Chapter 3
Convergence of the Finite-Difference Method for the $1-d$ Wave Equation with Homogeneous Dirichlet Boundary Conditions

3.1 Objectives

This chapter of the book is devoted to the study of the convergence of the numerical scheme

$$\begin{cases} \partial_{tt}\varphi_{j,h} - \dfrac{1}{h^2}\left(\varphi_{j+1,h} - 2\varphi_{j,h} + \varphi_{j-1,h}\right) = 0, \\ \qquad\qquad\qquad\qquad (t,j) \in (0,T)\times\{1,\ldots,N\}, \\ \varphi_{0,h}(t) = \varphi_{N+1,h}(t) = 0, \qquad\qquad t \in (0,T), \\ (\varphi_h(0), \partial_t\varphi_h(0)) = (\varphi_{0h}, \varphi_{1h}), \end{cases} \tag{3.1}$$

towards the continuous wave equation

$$\begin{cases} \partial_{tt}\varphi - \partial_{xx}\varphi = 0, & (t,x) \in (0,T)\times(0,1), \\ \varphi(t,0) = \varphi(t,1) = 0, & t \in (0,T), \\ (\varphi(0), \partial_t\varphi(0)) = (\varphi_0, \varphi_1). \end{cases} \tag{3.2}$$

Of course, first of all, one needs to explain how discrete and continuous solutions can be compared. This will be done in Sect. 3.2. In Sect. 3.3, we will present our main convergence result. We shall then present some further convergence results in Sect. 3.4 and illustrate them in Sect. 3.5.

3.2 Extension Operators

We first describe the extension operators we shall use. We will then explain how the obtained results can be interpreted in terms of the more classical extension operators.

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3.2.1 The Fourier Extension

For $h > 0$, given a discrete function $a_h = (a_{j,h})_{j\in\{1,\dots,N\}}$ (with $N+1 = 1/h$), since the sequence w_h^k is an orthonormal basis for the $h\langle\cdot,\cdot\rangle_{\ell^2(\mathbb{R}^N)}$-norm due to Lemma 2.1, there exist coefficients $\hat{a}_k$ such that

$$a_h = \sum_{k=1}^{N} \hat{a}_k w_h^k, \quad \text{[recall that } w_{j,h}^k = \sqrt{2}\sin(k\pi jh)] \tag{3.3}$$

in the sense that, for all $j \in \{1,\dots,N\}$,

$$a_{j,h} = \sum_{k=1}^{N} \hat{a}_k \sqrt{2}\sin(k\pi jh). \tag{3.4}$$

Of course, this yields a natural Fourier extension denoted by $\mathbb{F}_h$ for discrete functions a_h given by Eq. (3.3):

$$\mathbb{F}_h(a_h)(x) = \sum_{k=1}^{N} \hat{a}_k \sqrt{2}\sin(k\pi x), \quad x \in (0,1). \tag{3.5}$$

The advantage of this definition is that now $\mathbb{F}_h(a_h)$ is a smooth function of x.

The energy of a solution φ_h of Eq. (3.1) at time t, given by Eq. (2.2), is then equivalent, uniformly with respect to $h > 0$, to the $H_0^1(0,1)\times L^2(0,1)$-norm of $(\mathbb{F}_h(\varphi_h), \mathbb{F}_h(\varphi_h'))$. This issue will be discussed in Proposition 3.3 below.

Another interesting feature of this Fourier extension is that, due to the discrete orthogonality properties of the eigenvectors w^k proved in Lemma 2.1 and their usual $L^2(0,1)$-orthogonality, i.e., $\int_0^1 w^k(x)w^\ell(x)\,\mathrm{d}x = \delta_{k,\ell}$ for all $k,\ell \in \mathbb{N}$, for all discrete functions a_h, b_h, we have

$$h\sum_{j=1}^{N} a_{j,h} b_{j,h} = \int_0^1 \mathbb{F}_h(a_h)\mathbb{F}_h(b_h)\,\mathrm{d}x.$$

This fact will be used to simplify some expressions.

3.2.2 Other Extension Operators

When using finite-difference (or finite element) methods, the Fourier extension is not the most natural one. Given a discrete function $a_h = (a_{j,h})_{j\in\{1,\dots,N\}}$ (with $N + 1 = 1/h$), consider the classical extension operators $\mathbb{P}_h$ and $\mathbb{Q}_h$ defined by

$$\mathbb{P}_h(a_h)(x) = a_{j,h} + \left(\frac{a_{j+1,h} - a_{j,h}}{h}\right)(x - jh),$$
$$\text{for } x \in [jh,(j+1)h),\ j \in \{0,\dots,N\}, \tag{3.6}$$

$$\mathbb{Q}_h(a_h)(x) = \begin{cases} a_{j,h} \text{ for } x \in [(j-1/2)h, (j+1/2)h),\ j \in \{1,\ldots,N\}, \\ 0 \text{ for } x \in [0, h/2) \cup [(N+1/2)h, 1], \end{cases} \tag{3.7}$$

with the conventions $a_{0,h} = a_{N+1,h} = 0$.

The range of the extension operator $\mathbb{P}_h$ is the set of continuous, piecewise affine functions with (C^1) singularities in the points jh and vanishing on the boundary. This corresponds to the most natural approximation leading to $H_0^1(0,1)$ functions and to the point of view of the $P1$ finite element method. By the contrary, $\mathbb{Q}_h$ provides the simplest piecewise constant extension of the discrete function which, obviously, lies in $L^2(0,1)$ but not in $H_0^1(0,1)$.

Note that the extensions $\mathbb{F}_h(a_h)$ obtained using the Fourier representation (3.5) and $\mathbb{P}_h(a_h)$ do not coincide. However, they are closely related as follows:

Proposition 3.1. *For each $h = 1/(N+1) > 0$, let a_h be a sequence of discrete functions.*

Then, for $s \in \{0,1\}$, the sequence of Fourier extensions $(\mathbb{F}_h(a_h))_{h>0}$ converges strongly (respectively weakly) in $H^s(0,1)$ if and only if the sequence $(\mathbb{P}_h(a_h))_{h>0}$ converges strongly (respectively weakly) in $H^s(0,1)$. Besides, if one of these sequences converge, then they have the same limit.

Moreover, there exists a constant C independent of $h > 0$ such that

$$\frac{1}{C}\|\mathbb{F}_h(a_h)\|_{L^2} \leq \|\mathbb{P}_h(a_h)\|_{L^2} \leq C\|\mathbb{F}_h(a_h)\|_{L^2}, \tag{3.8}$$

$$\frac{1}{C}\|\mathbb{F}_h(a_h)\|_{H_0^1} \leq \|\mathbb{P}_h(a_h)\|_{H_0^1} \leq C\|\mathbb{F}_h(a_h)\|_{H_0^1}. \tag{3.9}$$

Proof. Let us begin with the case $s = 0$.
Let us first compare the $L^2(0,1)$-norms of the functions $\mathbb{F}_h(a_h)$ and $\mathbb{P}_h(a_h)$.

From the orthogonality properties of w^k (see Lemma 2.1), we have

$$\|\mathbb{F}_h(a_h)\|_{L^2(0,1)}^2 = \sum_{k=1}^{N} |\hat{a}_{k,h}|^2 = h\sum_{j=1}^{N} |a_{j,h}|^2. \tag{3.10}$$

Computing the $L^2(0,1)$-norm of $\mathbb{P}_h(a_h)$ is slightly more technical:

$$\begin{aligned} \int_0^1 |\mathbb{P}_h(a_h)(x)|^2\,\mathrm{d}x &= \sum_{j=0}^{N}\int_0^h \left| a_{j,h} + x\left(\frac{a_{j+1,h}-a_{j,h}}{h}\right)\right|^2 \mathrm{d}x \\ &= h\sum_{j=0}^{N}\left[a_{j,h}^2 + a_{j,h}(a_{j+1,h}-a_{j,h}) + \frac{1}{3}(a_{j+1,h}-a_{j,h})^2\right] \\ &= \frac{h}{3}\sum_{j=0}^{N}(a_{j,h}^2 + a_{j+1,h}^2 + a_{j,h}a_{j+1,h}) \\ &= \frac{h}{6}\sum_{j=0}^{N}(a_{j,h}^2 + a_{j+1,h}^2 + 2a_{j,h}a_{j+1,h}) + \frac{h}{6}\sum_{j=0}^{N}(a_{j,h}^2 + a_{j+1,h}^2) \\ &= \frac{h}{6}\sum_{j=0}^{N}(a_{j,h}+a_{j+1,h})^2 + \frac{h}{3}\sum_{j=1}^{N}|a_{j,h}|^2. \end{aligned} \tag{3.11}$$

It follows that the $L^2(0,1)$-norms of $\mathbb{F}_h(a_h)$ and $\mathbb{P}_h(a_h)$ are equivalent, hence implying Eq. (3.8), and then the boundedness properties for these sequences are equivalent.

