# Chapter 5
# Chemicals in the Environment, Turbulent Transport

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## Glossary

| | |
|---|---|
| Diffusion | The spreading of fluid constituents through the motion inherent to atoms and molecules. |
| Diffusion coefficient | A coefficient that describes the tendency of molecules to spread a constituent mass |
| Dirac delta | An impulse of a given quantity (mass) that occurs over an infinitely short time or space. |
| Kinematic viscosity | The fluid viscosity divided by the fluid density, resulting in units that are similar to a diffusion coefficient, or length squared per time. |
| Laminar flow | Flow that has no turbulent eddies, where the fluid flows in laminas and diffusion creates the mixing of the fluid. |
| Prandtl's mixing length | The mean length that the turbulence in the flow will transport mass, momentum, or energy. |
| Reynolds number | The ratio of inertial to viscous forces, resulting in a meaningful velocity times a meaningful distance divided by kinematic viscosity. |
| Turbulent diffusion | The mixing of fluids through turbulent eddies created by convection. |

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Turbulent diffusion coefficient — A coefficient that comes from the multiplication of two turbulent velocities of the flow, divided by density of the fluid. The coefficient's location in the mass transport equation is similar to diffusion coefficients, and the units are similar; so it is called a "turbulent diffusion coefficient."

## Definition of Turbulent Transport in the Environment

It is fairly safe to state that, except for flow through porous media, the environment experiences turbulent flow. To emphasize this point, the constriction of a water flow or airflow that would be required will be considered to have the other option, laminar flow.

An experimentally based rule of thumb is that laminar flow typically occurs when the pipe Reynolds number, $Vd/\nu$, is less than roughly 2,000, or when an open-channel Reynolds number, $Vh/\nu$, is less than roughly 500, where $V$ is the cross-sectional mean velocity, $d$ is the pipe diameter, $\nu$ is the kinematic viscosity of the fluid, and $h$ is the channel depth. The diameter or depth that would not be exceeded to have laminar flow by these experimental criteria is given in Table 5.1.

Table 5.1 shows that with the boundary conditions present in most environmental flows, that is, the earth's surface, ocean top and bottom, river or lake bottom, etc., turbulent flow would be the predominant condition. One exception that is important for interfacial mass transfer would be very close to an interface, such as air–solid, solid–liquid, or air–water interfaces, where the distance from the interface is too small for turbulence to occur due to the high viscous dissipation. Because turbulence is an important source of mass transfer, the lack of turbulence very near the interface is also significant for mass transfer, where diffusion once again becomes the predominant transport mechanism.

**Table 5.1** Maximum diameter or depth to have laminar flow, with the transition Reynolds number for a pipe at 2,000

| Water ($\nu = 10^{-6}$ m$^2$/s) | | Air ($\nu \sim 2 \times 10^{-5}$ m$^2$/s) | |
|---|---|---|---|
| $V$ (m/s) | $D$ (m) | $h$ (m) | $d$ (m) |
| 10 | $2 \times 10^{-4}$ | $5 \times 10^{-5}$ | 0.004 |
| 3 | $7 \times 10^{-4}$ | $1.5 \times 10^{-4}$ | 0.014 |
| 1 | 0.002 | 0.0005 | 0.04 |
| 0.3 | 0.007 | 0.0015 | 0.14 |
| 0.1 | 0.02 | 0.005 | 0.4 |
| 0.03 | 0.07 | 0.015 | 1.4 |
| 0.01 | 0.2 | 0.05 | 4.0 |

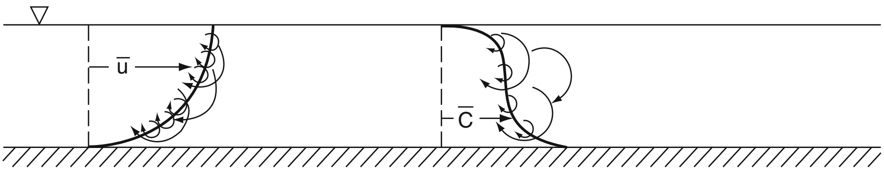



**Fig. 5.1** Turbulent eddies superimposed on a temporal-mean velocity and temporal-mean concentration profiles (From Gulliver [1])

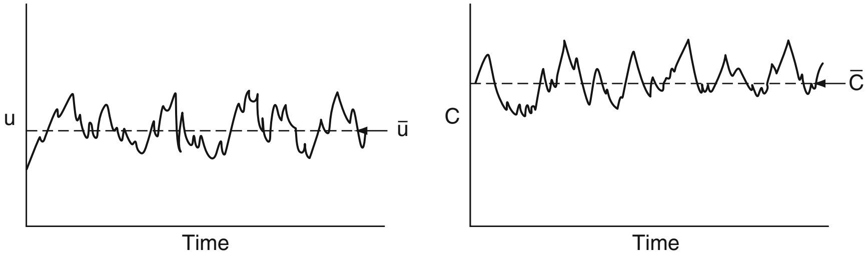



**Fig. 5.2** Time traces of typical measurements of velocity and concentration in a turbulent flow

## Introduction

What is turbulent flow? The simple illustration of a free-surface flow given in Fig. 5.1 is used to describe the essential points of the turbulence phenomena. Turbulent open-channel flow can be described with a temporal mean velocity profile which reaches a steady value with turbulent eddies superimposed upon it. These turbulent eddies are continually moving about in three dimensions, only restricted by the boundaries of the flow, such that they are eliminated from the temporal mean velocity profile, $\bar{u}$ in Fig. 2.1. It is this temporal mean velocity profile that is normally sketched in turbulent flows.

There will also be a temporal mean concentration. If there is a source or sink in the flow, or transport across the boundaries as in Fig. 5.1, then the temporal mean concentration profile will eventually reach a value such as that given in Fig. 5.1. This flux of compound seems to be from the bottom toward the top of the flow. Superimposed upon this temporal mean concentration profile will be short-term variations in concentration caused by turbulent transport. The concentration profile is “flatter” in the middle of the flow because the large turbulent eddies that transport mass quickly are not as constrained by the flow boundaries in this region. Now, if a concentration-velocity probe is placed into the flow at one location, the two traces of velocity and concentration versus time would look something like that shown in Fig. 5.2.

Turbulent diffusion is thus not really diffusion, but the mixing of chemicals through turbulent eddies created by convection. Turbulent diffusion is thus a form of convection. Although it has the appearance of diffusion in the end, that is, random mixing similar to diffusion, the causes of diffusion and turbulent diffusion are very different. Since the end products are similar, diffusion coefficients and turbulent diffusion coefficients are often simply added together.

