sum_{j=1}^{N} f_k \left(\frac{\mu_k}{\mu_k(h)}\right)^2 w^k. \tag{4.23}$$

But one easily has

$$\left\|\tilde{f}_h\right\|_{L^2} \le C\|f\|_{L^2}, \quad \left\|\tilde{f}_h - f\right\|_{H^{-1}} \le Ch\|f\|_{L^2}. \tag{4.24}$$

Indeed, from Eq. (4.17),

$$\|\tilde{f}_h - f_h\|_{H^{-1}}^2 = \sum_{k=1}^{N} \frac{|f_k|^2}{\mu_k^2}\left(1 - \left(\frac{\mu_k}{\mu_k(h)}\right)^2\right)^2 \le Ch^2\|f\|_{L^2}^2,$$

and thus Eq. (4.14) yields Eq. (4.24).

Therefore, using Eq. (4.21),

$$|\partial_x z(1) - \partial_x z_h(1)| \le C\left(\left\|f - \tilde{f}_h\right\|_{L^2}\left\|f - \tilde{f}_h\right\|_{H^{-1}}\right)^{1/2} \le C\sqrt{h}\|f\|_{L^2}. \tag{4.25}$$

Besides,

$$\partial_x z_h(1) + \frac{z_{N,h}}{h} = \sum_{k=1}^{N} \frac{f_k}{\mu_k(h)^2}(-1)^k\left(1 - \frac{\sin(k\pi h)}{k\pi h}\right)k\pi.$$

Note that this last expression coincides with the continuous normal derivative $\partial_x\tilde{z}(1)$ of the solution $\tilde{z}$ of the continuous problem

$$\begin{cases} -\partial_{xx}\tilde{z} = \tilde{g}_h, \text{ on } (0,1), \text{ where } \tilde{g}_h = \displaystyle\sum_{k=1}^{N} f_k \frac{\mu_k^2}{\mu_k(h)^2}\left(1 - \frac{\sin(k\pi h)}{k\pi h}\right)w^k, \\ \tilde{z}(0) = \tilde{z}(1) = 0. \end{cases} \tag{4.26}$$

Using that for some constant C independent of h and $k \in \{1, \dots, N\}$,

$$\left|\frac{\mu_k^2}{\mu_k(h)^2}\right| \le C, \qquad \left|1 - \frac{\sin(k\pi h)}{k\pi h}\right| \le Ck^2h^2,$$

we easily compute

$$\|\tilde{g}_h\|_{L^2} \le C\|f\|_{L^2}, \qquad \|\tilde{g}_h\|_{H^{-1}} \le Ch\|f\|_{L^2}. \tag{4.27}$$

Hence, from Eq. (4.20),

$$\left|\partial_x z_h(1) + \frac{z_{N,h}}{h}\right| = |\partial_x \tilde{z}(1)| \le C\sqrt{h}\|f\|_{L^2}.$$

Together with Eq. (4.25), this concludes the proof of Theorem 4.3. □

4.2.2 Stronger Norms

Recalling the definition of the functional spaces $H^{\ell}_{(0)}(0,1)$ in Eq. (3.34), we prove the counterparts of the above theorem within these spaces.

First, Proposition 4.1 can be modified into:

Proposition 4.2. *Let $\ell \in \mathbb{R}$. If f_h is a discrete function, then there exists a constant $C = C(\ell)$ independent of $h \in (0,1)$ such that*

$$\frac{1}{C}\|f_h\|_{H^{\ell}_{(0)}} \le \left\|(-\Delta_h)^{-1} f_h\right\|_{H^{\ell-2}_{(0)}} \le C\|f_h\|_{H^{\ell}_{(0)}}. \tag{4.28}$$

The proof of Proposition 4.2 follows line to line the one of Proposition 4.1 and is left to the reader.

The convergence results of Theorem 4.1 can be extended as follows:

Theorem 4.4. *Let $\ell \in \mathbb{R}$ and $f \in H^{\ell}_{(0)}(0,1)$ and $z = (-\partial_{xx})^{-1} f$ be the corresponding solution of the Laplace equation (4.12). With the notations of Theorem 4.1, setting f_h as in Eq. (4.11) and $z_h = (-\Delta_h)^{-1} f_h$, we have*

$$\|f - f_h\|_{H^{\ell-1}_{(0)}} + \|z - z_h\|_{H^{\ell+1}_{(0)}} \le Ch\|f\|_{H^{\ell}_{(0)}}, \tag{4.29}$$

$$\|z - z_h\|_{H^{\ell}_{(0)}} \le Ch^2\|f\|_{H^{\ell}_{(0)}}. \tag{4.30}$$

Here again, the proof of Theorem 4.4 is very similar to the one of Theorem 4.1 and is left to the reader.

We now focus on the convergence of the normal derivatives:

Theorem 4.5. *Let $\ell \ge 0$ and $f \in H^{\ell}_{(0)}(0,1)$ and $z = (-\partial_{xx})^{-1} f$ be the corresponding solution of the Laplace equation (4.12). With the notations of Theorem 4.1, setting f_h as in Eq. (4.11) and $z_h = (-\Delta_h)^{-1} f_h$, we have*

$$\left|\partial_x z(1) + \frac{z_{N,h}}{h}\right| \le Ch^{\min\{\ell+1/2,\ell/2+1,2\}}\|f\|_{H^{\ell}_{(0)}}. \tag{4.31}$$

Proof. The proof of Eq. (4.31) follows the one of Eq. (4.22), except for the estimates (4.24) on $\tilde{f}_h$ in Eqs. (4.23) and (4.27) on $\tilde{g}_h$ defined in Eq. (4.26).

Using that for all $h > 0$ and $k \in \{1, \dots, N\}$,

$$\left(1 - \left(\frac{\mu_k}{\mu_k(h)}\right)^2\right)^2 \leq Ck^4 h^4,$$

we easily derive that

$$\left\|f - \tilde{f}_h\right\|_{L^2}^2 \leq C\left(\frac{1}{N^{2\ell}} + Ch^4 \max\{1, N^{4-2\ell}\}\right) \|f\|_{H^\ell_{(0)}}^2 .$$

In particular, if $\ell \in (0,2]$, $\left\|f - \tilde{f}_h\right\|_{L^2} \leq Ch^\ell \|f\|_{H^\ell_{(0)}}$ and if $\ell \geq 2$, $\left\|f - \tilde{f}_h\right\|_{L^2} \leq Ch^2 \|f\|_{H^\ell_{(0)}}$, thus yielding

$$\left\|f - \tilde{f}_h\right\|_{L^2} \leq Ch^{\min\{\ell, 2\}} \|f\|_{H^\ell_{(0)}} .$$

Similarly,

$$\left\|f - \tilde{f}_h\right\|_{H^{-1}} \leq Ch^{\min\{\ell+1, 2\}} \|f\|_{H^\ell_{(0)}} .$$

We thus obtain, instead of Eq. (4.25),

$$|\partial_x z(1) - \partial_x z_h(1)| \leq Ch^{\min\{\ell+1/2, \ell/2+1, 2\}} \|f\|_{H^\ell_{(0)}} .$$

Estimates on $\partial_x z_h(1) + z_{N,h}/h$ can be deduced similarly from estimates on $\tilde{g}_h$ (defined in Eq. (4.26)) and are left to the reader. $\square$

Remark 4.2. Very likely, estimate (4.31) can be improved for $\ell > -1/2$ into

$$\left|\partial_x z(1) + \frac{z_{N,h}}{h}\right| \leq Ch^{\min\{\ell+1/2, 2\}} \|f\|_{H^\ell_{(0)}} . \tag{4.32}$$

For instance, using that, if $f = \sum_k f_k w^k$, the solution z of Eq. (4.12) can be expanded as $z = \sum_k f_k/\mu_k^2 w^k$ and we get

$$\partial_x z(1) = \sum_k f_k \frac{\partial_x w^k(1)}{\mu_k^2},$$

provided the sum converges. Since for all $k \in \mathbb{N}$,

$$\left|\frac{\partial_x w^k(1)}{\mu_k^2}\right| \leq \frac{C}{\mu_k},$$

by Cauchy–Schwarz, for any $\ell_0 > -1/2$, we obtain

$$|\partial_x z(1)| \leq C_{\ell_0} \|f\|_{H^{\ell_0}_{(0)}}$$

instead of Eq. (4.20).

Of course, we can get similar estimates for the discrete solutions $z_h = (-\Delta_h)^{-1} f_h$ and obtain, for all $\ell_{(0)} > -1/2$, a constant C_{ℓ_0} independent of $h > 0$ such that for all discrete function f_h and $z_h = (-\Delta_h)^{-1} f_h$,

$$\left| \frac{z_{N,h}}{h} \right| \leq C_{\ell_0} \|f_h\|_{H^{\ell_0}_{(0)}} .$$

instead of Eq. (4.21).

Using these two estimates instead of Eqs. (4.20) and (4.21) and following the proof of Theorem 4.5, we can obtain the following result: for all $\ell > -1/2$ and $\varepsilon > 0$, there exists a constant $C_{\ell,\varepsilon} = C(\ell,\varepsilon)$ such that $f \in H^{\ell}_{(0)}$,

$$\left| \partial_x z(1) + \frac{z_{N,h}}{h} \right| \leq C_{\ell,\varepsilon} h^{\min\{\ell+1/2-\varepsilon,2\}} \|f\|_{H^{\ell}_{(0)}} . \tag{4.33}$$

This last estimate is better than Eq. (4.31) when $\ell \in (-1/2, 0)$ and when $\ell \in (1,2)$.

4.2.3 Numerical Results

This section aims at giving numerical simulations and evidences of the convergence results Eq. (4.31) for the normal derivatives of solutions of the discrete Laplace equation. We do not present a systematic study of the convergence of the solution in $L^2(0,1)$ nor in $H^1_0(0,1)$ since these results are classical and can be found in many textbooks of numerical analysis; see, e.g., [4, 46].

In order to do that, we choose continuous functions f and z solving Eq. (4.12).

For $N \in \mathbb{N}$, we then discretize the source term f into f_h simply by taking $f_h(j) = f(jh)$ for $j \in \{1,\ldots,N\}$ and compute z_h the solution of $-\Delta_h z_h = f_h$ with $z_{0,h} = z_{N+1,h} = 0$. We then compute $\partial_x z(1) + z_{N,h}/h$.

Our first test function is

$$f(x) = -\sin(2\pi x) + 3\sin(\pi x), \text{ for } z(x) = \frac{\sin(2\pi x)}{4\pi^2} - \frac{3\sin(\pi x)}{\pi}. \tag{4.34}$$

The plot of $\left|\partial_x z(1) + z_{N,h}/h\right|$ versus N is represented in logarithmic scales in Fig. 4.1, left. Here, we have chosen $N \in [100, 300]$. The slope of the linear regression is -1.99 and completely corresponds to the result of Theorem 4.5.

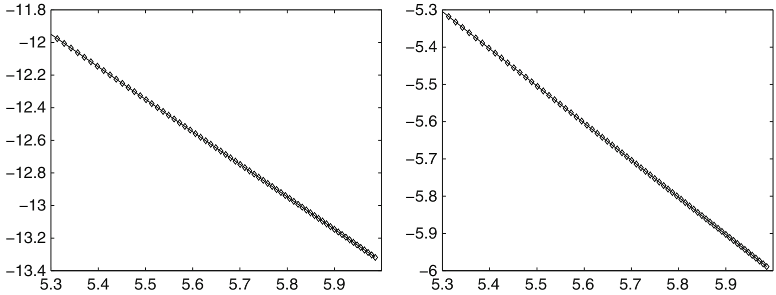

Fig. 4.1 Plot of $|\partial_x z(1) + z_{N,h}/h|$ versus N in logarithmic scales. *Left*, for f as in Eq. (4.34), the slope is -1.99. *Right*, for f as in Eq. (4.35), the slope is -1.00.