This also implies that the sequence $(\mathbb{F}_h(a_h))_{h>0}$ is a Cauchy sequence in $L^2(0,1)$ if and only if the sequence $(\mathbb{P}_h(a_h))$ is a Cauchy sequence in $L^2(0,1)$, and then one of these sequences converges strongly if and only if the other one does.

To guarantee that these sequences have the same limit when they converge, we have to check that their difference, if uniformly bounded, weakly converges to zero when $h \to 0$.

Let ψ denote a smooth test function. On one hand, we have

$$\int_0^1 \mathbb{F}_h(a_h)(x)\,\psi(x)\,\mathrm{d}x = \sum_{k=1}^{N} \hat{a}_{k,h} \int_0^1 w^k(x)\,\psi(x)\,\mathrm{d}x.$$

On the other one, we have

$$\begin{aligned}\int_0^1 \mathbb{P}_h(a_h)(x)\,\psi(x)\,\mathrm{d}x &= \sum_{j=1}^{N} \int_{jh}^{(j+1)h} \left(a_{j,h} + \frac{a_{j+1,h}-a_{j,h}}{h}(x-jh)\right)\psi(x)\,\mathrm{d}x \\ &= h\sum_{j=1}^{N} a_{j,h}\tilde{\psi}_{j,h},\end{aligned}$$

with

$$\begin{aligned}\tilde{\psi}_{j,h} &= \frac{1}{h}\int_{(j-1)h}^{jh} \psi(x)\left(\frac{x-(j-1)h}{h}\right)\mathrm{d}x + \frac{1}{h}\int_{jh}^{(j+1)h} \psi(x)\left(1-\frac{x-jh}{h}\right)\mathrm{d}x \\ &= \frac{1}{h}\int_{(j-1)h}^{(j+1)h} \psi(x)\left(1-\frac{|x-jh|}{h}\right)\mathrm{d}x.\end{aligned}$$

Using Eq. (3.4), we obtain

$$\int_0^1 \mathbb{P}_h(a_h)(x)\,\psi(x)\,\mathrm{d}x = \sum_{k=1}^{N} \hat{a}_{k,h}\left(h\sum_{j=1}^{N} w_j^k\,\tilde{\psi}_{j,h}\right). \tag{3.12}$$

Therefore,

$$\begin{aligned}&\int_0^1 \left(\mathbb{P}_h(a_h)(x) - \mathbb{F}_h(a_h)(x)\right)\psi(x)\,\mathrm{d}x \\ &\quad = \sum_{k=1}^{N} \hat{a}_{k,h}\left(h\sum_{j=1}^{N} w_j^k\,\tilde{\psi}_{j,h} - \int_0^1 w^k(x)\,\psi(x)\,\mathrm{d}x\right).\end{aligned} \tag{3.13}$$

Now, fix $\ell \in \mathbb{N}$, and choose $\psi(x) = w^\ell(x) = \sqrt{2}\sin(\ell\pi x)$. In this case, using Taylor's formula, we easily check that

$$\sup_{j\in\{1,\ldots,N\}} |\tilde{\psi}_{j,h} - \psi(jh)| \le \ell h\pi.$$

Since, for $\ell \le N$, see Lemma 2.1,

$$\int_0^1 w^k(x)\, w^\ell(x)\,\mathrm{d}x = h\sum_{j=1}^{N} w_j^k w^\ell(jh) = \delta_k^\ell,$$

we then obtain from Eq. (3.13) that for all $\ell \in \mathbb{N}$,

$$\int_0^1 (\mathbb{P}_h(a_h)(x) - \mathbb{F}_h(a_h)(x))\; w^\ell(x)\,\mathrm{d}x \underset{h\to 0}{\longrightarrow} 0.$$

Since the set $\{w^\ell\}_{l\in\mathbb{N}}$ spans the whole space $L^2(0,1)$, if one of the sequences $(\mathbb{F}_h(a_h))$ or $(\mathbb{P}_h(a_h))$ converges weakly in $L^2(0,1)$, then the other one also converges weakly in $L^2(0,1)$ and has the same limit.

This completes the proof in the case $s = 0$.

We now deal with the case $s = 1$. First remark that

$$\int_0^1 |\partial_x \mathbb{F}_h(a_h)|^2 \mathrm{d}x = \sum_{k=1}^{N} |\hat{a}_{k,h}|^2 k^2 \pi^2 \tag{3.14}$$

from the Fourier orthogonality properties, and, using Lemma 2.1,

$$\int_0^1 |\partial_x \mathbb{P}_h(a_h)(x)|^2 \mathrm{d}x = h\sum_{j=0}^{N} \left(\frac{a_{j+1,h} - a_{j,h}}{h}\right)^2 = \sum_{k=1}^{N} \lambda_k(h)|\hat{a}_{k,h}|^2. \tag{3.15}$$

Since $c_1k^2 \le \lambda_k(h) \le c_2k^2$, these two norms are equivalent, hence implying Eq. (3.9), and therefore the $H_0^1(0,1)$-boundedness properties of the sequences $(\mathbb{F}_h(a_h))$ and $(\mathbb{P}_h(a_h))$ are equivalent.

If one of these sequences weakly converges in $H_0^1(0,1)$, then the other one is bounded in $H_0^1(0,1)$ and weakly converges in $L^2(0,1)$ to the same limit from the previous result and then also weakly converges in $H_0^1(0,1)$.

Besides, if one of these sequences strongly converges in $H_0^1(0,1)$, it is a Cauchy sequence in $H_0^1(0,1)$, and then the other one also is a Cauchy sequence in $H_0^1(0,1)$ and therefore also strongly converges. □

Similarly, one can prove the following:

Proposition 3.2. *For each $h = 1/(N+1) > 0$, let a_h be a sequence of discrete functions.*

Then the sequence of Fourier extensions $(\mathbb{F}_h(a_h))_{h>0}$ converges strongly (respectively weakly) in $L^2(0,1)$ if and only if the sequence $(\mathbb{Q}_h(a_h))_{h>0}$ converges strongly (respectively weakly) in $L^2(0,1)$. Besides, when they converge, they have the same limit.

Moreover, there exists a constant C independent of $h > 0$ such that

$$\frac{1}{C}\|\mathbb{F}_h(a_h)\|_{L^2} \le \|\mathbb{Q}_h(a_h)\|_{L^2} \le C\|\mathbb{F}_h(a_h)\|_{L^2}. \tag{3.16}$$

The proof is very similar to the previous one and is left to the reader.

The above propositions show that the Fourier extension plays the same role as the classical extensions by continuous piecewise affine functions or by piecewise constant functions when considering convergence issues. We make the choice of considering this Fourier extension, rather than the usual ones, since it has the advantage of being smooth.