It is convenient to divide the velocity and concentration traces into temporal mean values and fluctuating components:

$$u = \bar{u} + u' \tag{5.1}$$

and

$$C = \bar{C} + C' \tag{5.2}$$

where $\bar{u}$ is the temporal mean velocity at a point location, $u'$ is the fluctuating component of velocity (variable over time), $\bar{C}$ is the temporal mean concentration at a point location, and $C'$ is the fluctuating concentration component of concentration which is also variable over time. Formal definitions of $\bar{u}$ and $\bar{C}$ are as follows:

$$\bar{u} = \frac{1}{\Delta t} \int_0^{\Delta t} u \, dt \tag{5.3}$$

and

$$\bar{C} = \frac{1}{\Delta t} \int_0^{\Delta t} C \, dt \tag{5.4}$$

where $\Delta t$ is long compared to the time period of the oscillating components.

## Mass Transport Equation with Turbulent Diffusion Coefficients

In this section the most common equations for dealing with mass transport in a turbulent flow will be derived. Beginning with the mass transport equation developed in the entry "▾Transport in the Environment,"

$$\begin{aligned} &\frac{\partial C}{\partial t} + \frac{\partial (uC)}{\partial x} + \frac{\partial (vC)}{\partial y} + \frac{\partial (wC)}{\partial z} \\ &= \frac{\partial}{\partial x}\left(D\frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial z}\left(D\frac{\partial C}{\partial z}\right) + S \end{aligned} \tag{5.5}$$

the temporal mean of the entire equation will be taken and eventually one will end up with an equation that incorporates turbulent diffusion coefficients.

In a turbulent flow field, Eq. 5.5 is difficult to apply because $C$, $u$, $v$, and $w$ are all highly variable functions of time and space. Osborne Reynolds [2] reduced the complexities of applying Eq. 5.5 to a turbulent flow by taking the temporal mean of each term (e.g., the entire equation). Then, the mean value of a fluctuating component will be equal to zero, or

$$\overline{\frac{\partial C}{\partial t}} = \overline{\frac{\partial(\bar{C} + C')}{\partial t}} = \overline{\frac{\partial \bar{C}}{\partial t}} + \overline{\frac{\partial C'}{\partial t}} = \frac{\partial \bar{C}}{\partial t} + 0 \tag{5.6}$$

Equation 5.6, the change of a temporal mean over time, may seem like a misnomer, but it will be left in to identify changes in $\bar{C}$ over a longer time period than $\Delta t$. Continuing,

$$\overline{\frac{\partial C}{\partial x}} = \overline{\frac{\partial(\bar{C} + C')}{\partial x}} = \overline{\frac{\partial \bar{C}}{\partial x}} + \overline{\frac{\partial C'}{\partial x}} = \frac{\partial \bar{C}}{\partial x} \tag{5.7}$$

$$\overline{\frac{\partial C}{\partial y}} = \overline{\frac{\partial(\bar{C} + C')}{\partial y}} = \overline{\frac{\partial \bar{C}}{\partial y}} + \overline{\frac{\partial C'}{\partial y}} = \frac{\partial \bar{C}}{\partial y} \tag{5.8}$$

$$\overline{\frac{\partial C}{\partial z}} = \overline{\frac{\partial(\bar{C} + C')}{\partial z}} = \overline{\frac{\partial \bar{C}}{\partial z}} + \overline{\frac{\partial C'}{\partial z}} = \frac{\partial \bar{C}}{\partial z} \tag{5.9}$$

However, the temporal mean value of two fluctuating components, multiplied by each other, will not necessarily be zero:

$$\overline{u'C'} \neq \bar{u}'\bar{C}' \tag{5.10}$$

This is similar to a least-square regression, where the mean error is zero, but the sum of square error is not. The $x$-component of our convective transport terms will be dealt with first:

$$\overline{uC} = \overline{(\bar{u} + u')(\bar{C} + c')} = \overline{\bar{u}\bar{C}} + \overline{\bar{u}C'} + \overline{u'\bar{C}} + \overline{u'C'} \tag{5.11}$$

Three of the four terms in Eq. 5.11 may be reduced to something known:

$$\overline{\bar{u}\bar{C}} = \bar{u}\bar{C} \tag{5.12}$$

$$\overline{\bar{u}C'} = 0 \tag{5.13}$$

$$\overline{u'\bar{C}} = 0 \tag{5.14}$$

but, the fourth term will take some additional consideration, because it is not equal to zero:

$$\overline{u'C'} \neq 0 \tag{5.15}$$

By inference, the following can be written for all three convective transport terms:

$$\overline{uC} = \bar{u}\bar{C} + \overline{u'C'} \tag{5.16}$$

$$\overline{vC} = \bar{v}\bar{C} + \overline{v'C'} \tag{5.17}$$

and

$$\overline{wC} = \bar{w}\bar{C} + \overline{w'C'} \tag{5.18}$$

Finally, applying continuity $(\bar{u} + \bar{v} + \bar{w} = 0)$ to Eq. 5.5 and taking the temporal mean results of Eqs. 5.6, 5.7, 5.8, 5.9, 5.16, 5.17, and 5.18

$$\begin{aligned} &\frac{\partial \bar{C}}{\partial t} + \bar{u}\frac{\partial \bar{C}}{\partial x} + \bar{v}\frac{\partial \bar{C}}{\partial y} + \bar{w}\frac{\partial \bar{C}}{\partial z} \\ &= -\frac{\partial}{\partial x}\overline{u'C'} - \frac{\partial}{\partial y}\overline{v'C'} - \frac{\partial}{\partial z}\overline{w'C'} + \frac{\partial}{\partial x}\left(D\frac{\partial \bar{C}}{\partial x}\right) \\ &\quad + \frac{\partial}{\partial y}\left(D\frac{\partial \bar{C}}{\partial y}\right) + \frac{\partial}{\partial z}\left(D\frac{\partial \bar{C}}{\partial z}\right) + \bar{S} \end{aligned} \tag{5.19}$$

where the turbulent convective transport term can be moved to the right-hand side, because the concentration distribution that results from these terms looks similar to diffusion.

With this temporal mean process, we have reduced the terms for which we will have difficulty defining boundary conditions in turbulent flow fields from seven in Eq. 5.5 to three in Eq. 5.19. We will now deal with these three terms.

The diffusion equation is a useful and convenient equation to describe mixing in environmental flows, where the boundaries are often not easily defined. It also lends itself to analytical solutions and is fairly straightforward in numerical solutions. Although an alternative technique for solutions to mixing problems is the mixed cell method described in the entry "▾Chemicals in the Environment, Dispersive Transport," there are complications when applied to multiple dimensions and to flows that vary with space and time. Finally, we are comfortable with the diffusion equation, so we would prefer to use that to describe turbulent mixing if possible.

Therefore, let us consider the following thought process: if the end result of turbulence, when visualized from sufficient distance, looks like diffusion with seemingly random fluctuations, then we should be able to identify the terms causing these fluctuations in Eq. 5.19. Once identified, they can be related to a "turbulent diffusion coefficient" that describes the diffusion caused by turbulent eddies. Looking over the terms in Eq. 5.19 from left to right, we see an unsteady term, three mean convective terms, the three "unknown" terms, the diffusive terms and the source/sink rate terms. The "unknown" terms are the only possibility to describe turbulent diffusion.