We then test

$$f(x) = \frac{1}{(x+1)^3}, \text{ corresponding to } z(x) = -\frac{1}{2(x+1)} + \frac{1}{2} - \frac{x}{4}. \tag{4.35}$$

Numerical simulations are represented in Fig. 4.1, right.

This function f is smooth, but it does not satisfy $f(0) = f(1) = 0$. Thus it is only in $\cap_{\varepsilon>0} H^{1/2-\varepsilon}_{(0)}(0,1)$ and the slope predicted by Theorem 4.5 is -1^- and completely agrees with the slope observed in Fig. 4.1 right.

These two examples indicate that the rates of convergence of the normal derivatives obtained in Theorem 4.5 are accurate.

4.3 Uniform Bounds on y_h

The goal of this section is to obtain uniform bounds on y_h in the natural space for the wave equation with nonhomogeneous Dirichlet control, that is $C([0,T];L^2(0,1)) \cap C^1([0,T];H^{-1}(0,1))$:

Theorem 4.6. *There exists a constant C independent of $h > 0$ such that any solution y_h of Eq.* (4.7) *with initial data (y_h^0, y_h^1) and source term $v_h \in L^2(0,T)$ satisfies*

$$\sup_{t\in[0,T]} \|(y_h(t), \partial_t y_h(t))\|_{L^2(0,1)\times H^{-1}(0,1)} \leq C\left(\left\|(y_h^0, y_h^1)\right\|_{L^2(0,1)\times H^{-1}(0,1)} + \|v_h\|_{L^2(0,T)}\right). \tag{4.36}$$

The proof of Theorem 4.6 is done in two steps: one focusing on the estimate on y_h and the other one on $\partial_t y_h$, respectively, corresponding to Propositions 4.3 and 4.4.

As we will see, each one of these propositions is based on a suitable duality argument for solutions of the adjoint system.

4.3.1 Estimates in $C([0,T]; L^2(0,1))$

We have the following:

Proposition 4.3. *There exists a constant C independent of $h>0$ such that any solution y_h of Eq. (4.7) satisfies*

$$\|y_h\|_{L^\infty(0,T;L^2(0,1))} \leq C\left(\left\|y_h^0\right\|_{L^2(0,1)} + \left\|y_h^1\right\|_{H^{-1}(0,1)} + \|v_h\|_{L^2(0,T)}\right). \tag{4.37}$$

We postpone the proof to the end of the section. As in the continuous case, Proposition 4.3 will be a consequence of a suitable duality argument.

Namely, let $f_h \in L^1(0,T;L^2(0,1))$ and define ϕ_h as being the solution of

$$\begin{cases} \partial_{tt}\phi_{j,h} - \dfrac{1}{h^2}\left[\phi_{j+1,h} + \phi_{j-1,h} - 2\phi_{j,h}\right] = f_{j,h}, & \\ & (t,j) \in (0,T)\times\{1,\ldots,N\}, \\ \phi_{0,h}(t) = \phi_{N+1,h}(t) = 0, & t \in (0,T), \\ \phi_{j,h}(T) = 0,\ \partial_t\phi_{j,h}(T) = 0, & j = 1,\ldots,N. \end{cases} \tag{4.38}$$

Then, multiplying Eq. (4.7) by ϕ_h solution of Eq. (4.38), we obtain

$$\begin{aligned} 0 &= h\sum_{j=1}^{N}\int_0^T \partial_{tt}y_{j,h}\phi_{j,h}\,\mathrm{d}t - h\sum_{j=1}^{N}\int_0^T \frac{1}{h^2}[y_{j+1,h} + y_{j-1,h} - 2y_{j,h}]\phi_{j,h}\,\mathrm{d}t \\ &= h\sum_{j=1}^{N}\int_0^T y_{j,h}\partial_{tt}\phi_{j,h}\,\mathrm{d}t - h\sum_{j=1}^{N}\int_0^T \frac{1}{h^2}y_{j,h}[\phi_{j+1,h} + \phi_{j-1,h} - 2\phi_{j,h}]\,\mathrm{d}t \\ &\quad + h\sum_{j=1}^{N}(\partial_t y_{j,h}\phi_{j,h} - y_{j,h}\partial_t\phi_{j,h})\Big|_0^T - \int_0^T y_{N+1,h}\frac{\phi_{N,h}}{h}\,\mathrm{d}t \\ &= h\sum_{j=1}^{N}\int_0^T y_{j,h}f_{j,h}\,\mathrm{d}t + h\sum_{j=1}^{N}(y_{j,h}^0\partial_t\phi_{j,h}(0) - y_{j,h}^1\phi_{j,h}(0)) \\ &\quad - \int_0^T v_h(t)\frac{\phi_{N,h}(t)}{h}\,\mathrm{d}t. \end{aligned} \tag{4.39}$$

Note that identity (4.39) is a discrete counterpart of the continuous identity (4.4). Remark that this can be used as a definition of solutions of Eq. (4.7) by transposition, even if in that case, solutions of Eq. (4.7) obviously exist due to the finite dimensional nature of system (4.7).

Formulation (4.39) will be used to derive estimates on solutions y_h by duality.

But we shall first prove the following lemma:

Lemma 4.1. *For ϕ_h solution of Eq. (4.38), there exists a constant C independent of $h>0$ such that*

$$\|\phi_h\|_{L^\infty(0,T;H_0^1(0,1))} + \|\partial_t\phi_h\|_{L^\infty(0,T;L^2(0,1))} \leq C\|f_h\|_{L^1(0,T;L^2(0,1))} \tag{4.40}$$

and

$$\left\| \frac{\phi_{N,h}}{h} \right\|_{L^2(0,T)} \leq C \|f_h\|_{L^1(0,T;L^2(0,1))} . \tag{4.41}$$

Proof. The first inequality (4.40) is an energy estimate, whereas Eq. (4.41) is a hidden regularity property.

Multiplying Eq. (4.38) by $\partial_t \phi_{j,h}$ and summing over j, we obtain

$$\begin{aligned} & h \sum_{j=1}^{N} \partial_{tt} \phi_{j,h} \partial_t \phi_{j,h} - h \sum_{j=1}^{N} \frac{1}{h^2} \left[\phi_{j+1,h} + \phi_{j-1,h} - 2\phi_{j,h} \right] \partial_t \phi_{j,h} \\ & \quad = h \sum_{j=1}^{N} f_{j,h} \partial_t \phi_{j,h}. \end{aligned} \tag{4.42}$$

The left-hand side of Eq. (4.42) is the derivative of the energy

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{h}{2} \sum_{j=1}^{N} |\partial_t \phi_{j,h}|^2 + \frac{h}{2} \sum_{j=1}^{N} \left(\frac{\phi_{j+1,h} - \phi_{j,h}}{h} \right)^2 \right) = \frac{1}{2} \frac{\mathrm{d}E_h[\phi_h]}{\mathrm{d}t},$$

whereas the right-hand side satisfies

$$\begin{aligned} \left| h \sum_{j=1}^{N} f_{j,h} \partial_t \phi_{j,h} \right| & \leq \left(h \sum_{j=1}^{N} |f_{j,h}|^2 \right)^{1/2} \left(h \sum_{j=1}^{N} |\partial_t \phi_{j,h}|^2 \right)^{1/2} \\ & \leq \left(h \sum_{j=1}^{N} |f_{j,h}|^2 \right)^{1/2} \sqrt{E_h[\phi_h](t)}. \end{aligned}$$

Equation (4.42) then implies

$$\left| \frac{\mathrm{d}\sqrt{E_h}}{\mathrm{d}t}(t) \right| \leq \left(h \sum_{j=1}^{N} |f_{j,h}(t)|^2 \right)^{1/2} . \tag{4.43}$$

Integrating in time, we obtain that for all $t \in [0,T]$,

$$\sqrt{E_h(t)} \leq \int_0^T \left(h \sum_{j=1}^{N} |f_{j,h}(t)|^2 \right)^{1/2} \mathrm{d}t.$$

Finally, recalling the properties of the Fourier extension operator in Sect. 3.2, we obtain Eq. (4.40).

Estimate (4.41) can be deduced from the multiplier approach developed in the proof of Theorem 2.2 by multiplying Eq. (4.38) by $j(\phi_{j+1,h} - \phi_{j-1,h})$:

$$\begin{aligned} &h\sum_{j=1}^{N}\int_0^T f_{j,h}\, jh\left(\frac{\phi_{j+1,h}-\phi_{j-1,h}}{h}\right)\mathrm{d}t \\ &\quad = h\sum_{j=1}^{N}\int_0^T \partial_{tt}\phi_{j,h}\, jh\left(\frac{\phi_{j+1,h}-\phi_{j-1,h}}{h}\right)\mathrm{d}t \\ &\quad\quad -h\sum_{j=1}^{N}\int_0^T \left[\frac{\phi_{j+1,h}+\phi_{j-1,h}-2\phi_{j,h}}{h^2}\right] jh\left(\frac{\phi_{j+1,h}-\phi_{j-1,h}}{h}\right)\mathrm{d}t. \end{aligned} \tag{4.44}$$

The right-hand side of Eq. (4.44) has already been dealt with in the proof of Theorem 2.2 and yields

$$\begin{aligned} &h\sum_{j=1}^{N}\int_0^T \partial_{tt}\phi_{j,h}\, jh\left(\frac{\phi_{j+1,h}-\phi_{j-1,h}}{h}\right)\mathrm{d}t \\ &\quad -h\sum_{j=1}^{N}\int_0^T \left[\frac{\phi_{j+1,h}+\phi_{j-1,h}-2\phi_{j,h}}{h^2}\right] jh\left(\frac{\phi_{j+1,h}-\phi_{j-1,h}}{h}\right) \\ &= \int_0^T \left|\frac{\phi_{N,h}(t)}{h}\right|^2 \mathrm{d}t + \frac{h^3}{2}\sum_{j=0}^{N}\int_0^T \left|\frac{\partial_t\phi_{j+1,h}-\partial_t\phi_{j,h}}{h}\right|^2 \mathrm{d}t \\ &\quad -\int_0^T E_h(t)\,\mathrm{d}t - X_h(t)\Big|_0^T, \end{aligned}$$

where, similarly as in Eq. (2.14), $X_h(t)$ is given by

$$X_h(t) = 2h\sum_{j=1}^{N} jh\left(\frac{\phi_{j+1,h}-\phi_{j-1,h}}{2h}\right)\partial_t\phi_{j,h}.$$

From the conditions $\phi_h(T) = \partial_t\phi_h(T) = 0$ in Eq. (4.38), $X_h(T) = 0$. Besides, as in Eq. (2.15), one has $|X_h(0)| \le E_h(0)$.