The following result is also relevant:

Proposition 3.3. *There exists a constant C independent of $h > 0$ such that for all solutions φ_h of Eq. (3.1):*

$$\frac{1}{C}\|(\mathbb{F}_h(\varphi_h), \mathbb{F}_h(\partial_t\varphi_h)\|_{H_0^1\times L^2} \le E_h[\varphi_h] \le C\|(\mathbb{F}_h(\varphi_h), \mathbb{F}_h(\partial_t\varphi_h))\|_{H_0^1\times L^2} \tag{3.17}$$

Proof. The discrete energy of a solution φ_h of Eq. (3.1) at time t exactly coincides with the $H_0^1(0,1)\times L^2(0,1)$-norm of $(\mathbb{P}_h(\varphi_h), \mathbb{Q}_h(\partial_t\varphi_h))$ at time t. Using the equivalences (3.9) and (3.16), we immediately obtain Eq. (3.17). □

In the following, we will often omit the operator $\mathbb{F}_h$ from explicit notations and directly identify the discrete function $a_h = (a_{j,h})_{j\in\{1,\dots,N\}}$ with its continuous Fourier extension $\mathbb{F}_h(a_h)$.

3.3 Orders of Convergence for Smooth Initial Data

In this section, we consider a solution φ of Eq. (3.2) with initial data $(\varphi^0, \varphi^1) \in H^2\cap H_0^1(0,1)\times H_0^1(0,1)$. The solution φ of Eq. (3.2) then belongs to the space

$$\varphi \in C([0,T]; H^2\cap H_0^1(0,1))\cap C^1([0,T]; H_0^1(0,1))\cap C^2([0,T]; L^2(0,1)).$$

In order to prove it, one can remark that the energy

$$E[\varphi](t) = \int_0^1 \left(|\partial_t\varphi(t,x)|^2 + |\partial_x\varphi(t,x)|^2\right)\,\mathrm{d}x$$

is constant in time for solutions of Eq. (3.2) with initial data in $H_0^1(0,1)\times L^2(0,1)$. We then apply it to $\partial_t\varphi$, which is a solution of Eq. (3.2) with initial data $(\varphi_1, \partial_{xx}\varphi_0) \in H_0^1(0,1)\times L^2(0,1)$.

The goal of this section is to prove the following result:

Proposition 3.4. *Let $(\varphi^0, \varphi^1) \in H^2\cap H_0^1(0,1)\times H_0^1(0,1)$. Then there exist a constant $C = C(T)$ independent of (φ^0, φ^1) and a sequence $(\varphi_h^0, \varphi_h^1)$ of discrete initial data such that for all $h > 0$,*

$$\left\|(\varphi_h^0,\varphi_h^1)-(\varphi^0,\varphi^1)\right\|_{H_0^1\times L^2}\leq Ch^{2/3}\left\|(\varphi^0,\varphi^1)\right\|_{H^2\cap H_0^1\times H_0^1} \tag{3.18}$$

and the solutions φ *of Eq. (3.2) with initial data* (φ^0,φ^1) *and* φ_h *of Eq. (3.1) with initial data* $(\varphi_h^0,\varphi_h^1)$ *satisfy, for all* $h>0$ *and* $t\in[0,T]$,

$$\|(\varphi_h(t),\partial_t\varphi_h(t))-(\varphi(t),\partial_t\varphi(t))\|_{H_0^1\times L^2}\leq Ch^{2/3}\left\|(\varphi^0,\varphi^1)\right\|_{H^2\cap H_0^1\times H_0^1}, \tag{3.19}$$

and

$$\left\|\frac{\varphi_{N,h}(\cdot)}{h}+\partial_x\varphi(\cdot,1)\right\|_{L^2(0,T)}\leq Ch^{2/3}\left\|(\varphi^0,\varphi^1)\right\|_{H^2\cap H_0^1\times H_0^1}. \tag{3.20}$$

Remark 3.1. The result in Eq. (3.18) may appear somewhat surprising since when approximating $(\varphi^0,\varphi^1)\in H^2\cap H_0^1(0,1)\times H_0^1(0,1)$ by the classical continuous piecewise affine approximations or truncated Fourier series, the approximations $(\varphi_h^0,\varphi_h^1)$ satisfy

$$\left\|(\varphi_h^0,\varphi_h^1)-(\varphi^0,\varphi^1)\right\|_{H_0^1\times L^2}\leq Ch\left\|(\varphi^0,\varphi^1)\right\|_{H^2\cap H_0^1\times H_0^1} \tag{3.21}$$

instead of Eq. (3.18).

However, the result in [45] indicates that, even if the convergence of the initial data is as in Eq. (3.21), one cannot obtain a better result than Eq. (3.19). This is due to the distance between the continuous and space semi-discrete semigroups generated by Eqs. (3.2) and (3.1), respectively, and their purely conservative nature. To be more precise, when looking at the dispersion diagram, the eigenvalues of the semi-discrete wave equation (3.1) are of the form

$$\sqrt{\lambda_k(h)}=\frac{2}{h}\sin\left(\frac{k\pi h}{2}\right),$$

whereas the ones of the continuous equation (3.2) are $\sqrt{\lambda_k}=k\pi$. In particular, for any $\varepsilon>0$,

$$\sup_{k\leq h^{-2/3+\varepsilon}}\left\{\left|\sqrt{\lambda_k(h)}-k\pi\right|\right\}=0,\quad \text{while}\quad \sup_{k\geq h^{-2/3-\varepsilon}}\left\{\left|\sqrt{\lambda_k(h)}-k\pi\right|\right\}=\infty.$$

Remark 3.2. The main issue in Proposition 3.4 is the estimate (3.20). Estimates (3.19) are rather classical in the context of finite element methods; see, e.g., [2] and the references therein.

Proof. Let $(\varphi^0,\varphi^1)\in H^2\cap H_0^1(0,1)\times H_0^1(0,1)$. Expanding these initial data on the Fourier basis (recall that $w^k(x)=\sqrt{2}\sin(k\pi x)$), we have

$$\varphi^0=\sum_{k=1}^{\infty}\hat{a}_k w^k,\quad \varphi^1=\sum_{k=1}^{\infty}\hat{b}_k w^k.$$

The solution φ of Eq. (3.2) can then be computed explicitly in Fourier:

$$\varphi(t,x) = \sum_{|k|=1}^{\infty} \hat{\varphi}_k \exp(i\mu_k t) w^{|k|}, \quad \mu_k = k\pi, \quad \hat{\varphi}_k = \frac{1}{2}\left(\hat{a}_{|k|} + \frac{i\hat{b}_{|k|}}{\mu_k}\right).$$

And the condition $(\varphi^0, \varphi^1) \in H^2 \cap H_0^1(0,1) \times H_0^1(0,1)$ can be written as

$$\sum_{k=1}^{\infty} \left(k^4 \left|\hat{\varphi}_k^0\right|^2 + k^2 \left|\hat{\varphi}_k^1\right|^2\right) < \infty \quad \text{or, equivalently,} \sum_{|k|=1}^{\infty} k^4 |\hat{\varphi}_k|^2 < \infty, \tag{3.22}$$

and both these quantities are equivalent to the $H^2 \cap H_0^1(0,1) \times H_0^1(0,1)$-norm of the initial data (φ^0, φ^1).

We now look for a solution φ_h of Eq. (3.1) on the Fourier basis. Using that the functions w^k correspond to eigensolutions of the discrete Laplace operator for $k \le N$, one easily checks that any solution of Eq. (3.1) can be written as $\sum_{|k|=1}^{N} a_k w^{|k|} \exp(i\mu_k(h)t)$ with $\mu_k(h) = 2\sin(k\pi h/2)/h$. Keeping this in mind, we take

$$\varphi_h(t) = \sum_{|k|=1}^{n(h)} \hat{\varphi}_k \exp(i\mu_k(h)t) w^{|k|}, \tag{3.23}$$

where $n(h)$ is an integer smaller than N that will be fixed later on.