In the late nineteenth century, Boussinesq [3] probably went through something similar to the thought process described above. The end result was the *Boussinesq eddy diffusion coefficient*:

$$-\overline{u'C'} = \varepsilon_x \frac{\partial \bar{C}}{\partial x} \tag{5.20a}$$

$$-\overline{v'C'} = \varepsilon_y \frac{\partial \bar{C}}{\partial y} \tag{5.20b}$$

$$-\overline{w'C'} = \varepsilon_z \frac{\partial \bar{C}}{\partial z} \tag{5.20c}$$

where $\varepsilon_x$, $\varepsilon_y$, and $\varepsilon_z$ are the turbulent (or eddy) diffusion coefficients, with units of $m^2/s$ similar to the (molecular) diffusion coefficients.

Then Eq. 5.19 with Eqs. 5.20a, 5.20b, and 5.20c becomes

$$\begin{aligned}\frac{\partial \bar{C}}{\partial t} + \bar{u}\frac{\partial \bar{C}}{\partial x} + \bar{v}\frac{\partial \bar{C}}{\partial y} + \bar{w}\frac{\partial \bar{C}}{\partial z} &= \frac{\partial}{\partial x}\left[(D + \varepsilon_x)\frac{\partial \bar{C}}{\partial x}\right] \\ &+ \frac{\partial}{\partial y}\left[(D + \varepsilon_y)\frac{\partial \bar{C}}{\partial y}\right] + \frac{\partial}{\partial z}\left[(D + \varepsilon_z)\frac{\partial \bar{C}}{\partial z}\right] + \bar{S}\end{aligned} \tag{5.21}$$

Turbulent diffusion is created by the flow field, which can vary with distance. Hence, turbulent diffusion coefficient cannot be assumed constant with distance. Removing that assumption leaves turbulent diffusion coefficient inside of the brackets.

## *Character of Turbulent Diffusion Coefficients*

A turbulent eddy can be visualized as a large number of differently sized rotating spheres or ellipsoids. Each sphere has sub-spheres, and so on until the smallest eddy size is reached. The smallest eddies are dissipated by viscosity, which explains why turbulence does not occur in narrow passages: there is simply no room for eddies that will not be dissipated by viscosity.

The cause of the rotation is shear forces created by solid boundaries or variations in velocity lateral to the primary flow direction. A buoyant plume of smoke or steam, for example, will have a temporal mean velocity profile develop laterally to the plume, as the rising plume mixes with the ambient air. Turbulent eddies are formed by this velocity gradient, and can be seen at the edge of the smoke or steam plume. The magnitude of turbulent diffusion coefficients is primarily dependent

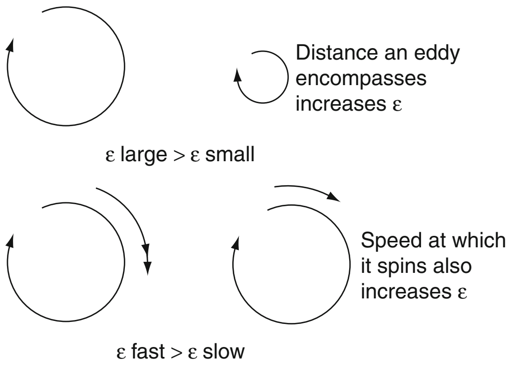



**Fig. 5.3** Character of turbulent diffusion coefficients (From Gulliver [1])

upon the scale of turbulent eddies and the speed of the eddy rotation. As illustrated in Fig. 5.3, a large eddy will have greater eddy diffusion coefficient than a small eddy because it will transport a compound (or solute) farther in one rotation. Likewise, a faster spinning eddy will have a larger eddy diffusion coefficient than one which is the same size but spinning more slowly because the solute simply gets there faster. These two facts provide meaning to the following observations:

1. The largest scale of turbulence is roughly equal to the smallest overall scale of the flow field. This may be seen in comparing the size of eddies at the edge of the smoke or steam plume to the width of the plume.
2. The rotational eddy velocity is roughly proportional to the velocity gradient times the eddy scale.
3. Eddy size decreases near boundaries to the flow field. Since the eddy size is zero at a solid boundary, and often close to zero at a fluid density interface (like an air–water interface), the turbulent eddy size has to decrease as one approaches the boundaries. In addition, since the flow cannot go through a boundary, the largest eddy size cannot be greater than the distance from the center of the eddy to the boundary.
4. Turbulent diffusion occurs because turbulent eddies are transporting mass, momentum, and energy over the eddy scale at the rotational velocity. This transport rate is generally orders of magnitude greater than the transport rate due to molecular motion. Thus, when a flow is turbulent, diffusion is normally ignored because $\varepsilon \gg D$. The exception is very near the flow boundaries, where the eddy size (and turbulent diffusion coefficient) decreases to zero.

Thus, what influences the velocity and scale of eddies? For the most part, it is the velocity gradients and scale of the flow. Velocity gradients are the change in velocity over distance. If we have a high velocity, we typically have a large velocity gradient somewhere in the flow field. At solid walls, for example, the velocity must go to zero. Thus, *the large velocity difference results in large velocity gradients, which results in faster spinning eddies and a larger turbulent diffusion coefficient*. This process is illustrated in Fig. 5.4.

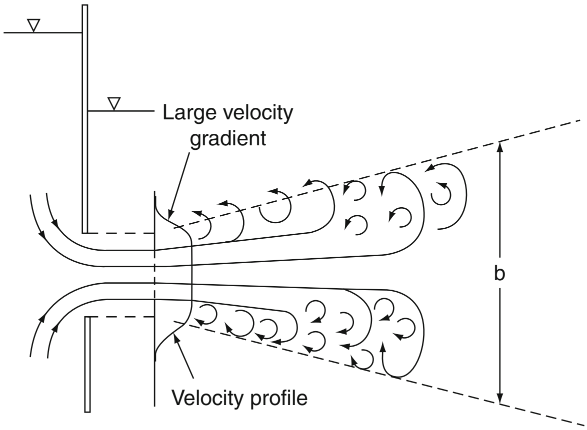



**Fig. 5.4** Eddy formation at the edge of a jet issuing into a tank illustrates the importance of velocity gradients in eddy diffusion coefficient (From Gulliver [1])

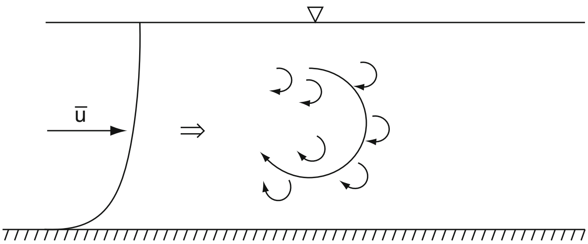



**Fig. 5.5** Large and small eddies in an open-channel flow. The large eddies perform most of the top-to-bottom transport (From Gulliver [1])

The scale of the flow field is also important because *the larger eddies perform most of the transport*. The small eddies are always there in a turbulent flow, and their existence is important for local mixing. It is the large eddies, however, that are the most responsible for transport, as illustrated in Fig. 5.5.