On the other hand,

$$\begin{aligned} &\left|h\sum_{j=1}^{N}\int_0^T f_{j,h}\, jh\left(\frac{\phi_{j+1,h}-\phi_{j-1,h}}{h}\right)\mathrm{d}t\right| \\ &\quad \le \int_0^T \left(h\sum_{j=1}^{N}|f_{j,h}|^2\right)^{1/2}\sqrt{E_h(t)}\,\mathrm{d}t \\ &\quad \le \sup_{t\in[0,T]}\left\{\sqrt{E_h(t)}\right\}\int_0^T \left(h\sum_{j=1}^{N}|f_{j,h}|^2\right)^{1/2}\mathrm{d}t. \end{aligned}$$

Therefore, from Eq. (4.40), there exists a constant independent of h such that

$$\int_0^T \left|\frac{\phi_{N,h}(t)}{h}\right|^2 \mathrm{d}t + \frac{h^3}{2}\sum_{j=0}^{N}\int_0^T \left|\frac{\partial_t\phi_{j+1,h}-\partial_t\phi_{j,h}}{h}\right|^2 \mathrm{d}t$$
$$\leq C\left(\int_0^T\left(h\sum_{j=1}^{N}|f_{j,h}|^2\right)^{1/2}\mathrm{d}t\right)^2,$$

which implies Eq. (4.41). □

Proof (Proposition 4.3). Lemma 4.1 and identity (4.39) allow us to deduce bounds on y_h. Indeed,

$$\|y_h\|_{L^\infty(0,T;L^2(0,1))} = \sup_{\substack{f\in L^1(0,T;L^2(0,1))\\ \|f\|_{L^1((0,T);L^2(0,1))}}} \int_0^1 y_h(x)f(x)\,\mathrm{d}x.$$

But there y_h is the Fourier extension $\mathbb{F}_h(y_h)$ (recall Sect. 3.2); hence it involves only Fourier modes smaller than N. We thus only have to consider the projection of f onto the first N Fourier modes. But this exactly corresponds to discrete functions f_h. Therefore,

$$\|y_h\|_{L^\infty(0,T;L^2(0,1))} = \sup_{\substack{f_h\in L^1(0,T;L^2(0,1))\\ \|f_h\|_{L^1((0,T);L^2(0,1))}\leq 1}} \left\{h\sum_{j=1}^{N}\int_0^T y_{j,h}f_{j,h}\,\mathrm{d}t\right\}.$$

But, introducing ϕ_h, the solution of Eq. (4.38) with source term f_h, using Lemma 4.1, we obtain:

$$\begin{aligned}
h\sum_{j=1}^{N}\int_0^T y_{j,h}f_{j,h}\,\mathrm{d}t &= -h\sum_{j=1}^{N}(y^0_{j,h}\partial_t\phi_{j,h}(0)-y^1_{j,h}\phi_{j,h}(0)) + \int_0^T v_h(t)\frac{\phi_{N,h}(t)}{h}\,\mathrm{d}t\\
&\leq C\left\|y^0_h\right\|_{L^2(0,1)}\|\partial_t\phi_h(0)\|_{L^2(0,1)} + C\left\|y^1_h\right\|_{H^{-1}(0,1)}\|\phi_h(0)\|_{H^1_0(0,1)}\\
&\quad + \|v_h\|_{L^2(0,T)}\left\|\frac{\phi_{N,h}}{h}\right\|_{L^2(0,T)}\\
&\leq C\left(\left\|y^0_h\right\|_{L^2(0,1)} + \left\|y^1_h\right\|_{H^{-1}(0,1)} + \|v_h\|_{L^2(0,T)}\right)\|f_h\|_{L^1(0,T;L^2(0,1))}.
\end{aligned}$$

This yields in particular Eq. (4.37). □

4.3.2 Estimates on $\partial_t y_h$

We now focus on getting estimates on $\partial_t y_h$.

Proposition 4.4. *There exists a constant C independent of $h>0$ such that any solution y_h of Eq.* (4.7) *satisfies*

$$\|\partial_t y_h\|_{L^\infty(0,T;H^{-1}(0,1))} \leq C\left(\left\|y_h^0\right\|_{L^2(0,1)} + \left\|y_h^1\right\|_{H^{-1}(0,1)} + \|v_h\|_{L^2(0,T)}\right). \tag{4.45}$$

Similarly as for Proposition 4.3, this result is obtained by duality, based on the following identity: if ϕ_h solves the adjoint wave equation (4.38) with source term $f_h = \partial_t g_h$ with $g_h \in L^1(0,T;H_0^1(0,1))$, we have:

$$\begin{aligned} h\sum_{j=1}^N \int_0^T y_{j,h}\partial_t g_{j,h}\,\mathrm{d}t &= -h\sum_{j=1}^N (y_{j,h}^0 \partial_t \phi_{j,h}(0) - y_{j,h}^1 \phi_{j,h}(0)) \\ &\quad + \int_0^T v_h(t)\frac{\phi_{N,h}(t)}{h}\,\mathrm{d}t. \end{aligned} \tag{4.46}$$

The proof of Proposition 4.4 is sketched at the end of the section, since it is very similar to the one of Proposition 4.3.

Hence, we focus on the following adjoint problem:

$$\begin{cases} \partial_{tt}\phi_{j,h} - \dfrac{1}{h^2}\left[\phi_{j+1,h} + \phi_{j-1,h} - 2\phi_{j,h}\right] = \partial_t g_{j,h}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,j) \in (0,T)\times\{1,\ldots,N\}, \\ \phi_{0,h}(t) = \phi_{N+1,h}(t) = 0, \qquad\qquad\quad t\in(0,T), \\ \phi_{j,h}(T) = 0,\ \partial_t\phi_{j,0}(T) = 0, \qquad\quad j = 1,\ldots,N. \end{cases} \tag{4.47}$$

We shall thus prove the following:

Lemma 4.2. *For ϕ_h solution of Eq. (4.47), there exists a constant C independent of $h > 0$ such that*

$$\|\phi_h\|_{L^\infty(0,T;H_0^1(0,1))} + \|\partial_t\phi_h(0)\|_{L^2(0,1)} \leq C\,\|g_h\|_{L^1(0,T;H_0^1(0,1))} \tag{4.48}$$

and

$$\left\|\frac{\phi_{N,h}}{h}\right\|_{L^2(0,T)} \leq C\,\|g_h\|_{L^1(0,T;H_0^1(0,1))}. \tag{4.49}$$

Proof. To study solutions ϕ_h of Eq. (4.47), it is convenient to first assume that g_h is compactly supported in time in $(0,T)$ and use the density of compactly supported functions in time in $L^1(0,T;H_0^1(0,1))$.

Let us introduce ψ_h satisfying $\partial_t\psi_h = \phi_h$, which satisfies

$$\begin{cases} \partial_{tt}\psi_{j,h} - \dfrac{1}{h^2}\left[\psi_{j+1,h} + \psi_{j-1,h} - 2\psi_{j,h}\right] = g_{j,h}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,j) \in (0,T)\times\{1,\ldots,N\}, \\ \psi_{0,h}(t) = \psi_{N+1,h}(t) = 0, \qquad\qquad\quad t\in(0,T), \\ \psi_{j,h}(T) = 0,\ \partial_t\psi_{j,h}(T) = 0, \qquad\quad j = 1,\ldots,N. \end{cases} \tag{4.50}$$

Obviously, using Lemma 4.1, we immediately obtain

$$\|\psi_h\|_{L^\infty(0,T;H_0^1(0,1))} + \|\partial_t \psi_h\|_{L^\infty(0,T;L^2(0,1))} + \left\| \frac{\psi_{N,h}}{h} \right\|_{L^2(0,T)} \leq C \|g_h\|_{L^1(0,T;L^2(0,1))}$$
$$\leq C \|g_h\|_{L^1(0,T;H_0^1(0,1))}.$$

To derive more precise estimates on ϕ_h, we multiply Eq. (4.50) by $-(\partial_t \psi_{j+1,h} + \partial_t \psi_{j-1,h} - 2\partial_t \psi_{j,h})/h^2$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{h}{2} \sum_{j=0}^{N} \left(\frac{\partial_t \psi_{j+1,h} - \partial_t \psi_{j,h}}{h} \right)^2 + \frac{h}{2} \sum_{j=1}^{N} \left(\frac{\psi_{j+1,h} + \psi_{j-1,h} - 2\psi_{j,h}}{h^2} \right)^2 \right)$$
$$= h \sum_{j=1}^{N} \left(\frac{g_{j+1,h} - g_{j,h}}{h} \right) \left(\frac{\partial_t \psi_{j+1,h} - \partial_t \psi_{j,h}}{h} \right).$$

Arguing as in Eq. (4.43), this allows to conclude that

$$\sup_{t\in[0,T]} \left\{ \frac{h}{2} \sum_{j=0}^{N} \left(\frac{\partial_t \psi_{j+1,h} - \partial_t \psi_{j,h}}{h} \right)^2 + \frac{h}{2} \sum_{j=1}^{N} \left(\frac{\psi_{j+1,h} + \psi_{j-1,h} - 2\psi_{j,h}}{h^2} \right)^2 \right\}$$
$$\leq C \left(\int_0^T \left(h \sum_{j=0}^{N} \left(\frac{g_{j+1,h} - g_{j,h}}{h} \right)^2 \right)^{1/2} \mathrm{d}t \right)^2. \tag{4.51}$$

Using Eq. (4.38) and $\partial_t \psi_h = \phi_h$ and again the equivalences proven in Sect. 3.2, we deduce

$$\|\phi_h\|_{L^\infty(0,T;\, H_0^1(0,1))} + \|\partial_{tt} \psi_h + g_h\|_{L^\infty((0,T);L^2(0,1))} \leq C \|g_h\|_{L^1(0,T;H_0^1(0,1))},$$

where we use the equation of ψ_h. In order to get Eq. (4.48), we only use the fact that $g_h(0) = 0$.

To deduce Eq. (4.49), we need to apply a multiplier technique on the Eq. (4.47) directly.

Multiplying Eq. (4.47) by $j(\phi_{j+1,h} - \phi_{j-1,h})$, we obtain, similarly as in Eq. (2.13),

$$\int_0^T \left| \frac{\phi_{N,h}(t)}{h} \right|^2 \mathrm{d}t + \frac{h^3}{2} \sum_{j=0}^{N} \int_0^T \left| \frac{\partial_t \phi_{j+1,h} - \partial_t \phi_{j,h}}{h} \right|^2 \mathrm{d}t$$
$$= \int_0^T E_h(t)\,\mathrm{d}t - X_h(0) - h \int_0^T \sum_{j=1}^{N} jh \left(\frac{\phi_{j+1,h} - \phi_{j-1,h}}{h} \right) \partial_t g_{j,h}\,\mathrm{d}t, \tag{4.52}$$

where X_h is as in Eq. (2.14). To derive Eq. (4.49), it is then sufficient to bound each term in the right-hand side of this identity.