We now compute how this solution approximates φ:

$$\begin{aligned}
&\|\varphi_h(t) - \varphi(t)\|_{H_0^1}^2 \\
&= \sum_{|k|=n(h)+1}^{\infty} k^2\pi^2 |\hat{\varphi}_k|^2 + \sum_{|k|=1}^{n(h)} k^2\pi^2 |\hat{\varphi}_k|^2 4\sin^2\left(\frac{(\mu_k(h) - \mu_k)t}{2}\right) \\
&\quad \le \frac{C}{n(h)^2} \sum_{|k|=n(h)+1}^{\infty} k^4\pi^4 |\hat{\varphi}_k|^2 + C \sum_{|k|=1}^{n(h)} (k^4h^4) k^4\pi^4 |\hat{\varphi}_k|^2 \\
&\quad \le C\left(n(h)^4 h^4 + \frac{1}{n(h)^2}\right) \left\|(\varphi^0, \varphi^1)\right\|_{H^2 \cap H_0^1 \times H_0^1}^2,
\end{aligned} \tag{3.24}$$

where we have used that for some constant C independent of $h > 0$ and $k \in \{1, \ldots, N\}$,

$$|\mu_k(h) - \mu_k| = \left|\frac{2}{h}\sin\left(\frac{k\pi h}{2}\right) - k\pi\right| \le Ck^3h^2,$$

and

$$\left|\sin\left(\frac{(\mu_k(h) - \mu_k)t}{2}\right)\right| \le CT |\mu_k(h) - \mu_k|.$$

The same can be done for $\partial_t \varphi_h$:

$$\begin{aligned}
&\|\partial_t \varphi_h(t) - \partial_t \varphi(t)\|_{L^2}^2 \\
&= \sum_{|k|=n(h)+1}^{\infty} k^2\pi^2 \left|\hat{\varphi}_k\right|^2 + \sum_{|k|=1}^{n(h)} \left|\hat{\varphi}_k\right|^2 \left|\mu_k(h) \mathrm{e}^{i\mu_k(h)t} - \mu_k \mathrm{e}^{i\mu_k t}\right|^2 \\
&\quad \leq C\left(n(h)^4 h^4 + \frac{1}{n(h)^2}\right) \left\|(\varphi^0, \varphi^1)\right\|_{H^2\cap H_0^1 \times H_0^1}^2,
\end{aligned} \tag{3.25}$$

where we used that

$$\left|\mu_k(h)\mathrm{e}^{i\mu_k(h)t} - \mu_k \mathrm{e}^{i\mu_k t}\right| \leq \left|2k\pi \sin\left(\frac{(\mu_k(h)-\mu_k)t}{2}\right)\right| + |\mu_k(h) - \mu_k| \leq Ck^4h^2.$$

Estimates (3.24) and (3.25) then imply Eqs. (3.18) and (3.19) when choosing $n(h) \simeq h^{-2/3}$, a choice that, as we will see below, also optimizes the convergence of the normal derivatives.

We shall now prove Eq. (3.20). This will be done in two main steps, computing separately the integrals

$$I_1 = \int_0^T \left|\partial_x \varphi_h(t,1) + \frac{\varphi_{N,h}(t)}{h}\right|^2 \mathrm{d}t, \quad \text{and} \quad I_2 = \int_0^T \left|\partial_x \varphi(t,1) - \partial_x \varphi_h(t,1)\right|^2 \mathrm{d}t. \tag{3.26}$$

Estimates on I_1. We shall first write the admissibility inequality proved in Theorem 2.1 in terms of Fourier series.

Consider a solution ϕ_h of Eq. (3.1) and write it as

$$\phi_h(t) = \sum_{|k|=1}^{N} \hat{\phi}_{k,h} \mathrm{e}^{i\mu_k(h)t} w^{|k|},$$

where

$$\hat{\phi}_{k,h} = \frac{1}{2}\left(\hat{\phi}_{k,h}^0 + \frac{\hat{\phi}_{k,h}^1}{i\mu_k(h)}\right).$$

The energy of the solution is then given by

$$E_h = 2\sum_{|k|=1}^{N} \lambda_{|k|}(h) \left|\hat{\phi}_{k,h}\right|^2.$$

Hence the admissibility result in Theorem 2.1 reads as follows: for any sequence $(\hat{\phi}_{k,h})$,

$$\int_0^T \left|\sum_{|k|=1}^{N} \hat{\phi}_{k,h} \mathrm{e}^{i\mu_k(h)t} \frac{w_N^{|k|}}{h}\right|^2 \mathrm{d}t \leq C \sum_{|k|=1}^{N} \lambda_{|k|}(h) \left|\hat{\phi}_{k,h}\right|^2. \tag{3.27}$$

But the difference $\partial_x \varphi_h(t,1) + \varphi_{N,h}/h$ reads as

$$\partial_x \varphi_h(t,1) + \frac{\varphi_{N,h}}{h}(t) = \sum_{|k|=1}^{n(h)} \hat{\varphi}_k e^{i\mu_k(h)t} \left(\partial_x w^{|k|}(1) + \frac{w_N^{|k|}}{h} \right)$$
$$= \sum_{|k|=1}^{n(h)} \hat{\varphi}_k \left(1 + \frac{h\partial_x w^{|k|}(1)}{w_N^{|k|}} \right) e^{i\mu_k(h)t} \frac{w_N^{|k|}}{h}.$$

Thus, applying Eq. (3.27), we get

$$\int_0^T \left| \partial_x \varphi_h(t,1) + \frac{\varphi_{N,h}(t)}{h} \right|^2 dt \le C \sum_{|k|=1}^{n(h)} \lambda_{|k|}(h) \, |\hat{\varphi}_k|^2 \left(1 + \frac{h\partial_x w^{|k|}(1)}{w_N^{|k|}} \right)^2. \tag{3.28}$$

But for all $k \in \{1, \ldots, N\}$,

$$\frac{h\partial_x w^k(1)}{w_N^k} = -\frac{k\pi h \cos(k\pi)}{\sin(k\pi h)\cos(k\pi)} = -\frac{k\pi h}{\sin(k\pi h)},$$

and we thus have, for some explicit constant C independent of h and k, that for all $h > 0$ and $k \in \{1, \ldots, N\}$,

$$\left| 1 + \frac{h\partial_x w^k(1)}{w_N^k} \right| \le C(k\pi h)^2.$$

Plugging this last estimate into Eq. (3.28) and using $\lambda_k(h) \le Ck^2$, we obtain

$$I_1 = \int_0^T \left| \partial_x \varphi_h(t,1) + \frac{\varphi_{N,h}(t)}{h} \right|^2 dt \le C \sum_{|k|=1}^{n(h)} |\hat{\varphi}_k|^2 k^6 h^4$$
$$\le Cn(h)^2 h^4 \sum_{|k|=1}^{n(h)} k^4 \, |\hat{\varphi}_k|^2$$
$$\le Cn(h)^2 h^4 \left\| (\varphi^0, \varphi^1) \right\|^2_{H^2 \cap H_0^1 \times H_0^1}. \tag{3.29}$$

Estimates on I_2. The idea now is to see φ_h as a solution of Eq. (3.1) up to a perturbation. Note that this is a classical technique in numerical analysis and more particularly in *a posteriori* error analysis.

Indeed, recall that

$$\varphi_h = \sum_{|k|=1}^{n(h)} \hat{\varphi}_k e^{i\mu_k(h)t} w^{|k|}(x).$$

This implies that

$$\partial_{tt}\varphi_h - \partial_{xx}\varphi_h = f_h, \quad (t,x) \in \mathbb{R} \times (0,1)$$

with

$$f_h(x,t) = \sum_{|k|=1}^{n(h)} \hat{\varphi}_k \mathrm{e}^{i\mu_k(h)t} w^{|k|}(x) \left(-\lambda_{|k|}(h) + k^2\pi^2\right).$$

In particular, for all $t \in \mathbb{R}$,

$$\begin{aligned}
\|f_h(t)\|_{L^2(0,1)}^2 &\leq \sum_{|k|=1}^{n(h)} k^4\pi^4 |\hat{\varphi}_k|^2 \left(1 - \frac{4}{k^2\pi^2h^2}\sin^2\left(\frac{k\pi h}{2}\right)\right)^2 \\
&\leq C \sum_{|k|=1}^{n(h)} k^4\pi^4 |\hat{\varphi}_k|^2 (k\pi h)^4 \\
&\leq Cn(h)^4h^4 \sum_{|k|=1}^{n(h)} k^4\pi^4 |\hat{\varphi}_k|^2 \\
&\leq Cn(h)^4h^4 \left\|(\varphi^0,\varphi^1)\right\|_{H^2\cap H_0^1\times H_0^1}^2,
\end{aligned}$$

where the constant C is independent of $h > 0$.