The four observations, listed above, were enough for Ludwig Prandtl [4] to hypothesize a simple model for describing turbulent transport that works surprisingly well, considering the complexity of turbulent flow.

## Prandtl's Mixing Length Hypothesis for Turbulent Flow

Prandtl's mixing length hypothesis was developed for momentum transport, instead of mass transport. The end result was a turbulent viscosity, instead of a turbulent diffusivity. However, since both turbulent viscosity and turbulent diffusion coefficient are properties of the flow field, they are related. Turbulent viscosity describes the transport of momentum by turbulence, and turbulent diffusivity describes the transport of mass by the same turbulence. Thus,

$$\varepsilon_x = \mu_{tx}/\rho,\ \varepsilon_y = \mu_{ty}/\rho,\ \text{and}\ \ \varepsilon_z = \mu_{tz}/\rho$$

where $\mu_{tx}$, $\mu_{ty}$, and $\mu_{tz}$ are the turbulent viscosity in the $x$, $y$, and $z$ directions. Now, for the $x$-component of momentum ($\rho u$), the Boussinesq approximation is

$$-\rho\,\overline{u'u'} = \mu_{tx}\frac{\partial \bar{u}}{\partial x} \tag{5.22}$$

$$-\rho\,\overline{v'u'} = \mu_{ty}\frac{\partial \bar{u}}{\partial y} \tag{5.23}$$

$$-\rho\,\overline{w'u'} = \mu_{tz}\frac{\partial \bar{u}}{\partial z} \tag{5.24}$$

Let us consider the fully developed velocity profile in the middle of a wide-open channel, with $x$-, $y$-, and $z$-components in the longitudinal, lateral, and vertical directions, respectively. It is fully developed because $\partial\bar{u}/\partial x$ is close to zero. The fact that it is a wide channel means that $\partial\bar{u}/\partial y$ also is very small in the middle. From Eqs. 5.22 and 5.23, we can see that the turbulent transport of momentum in the $x$- and $y$-directions will be small because the gradients are small. Equation 5.24 indicates that there will be a net turbulent transport of momentum in the z-direction.

$$-\overline{w'u'} = \varepsilon_z\frac{\partial \bar{u}}{\partial z} \neq 0 \tag{5.25}$$

Now, half of the $w'$ values will be positive, and the other half will be negative. We will use this criterion to divide them into two parts:

$$\overline{w'u'} = \overline{w'u'}^{+} + \overline{w'u'}^{-} \tag{5.26}$$

where $w'u'^{+}$ has a value when $w'$ is positive and is equal to zero when $w'$ is negative. $w'u'^{-}$ has a value when $w'$ is negative and is equal to zero when $w'$ is positive. Consider the cases when $w'$ is positive. Then Eq. 5.26 becomes

$$\overline{w'u'} = \overline{w'u'}^{+} + 0 \tag{5.27}$$

Let us assume that an eddy of length $L$ is pulling a blob of fluid upward, as illustrated in Fig. 5.6. On average, the blob will have an x-component of velocity equal to $\bar{u}(z - L/2)$, where $z$ is the location where $u'$ is to be estimated. Thus, the eddy pulls up, on average, the $u$ value that is at $z - L/2$. This will become the deviation from the temporal mean velocity at location $z$:

$$u' = u - \bar{u} \approx \bar{u}(z - L/2) - \bar{u}(z) \cong \frac{1}{2}(\bar{u}(z - L) - \bar{u}(z)) \tag{5.28}$$

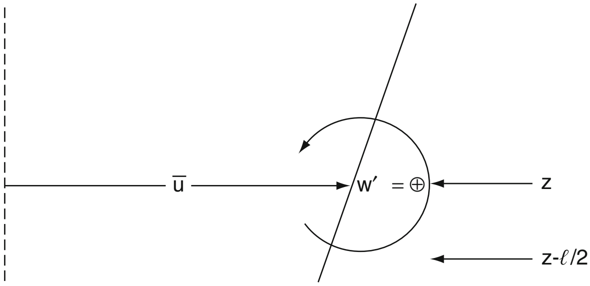



**Fig. 5.6** Illustration of the relationship between velocity profile, turbulent eddies, and mixing length

Equation 5.28 is a relation for a difference in velocity, which can be written as a velocity gradient times a distance:

$$u' = -\frac{\partial \bar{u}}{\partial x}\left(\frac{L}{2}\right) \tag{5.29}$$

| Velocity difference | = | velocity gradient | × | distance |
|---|---|---|---|---|

Then,

$$\overline{w'u'}^{+} \approx \overline{\frac{w'}{2}[\bar{u}(z-L) - \bar{u}(z)]} \approx -\frac{\overline{w'}}{2}L\frac{\partial \bar{u}}{\partial z} \tag{5.30}$$

the development is similar for $\overline{w'u'}^{-}$:

$$\overline{w'u'}^{-} \approx \overline{w'u'}^{+} \approx -\frac{\overline{w'}}{2}L\frac{\partial \bar{u}}{\partial z} \tag{5.31}$$

Now combining Eqs. 5.26, 5.30, and 5.31 gives

$$\overline{w'u'} = -\overline{w'L\frac{\partial \bar{u}}{\partial z}} \tag{5.32}$$

Because turbulent eddies tend to be close to spherical in shape:

$$|w'| \approx |u'| \tag{5.33}$$

and from Eq. 5.29:

$$w' \sim L\frac{\partial \bar{u}'}{\partial z} \tag{5.34}$$

**Table 5.2** Dynamic roughness lengths, $z_0$, for typical atmospheric surfaces (Turner [7])

| Surface Type | $z_0$ (m) |
|---|---|
| Urban | 1.-3 |
| Forest | 1.3 |
| Deciduous forest in winter | 0.5 |
| Desert shrubland | 0.3 |
| Wetland | 0.3 |
| Cropland (summer) | 0.2 |
| Cropland (winter) | 0.01 |
| Grassland (summer) | 0.1 |
| Grassland (winter) | 0.001 |
| Water | ~0.0001 |

If we substitute Eq. 5.34 into Eq. 5.32, and then substitute the result into Eq. 5.24, we get

$$-\overline{w'u'} = \varepsilon_z \frac{\partial \bar{u}}{\partial z} = L^2 \left( \frac{\partial \bar{u}}{\partial z} \right)^2 \tag{5.35}$$

or

$$\varepsilon_z = L^2 \left| \frac{\partial \bar{u}}{\partial z} \right| \tag{5.36}$$

Equation 5.36 is Prandtl's *mixing length hypothesis*, and it works well, considering that the basis for the equation is so empirical. However, Eq. 5.36 does present a challenge for us that mixing length, $L$, still needs to be specified. Measurements have shown us the following:

1. Near a wall, $L = \kappa z$, where κ is von Kármán's constant [5] and is very close to 0.4, and $z$ is the distance from the closest wall.