First remark that

$$\int_0^T E_h(t)\,\mathrm{d}t = h\int_0^T \sum_{j=0}^N \left(\frac{\phi_{j+1,h}-\phi_{j,h}}{h}\right)^2 \mathrm{d}t + h\int_0^T \sum_{j=0}^N |\partial_t \phi_{j,h}|^2\,\mathrm{d}t$$

$$= h\int_0^T \sum_{j=0}^N \left(\frac{\partial_t \psi_{j+1,h}-\partial_t\psi_{j,h}}{h}\right)^2 \mathrm{d}t + h\int_0^T \sum_{j=0}^N |\partial_{tt}\psi_{j,h}|^2\,\mathrm{d}t$$

$$= h\int_0^T \sum_{j=0}^N \left(\frac{\partial_t \psi_{j+1,h}-\partial_t\psi_{j,h}}{h}\right)^2 \mathrm{d}t + h\int_0^T \sum_{j=1}^N \left(\frac{\psi_{j+1,h}+\psi_{j-1,h}-2\psi_{j,h}}{h^2}\right)^2 \mathrm{d}t$$

$$+h\int_0^T \sum_{j=0}^N g_{j,h}^2\,\mathrm{d}t + 2h\int_0^T \sum_{j=1}^N \left(\frac{\psi_{j+1,h}+\psi_{j-1,h}-2\psi_{j,h}}{h^2}\right) g_{j,h}\,\mathrm{d}t.$$

In particular, from Eq. (4.51), we obtain

$$\left|\int_0^T E_h(t)\,\mathrm{d}t - h\int_0^T \sum_{j=0}^N g_{j,h}^2\,\mathrm{d}t\right| \le C\,\|g\|^2_{L^1(0,T;H_0^1(0,1))}.$$

Let us then bound $X_h(0)$. Since $g_h(0)=0$,

$$X_h(0) = 2h\sum_{j=1}^N jh\left(\frac{\phi_{j+1,h}(0)-\phi_{j-1,h}(0)}{2h}\right)\partial_t\phi_j(0)$$

$$= 2h\sum_{j=1}^N jh\left(\frac{\phi_{j+1,h}(0)-\phi_{j-1,h}(0)}{2h}\right)\partial_{tt}\psi_j(0)$$

$$= 2h\sum_{j=1}^N jh\left(\frac{\phi_{j+1,h}(0)-\phi_{j-1,h}(0)}{2h}\right)\left(\frac{\psi_{j+1,h}(0)+\psi_{j-1,h}(0)-2\psi_{j,h}(0)}{h^2}\right).$$

It follows then from Eq. (4.51) that

$$|X_h(0)| \le C\,\|g_h\|^2_{L^1(0,T;H_0^1(0,1))}.$$

We now deal with the last term in Eq. (4.52):

$$I := 2h\int_0^T \sum_{j=1}^N jh\left(\frac{\phi_{j+1,h}-\phi_{j-1,h}}{2h}\right)\partial_t g_{j,h}\,\mathrm{d}t.$$

Integrating by parts we get

$$I = -h\int_0^T \sum_{j=1}^N \phi_{j,h}\left((j+1)\partial_t g_{j+1,h}-(j-1)\partial_t g_{j-1,h}\right)\mathrm{d}t$$

$$= -h\int_0^T \sum_{j=1}^N \phi_{j,h}\left((\partial_t g_{j-1,h}+\partial_t g_{j+1,h}) + jh\left(\frac{\partial_t g_{j+1,h}-\partial_t g_{j-1,h}}{h}\right)\right)\mathrm{d}t.$$

Taking into account that, by assumption, $g_h(0) = g_h(T) = 0$,

$$I = h \int_0^T \sum_{j=1}^{N} \partial_t \phi_{j,h} \left((g_{j-1,h} + g_{j+1,h}) + jh \left(\frac{g_{j+1,h} - g_{j-1,h}}{h} \right) \right) \mathrm{d}t.$$

But $\partial_t \phi_{j,h} = \partial_{tt} \psi_{j,h}$, and then Eq. (4.50) yields:

$$\begin{aligned} I = {} & h \int_0^T \sum_{j=1}^{N} g_{j,h} \left((g_{j-1,h} + g_{j+1,h}) + jh \left(\frac{g_{j+1,h} - g_{j-1,h}}{h} \right) \right) \mathrm{d}t \\ & + h \int_0^T \sum_{j=1}^{N} \left(\frac{\psi_{j+1,h} + \psi_{j-1,h} - 2\psi_{j,h}}{h^2} \right) (g_{j-1,h} + g_{j+1,h}) \, \mathrm{d}t. \\ & + h \int_0^T \sum_{j=1}^{N} \left(\frac{\psi_{j+1,h} + \psi_{j-1,h} - 2\psi_{j,h}}{h^2} \right) jh \left(\frac{g_{j+1,h} - g_{j-1,h}}{h} \right) \mathrm{d}t. \end{aligned}$$

Since

$$\begin{aligned} & h \int_0^T \sum_{j=1}^{N} g_{j,h} \left((g_{j-1,h} + g_{j+1,h}) + jh \left(\frac{g_{j+1,h} - g_{j-1,h}}{h} \right) \right) \mathrm{d}t \\ & \quad = h \int_0^T \sum_{j=1}^{N} g_{j,h} g_{j+1,h} \, \mathrm{d}t, \end{aligned}$$

due to estimates (4.51), we obtain

$$\left| I - h \int_0^T \sum_{j=1}^{N} g_{j,h} g_{j+1,h} \, \mathrm{d}t \right| \leq C \, \|g\|^2_{L^1(0,T;H_0^1(0,1))}.$$

These estimates, combined with Eq. (4.52), finally give

$$\begin{aligned} & \left| \int_0^T \left| \frac{\phi_{N,h}(t)}{h} \right|^2 \mathrm{d}t + \frac{h^3}{2} \sum_{j=0}^{N} \int_0^T \left| \frac{\partial_t \phi_{j+1,h} - \partial_t \phi_{j,h}}{h} \right|^2 \mathrm{d}t \right. \\ & \qquad \left. - h \int_0^T \sum_{j=1}^{N} \left(|g_{j,h}|^2 - g_{j,h} g_{j+1,h} \right) \mathrm{d}t \right| \leq C \, \|g\|^2_{L^1(0,T;H_0^1(0,1))}, \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \left| \int_0^T \left| \frac{\phi_{N,h}(t)}{h} \right|^2 \mathrm{d}t + \frac{h}{2} \sum_{j=0}^{N} \int_0^T \left| \partial_t \phi_{j+1,h} - \partial_t \phi_{j,h} \right|^2 \mathrm{d}t \right. \\ & \qquad \left. - \frac{h}{2} \int_0^T \sum_{j=0}^{N} |g_{j+1,h} - g_{j,h}|^2 \, \mathrm{d}t \right| \leq C \, \|g\|^2_{L^1(0,T;H_0^1(0,1))}. \end{aligned} \tag{4.53}$$

Remark then that

$$
\begin{aligned}
& h\sum_{j=0}^{N}\int_0^T \left|\partial_t\phi_{j+1,h}-\partial_t\phi_{j,h}\right|^2 \mathrm{d}t - h\int_0^T\sum_{j=0}^{N}|g_{j+1,h}-g_{j,h}|^2\,\mathrm{d}t \\
&\quad = h\sum_{j=0}^{N}\int_0^T \left|\partial_{tt}\psi_{j+1,h}-\partial_{tt}\psi_{j,h}\right|^2 \mathrm{d}t - h\int_0^T\sum_{j=0}^{N}|g_{j+1,h}-g_{j,h}|^2\,\mathrm{d}t \\
&\quad = h\sum_{j=0}^{N}\int_0^T \left(\frac{\psi_{j+2,h}+\psi_{j,h}-2\psi_{j+1,h}}{h^2}-\frac{\psi_{j+1,h}+\psi_{j-1,h}-2\psi_{j,h}}{h^2}\right)^2 \mathrm{d}t \\
&\qquad +2h\sum_{j=0}^{N}\int_0^T \left(\frac{\psi_{j+2,h}+\psi_{j,h}-2\psi_{j+1,h}}{h^2}\right)(g_{j+1,h}-g_{j,h})\,\mathrm{d}t, \\
&\qquad -2h\sum_{j=0}^{N}\int_0^T \left(\frac{\psi_{j+1,h}+\psi_{j-1,h}-2\psi_{j,h}}{h^2}\right)(g_{j+1,h}-g_{j,h})\,\mathrm{d}t,
\end{aligned}
$$

with the notation $\psi_{-1,h}=-\psi_{1,h}$ and $\psi_{N+2,h}=-\psi_{N,h}$.

In view of Eq. (4.51), we have

$$
\begin{aligned}
& \left|h\sum_{j=0}^{N}\int_0^T \left|\partial_t\phi_{j+1,h}-\partial_t\phi_{j,h}\right|^2 \mathrm{d}t - h\int_0^T\sum_{j=0}^{N}|g_{j+1,h}-g_{j,h}|^2\,\mathrm{d}t\right| \\
&\quad \le C\,\|g\|^2_{L^1(0,T;H_0^1(0,1))}.
\end{aligned}
$$

Estimate (4.49) then follows directly from Eq. (4.53). □

Proof (Proposition 4.4). Since y_h is a smooth function of time and space (recall that y_h has been identified with its Fourier extension; see Sect. 3.2),

$$
\|\partial_t y_h\|_{L^\infty((0,T);H^{-1}(0,1))} = \sup_{\substack{g\in L^1((0,T);H_0^1(0,1))\\ \|g\|_{L^1((0,T);H_0^1(0,1))}\le 1}} \int_0^T \partial_t y_h g.
$$

As in the proof of Proposition 4.3, we can take the supremum of the functions $g\in L^1(0,T;H_0^1(0,1))$ that are Fourier extensions of discrete functions. Therefore, using Lemma 4.2 together with the duality identity (4.46), we immediately obtain Proposition 4.4. □

4.4 Convergence Rates for Smooth Data

4.4.1 Main Convergence Result

Our goal is to show the following result:

Theorem 4.7. *Let $(y^0, y^1) \in H_0^1(0,1) \times L^2(0,1)$ and $v \in H^1(0,T)$ be such that $v(0) = 0$ and y the corresponding solution of Eq. (4.1) with initial data (y^0, y^1) and boundary condition v.*

Then there exists a discrete sequence of initial data (y_h^0, y_h^1) such that the solution y_h of Eq. (4.7) with initial data (y_h^0, y_h^1) and boundary data v satisfies the following convergence rates:

- *Convergence of y_h: the following convergence estimates hold:*

$$\sup_{t\in[0,T]} \|y_h(t) - y(t)\|_{L^2} \le C\left(h^{2/3}\left\|(y^0, y^1)\right\|_{H_0^1\times L^2} + h^{1/2}\|v\|_{H^1}\right). \tag{4.54}$$

If we furthermore assume that $v(T) = 0$,

$$\|y_h(T) - y(T)\|_{L^2} \le Ch^{2/3}\left(\left\|(y^0, y^1)\right\|_{H_0^1\times L^2} + \|v\|_{H^1}\right). \tag{4.55}$$

- *Convergence of $\partial_t y_h$: the following convergence estimates hold:*

$$\sup_{t\in[0,T]} \|\partial_t y_h(t) - \partial_t y(t)\|_{H^{-1}} \le Ch^{2/3}\left(\left\|(y^0, y^1)\right\|_{H_0^1\times L^2} + \|v\|_{H^1}\right). \tag{4.56}$$

Remark 4.3. The above convergences (4.54) and (4.56) may appear surprising since the rates of convergence of the displacement and of the velocity are not the same except when $v(T) = 0$. We refer to Sect. 4.4.2 for the details of the proof.

More curiously, the rates of convergence for the displacement are not the same depending on the fact that $v(T) = 0$ or not. This definitely is a surprise. In the proof below, we will see that this is due to the rate Eq. (4.22) of convergence of the normal derivative for solutions of the Laplace operator.

The proof is divided in two main steps, namely one focusing on the convergence of y_h towards y and the other one on the convergence of $\partial_t y_h$ to $\partial_t y$, these two estimates being the object of the next sections.

Also, recall that under the assumptions of Theorem 4.7, the solution y of Eq. (4.1) lies in $C([0,T]; H^1(0,1))$, its time derivative $\partial_t y$ belongs to $C([0,T]; L^2(0,1))$ and Δy to $C([0,T]; H^{-1}(0,1))$.