Now, consider $z_h = \varphi_h - \varphi$. Then z_h satisfies the following system of equations:

$$\begin{cases}
\partial_{tt} z_h - \partial_{xx} z_h = f_h, & t \in \mathbb{R}, x \in (0,1) \\
z_h(t,0) = z_h(t,1) = 0, & t \in \mathbb{R}, \\
z_h(0,x) = z_h^0(x),\ \partial_t z_h(0,x) = z^1(x), & 0 < x < 1,
\end{cases} \tag{3.30}$$

with $(z_h^0, z_h^1) = (\varphi_h^0, \varphi_h^1) - (\varphi^0, \varphi^1)$, which satisfies, according to Eqs. (3.24) and (3.25) for $t = 0$,

$$\left\|(z_h^0, z_h^1)\right\|_{H_0^1\times L^2}^2 \leq C\left(\frac{1}{n(h)^2} + n(h)^4h^4\right)\left\|(\varphi^0,\varphi^1)\right\|_{H^2\cap H_0^1\times H_0^1}^2.$$

But this is now the continuous wave equation and one can easily check that the normal derivative of z_h then satisfies the following admissibility result: for some constant C independent of $h > 0$,

$$\int_0^T |\partial_x z_h(t,1)|^2 \,\mathrm{d}t \leq C\left(\|f_h\|_{L^1(0,T;L^2(0,1))}^2 + \left\|(z_h^0, z_h^1)\right\|_{H_0^1\times L^2}^2\right).$$

For a proof of that fact we refer to the book of Lions [36] and the article [34].

This gives

$$\begin{aligned}
I_2 &= \int_0^T |\partial_x\varphi(t,1) - \partial_x\varphi_h(t,1)|^2 \,\mathrm{d}t \\
&\leq C\left(\frac{1}{n(h)^2} + n(h)^4h^4\right)\left\|(\varphi^0,\varphi^1)\right\|_{H^2\cap H_0^1\times H_0^1}^2.
\end{aligned} \tag{3.31}$$

Combining the estimates (3.29) and (3.31), we obtain

$$\int_0^T \left| \partial_x \varphi(t,1) + \frac{\varphi_{N,h}(t)}{h} \right|^2 \mathrm{d}t \le C \left(\frac{1}{n(h)^2} + n(h)^4 h^4 \right) \left\| (\varphi^0, \varphi^1) \right\|^2_{H^2 \cap H_0^1 \times H_0^1}.$$

The choice $n(h) \simeq h^{-2/3}$ optimizes this estimate and yields Eq. (3.20). This choice also optimizes estimates (3.24) and (3.25) and implies Eqs. (3.18) and (3.19) and thus completes the proof. □

3.4 Further Convergence Results

3.4.1 Strongly Convergent Initial Data

As a corollary to Proposition 3.4, we can give convergence results for *any* sequence of discrete initial data $(\varphi_h^0, \varphi_h^1)$ satisfying

$$\lim_{h \to 0} \left\| (\varphi_h^0, \varphi_h^1) - (\varphi^0, \varphi^1) \right\|_{H_0^1 \times L^2} = 0. \tag{3.32}$$

Proposition 3.5. *Let $(\varphi^0, \varphi^1) \in H_0^1(0,1) \times L^2(0,1)$ and consider a sequence of discrete initial data $(\varphi_h^0, \varphi_h^1)$ satisfying Eq. (3.32). Then the solutions φ_h of Eq. (3.1) with initial data $(\varphi_h^0, \varphi_h^1)$ converge strongly in $C([0,T]; H_0^1(0,1)) \cap C^1([0,T]; L^2(0,1))$ towards the solution φ of Eq. (3.2) with initial data (φ^0, φ^1) as $h \to 0$. Moreover, we have*

$$\lim_{h \to 0} \int_0^T \left| \partial_x \varphi(t,1) + \frac{\varphi_{N,h}}{h} \right|^2 \mathrm{d}t = 0. \tag{3.33}$$

Proof. Let $(\varphi^0, \varphi^1) \in H_0^1(0,1) \times L^2(0,1)$ and, given $\varepsilon > 0$, choose $(\psi^0, \psi^1) \in H^2 \cap H_0^1(0,1) \times H_0^1(0,1)$ so that

$$\left\| (\varphi^0, \varphi^1) - (\psi^0, \psi^1) \right\|_{H_0^1 \times L^2} \le \varepsilon.$$

We now use the discrete initial data (ψ_h^0, ψ_h^1) provided by Proposition 3.4. The solutions ψ_h of Eq. (3.1) with initial data (ψ_h^0, ψ_h^1) thus converge to the solution ψ of Eq. (3.2) with initial data (ψ^0, ψ^1) in the sense of Eqs. (3.19)–(3.20).

We now denote by φ_h the solutions of Eq. (3.1) with initial data $(\varphi_h^0, \varphi_h^1)$ and φ the solution of Eq. (3.2) with initial data (φ^0, φ^1).

Since $\varphi_h - \psi_h$ is a solution of Eq. (3.1), the conservation of the energy and the uniform admissibility property (2.12) yield

$$\begin{aligned}
&\sup_{t\in[0,T]} \|(\varphi_h,\partial_t\varphi_h)(t)-(\psi_h,\partial_t\psi_h)(t)\|_{H_0^1\times L^2} + \left\|\frac{\varphi_{N,h}-\psi_{N,h}}{h}\right\|_{L^2(0,T)}\\
&\leq C\left\|(\varphi_h^0,\varphi_h^1)-(\psi_h^0,\psi_h^1)\right\|_{H_0^1\times L^2}\\
&\leq C(\left\|(\varphi_h^0,\varphi_h^1)-(\varphi^0,\varphi^1)\right\|_{H_0^1\times L^2}+\left\|(\varphi^0,\varphi^1)-(\psi^0,\psi^1)\right\|_{H_0^1\times L^2}\\
&\quad +\left\|(\psi^0,\psi^1)-(\psi_h^0,\psi_h^1)\right\|_{H_0^1\times L^2})\\
&\leq C(\left\|(\varphi_h^0,\varphi_h^1)-(\varphi^0,\varphi^1)\right\|_{H_0^1\times L^2}+\varepsilon+C_\varepsilon h^{2/3}\left\|(\psi^0,\psi^1)\right\|_{H^2\cap H_0^1\times H_0^1}).
\end{aligned}$$

Besides, recalling that ψ_h converge to ψ in the sense of Eqs. (3.19)–(3.20), we have

$$\lim_{h\to 0}\sup_{t\in[0,T]} \|(\psi_h,\partial_t\psi_h)(t)-(\psi,\partial_t\psi)(t)\|_{H_0^1\times L^2}+\left\|\partial_x\psi(t,1)+\frac{\psi_{N,h}}{h}\right\|_{L^2(0,T)}=0.$$

We also use that the energy of the continuous wave equation (3.2) is constant in time and the admissibility result of the continuous wave equation and apply it to $\varphi-\psi$:

$$\sup_{t\in[0,T]} \|(\varphi,\partial_t\varphi)(t)-(\psi,\partial_t\psi)(t)\|_{H_0^1\times L^2}+\|\partial_x\varphi(t,1)-\partial_x\psi(t,1)\|_{L^2(0,T)}\leq C\varepsilon.$$

Combining these three estimates and taking the limsup as $h\to 0$, for all $\varepsilon>0$, we get

$$\begin{aligned}
\limsup_{h\to 0}\Bigg(&\sup_{t\in[0,T]} \|(\varphi_h,\partial_t\varphi_h)(t)-(\varphi,\partial_t\varphi)(t)\|_{H_0^1\times L^2}\\
&+\left\|\frac{\varphi_{N,h}(t)}{h}+\partial_x\varphi(t,1)\right\|_{L^2(0,T)}\Bigg)\leq C\varepsilon.
\end{aligned}$$

This concludes the proof of Proposition 3.5 since $\varepsilon>0$ was arbitrary. □

3.4.2 Smooth Initial Data

In this section, we derive higher convergence rates when the initial data are smoother.