   Prandtl also made another assumption in this region, that $w'u'$ could be approximated by a constant equal to the mean wall shear stress, or

$$-\overline{w'u'} = \tau/\rho = u_*^2 \tag{5.37}$$

   Then, eliminating $\overline{w'u'}$ from Eqs. 5.35 and 5.37 results in the well-known *logarithmic velocity profile*:

$$\frac{\bar{u}}{u_*} = \frac{1}{k} \ell n \left( \frac{z}{z_0} \right) \tag{5.38}$$

where $u_*$ is the shear velocity at the wall, τ is the wall shear stress, and $z_0$ *is* an integration constant, often *called the dynamic roughness*. Table 5.2 provides some typical dynamic roughness lengths for atmospheric boundary layers. Applying Eq. 5.36 to 5.37 results in an equation for $\varepsilon_z$ in this region:

$$\varepsilon_z = \kappa u_* z \tag{5.39}$$

2. Very near a wall (approaching the laminar sublayer where the turbulence is so small that it is eliminated by the viscosity of the fluid), that is, for $zu_*/\nu < 35$, $L \sim y^2$ [6].

   Making the same assumption that $u'w'$ is approximately equal to wall shear stress, this relation for $L$ results in the following relation for velocity profile very near the wall:

$$\frac{\bar{u}}{u_*} = \beta \frac{\nu}{u_{**} z} \tag{5.40}$$

   Equation 5.40 is not used in mass transport calculations near a wall or interface because the unsteady character of mass transport in this region is very important, and Eq. 5.39 is for a temporal mean velocity profile.
3. Away from a wall, where the closest wall does not influence the velocity profile, $L$ is a function of another variable of the flow field (Prandtl [8]). For example, consider the jet mixer given in Fig. 5.4. In this case, the mixing length, $L$, is a function of the width of the jet or plume. As the jet/plume grows larger, the value of $L$ is larger.

   Here, it is easier to simply give the experimental relation for eddy diffusivity:

$$\varepsilon_z = \beta\, u_{\max} b \tag{5.41}$$

where $b$ is the width of the mixing zone, $\beta$ is a constant, and $u_{\max}$ is the maximum velocity in the jet at the given location, $x$.

Figure 5.7 gives some relationships for eddy diffusion coefficient profiles under different conditions that will be handy in applications of turbulent diffusive transport.

## Example Applications

***Example 5.1: Profile of eddy diffusion coefficient*** Estimate the eddy diffusivity profile for a wind velocity of 18 m/s measured at 10 m over a large lake (Fig. 5.8), and calculate the elevation above the water surface where $\varepsilon_z = D$ for water vapor. There is only one assumption needed:

1. The wind fetch is sufficient so that $\overline{U}_{10}$ is influenced by shear at the water surface (10 m is inside the boundary layer of the lake surface at this point).

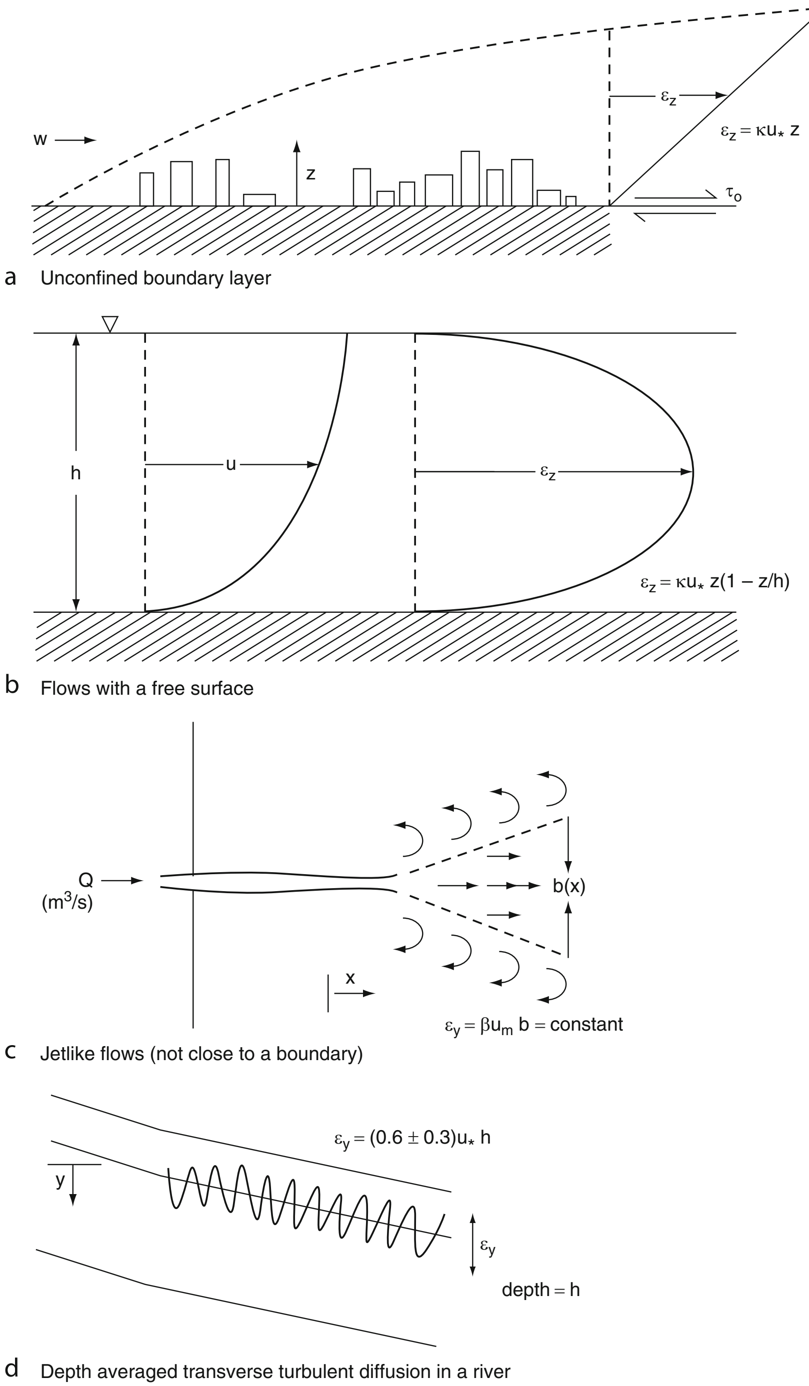
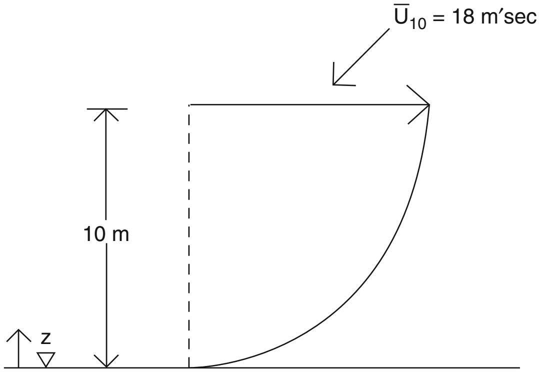