As in the case of homogeneous Dirichlet boundary conditions, we will write down

$$y^0 = \sum_{k=1}^{\infty} \hat{y}_k^0 w^k, \quad y^1 = \sum_{k=1}^{\infty} \hat{y}_k^1 w^k, \tag{4.57}$$

whose $H_0^1(0,1) \times L^2(0,1)$-norm coincides with

$$\left\|(y^0, y^1)\right\|_{H_0^1\times L^2}^2 = \sum_{k=1}^{\infty} k^2\pi^2|\hat{y}_k^0|^2 + \sum_{k=1}^{\infty} |\hat{y}_k^1|^2 < \infty.$$

We will then choose the initial data (y_h^0, y_h^1) of the form

$$y_h^0 = \sum_{k=1}^{N} \hat{y}_k^0 w^k, \quad y_h^1 = \sum_{k=1}^{N} \hat{y}_k^1 w^k. \tag{4.58}$$

4.4.2 Convergence of y_h

Proposition 4.5. *Under the assumptions of Theorem 4.7, taking (y_h^0, y_h^1) as in Eq. (4.58), we have the convergences (4.54) and Eq. (4.55).*

Proof. To estimate the convergence of y_h to y at time T, we write

$$\|y_h(T) - y(T)\|_{L^2} = \sup_{\substack{\phi_T \in L^2(0,1) \\ \|\phi_T\|_{L^2(0,1)} \leq 1}} \left\{ \int_0^1 (y_h(T) - y(T))\phi_T \right\}. \tag{4.59}$$

We thus fix $\phi_T \in L^2(0,1)$ and compute

$$\int_0^1 (y_h(T) - y(T))\phi_T. \tag{4.60}$$

We expand ϕ_T on its Fourier basis:

$$\phi_T = \sum_{k=1}^{\infty} \hat{\phi}_k w^k, \quad \sum_{k=1}^{\infty} |\hat{\phi}_k|^2 < \infty. \tag{4.61}$$

4.4.2.1 Computation of $\int_0^1 y(T)\phi_T$

Let us now compute $\int_0^1 y(T)\phi_T$. In order to do that, we introduce φ solution of

$$\begin{cases} \partial_{tt}\varphi - \partial_{xx}\varphi = 0, & (t,x) \in (0,T) \times (0,1), \\ \varphi(t,0) = \varphi(t,1) = 0, & t \in (0,T), \\ \varphi(T) = 0, \ \partial_t \varphi(T) = \phi_T. \end{cases} \tag{4.62}$$

Then, multiplying Eq. (4.1) by φ, we easily obtain

$$\int_0^1 y(T)\phi_T = \int_0^T v(t)\partial_x \varphi(t,1)\,\mathrm{d}t + \int_0^1 y^0 \partial_t \varphi(0) - \int_0^1 y^1 \varphi(0). \tag{4.63}$$

But $v(t) = \int_0^t \partial_t v(s)\,\mathrm{d}s$, thus yielding

$$\int_0^T v(t)\partial_x \varphi(t,1)\,\mathrm{d}t = \int_0^T \partial_t v(t) \left(\int_t^T \partial_x \varphi(s,1)\,\mathrm{d}s \right) \mathrm{d}t.$$

We therefore introduce $\Phi(t) = \int_t^T \varphi(s)\,ds$. One then easily checks that

$$\int_0^1 y(T)\phi_T = \int_0^T \partial_t v(t)\partial_x \Phi(t,1)\,\mathrm{d}t - \int_0^1 y^0 \partial_{tt}\Phi(0) + \int_0^1 y^1 \partial_t \Phi(0), \tag{4.64}$$

where Φ solves

$$\begin{cases} \partial_{tt}\Phi - \partial_{xx}\Phi = -\phi_T, & (t,x) \in (0,T)\times(0,1), \\ \Phi(t,0) = \Phi(t,1) = 0, & t \in (0,T), \\ \Phi(T) = 0,\ \partial_t \Phi(T) = 0. \end{cases} \tag{4.65}$$

We also introduce z_T the solution of

$$-\partial_{xx} z_T = \phi_T, \quad \text{on}\,(0,1), \qquad z_T(0) = z_T(1) = 0, \tag{4.66}$$

so that

$$\Psi = \Phi - z_T \tag{4.67}$$

satisfies

$$\begin{cases} \partial_{tt}\Psi - \partial_{xx}\Psi = 0, & (t,x) \in (0,T)\times(0,1) \\ \Psi(t,0) = \Psi(t,1) = 0, & t \in (0,T), \\ \Psi(T) = z_T,\ \partial_t \Psi(T) = 0. \end{cases} \tag{4.68}$$

and

$$\begin{aligned} \int_0^1 y(T)\phi_T = &\int_0^T \partial_t v(t)\partial_x \Psi(t,1)\,\mathrm{d}t - \int_0^1 y^0 \partial_{tt}\Psi(0) + \int_0^1 y^1 \partial_t \Psi(0) \\ &+ \int_0^T \partial_t v(t)\partial_x z_T(1)\,\mathrm{d}t, \end{aligned}$$

and, using that z_T is independent of time,

$$\begin{aligned} \int_0^1 y(T)\phi_T = &\int_0^T \partial_t v(t)\partial_x \Psi(t,1)\,\mathrm{d}t - \int_0^1 y^0 \partial_{tt}\Psi(0) + \int_0^1 y^1 \partial_t \Psi(0) \\ &+ v(T)\partial_x z_T(1). \end{aligned} \tag{4.69}$$

4.4.2.2 Computation of $\int_0^1 y_h(T)\phi_T$

Expanding $y_h(T)$ in discrete Fourier series, we get

$$\int_0^1 y_h(T)\phi_T = \int_0^1 y_h(T)\phi_{T,h} = h\sum_{j=1}^N y_{j,h}(T)\phi_{j,T,h},$$

where

$$\phi_{j,T,h} = \sum_{k=1}^{N} \hat{\phi}_k w_j^k, \quad j \in \{1,\ldots,N\}. \tag{4.70}$$

Then, similarly as in Eq. (4.64), we can prove

$$\int_0^1 y_h(T)\phi_T = -\int_0^T \partial_t v(t) \frac{\Phi_{N,h}}{h}\,\mathrm{d}t - h\sum_{j=1}^{N} y_{j,h}^0 \partial_{tt}\Phi_{j,h}(0) + h\sum_{j=1}^{N} y_{j,h}^1 \partial_t \Phi_{j,h}(0), \tag{4.71}$$

where Φ_h is the solution of

$$\begin{cases} \partial_{tt}\Phi_{j,h} - \frac{1}{h^2}\left(\Phi_{j+1,h} - 2\Phi_{j,h} + \Phi_{j-1,h}\right) = -\phi_{j,T,h}, \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,j) \in (0,T)\times\{1,\ldots,N\}, \\ \Phi_{0,h}(t) = \Phi_{N+1,h}(t) = 0, \qquad t \in (0,T), \\ \Phi_h(T) = 0,\ \partial_t \Phi_h(T) = 0. \end{cases} \tag{4.72}$$

Note that, due to the orthogonality properties of the Fourier basis, we can write

$$\begin{aligned} -h\sum_{j=1}^{N} y_{j,h}^0 \partial_{tt}\Phi_{j,h}(0) + h\sum_{j=1}^{N} y_{j,h}^1 \partial_t \Phi_{j,h}(0) &= -\int_0^1 y_h^0 \partial_{tt}\Phi_h(0) + \int_0^1 y_h^1 \partial_t \Phi_h(0) \\ &= -\int_0^1 y^0 \partial_{tt}\Phi_h(0) + \int_0^1 y^1 \partial_t \Phi_h(0), \end{aligned}$$

and thus Eq. (4.71) can be rewritten as

$$\int_0^1 y_h(T)\phi_T = -\int_0^T \partial_t v(t) \frac{\Phi_{N,h}}{h}\,\mathrm{d}t - \int_0^1 y^0 \partial_{tt}\Phi_h(0) + \int_0^1 y^1 \partial_t \Phi_h(0). \tag{4.73}$$

Then setting

$$z_{T,h} = (-\Delta_h)^{-1}\phi_{T,h}, \tag{4.74}$$

we obtain

$$\begin{aligned} \int_0^1 y_h(T)\phi_T = &-\int_0^T \partial_t v(t) \frac{\Psi_{N,h}}{h}\,\mathrm{d}t - \int_0^1 y^0 \partial_{tt}\Psi_h(0) + \int_0^1 y^1 \partial_t \Psi_h(0) \\ &- v(T)\frac{z_{N,T,h}}{h}, \end{aligned} \tag{4.75}$$

where Ψ_h is the solution of

$$\begin{cases} \partial_{tt}\Psi_{j,h} - \frac{1}{h^2}\left(\Psi_{j+1,h} - 2\Psi_{j,h} + \Psi_{j-1,h}\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,j) \in (0,T)\times\{1,\ldots,N\}, \\ \Psi_{0,h}(t) = \Psi_{N+1,h}(t) = 0, \qquad t \in (0,T) \\ \Psi_h(T) = z_{T,h},\ \partial_t \Psi_h(T) = 0. \end{cases} \tag{4.76}$$

4.4.2.3 Estimating the Difference $\int_0^1 y(T)\phi_T - \int_0^1 y_h(T)\phi_T$

First, since z_T solves the Laplace equation (4.66), $z_T \in H^2 \cap H_0^1(0,1)$ and

$$\|z_T\|_{H^2\cap H_0^1} \simeq \|\phi_T\|_{L^2}.$$

Since $\phi_T \in L^2(0,1)$, using Theorems 4.1 and 4.3,

$$\left\|z_{T,h} - z_T\right\|_{H_0^1} \leq Ch\|\phi_T\|_{L^2}, \tag{4.77}$$

$$\left|\partial_x z_T(1) + \frac{z_{N,T,h}}{h}\right| \leq C\sqrt{h}\|\phi_T\|_{L^2}. \tag{4.78}$$

Hence using Proposition 3.8, we obtain

$$\begin{aligned}\sup_{t\in[0,T]} &\|(\Psi_h, \partial_t \Psi_h, \partial_{tt}\Psi_h) - (\Psi, \partial_t\Psi, \partial_{tt}\Psi)\|_{H_0^1\times L^2\times H^{-1}} \\ &+ \left\|\partial_x\Psi(t,1) + \frac{\Psi_{N,h}}{h}(t)\right\|_{L^2(0,T)} \leq Ch^{2/3}\|\phi_T\|_{L^2}.\end{aligned} \tag{4.79}$$

We thus deduce that

$$\begin{aligned}\bigg|\int_0^T \partial_t v(t)\left(\frac{\Psi_{N,h}}{h} + \partial_x\Psi(t,1)\right)\mathrm{d}t + \int_0^1 y^0(\partial_{tt}\Psi_h(0) - \partial_{tt}\Psi(0)) \\ - \int_0^1 y^1(\partial_t\Psi_h(0) - \partial_t\Psi(0))\bigg| \leq Ch^{2/3}\|\phi_T\|_{L^2}\left(\left\|(y^0,y^1)\right\|_{H_0^1\times L^2} + \|v\|_{H^1}\right).\end{aligned}$$

According to Eqs. (4.69), (4.75), and the bound Eq. (4.78), this implies

$$\begin{aligned}&\left|\int_0^1 (y_h(T) - y(T))\phi_T\right| \\ &\quad\leq C\left(\sqrt{h}|v(T)| + h^{2/3}(\left\|(y^0,y^1)\right\|_{H_0^1\times L^2} + \|v\|_{H^1})\right)\|\phi_T\|_{L^2}.\end{aligned}$$