In order to do that, we introduce, for $\ell\in\mathbb{R}$, the functional space $H^\ell_{(0)}$ defined by

$$H^\ell_{(0)}(0,1)=\left\{\varphi=\sum_{k=1}^{\infty}\hat{\varphi}_k w^k,\ \text{with}\ \sum_{k=1}^{\infty}k^{2\ell}|\hat{\varphi}_k|^2<\infty\right\}$$

$$\text{endowed with the norm } \|\varphi\|^2_{H^\ell_{(0)}}=\sum_{k=1}^{\infty}k^{2\ell}|\hat{\varphi}_k|^2. \tag{3.34}$$

These functional spaces correspond to the domains $\mathscr{D}((-\Delta_d)^{\ell/2})$ of the fractional powers of the Dirichlet Laplace operator $-\Delta_d$. In particular, we have $H^0_{(0)}(0,1) = L^2(0,1)$, $H^1_{(0)}(0,1) = H^1_0(0,1)$ and $H^{-1}_{(0)}(0,1) = H^{-1}(0,1)$.

As an extension of Proposition 3.4, we obtain:

Proposition 3.6. *Let $\ell \in (0,3]$ and $(\varphi^0, \varphi^1) \in H^{\ell+1}_{(0)}(0,1) \times H^{\ell}_{(0)}(0,1)$. Denote by φ the solution of Eq. (3.2) with initial data (φ^0, φ^1). Then there exists a constant $C = C(T,\ell)$ independent of (φ^0, φ^1) such that the sequence φ_h of solutions of Eq. (3.1) with initial data $(\varphi^0_h, \varphi^1_h)$ constructed in Proposition 3.4 satisfies, for all $h > 0$,*

$$\sup_{t\in[0,T]} \|(\varphi_h(t), \partial_t \varphi_h(t)) - (\varphi(t), \partial_t \varphi(t))\|_{H^1_0 \times L^2} \leq C h^{2\ell/3} \left\|(\varphi^0, \varphi^1)\right\|_{H^{\ell+1}_{(0)} \times H^{\ell}_{(0)}}, \tag{3.35}$$

and

$$\left\| \frac{\varphi_{N,h}(\cdot)}{h} + \partial_x \varphi(\cdot, 1) \right\|_{L^2(0,T)} \leq C h^{2\ell/3} \left\|(\varphi^0, \varphi^1)\right\|_{H^{\ell+1}_{(0)} \times H^{\ell}_{(0)}}. \tag{3.36}$$

In particular, for $\ell = 3$, this result reads as follows: if $(\varphi^0, \varphi^1) \in H^4_{(0)}(0,1) \times H^3_{(0)}(0,1)$, the sequence φ_h constructed in Proposition 3.4 satisfies the following convergence results:

$$\sup_{t\in[0,T]} \|(\varphi_h(t), \partial_t \varphi_h(t)) - (\varphi(t), \partial_t \varphi(t))\|_{H^1_0 \times L^2} \leq C h^2 \left\|(\varphi^0, \varphi^1)\right\|_{H^4_{(0)} \times H^3_{(0)}}, \tag{3.37}$$

and

$$\left\| \frac{\varphi_{N,h}(\cdot)}{h} + \partial_x \varphi(\cdot, 1) \right\|_{L^2(0,T)} \leq C h^2 \left\|(\varphi^0, \varphi^1)\right\|_{H^4_{(0)} \times H^3_{(0)}}. \tag{3.38}$$

Note that we cannot expect to go beyond the rate h^2 since the method is consistent of order 2.

Proof (Sketch). The proof of these convergence results follows line to line the one of Proposition 3.4.

Let us for instance explain how it has to be modified to get Eq. (3.37). First remark that Eq. (3.22) now reads

$$\sum_{|k|=1}^{\infty} k^{2\ell+2} |\hat{\varphi}_k|^2 \simeq \left\|(\varphi^0, \varphi^1)\right\|^2_{H^{\ell+1}_{(0)} \times H^{\ell}_{(0)}}.$$

Estimates (3.24)–(3.25) can then be modified into

$$\|\varphi_h(t) - \varphi(t)\|^2_{H^1_0} + \|\partial_t \varphi_h(t) - \partial_t \varphi(t)\|^2_{L^2} \leq C \left(h^4 n(h)^{6-2\ell} + \frac{1}{n(h)^{2\ell}} \right) \left\|(\varphi^0, \varphi^1)\right\|^2_{H^{\ell+1}_{(0)} \times H^{\ell}_{(0)}},$$

thus implying Eq. (3.35) immediately when taking $n(h) \simeq h^{-2/3}$.

The proof of the strong convergence (3.36) also relies upon the estimate

$$I_1 + I_2 \leq C\left(h^4 n(h)^{6-2\ell} + \frac{1}{n(h)^{2\ell}}\right) \left\|(\varphi^0, \varphi^1)\right\|^2_{H^{\ell+1}_{(0)} \times H^{\ell}_{(0)}},$$

where I_1 and I_2 are, respectively, given as above by Eq. (3.26). Details are left to the reader. □

3.4.3 General Initial Data

In Propositions 3.4 and 3.6, the discrete initial data are very special ones constructed during the proof. In this section, we explain how this yields convergence rates even for other initial data.

Proposition 3.7. *Let* $\ell \in (0,3]$ *and* $(\varphi^0, \varphi^1) \in H^{\ell+1}_{(0)}(0,1) \times H^{\ell}_{(0)}(0,1)$ *and consider a sequence* (ϕ_h^0, ϕ_h^1) *satisfying, for some constants* $C_0 > 0$ *and* $\theta > 0$ *independent of* $h > 0$*,*

$$\left\|(\phi_h^0, \phi_h^1) - (\varphi^0, \varphi^1)\right\|_{H_0^1 \times L^2} \leq C_0 h^\theta. \tag{3.39}$$

Denote by ϕ_h *(respectively* φ*) the solution of Eq.* (3.1) *(resp. Eq.* (3.2)*) with initial data* (ϕ_h^0, ϕ_h^1) *(resp.* (φ^0, φ^1)*).*

Then the following estimates hold:

$$\begin{aligned} &\sup_{t \in [0,T]} \|(\phi_h(t), \partial_t \phi_h(t)) - (\varphi(t), \partial_t \varphi(t))\|_{H_0^1 \times L^2} \\ &\qquad \leq C\left(h^{2\ell/3} \left\|(\varphi^0, \varphi^1)\right\|_{H^{\ell+1}_{(0)} \times H^{\ell}_{(0)}} + C_0 h^\theta\right), \end{aligned} \tag{3.40}$$

and

$$\left\|\frac{\phi_{N,h}(\cdot)}{h} + \varphi_x(\cdot, 1)\right\|_{L^2(0,T)} \leq C\left(h^{2\ell/3} \left\|(\varphi^0, \varphi^1)\right\|_{H^{\ell+1}_{(0)} \times H^{\ell}_{(0)}} + C_0 h^\theta\right). \tag{3.41}$$

Proof. The proof easily follows from Proposition 3.6 since it simply consists in comparing φ_h, the solution of Eq. (3.1) given by Proposition 3.4, and ϕ_h, the solution of Eq. (3.1) with initial data (ϕ_h^0, ϕ_h^1). But $\varphi_h - \phi_h$ solves Eq. (3.1) with an initial data of $H_0^1(0,1) \times L^2(0,1)$-norm less than $Ch^{2\ell/3} \|(\varphi^0, \varphi^1)\|_{H^{\ell+1}_{(0)} \times H^{\ell}_{(0)}} + CC_0 h^\theta$.