**Fig. 5.7** Profiles of eddy diffusion coefficient for various types of applications (From Gulliver [1])

**Fig. 5.8** Velocity profile over a large lake




Then, mixing length theory may be used with momentum transport to derive:

$$\frac{\partial \bar{u}}{\partial z} = \frac{u_*}{\kappa z} \tag{5.42}$$

and,

$$\varepsilon_z = \kappa\, u_* z \tag{5.43}$$

Now, Wu [9] has provided the following equation from a fit of field data:

$$u_* = 0.01\, \overline{U}_{10}\left(8 + 0.65\, \overline{U}_{10}\right)^{1/2} \tag{5.44}$$

which indicates that as the waves get larger at high wind speeds the boundary roughness effect upon $u_*$ increases by the factor $\left(8 + 0.65\, \overline{U}_{10}\right)^{1/2}$, where $\overline{U}_{10}$ is given in m/s.

Then,

$$\varepsilon_z(\mathrm{m}^2/\mathrm{s}) = 0.01\ \kappa\, \bar{u}_{10} z (8 + 0.65\, \bar{u}_{10})^{1/2} = 0.32\, z$$

when $z$ is given in meters. Now, the diffusion coefficient of water vapor in air is calculated to be $D = 2.6 \times 10^{-5}$ m$^2$/s. Then, the elevation at which the diffusivity of water vapor would equal eddy diffusivity in this case would be,

$$0.32z = 2.6 \times 10^{-5}$$

or,

$$z = 8 \times 10^{-5}\ \mathrm{m} = 0.08\ \mathrm{mm} = 80\ \ \mu\mathrm{m}$$

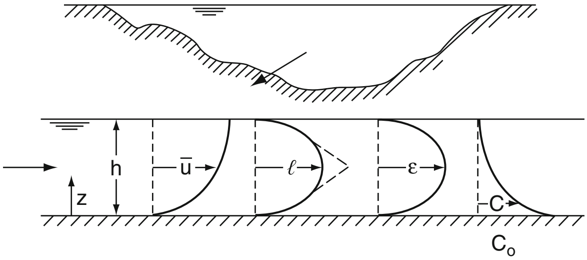



**Fig. 5.9** Lateral and longitudinal cross sections of a typical river

Thus at $z = 80$ µm elevation above the water surface, eddy diffusivity will be equal to the diffusivity of water. A similarly small elevation would result for almost any environmentally relevant compound. We can thus see that both ε and $D$ need to be considered simultaneously in Eq. 5.20 only very close to surfaces in turbulent flow, where ε approaches the diffusion coefficient. Otherwise, diffusivities can be ignored in solving turbulent flow transport problems, since $\varepsilon + D$ is essentially equal to ε.

***Example 5.2: Concentration profile of suspended sediment in a river (assuming*** $\boldsymbol{\varepsilon_z}$ ***is constant)*** We will apply Eq. 5.20 to solve for the concentration profile of suspended sediment in a river, with some simplifying assumptions. Suspended sediment is generally considered similar to a solute, in that it is a scalar quantity in Eq. 5.20, except that it has a settling velocity. We will also change our notation, in that the bars over the temporal mean values will be dropped. This is a common protocol in turbulent transport, and will be followed here for conformity. Thus, if an eddy diffusion coefficient,ε, is in the transport equation,

$$u \quad \text{means} \quad \bar{u}$$

$$v \quad \text{means} \quad \bar{v}$$

$$w \quad \text{means} \quad \bar{w} \text{ and}$$

$$C \quad \text{means} \quad \bar{C}$$

throughout the remainder of this entry. Fig. 5.9 gives a longitudinal and lateral cross section of our river. We will make the following assumptions:

1. The flow is steady over the long term, so that $\partial C/\partial t = 0$.
2. The flow is fully developed, such that any gradient with respect to $x$ is equal to zero ($\partial C/\partial x = 0$).
3. The river can be divided into a series of longitudinal planes with no significant interaction, such that $v = 0$ and $\varepsilon_y = 0$ (this is the assumption of the stream-tube computational models).
4. The vertical eddy diffusivity, $\varepsilon_z$, is a constant value.

Assumptions 3 and 4 are more difficult to justify.

The solute will have a vertical velocity, $w = -v_s$, where $v_s$ is the settling velocity of the suspended sediment.

Then, Eq. 5.20 becomes

$$-v_s \frac{\partial C}{\partial z} = \frac{\partial}{\partial z}\left[(D + \varepsilon_z)\frac{\partial C}{\partial z}\right] \tag{5.45}$$

where we have not yet applied assumption 4. We can move the settling velocity into the partial term:

$$\frac{\partial(-v_s C)}{\partial z} = \frac{\partial}{\partial z}\left[(D + \varepsilon_z)\frac{\partial C}{\partial z}\right] \tag{5.46}$$

and since both sides of Eq. 5.46 are a gradient with respect to $z$, the terms inside of the gradients must also be equal:

$$-v_s C = (D + \varepsilon_z)\frac{dC}{dz} \tag{5.47}$$

Equation 5.47 is converted to an ordinary differential equation because all variables are only a function of $z$. Now, we will deal with assumption 4. Fig. 5.7 gives the equation developed by Rouse [10] for $\varepsilon_z$:

$$\varepsilon_z = \kappa\, u_* z(1 - z/h) \tag{5.48}$$

where $u_*$ is the shear velocity at the bottom of the channel, or

$$u_* = \sqrt{\tau/\rho} \tag{5.49}$$

where $\tau$ is the shear stress at the wall. For a fully developed open-channel flow in a wide channel, the following relation is easily derived:

$$u_* = \sqrt{ghS} \tag{5.50}$$

This derivation can be found in a text on fluid mechanics or open-channel flow. Assumption 4 states that $\varepsilon_z = \bar{\varepsilon}_z$ for all values of $z$, where $\varepsilon_z$ is the depth average, or

$$\bar{\varepsilon}_z = \frac{1}{h}\int_0^h \varepsilon_z dz = \frac{\kappa u_*}{h}\int_o^h z(1 - z/h)dz = 0.067\, u_* h \tag{5.51}$$