Using now identity (4.59), we obtain the following result:

$$\|y_h(T) - y(T)\|_{L^2} \leq C\left(\sqrt{h}|v(T)| + h^{2/3}(\left\|(y^0,y^1)\right\|_{H_0^1\times L^2} + \|v\|_{H^1})\right),$$

which implies that, if $v(T) = 0$,

$$\|y_h(T) - y(T)\|_{L^2} \leq Ch^{2/3}\left(\left\|(y^0,y^1)\right\|_{H_0^1\times L^2} + \|v\|_{H^1}\right),$$

whereas otherwise

$$\|y_h(T) - y(T)\|_{L^2} \leq C\left(h^{2/3}\left\|(y^0,y^1)\right\|_{H_0^1\times L^2} + \sqrt{h}\|v\|_{H^1}\right).$$

4.4.2.4 Conclusion

Note that all the above estimates hold uniformly for T in bounded intervals of time. This concludes the proof of Proposition 4.5. □

4.4.3 Convergence of $\partial_t y_h$

Proposition 4.6. *Under the assumptions of Theorem 4.7, taking (y_h^0, y_h^1) as in Eq.* (4.58)*, we have the convergence* (4.56)*.*

Proof. The proof of Proposition 4.6 closely follows the one of Proposition 4.5 and actually it is easier. We first begin by the following remark:

$$\|\partial_t y_h(T) - \partial_t y(T)\|_{H^{-1}} = \sup_{\substack{\phi_T \in H_0^1 \\ \|\phi_T\|_{H_0^1} \le 1}} \left\{ \int_0^1 \partial_t y_h(T)\phi_T - \int_0^1 \partial_t y(T)\phi_T \right\}.$$

Hence we fix $\phi_T \in H_0^1(0,1)$. We expand it in Fourier series:

$$\phi_T = \sum_{k=1}^{\infty} \hat{\phi}_k w^k, \quad \text{with } \|\phi_T\|_{H_0^1}^2 = \sum_{k=1}^{\infty} k^2 \pi^2 |\hat{\phi}_k|^2. \tag{4.80}$$

We thus introduce

$$\phi_{T,h} = \sum_{k=1}^{N} \hat{\phi}_k w^k.$$

Using the fact that $\partial_t y_h$ belongs to the span of the N-first Fourier modes,

$$\int_0^1 \partial_t y_h(T)\phi_T = \int_0^1 \partial_t y_h(T)\phi_{T,h}. \tag{4.81}$$

Hence we are reduced to show

$$\left| \int_0^1 \partial_t y(T)\phi_T - \int_0^1 \partial_t y_h(T)\phi_{T,h} \right| \\ \le C h^{2/3} \left(\left\| (y^0, y^1) \right\|_{H_0^1 \times L^2} + \|v\|_{H^1} \right) \|\phi_T\|_{H_0^1}. \tag{4.82}$$

Again, we will express each of these quantities by an adjoint formulation and then relate the proof of Eq. (4.82) to convergence results for the adjoint system.

Indeed,

$$\int_0^1 \partial_t y(T)\phi_T = \int_0^T v(t)\partial_x \varphi(t,1)\,\mathrm{d}t - \int_0^1 y^0 \partial_t \varphi(0) + \int_0^1 y^1 \varphi(0), \tag{4.83}$$

where φ solves

$$\begin{cases} \partial_{tt}\varphi - \partial_{xx}\varphi = 0, & (t,x) \in (0,T)\times(0,1), \\ \varphi(t,0) = \varphi(t,1) = 0, & t \in (0,T), \\ (\varphi(T), \partial_t\varphi(T)) = (\phi_T, 0). \end{cases} \tag{4.84}$$

Then, introducing $\Phi(t) = \int_t^T \varphi(s)\,\mathrm{d}s$, we easily check that Φ solves

$$\begin{cases} \partial_{tt}\Phi - \partial_{xx}\Phi = 0, & (t,x) \in (0,T)\times(0,1), \\ \Phi(t,0) = \Phi(t,1) = 0, & t \in (0,T), \\ (\Phi(T), \partial_t\Phi(T)) = (0, -\phi_T). \end{cases} \tag{4.85}$$

Besides, identity (4.83) then becomes

$$\int_0^1 \partial_t y(T)\phi_T = \int_0^T \partial_t v(t)\partial_x\Phi(t,1)\,\mathrm{d}t + \int_0^1 y^0\partial_{tt}\Phi(0) - \int_0^1 y^1\partial_t\Phi(0). \tag{4.86}$$

Similarly, we have

$$\int_0^1 \partial_t y_h(T)\phi_{T,h} = -\int_0^T \partial_t v(t)\frac{\Phi_{N,h}}{h}(t)\,\mathrm{d}t + \int_0^1 y_h^0\partial_{tt}\Phi_h(0) - \int_0^1 y_h^1\partial_t\Phi_h(0), \tag{4.87}$$

where Φ_h solves

$$\begin{cases} \partial_{tt}\Phi_{j,h} - \dfrac{1}{h^2}\left(\Phi_{j+1,h} + \Phi_{j-1,h} - 2\Phi_{j,h}\right) = 0, & \\ & (t,j) \in (0,T)\times\{1,\dots,N\}, \\ \Phi_{0,h}(t) = \Phi_{N+1,h}(t) = 0, & t \in (0,T), \\ (\Phi_h(T), \partial_t\Phi_h(T)) = (0, -\phi_{T,h}). \end{cases} \tag{4.88}$$

Also remark that, since $\phi_{T,h}$ is formed by Fourier modes smaller than N, Φ_h has this same structure. Due to the orthogonality properties of the Fourier basis and the choice of the initial data in Eq. (4.58), we have

$$\int_0^1 \partial_t y_h(T)\phi_{T,h} = -\int_0^T \partial_t v(t)\frac{\Phi_{N,h}}{h}(t)\,\mathrm{d}t + \int_0^1 y^0\partial_{tt}\Phi_h(0) - \int_0^1 y^1\partial_t\Phi_h(0). \tag{4.89}$$

We are thus in the setting of Proposition 3.8 since $\phi_T \in H_0^1$ and one easily checks

$$\left\|\phi_T - \phi_{T,h}\right\|_{L^2} \leq Ch\left\|\phi_T\right\|_{H_0^1}.$$

We thus obtain

$$\begin{aligned} &\sup_{t\in[0,T]} \left\|(\partial_t\Phi_h, \partial_{tt}\Phi_h) - (\partial_t\Phi, \partial_{tt}\Phi)\right\|_{L^2\times H^{-1}} + \left\|\partial_x\Phi(t,1) + \frac{\Phi_{N,h}}{h}(t)\right\|_{L^2(0,T)} \\ &\quad \leq Ch^{2/3}\left\|\phi_T\right\|_{H_0^1}. \end{aligned} \tag{4.90}$$

Then, using the identities (4.86) and (4.89), we get

$$\left| \int_0^1 \partial_t y(T)\phi_T - \int_0^T \partial_t y_h(T)\phi_{T,h} \right| \\ \leq Ch^{2/3} \|\phi_T\|_{H_0^1} \left(\left\|(y^0, y^1)\right\|_{H_0^1 \times L^2} + \|v\|_{H^1} \right). \tag{4.91}$$

Combined with Eq. (4.81), this easily yields Eq. (4.82). □

4.4.4 More Regular Data

In this section, our goal is to explain what happens for smoother initial data (y^0, y^1) and v, for instance, for $(y^0, y^1) \in H^2 \cap H_0^1(0,1) \times H_0^1(0,1)$ and $v \in H^2(0,T)$ with $v(0) = \partial_t v(0) = 0$. More precisely, we are going to prove the following:

Theorem 4.8. *Let $\ell_0 \in \{1,2\}$ and fix $(y^0, y^1) \in H_{(0)}^{\ell_0+1}(0,1) \times H_{(0)}^{\ell_0}(0,1)$ and $v \in H^{\ell_0+1}(0,T)$ satisfying $v(0) = \partial_t v(0) = 0$ if $\ell_0 = 1$, or $v(0) = \partial_t v(0) = \partial_{tt} v(0) = 0$ if $\ell_0 = 2$. Let (y_h^0, y_h^1) be as in Eq. (4.58) and y_h the corresponding solution of Eq. (4.7) with Dirichlet boundary conditions $v_h = v$.*

Then there exists a constant $C > 0$ independent of $h > 0$ and $t \in [0,T]$ such that:

- *For the displacement y_h, for all $t \in [0,T]$,*

$$\|y_h(t) - y(t)\|_{L^2} \leq Ch^{2(\ell_0+1)/3} \left(\left\|(y^0, y^1)\right\|_{H_{(0)}^{\ell_0+1} \times H_{(0)}^{\ell_0}} + \|v\|_{H^{\ell_0+1}(0,T)} \right) \\ + Ch^{1/2}|v(t)|. \tag{4.92}$$

- *For the velocity $\partial_t y_h$, for all $t \in [0,T]$,*

$$\|\partial_t y_h(t) - \partial_t y(t)\|_{H^{-1}} \leq Ch^{2(\ell_0+1)/3} \left(\left\|(y^0, y^1)\right\|_{H_{(0)}^{\ell_0+1} \times H_{(0)}^{\ell_0}} + \|v\|_{H^{\ell_0+1}(0,T)} \right) \\ + Ch^{3/2}|\partial_t v(t)|. \tag{4.93}$$

Proof. The proof follows the one of Theorem 4.7.

Let us then focus on the convergence of the displacement and follow the proof of Proposition 4.5. We introduce $\phi_T \in L^2(0,1)$, z_T as in Eq. (4.66), Ψ the solution of the homogeneous wave equation (4.68) with initial data $(z_T, 0)$ and, similarly, $\phi_{T,h}$ as in Eq. (4.70), $z_{T,h}$ as in Eq. (4.74), and Ψ_h the solution of the discrete homogeneous wave equation (4.76) with initial data $(z_{T,h}, 0)$. Since $z_T \in H_{(0)}^2(0,1)$ and $\|z_T\|_{H_{(0)}^2} \simeq \|\phi_T\|_{L^2}$, applying (4.15), we get

$$\left\|z_{T,h} - z_T\right\|_{L^2} \leq Ch^2 \|\phi_T\|_{L^2}. \tag{4.94}$$

Proposition 3.8 then applies and yields

$$\|(\partial_t \Psi_h, \partial_{tt} \Psi_h) - (\partial_t \Psi, \partial_{tt} \Psi)\|_{H^{-\ell_0} \times H^{-\ell_0 - 1}} \leq C h^{2(\ell_0+1)/3} \|\phi_T\|_{L^2}.$$

In particular,

$$\left| \int_0^1 y^0 (\partial_{tt} \Psi_h(0) - \partial_{tt} \Psi(0)) - \int_0^1 y^1 (\partial_t \Psi_h(0) - \partial_t \Psi(0)) \right|$$
$$\leq C h^{2(\ell_0+1)/3} \|\phi_T\|_{L^2} \left\|(y^0, y^1)\right\|_{H^{\ell_0+1}_{(0)} \times H^{\ell_0}_{(0)}}. \tag{4.95}$$

According to identities (4.69) and (4.75), we shall then derive a convergence estimate on

$$\int_0^T \partial_t v \left(\partial_x \Psi(t,1) + \frac{\Psi_{N,h}(t)}{h} \right) \mathrm{d}t.$$

In order to do that, we write $\partial_t v = \int_0^t \partial_{tt} v$ and introduce

$$\xi(t) = \int_t^T \Psi(s) \, \mathrm{d}s, \quad \xi_h(t) = \int_t^T \Psi_h(s) \, \mathrm{d}s,$$

so that

$$\int_0^T \partial_t v \left(\partial_x \Psi(t,1) + \frac{\Psi_{N,h}(t)}{h} \right) \mathrm{d}t = \int_0^T \partial_{tt} v \left(\partial_x \xi(t,1) + \frac{\xi_{N,h}(t)}{h} \right) \mathrm{d}t.$$