The first estimate (3.40) then follows immediately from the fact that the discrete energy is constant for solutions of Eq. (3.1), whereas estimate (3.41) is based on the uniform admissibility results proved in Theorem 2.1. □

3.4.4 Convergence Rates in Weaker Norms

For later use, we also give the following result:

Proposition 3.8. *Let* $(\varphi^0,\varphi^1)\in H^2_{(0)}(0,1)\times H^1_{(0)}(0,1)$. *Denote by* φ *the solution of Eq.* (3.2) *with initial data* (φ^0,φ^1). *Then for all* $\ell\in(0,3]$, *there exists a constant* $C=C(T,\ell)$ *independent of* (φ^0,φ^1) *such that the sequence* φ_h *of solutions of Eq.* (3.1) *with initial data* $(\varphi_h^0,\varphi_h^1)$ *constructed in Proposition 3.4 satisfies, for all* $h>0$,

$$\begin{aligned}\sup_{t\in[0,T]}&\|(\varphi_h(t),\partial_t\varphi_h(t),\partial_{tt}\varphi_h(t))-(\varphi(t),\partial_t\varphi(t),\partial_{tt}\varphi(t))\|_{H^{2-\ell}_{(0)}\times H^{1-\ell}_{(0)}\times H^{-\ell}_{(0)}}\\&\leq Ch^{2\ell/3}\left\|(\varphi^0,\varphi^1)\right\|_{H^2_{(0)}\times H^1_{(0)}}.\end{aligned}\tag{3.42}$$

In particular, if (ϕ_h^0,ϕ_h^1) *are discrete functions such that for some* $\ell_0\in(0,3]$, C_0 *independent of* $h>0$ *and* $\theta>0$,

$$\left\|(\phi_h^0,\phi_h^1)-(\varphi^0,\varphi^1)\right\|_{H^{2-\ell_0}_{(0)}\times H^{1-\ell_0}_{(0)}}\leq C_0h^\theta,\tag{3.43}$$

then denoting by ϕ_h *the corresponding solution of Eq.* (3.1), *we have*

$$\begin{aligned}\sup_{t\in[0,T]}&\|(\phi_h(t),\partial_t\phi_h(t),\partial_{tt}\phi_h(t))-(\varphi(t),\partial_t\varphi(t),\partial_{tt}\varphi(t))\|_{H^{2-\ell_0}_{(0)}\times H^{1-\ell_0}_{(0)}\times H^{-\ell_0}_{(0)}}\\&\leq C\left(h^{2\ell_0/3}\left\|(\varphi^0,\varphi^1)\right\|_{H^2_{(0)}\times H^1_{(0)}}+C_0h^\theta\right).\end{aligned}\tag{3.44}$$

Proof. The proof of Eq. (3.42) again follows the one of Proposition 3.4. This time, following Eqs. (3.24)–(3.25), we get

$$\begin{aligned}&\|\varphi_h(t)-\varphi(t)\|^2_{H^{2-\ell}_{(0)}}+\|\partial_t\varphi_h(t)-\partial_t\varphi(t)\|^2_{H^{1-\ell}_{(0)}}\\&\leq C\left(n(h)^{6-2\ell}h^4+\frac{1}{n(h)^{2\ell}}\right)\left\|(\varphi^0,\varphi^1)\right\|^2_{H^2_{(0)}\times H^1_{(0)}}.\end{aligned}$$

The proof of the estimate

$$\sup_{t\in[0,T]}\|\partial_{tt}\varphi_h(t)-\partial_{tt}\varphi(t)\|_{H^{-\ell}_{(0)}}\leq Ch^{2\ell/3}\left\|(\varphi^0,\varphi^1)\right\|_{H^2_{(0)}\times H^1_{(0)}}$$

can be done by writing

$$\partial_{tt}\varphi_h(t)-\partial_{tt}\varphi(t)=\sum_{|k|=1}^{n(h)}\hat{\varphi}_kw^{|k|}\left(-\mu_k(h)^2e^{i\mu_k(h)t}+\mu_k^2e^{i\mu_kt}\right)+\sum_{n(h)+1}^{\infty}\hat{\varphi}_kw^{|k|}\mu_k^2e^{i\mu_kt}$$

and by using the estimate

$$\left| -\mu_k(h)^2 e^{i\mu_k(h)t} + \mu_k^2 e^{i\mu_k t} \right| \le C k^5 h^2.$$

The complete proof of Eq. (3.42) is left to the reader.

The proof of Eq. (3.44) for initial data satisfying Eq. (3.43) is very similar to the one of Proposition 3.7 and is based on the following facts:

- For any ψ_h solution of the discrete wave equation (3.1), for all $\ell \in \mathbb{Z}$, the $H^{2-\ell}_{(0)}(0,1) \times H^{1-\ell}_{(0)}(0,1)$-norm of $(\psi_h(t), \partial_t \psi_h(t))$ is independent of the time $t \ge 0$, as one easily checks by writing the solutions under the form

$$\psi_h(t) = \sum_{k=1}^{N} w^k \left(\hat{\psi}_k e^{i\mu_k(h)t} + \hat{\psi}_{-k} e^{-i\mu_k(h)t} \right).$$

  Applying this remark to $(\psi_h, \partial_t \psi_h)$ and to $(\partial_t \psi_h, \partial_{tt} \psi_h)$ for $\psi_h = \phi_h - \varphi_h$, we get

$$\begin{aligned} &\sup_{t\in[0,T]} \|(\phi_h(t), \partial_t \phi_h(t), \partial_{tt}\phi_h(t)) - (\varphi(t), \partial_t \varphi(t), \partial_{tt}\varphi(t))\|_{H^{2-\ell_0}_{(0)} \times H^{1-\ell_0}_{(0)} \times H^{-\ell_0}_{(0)}} \\ &\le C \Bigg(h^{2\ell_0/3} \left\|(\varphi^0, \varphi^1)\right\|_{H^2_{(0)} \times H^1_{(0)}} \\ &\quad + \left\|(\phi_h^0, \phi_h^1, \Delta_h \phi_h^0) - (\varphi_h^0, \varphi_h^1, \Delta_h \varphi_h^0)\right\|_{H^{2-\ell_0}_{(0)} \times H^{1-\ell_0}_{(0)} \times H^{-\ell_0}_{(0)}} \Bigg). \end{aligned}$$

- By construction,

$$\left\|\Delta_h \phi_h^0 - \Delta_h \varphi_h^0\right\|_{H^{-\ell_0}_{(0)}} \le C \left\|\phi_h^0 - \varphi_h^0\right\|_{H^{2-\ell_0}_{(0)}};$$

  hence

$$\begin{aligned} &\left\|(\phi_h^0, \phi_h^1, \Delta_h \phi_h^0) - (\varphi_h^0, \varphi_h^1, \Delta_h \varphi_h^0)\right\|_{H^{2-\ell_0}_{(0)} \times H^{1-\ell_0}_{(0)} \times H^{-\ell_0}_{(0)}} \\ &\le C \left\|(\phi_h^0, \phi_h^1) - (\varphi_h^0, \varphi_h^1)\right\|_{H^{2-\ell_0}_{(0)} \times H^{1-\ell_0}_{(0)}}. \end{aligned}$$

- We finally conclude Eq. (3.44) by using Eq. (3.43) and the estimate (3.42) for $t = 0$. □

3.5 Numerics

In this section, we briefly illustrate the above convergence results on the normal derivatives. The rate of convergence of the discrete trajectories towards the continuous ones is well known and well illustrated in the literature.

We thus choose an initial data $(\varphi^0, \varphi^1) \in H^1_0(0,1) \times L^2(0,1)$.

For $N \in \mathbb{N}$, we set $h = 1/(N+1)$ and take $(\varphi_h^0, \varphi_h^1)$ defined by $\varphi_{j,h}^0 = \varphi^0(jh)$ and $\varphi_{j,h}^1 = \int_{((j-1/2)h,(j+1/2)h)} \varphi^1(jh)$ for all $j \in \{1, \dots, N\}$. We then compute φ_h the corresponding solution of Eq. (3.1) and the corresponding discrete derivative at $x = 1$, i.e., $-\varphi_{N,h}(t)/h$.