where h is the depth of the stream. The term $\bar{\varepsilon}_z$ is almost always much greater than $D$ in a turbulent flow. Thus,

$$D + \overline{\varepsilon_z} \cong \overline{\varepsilon_z}$$

Now, substituting these equations into Eq. 5.47 results in

$$\overline{\varepsilon_z}\frac{dC}{dz} + v_s C = 0$$

We will solve this by separating variables,

$$\frac{dC}{C} = \frac{-v_s}{\bar{\varepsilon}_z} dz$$

integrating, and taking both sides of the solution to the power of $e$:

$$C = \beta_1\ e^{\frac{-v_s}{\bar{\varepsilon}_z} z}$$

Now, we need a boundary condition to determine $\beta_1$. This is difficult with suspended sediment profiles. We can develop a fairly good estimate of the distribution of suspended sediment once we have a known concentration at some location in the flow field. In the sediment transport field bed load and suspended load are often discussed. The relation between the two, and some experience and measurements of both simultaneously, can be used to predict an equivalent suspended sediment concentration at the bed. Then, the relevant boundary condition is

1. At

$$z = 0,\ C = C_0.$$

where $C_0$ is the concentration that has been determined from the bed load–suspended load relationship. Applying this boundary condition gives $\beta_1$ = $C_0$, and our solution is

$$C = C_0\ e^{-v_s z/\bar{\varepsilon}_z}$$

The result is illustrated in Fig. 5.10. This problem can also be solved *without* assumption 4 [10].

***Example 5.3: Concentration of organic compounds released into the air by an industrial plant (application of the product rule to error function solutions)*** There is some concern about the emissions from the adhesives produced in an industrial plant. Specifically, the town of Scream Hollow is 1 km away from the plant, where citizens have begun to complain about odors from the plant and of headaches.

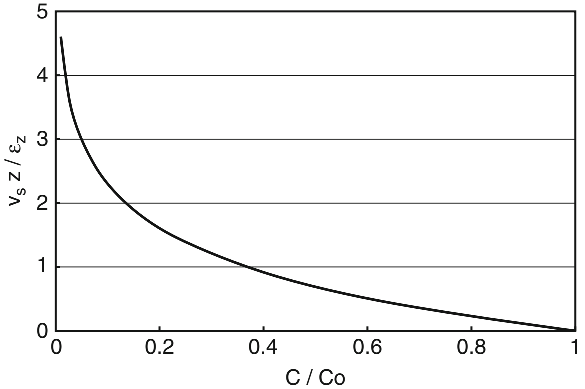



**Fig. 5.10** Suspended sediment concentration profile for Example 2

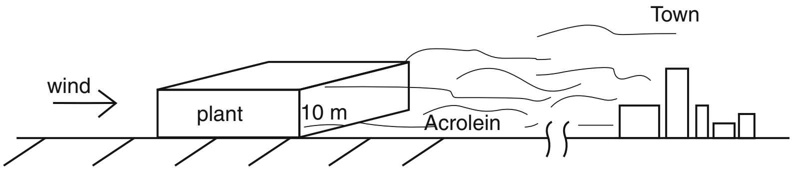



**Fig. 5.11** Illustration of toxic chemical release into the atmosphere, with the wind blowing toward a town

One culprit, aside from a haunting, may be the release of Acrolein, $C_3H_4O$, a priority pollutant that is an intermediary of many organic reactions. The average release from the 200 m × 200 m × 10 m plant sketched in Fig. 5.11 is assumed to be 20 g/h. If the wind is blowing directly toward Scream Hollow, at 3 m/s measured at 3 m height, with a dynamic roughness of 0.2 m for the farmland, what concentrations will the Scream Hollow inhabitants experience? Is this above the EPA threshold limit of 0.1 ppm(*v*)?

We will need to make some assumptions to formulate this problem. They are:

1. The Acrolein release is distributed over the most downwind plane of the building. With the important concentration being 1 km away, this is not a bad assumption. Then, the Acrolein will be released over the plane that is 200 m × 10 m. If 20 g/h = 0.0056 g/s are released into a wind moving at 3 m/s, the initial concentration is

$$C_0 = \frac{0.0056 \text{ g/s}}{3 \text{ m/s}(200 \text{ m})(10 \text{ m})} = 9.3 \times 10^{-7} \text{ g/m}^3$$

2. We will use a cross-sectional mean velocity of $U = \bar{u}$ at 3 m height, or $U = 3$ m/s.
3. We will use $\bar{\varepsilon}_z = \bar{\varepsilon}_y = \varepsilon_z$ at 3 m height.
4. We will not consider any of the source or sink terms for Acrolein.

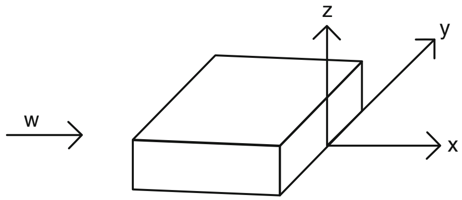



**Fig. 5.12** Illustration of the coordinate system for Example 5

We will also set up the coordinates so that $(x,y,z) = (0,0,0)$ occurs on the ground at mid-plant width, and will orient the wind in the $x$-direction.

With these assumptions, the governing equation becomes

$$U\frac{\partial C}{\partial x} = \bar{\varepsilon}_y \frac{\partial^2 C}{\partial y^2} + \bar{\varepsilon}_z \frac{\partial^2 C}{\partial z^2} \tag{5.52}$$

The boundary conditions are

1. At $(x,y,z) = (0, -100\text{ m} \Rightarrow 100\text{ m}, 0 \Rightarrow 10\text{ m})$, $C = C_0$.
2. As $x \Rightarrow \infty$, $y \Rightarrow \infty$, or $z \Rightarrow \infty$, $C \Rightarrow 0$.
3. Zero mass flux at $z = 0$.

These boundary conditions, illustrated in Fig. 5.12, will give us a concentration front, but in two dimensions. In addition, we have a zero flux condition that will require an image solution. We will use the solution of Example 5 in the entry "▾Transport in the Environment" to develop a solution for this problem. The solution, before applying boundary conditions, was

$$C = \beta_o + \beta_1 erf\left(\frac{z}{\sqrt{4Dt}}\right)$$

Now, we need an image to the concentration front about the $z = 0$ plane. In the y-direction we have a step-up at $y = -\Delta y$ and a step down at $y = \Delta y$. We will also use the product rule (Example 3, Transport in the Environment) to indicate that the solution to our governing equation for the $y$-direction should be multiplied times the solution in the $z$-direction. Then the solution can be given as

$$\frac{C}{C_o} = \beta_o + \left[\beta_1\, erf\left(\frac{(z+\Delta z)}{\sqrt{4\bar{\varepsilon}_z x/U}}\right) + \beta_2\, erf\left(\frac{(z-\Delta z)}{\sqrt{4\bar{\varepsilon}_z x/U}}\right)\right] \times \left[\beta_3\, erf\left(\frac{(y+\Delta y)}{\sqrt{4\bar{\varepsilon}_y x/U}}\right) + \beta_4\, erf\left(\frac{(y-\Delta y)}{\sqrt{4\bar{\varepsilon}_y x/U}}\right)\right]$$

where $\Delta y = 100$ m and $\Delta z = 10$ m.