Of course, ξ and ξ_h can be interpreted as solutions of continuous and discrete wave equations: ξ solves

$$\begin{cases} \partial_{tt} \xi - \partial_{xx} \xi = 0, & (t,x) \in (0,T) \times (0,1) \\ \xi(t,0) = \xi(t,1) = 0, & t \in (0,T), \\ \xi(T) = 0, \; \partial_t \xi(T) = -z_T, \end{cases} \tag{4.96}$$

whereas ξ_h solves

$$\begin{cases} \partial_{tt} \xi_{j,h} - \dfrac{1}{h^2} \left(\xi_{j+1,h} - 2\xi_{j,h} + \xi_{j-1,h} \right) = 0, & \\ & (t,j) \in (0,T) \times \{1, \ldots, N\}, \\ \xi_{0,h}(t) = \xi_{N+1,h}(t) = 0, & t \in (0,T), \\ \xi_h(T) = 0, \; \partial_t \xi_h(T) = -z_{T,h}. \end{cases} \tag{4.97}$$

Then, due to Eq. (4.94), the convergence results in Proposition 3.7 yield

$$\left\| \partial_x \xi(t,1) + \frac{\xi_{N,h}(t)}{h} \right\|_{L^2(0,T)} \leq C h^{4/3} \|\phi_T\|_{L^2}.$$

This implies in particular that

$$\left| \int_0^T \partial_t v \left(\partial_x \Psi(t,1) + \frac{\Psi_{N,h}(t)}{h} \right) \mathrm{d}t \right| \leq C h^{4/3} \|\phi_T\|_{L^2} \|\partial_{tt} v\|_{L^2(0,T)}. \tag{4.98}$$

Hence, if $\ell_0 = 1$, i.e., $(y^0, y^1) \in H^2_{(0)}(0,1) \times H^1_{(0)}(0,1)$ and $v \in H^2(0,T)$ with $v(0) = \partial_t v(0) = 0$, combining Eqs. (4.95) and (4.98) in identities (4.69) and (4.75), we get

$$\|y_h(T) - y(T)\|_{L^2(0,1)} \leq C h^{4/3} \left(\left\|(y^0, y^1)\right\|_{H^2_{(0)} \times H^1_{(0)}} + \|v\|_{H^2(0,T)} \right) + C h^{1/2} |v(T)|. \tag{4.99}$$

The Case $\ell_0 = 2$. In this case, $v \in H^3(0,T)$, we introduce $\zeta = \int_t^T \xi$ and $\zeta_h = \int_t^T \xi_h$, so that

$$\int_0^T \partial_t v \left(\partial_x \Psi(t,1) + \frac{\Psi_{N,h}(t)}{h} \right) \mathrm{d}t = \int_0^T \partial_{ttt} v \left(\partial_x \zeta(t,1) + \frac{\zeta_{N,h}(t)}{h} \right) \mathrm{d}t. \tag{4.100}$$

Obviously, the function ζ can be characterized as the solution of a wave equation, namely,

$$\begin{cases} \partial_{tt} \zeta - \partial_{xx} \zeta = z_T, & (t,x) \in (0,T) \times (0,1) \\ \zeta(t,0) = \zeta(t,1) = 0, & t \in (0,T), \\ \zeta(T) = 0, \ \partial_t \zeta(T) = 0. \end{cases} \tag{4.101}$$

We thus introduce w_T solution of

$$\partial_{xx} w_T = z_T, \quad \text{on } (0,1), \quad w_T(0) = w_T(1) = 0, \tag{4.102}$$

so that

$$\tilde{\zeta} = \zeta - w_T$$

solves

$$\begin{cases} \partial_{tt} \tilde{\zeta} - \partial_{xx} \tilde{\zeta} = 0, & (t,x) \in (0,T) \times (0,1) \\ \tilde{\zeta}(t,0) = \tilde{\zeta}(t,1) = 0, & t \in (0,T), \\ \tilde{\zeta}(T) = w_T, \ \partial_t \tilde{\zeta}(T) = 0. \end{cases} \tag{4.103}$$

Doing that

$$\int_0^T \partial_{ttt} v \, \partial_x \zeta(t,1) \, \mathrm{d}t = \int_0^T \partial_{ttt} v \, \partial_x \tilde{\zeta}(t,1) \, \mathrm{d}t - \partial_x w_T(1) \partial_{tt} v(T). \tag{4.104}$$

Similar computations can be done for ζ_h. We thus obtain that

$$\int_0^T \partial_{ttt} v \, \frac{\zeta_{N,h}(t)}{h} \, \mathrm{d}t = \int_0^T \partial_{ttt} v \, \frac{\tilde{\zeta}_{N,h}(t)}{h} \, \mathrm{d}t - \frac{w_{N,T,h}}{h} \partial_{tt} v(T), \tag{4.105}$$

where $w_{T,h} = (\Delta_h)^{-1} z_{T,h}$ and $\tilde{\zeta}_h$ solves

$$\begin{cases} \partial_{tt}\tilde{\zeta}_{j,h} - \dfrac{1}{h^2}\left(\tilde{\zeta}_{j+1,h} - 2\tilde{\zeta}_{j,h} + \tilde{\zeta}_{j-1,h}\right) = 0, & \\ & (t,j) \in (0,T) \times \{1,\dots,N\}, \\ \tilde{\zeta}_{0,h}(t) = \tilde{\zeta}_{N+1,h}(t) = 0, & t \in (0,T) \\ \tilde{\zeta}_h(T) = w_{T,h},\ \partial_t \tilde{\zeta}_h(T) = 0. & \end{cases} \tag{4.106}$$

We now derive convergence estimates. Recall first that $z_T \in H^2_{(0)}(0,1)$ and the convergences (4.94). Since $z_T \in H^2_{(0)}$, setting $\tilde{z}_{T,h}$ its projection on the N-first Fourier modes, we have

$$\left\| \tilde{z}_{T,h} - z_T \right\|_{L^2} \le Ch^2 \left\| z_T \right\|_{H^2_{(0)}} \le Ch^2 \left\| \phi_T \right\|_{L^2}. \tag{4.107}$$

Setting $\tilde{w}_{T,h} = (\Delta_h)^{-1} \tilde{z}_{T,h}$, Theorems 4.4 and 4.5 yield

$$\begin{aligned} \left\| w_T - \tilde{w}_{T,h} \right\|_{H^1_0} \le &\quad Ch^2 \left\| z_T \right\|_{H^2_{(0)}} \le Ch^2 \left\| \phi_T \right\|_{L^2}, \\ \left| \partial_x w_T(1) + \frac{\tilde{w}_{N,T,h}}{h} \right| \le &\quad Ch^2 \left\| z_T \right\|_{H^2_{(0)}} \le Ch^2 \left\| \phi_T \right\|_{L^2}. \end{aligned} \tag{4.108}$$

According to the estimate (4.94), we thus have

$$\left\| \tilde{z}_{T,h} - z_{T,h} \right\|_{L^2} \le Ch^2 \left\| z_T \right\|_{H^2_{(0)}} \le Ch^2 \left\| \phi_T \right\|_{L^2}.$$

Using then estimate (4.21),

$$\left| \frac{\tilde{w}_{N,T,h}}{h} - \frac{w_{N,T,h}}{h} \right| \le Ch^2 \left\| \phi_T \right\|_{L^2},$$

and thus

$$\left| \partial_x w_T(1) + \frac{w_{N,T,h}}{h} \right| \le Ch^2 \left\| \phi_T \right\|_{L^2}. \tag{4.109}$$

Besides, due to Eqs. (4.94) and (4.107),

$$\left\| z_{T,h} - \tilde{z}_{T,h} \right\|_{L^2} \le Ch^2 \left\| \phi_T \right\|_{L^2},$$

which readily implies

$$\left\| w_{T,h} - \tilde{w}_{T,h} \right\|_{H^1_0} \le Ch^2 \left\| \phi_T \right\|_{L^2},$$

and thus, by Eq. (4.108),

$$\left\| w_{T,h} - w_T \right\|_{H^1_0} \le Ch^2 \left\| \phi_T \right\|_{L^2}.$$

Using then Proposition 3.6,

$$\left\| \partial_x \zeta(\cdot,1) + \frac{\zeta_{N,h}}{h}(\cdot) \right\|_{L^2(0,T)} \le Ch^2 \left\| \phi_T \right\|_{L^2}. \tag{4.110}$$

Combined with the convergences (4.109) and (4.110), identities (4.100), (4.104), and (4.105) then imply

$$\left| \int_0^T \partial_t v \left(\partial_x \Psi(t,1) + \frac{\Psi_{N,h}(t)}{h} \right) \mathrm{d}t \right|$$
$$\leq Ch^2 \left\| \phi_T \right\|_{L^2} \left\| \partial_{ttt} v \right\|_{L^2} + Ch^2 \left\| \phi_T \right\|_{L^2} \left| \partial_{tt} v(T) \right| \leq Ch^2 \left\| \phi_T \right\|_{L^2} \left\| v \right\|_{H^3}. \quad (4.111)$$

Combining Eqs. (4.95) and (4.111) in identities (4.69) and (4.75), we get Eq. (4.92) when $\ell_0 = 2$.

The proof of the estimate (4.93) on the rate of convergence for $\partial_t y_h$ relies on very similar estimates which are left to the reader. □

4.5 Further Convergence Results

As a corollary to Theorems 4.6 and 4.7, we can give convergence results for *any* sequence of discrete initial data (y_h^0, y_h^1) and boundary data v_h satisfying

$$\lim_{h \to 0} \left\| (y_h^0, y_h^1) - (y^0, y^1) \right\|_{L^2 \times H^{-1}} = 0 \quad \text{and} \quad \lim_{h \to 0} \left\| v_h - v \right\|_{L^2(0,T)} = 0. \quad (4.112)$$

Proposition 4.7. *Let $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$ and $v \in L^2(0,T)$. Then consider sequences of discrete initial data (y_h^0, y_h^1) and v_h satisfying Eq. (4.112). Then the solutions y_h of Eq. (4.7) with initial data (y_h^0, y_h^1) and boundary data v_h converge strongly in $C([0,T]; L^2(0,1)) \cap C^1([0,T]; H^{-1}(0,1))$ towards the solution y of Eq. (4.1) with initial data (y^0, y^1) and boundary data v as $h \to 0$.*

Proof. Similarly as in the proof of Proposition 3.5, this result is obtained by using the density of $H_0^1(0,T)$ in $L^2(0,T)$ and of $H_0^1(0,1) \times L^2(0,1)$ in $L^2(0,1) \times H^{-1}(0,1)$. We then use Theorem 4.7 for smooth solutions and the uniform stability results in Theorem 4.6 to obtain Proposition 4.7. Details of the proof are left to the reader. □

Another important corollary of Theorem 4.7 is the fact that, if the initial data (y^0, y^1) belong to $H_0^1(0,1) \times L^2(0,1)$ and the Dirichlet data v lies in $H_0^1(0,T)$, *any* sequence of discrete initial (y_h^0, y_h^1) and Dirichlet data v_h satisfying

$$\left\| (y_h^0, y_h^1) - (y^0, y^1) \right\|_{L^2 \times H^{-1}} + \left\| v - v_h \right\|_{L^2(0,T)} \leq C_0 h^\theta, \quad (4.113)$$

for some constant C_0 uniform in $h > 0$ and $\theta > 0$, yield solutions y_h of Eq. (4.7) such that $y_h(T)$ approximates at a rate $h^{\min\{2/3, \theta\}}$ the state $y(T)$, where y is the continuous trajectory corresponding to initial data (y^0, y^1) and source term v.