Note that, actually, this discrete solution should rather be denoted as $\varphi_{h,\Delta t}$ since we also discretize in time using an explicit scheme. More precisely, if $\varphi_{h,\Delta t}^k$ denotes the approximation of φ_h at time $k\Delta t$, we solve

$$\varphi_h^{k+1} = 2\varphi_h^k - \varphi_h^{k-1} - (\Delta t)^2 \Delta_h \varphi_h^k. \tag{3.45}$$

The CFL condition is chosen such that $\Delta t/h = 0.2$ so that the convergence of the scheme (in what concurs solving the boundary–initial value problem) is ensured.

Since our goal is to estimate rates of convergence, we also need a reference data. In order to do that, we expand the initial data (φ^0, φ^1) in Fourier:

$$\varphi^0 = \sum_{k=1}^{\infty} \hat{a}_k w^k, \qquad \varphi^1 = \sum_{k=1}^{\infty} \hat{b}_k w^k.$$

The corresponding solution φ of Eq. (3.2) is then explicitly given by

$$\varphi(t) = \sum_{k=1}^{\infty} \left(\hat{a}_k \cos(k\pi t) + \hat{b}_k \frac{\sin(k\pi t)}{k\pi} \right) w^k,$$

so that

$$\partial_x \varphi(t,1) = \sum_{k=1}^{\infty} \left(\hat{a}_k \cos(k\pi t) + \hat{b}_k \frac{\sin(k\pi t)}{k\pi} \right) \sqrt{2}(-1)^k k\pi. \tag{3.46}$$

Of course, we cannot compute numerically these Fourier series for the continuous solutions of Eq. (3.2) since they involve infinite sums. So we take a reference number N_{ref} large enough and replace the infinite sum in formula (3.46) by a truncated version up to N_{ref}. N_{ref} is taken to be large compared to N, the number of nodes in the space discretization involved in the computations of $\varphi_{N,h}(t)/h$. We thus approximate the normal derivative by

$$(\partial_x \varphi(t,1))_{\mathrm{ref}} = \sum_{k=1}^{N_{\mathrm{ref}}} \left(\hat{a}_k \cos(k\pi t) + \hat{b}_k \frac{\sin(k\pi t)}{k\pi} \right) \sqrt{2}(-1)^k k\pi.$$

In the computations below, we take $N_{\mathrm{ref}} = 1{,}000$ for N varying between 200 and 400.

In Fig. 3.1 (left), we have chosen (φ^0, φ^1) as follows:

$$\varphi^0(x) = \sin(\pi x), \quad \varphi^1(x) = 0. \tag{3.47}$$

In this particular case, the continuous solution involves one single Fourier mode. So, we could have taken $N_{\mathrm{ref}} = 1$. Figure 3.1 (left) represents the $L^2(0,T)$-norm of $(\partial_x \varphi(t,1))_{\text{ref}} + \varphi_{N,h}(t)/h$ for $T = 1$ versus N in logarithmic scales. The slope of the linear regression is -1.99, thus very close to -2, the rate predicted by Proposition 3.7.

We then test the initial data

$$\varphi^0(x) = 0, \quad \varphi^1(x) = \begin{cases} -x & \text{if } x < 1/2, \\ -x+1 & \text{if } x > 1/2, \end{cases} \tag{3.48}$$

and plot the error in Fig. 3.1 (middle). The initial data velocity only belongs to $\cap_{\varepsilon>0} H^{1/2-\varepsilon}_{(0)}(0,1)$, so the predicted rate of convergence given by Proposition 3.7 is $-(1/3)^-$. This is indeed very close to the slope -0.31 observed in Fig. 3.1 (right).

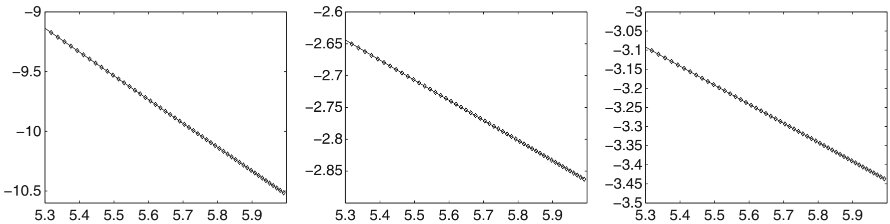

Fig. 3.1 Plot of $|(\partial_x \varphi(t,1))_{\text{ref}} + \varphi_{N,h}(t)/h|_{L^2(0,T)}$ versus $\log(N)$ for $N \in \{200,\ldots,400\}$, $N_{\text{ref}} = 1{,}000$ and $T = 1$. *Left*: for the initial data (φ^0, φ^1) in Eq. (3.47), slope of the linear regression $= -1.99$. *Middle*: for the initial data (φ^0, φ^1) in Eq. (3.48), slope $= -0.31$. *Right*: for the initial data (φ^0, φ^1) in Eq. (3.49), with $(\partial_x \varphi(t,1))_{\text{ref}} = -1+t$ in this case, slope $= -0.5$.

These numerical experiments both confirm the accuracy of the rates of convergence derived in Proposition 3.7.

We then test the initial data

$$\varphi^0(x) = 0, \quad \varphi^1(x) = x. \tag{3.49}$$

These data are smooth but $\varphi^1(1) \neq 0$. Hence φ^1 only belongs to $\cap_{\varepsilon>0} H^{1/2-\varepsilon}_{(0)}(0,1)$ and we thus expect a convergence rate of order $h^{1/3}$. Note that in this case, the normal derivative of the solution at $x = 1$ can be computed explicitly using Fourier series and $\partial_x \varphi(t,1) = -1+t$ (recall the formula (3.46)). Of course, we are thus going to use this explicit expression to compute $(\partial_x \varphi(t,1))_{\text{ref}} = -1+t$ in this case.

Note that the numerical simulations yield the slope -0.5 for the linear regression (see Fig. 3.1 (right)). This error term mainly comes from the fact that the continuous solution φ of Eq. (3.2) does not satisfy $\partial_x \varphi^0(x) = -1$ as the computation $(\partial_x \varphi(t,1))_{\text{ref}} = -1+t$ would imply for $t = 0$. This creates a layer close to $t = 0$ that the numerical method has some difficulties to handle. In Fig. 3.2, we represent the normal derivative computed numerically for $N = 300$ and compare it with the continuous normal derivative $\partial_x \varphi(t,1) = -1+t$. As one can see, there is a boundary layer close to $t = 0$.

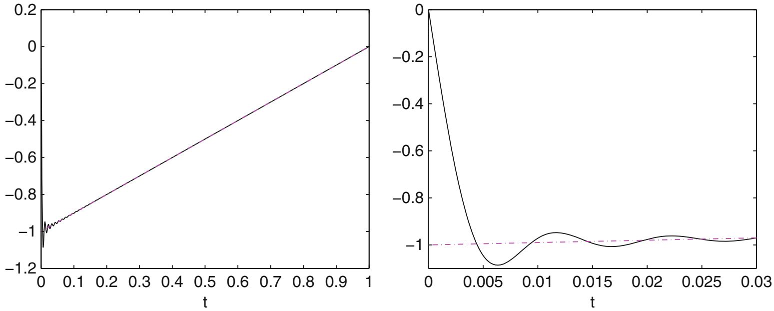

Fig. 3.2 Plot of $-\varphi_{N,h}(t)/h$ computed for $N = 300$ (*black solid line*) and of $(\partial_x \varphi(t,1))_{\text{ref}} = -1+t$ (*red dash dot line*) for (φ^0, φ^1) in Eq. (3.49). *Left*: on the time interval $(0,1)$. *Right*: a zoom on the time interval $(0,0.03)$.

This last example illustrates the fact that the boundary conditions play an important role for the regularity properties of the trajectory of the continuous model (3.2) and therefore also have an influence on the rates of convergence of the corresponding approximations given by Eq. (3.1). The above example also confirms the good accuracy of the rates of convergence given in Proposition 3.7 when the regularity properties are limited by the boundary conditions.