Now, to see if our boundary conditions can be satisfied with the form of the solution:

$$@x \Rightarrow \infty, C \Rightarrow 0. \quad \text{Thus} \quad \beta_0 = 0.$$

$$@z \Rightarrow \infty, C \Rightarrow 0. \quad \text{Thus} \quad \beta_1 = -\beta_2.$$

$$@y \Rightarrow \infty, C \Rightarrow 0. \quad \text{Thus} \quad \beta_3 = -\beta_4.$$

$$@x \Rightarrow 0, \text{ and } (y,z) = (0,0), C/C_0 = 1.$$

With the last boundary condition, Eq. E5.5.3 becomes

$$1 = (\beta_1 - \beta_2)(\beta_3 - \beta_4)$$

or,

$$1 = 2\beta_1 \times 2\beta_3$$

or,

$$1 = 2\beta_2 \times 2\beta_4$$

Finally, @ $x \Rightarrow 0$ and $(y,z) = (0,\Delta z)$, $C/C_0 = ½$. Thus $\beta_1 = ½$.

Applying this last boundary condition results in $\beta_3 = ½$, $\beta_2 = -½$, and $\beta_4 = -½$. Thus, the solution to Eq. 5.52 is

$$\frac{C}{C_o} = \frac{1}{2}\left\{erf\left(\frac{(z+\Delta z)}{\sqrt{4\bar{\varepsilon}_z x/U}}\right) - erf\left(\frac{(z-\Delta z)}{\sqrt{4\bar{\varepsilon}_z x/U}}\right)\right\} \times\left\{erf\left(\frac{(y+\Delta y)}{\sqrt{4\bar{\varepsilon}_y x/U}}\right) - erf\left(\frac{(y-\Delta y)}{\sqrt{4\bar{\varepsilon}_y x/U}}\right)\right\} \tag{5.53}$$

Now, if we use $\Delta z = 10$ m, $\Delta y = 100$ m, $U = 3$ m/s, the only remaining parameter to find is $\bar{\varepsilon}$. Using Eq. 5.43 given in Example 1:

$$\bar{\varepsilon}_z = \bar{\varepsilon}_y = \kappa u_* z \tag{5.43}$$

Note that the logarithmic boundary equation can be written as

$$\frac{\bar{u}}{u_*} = \ell n\left(\frac{z}{z_o}\right) \tag{5.54}$$

where $z_0$ is the dynamic roughness, assumed to be 0.2 m for the crop land between the plant and Scream Hollow. Then,

$$u_* = \frac{\bar{u}}{\ell n(z/z_o)} = \frac{3\ \text{m/s}}{\ell n\left(\frac{3\ \text{m}}{0.2\ \text{m}}\right)} = 1.1\ \text{m/s}$$

and

$$\bar{\varepsilon}_z = 0.4(1.1m/s)(3m) = 1.3\ m^2/s$$

If we now plug all of the parameters for the industrial plant into Eq. 5.53, we get $C = 0.25\ \mu\text{g/m}^3 = 2.5 \times 10^{-7}\ \text{g/m}^3$. In terms of ppm(v), we will use $\rho_{\text{air}} = 1.2\ \text{g/m}^3$, and the molecular weights of air and Acrolein of 29 and 56 g/mole, respectively. Then,

$$\begin{aligned} C &= \frac{2.5 \times 10^{-7}\ \text{g/m}^3}{\rho_{\text{air}}} \frac{MW_{\text{air}}}{MW_{C_3H_40}} \\ &= \frac{2.5 \times 10^{-7}\ \text{g/m}^3}{1.2\ \text{g/m}^3} \frac{20\ \text{g/mole}}{56\ \text{g/mole}} \\ &= 1.08 \times 10^{-7} \frac{\text{moles}\ C_3H_40}{\text{mole air}} \end{aligned}$$

This is right at the threshold for continuous exposure, and the pollution from the plant should be investigated in more detail.

## Conclusions

1. Although turbulent diffusion is a convection transport, and not a diffusive transport, the result looks similar to diffusion, and can be described by a turbulent diffusion coefficient.
2. Most environmental flows are turbulent. The exceptions are flow through porous media and flows that are very close to an interface.
3. Reynolds averaging and the Boussinesq assumption result in a turbulent transport equation that contains many features of the diffusive mass transport equation, and can be solved by similar techniques.
4. Prandtl's mixing length is a relatively accurate simplification for many turbulent flows.

## Future Directions

The future for turbulent transport in the environment is in the direction of computational mass transport. This requires a simultaneous fluid dynamics–mass transport solution. On typical environmental scales, the computational power of our computers still must be advanced to solve these large problems while resolving the scale of the smallest turbulent eddies. Direct numerical simulation cannot deal with the scale of these problems, and large eddy simulation cannot keep both the scale and grid refinement required.

## Bibliography

### Primary Literature

1. Gulliver JS (2007) Introduction to chemical transport in the environment. Cambridge University Press, Cambridge, UK, 288 pp
2. Reynolds O (1895) On the dynamical theory of incompressible viscous fluids and the determination of the criterion. Philos Trans R Soc 186:123–164
3. Boussinesq J (1877) Essai sur la théorie des eaux courantes. Mem Pres Acad Sci Paris 23:46
4. Prandtl L (1925) Bericht iiber Untersuchungen zur ausgebildeten Turbulenz. Z Angew Math Mech 5:136–139
5. von Kármán T (1930) Mechanische Ahnlichkeit und Turbulenz. Nachr Ges Wiss Gottingen 5:58–76. Also Proceedings of third international congress on applied mechanics, vol I, Stockholm, pp 85–93, 1930
6. Reichardt H (1951) Vollstandige Darstellung der turbulenten. Geschwindigkeitsverteilung. Ann Angew Math Mech 31:7
7. Turner DB (1994) Workbook of atmospheric dispersion estimates, 2nd edn. Lewis, Boca Raton
8. Prandtl L (1942) Bemerkungen zur Theorie der freien Turbulenz. Z Angew Math Mech 22:241–243
9. Wu J (1980) Wind-stress coefficients over sea surface. J Geophys Res 74:444
10. Rouse H (1937) Modern conceptions of the mechanics of fluid turbulence. Trans Am Soc Civil Eng 102(1965):463–543

### Books and Reviews

Nezu I, Nakagawa H (1993) Turbulence in open channel flow. Balkema, Rotterdam
White FM (1974) Viscous fluid flow. McGraw-Hill, New York