Proposition 4.8. *Let $(y^0, y^1) \in H_0^1(0,1) \times L^2(0,1)$ and $v \in H_0^1(0,T)$ and consider sequences (y_h^0, y_h^1) and v_h satisfying Eq. (4.113).*

Denote by y_h (respectively y) the solution of Eq. (4.7) *(resp.* (4.1)*) with initial data* (y_h^0, y_h^1) *(resp.* (y^0, y^1)*) and Dirichlet boundary data* v_h*, (resp.* v*).*

Then the following estimates hold:

$$\begin{aligned} &\|(y_h(T), \partial_t y_h(T)) - (y(T), \partial_t y(T))\|_{L^2 \times H^{-1}} \\ &\quad \leq C h^{2/3} \left(\left\| (y^0, y^1) \right\|_{H_0^1 \times L^2} + \|v\|_{H_0^1(0,T)} \right) + C C_0 h^\theta. \end{aligned} \tag{4.114}$$

Remark 4.4. In the convergence result Eq. (4.114), we keep explicitly the dependence in the constant C_0 coming into play in Eq. (4.113). In many situations, this constant can be chosen proportional to $\left\| (y^0, y^1) \right\|_{H_0^1 \times L^2} + \|v\|_{H_0^1(0,T)}$. In particular, in the control theoretical setting of Chap. 1 and its application to the wave equation in Sect. 1.7, this dependence on C_0 is important to derive Assumption 1 and more specifically estimate (1.29).

Proof. The proof follows the one of Proposition 3.7. The idea is to compare y with $\tilde{y}_h$, the solution of Eq. (4.7) constructed in Theorem 4.7 and then to compare $\tilde{y}_h$ and y_h by using Propositions 4.3 and 4.6. □

Remark 4.5. Note that under the assumptions of Proposition 4.8, the trajectories y_h converge to y in the space $C([0,T]; L^2(0,1)) \cap C^1([0,T]; H^{-1}(0,1))$ with the rates (4.54)–(4.56) in addition to the error $C_0 h^\theta$.

Of course, Proposition 4.8 is based on the convergence result obtained in Theorem 4.7. Similar results can be stated based on Theorem 4.8, for instance:

Proposition 4.9. *Let* $\ell_0 \in \{0, 1, 2\}$*. Let* $(y^0, y^1) \in H_{(0)}^{\ell_0+1}(0,1) \times H_{(0)}^{\ell_0}(0,1)$ *and* $v \in H_0^{\ell_0+1}(0,T)$ *and consider sequences* (y_h^0, y_h^1) *and* v_h *satisfying Eq.* (4.113)*.*

Let (y_h^0, y_h^1) *as in Eq.* (4.58) *and* y_h *the corresponding solution of Eq.* (4.7) *with Dirichlet boundary conditions* v_h*.*

Denote by y_h (respectively y) the solution of Eq. (4.7) *(resp. Eq.* (4.1)*) with initial data* (y_h^0, y_h^1) *(resp.* (y^0, y^1)*) and Dirichlet boundary data* v_h *(resp.* v*).*

Then the following estimates hold:

$$\begin{aligned} &\|(y_h(T), \partial_t y_h(T)) - (y(T), \partial_t y(T))\|_{L^2 \times H^{-1}} \\ &\quad \leq C h^{2(\ell_0+1)/3} \left(\left\| (y^0, y^1) \right\|_{H_{(0)}^{\ell_0+1} \times H_{(0)}^{\ell_0}} + \|v\|_{H_0^{\ell_0+1}(0,T)} \right) + C C_0 h^\theta. \end{aligned} \tag{4.115}$$

Remark 4.6. Proposition 4.9 can then be slightly generalized for $\ell_0 \in [0,2]$ by interpolation.

4.6 Numerical Results

In this section, we present numerical simulations and evidences of Proposition 4.9. Since our main interest is in the non-homogeneous boundary condition, we focus on the case $(y^0, y^1) = (0,0)$ and $(y_h^0, y_h^1) = (0,0)$.

We fix $T = 2$. This choice is done for convenience to explicitly compute the solution y of Eq. (4.1) with initial data $(0,0)$ and source term v. Indeed, for $T = 2$, multiplying the equation (4.1) by φ solution of Eq. (3.2) with initial data $(\varphi^0, \varphi^1) \in H_0^1(0,1) \times L^2(0,1)$ and using the two-periodicity of the solutions of the wave equation (3.2), we obtain

$$\int_0^1 y(2,x)\varphi^1(x)\,\mathrm{d}x - \int_0^1 \partial_t y(2,x)\varphi^0(x)\,\mathrm{d}x = \int_0^2 v(t)\partial_x\varphi(t,1)\,\mathrm{d}t.$$

Based on this formula, taking successively $(\varphi^0, \varphi^1) = (w^k, 0)$ and $(0, w^k)$ and solving explicitly the equation (3.2) satisfied by φ, we obtain

$$y(2) = \sum_k \left(\sqrt{2}(-1)^k \int_0^2 v(t)\sin(k\pi t)\,\mathrm{d}t\right) w^k,$$
$$\partial_t y(2) = \sum_k \left(\sqrt{2}(-1)^{k+1} k\pi \int_0^2 v(t)\cos(k\pi t)\,\mathrm{d}t\right) w^k.$$

We will numerically compute the reference solutions using these formulae by restricting the sums over $k \in \{1, \ldots, N_{\mathrm{ref}}\}$ for a large enough N_{ref}. We will choose $N_{\mathrm{ref}} = 300$ for N varying between 50 and 200.

We then compute numerically the solution y_h of Eq. (4.7) with initial data $(y_h^0, y_h^1) = (0,0)$ and source term $v(t)$.

Of course, we also discretize the equation (4.7) in time. We do it in an explicit manner similarly as in Eq. (3.45). If y_h^k denotes the approximation of y_h solution of Eq. (4.7) at time $k\Delta t$, we solve

$$y_h^{k+1} = 2y_h^k - y_h^{k-1} - (\Delta t)^2 \Delta_h y_h^k - \left(\frac{\Delta t}{h}\right)^2 F^k, \quad F^k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v(k\Delta t) \end{pmatrix}.$$

The time discretization parameter Δt is chosen such that the CFL condition is $\Delta t / h = 0.3$. With such low CFL condition, the effects of the time-discretization can be neglected.

We run the tests for several choices of v and for $N \in \{50, \ldots, 200\}$:

$$\begin{aligned} &v_1(t) = \sin(\pi t)^3, \quad t \in (0,2), && v_2(t) = \sin(\pi t)^2, \quad t \in (0,2),\\ &v_3(t) = \sin(\pi t), \quad t \in (0,2), && v_4(t) = t, \quad t \in (0,2),\\ &v_5(t) = t\sin(\pi t), \quad t \in (0,2). && \end{aligned}$$

In each case, we plot the L^2-norm of the error on the displacement and the H^{-1}-norm of the error on the velocity versus N in logarithmic scales: Fig. 4.2 corresponds to the data v_1. We then compute the slopes of the linear regression for the L^2-error on the displacement and for the H^{-1}-error on the velocity. We put all these data in Table 4.1.

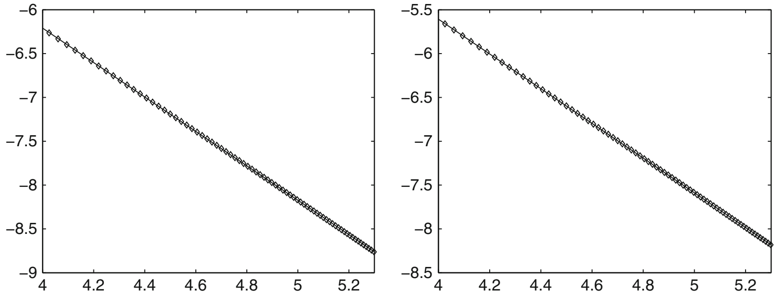

Fig. 4.2 Plots of the errors versus N in logarithmic scales for v_1. *Left*, the $L^2(0,1)$-error $\|y_h(T)-y(T)\|_{L^2}$ for $T=2$: the slope of the linear regression is -1.96. *Right*, the $H^{-1}(0,1)$-error $\|\partial_t y_h(T)-\partial_t y(T)\|_{H^{-1}}$ for $T=2$: the slope of the linear regression is -1.98.

Table 4.1 Numerical investigation of the convergence rates.

Data	Computed L^2 slope	Computed H^{-1} slope	Exp. L^2 slope	Exp. H^{-1} slope
v_1	-1.96	-1.98	-2	-2
v_2	-1.87	-1.70	$-5/3^-$	$-5/3^-$
v_3	-0.99	-0.95	-1^-	-1^-
v_4	-0.97	-0.95	$-1/2$	-1^-
v_5	-1.82	-1.47	$-5/3^-$	$-3/2$

Columns 2 and 3 give the slopes observed numerically (respectively, for the L^2-error on the displacement, for the H^{-1}-error on the velocity), whereas columns 4 and 5 provide the slopes (respectively, for the L^2-error on the displacement, for the H^{-1}-error on the velocity) expected from our theoretical results

Table 4.1 is composed of five columns. The first one is the data under consideration. The second and third ones, respectively, are the computed slopes of the linear regression of, respectively, the L^2-error on the displacement and for the H^{-1}-error on the velocity. The fourth and fifth columns are the rates expected from the analysis of the data v and Proposition 4.9:

- $v_1 \in H_0^3(0,2)$: we thus expect from Eq. (4.115) a convergence of the order of h^2. This is indeed what is observed numerically.
- v_2 is smooth but its boundary condition vanishes only up to order 1. Hence $v_2 \in H_0^{5/2-\varepsilon}(0,2)$ for all $\varepsilon>0$ due to the boundary conditions. Using Remark 4.6, the expected slopes are $-5/3^-$, which is not far from the slopes computed numerically.
- The same discussion applies for v_3, which belongs to $H_0^{3/2-\varepsilon}(0,2)$ for all $\varepsilon>0$. Hence the expected slopes are -1^-, which again are confirmed by the numerical experiments.
- v_4 almost belongs to $H_0^{3/2-\varepsilon}(0,2)$ except for what concerns its nonzero value at $t=2$. But the value of v is an impediment for the order of convergence only for the displacement; see Theorem 4.8. We therefore expect a convergence of the L^2-norm of the error on the displacement like $\sqrt{h}$, whereas the convergence of the H^{-1}-norm of the error on the velocity is expected to go much faster, as h^{1^-}.

The numerical test indicates a good accuracy on the convergence of the H^{-1}-norm on the velocity error. The convergence of the L^2-norm of the displacement is better than expected.

- v_5 is smooth and satisfies $v_5(0) = \partial_t v_5(0) = 0$ and $v_5(2) = 0$ but $\partial_t v_5(2) \neq 0$. According to Theorem 4.8, we thus expect that the L^2-norm of the error on the displacement behaves as when v_5 belongs to $H_0^{5/2^-}(0,1)$, i.e., as $h^{5/3^-}$. However, the H^{-1}-norm of the error on the velocity should behave like $h^{3/2}$ according to Eq. (4.93). This is completely consistent with the slopes observed numerically.

In each case, the numerical results indicate good accuracy of the theoretical results derived in Theorem 4.8 and Proposition 4.9.