Chapter 10
Sets of Zero Discrete Harmonic Density

Sets with zdhd and zhd are defined. Finite unions of I_0 sets have zdhd. A "Hadamard gap" theorem holds for sets with zhd.

10.1 Introduction

Two important themes have motivated much of the research on Sidon and related special sets: determining which classes of special sets have the property that every Sidon set is a finite union of sets from the class and understanding the "size" of Sidon sets. Much progress has been made on these themes, as discussed in Chaps. 6–9. Two specific problems which remain outstanding are:

1. Is every Sidon set a finite union of I_0 sets? **[P 1]**
2. Can a Sidon set be dense in $\overline{\mathbf{\Gamma}}$? **[P 2]**

These questions are not independent. A finite union of I_0 sets cannot be dense in $\overline{\mathbf{\Gamma}}$ (Theorem 3.5.1), and hence, a "yes" answer to the first question implies "no" to the second.

In this chapter we approach these two questions by introducing the notion of *zero (discrete) harmonic density*, abbreviated *z(d)hd*. A set $\mathbf{E} \subseteq \mathbf{\Gamma}$ is said to have property z(d)hd if for each non-empty, open $U \subseteq G$, the Fourier transform of every (discrete) measure agrees on $\mathbf{E}$ with the Fourier transform of a (discrete) measure concentrated on U. The characterizations given of z(d)hd in Sect. 10.2 will make it easy to show that property zdhd implies zhd.

The definition is motivated by the facts that when G is connected every Sidon (or I_0) set is Sidon(U) (resp., $I_0(U)$) (see Corollary 6.3.7 and Theorem 5.3.6) for all non-empty, open U. It follows that Sidon sets have the property zhd and I_0 sets have the property zdhd. In particular, every finite set has zdhd. But there are also non-Sidon sets with property zdhd; several examples

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are given in Sect. 10.4. It is unknown if every Sidon set has zdhd or even if every ε-Kronecker (for $\varepsilon \geq \sqrt{2}$) or dissociate set has zdhd **[P 11]**.

It is easy to see that a set with the zdhd property cannot be dense in $\overline{\mathbf{\Gamma}}$ (Proposition 10.2.6). A deeper result, Theorem 10.3.5, is that finite unions of I_0 sets have property zdhd. Thus, if it could be resolved whether every Sidon set has zdhd (either way), then one of the two questions stated in the opening paragraph could be answered: If every Sidon set has zdhd, then a Sidon set cannot be dense in $\overline{\mathbf{\Gamma}}$. If, instead, there is a Sidon set which does not have zdhd, then that Sidon set is not a finite union of I_0 sets.

Another motivation for the study of the z(d)hd property is a "globalization principle", which is illustrated in Sect. 10.4.1 by a novel proof of the classical Hadamard gap Theorem 1.2.2.

10.2 Characterizations and Closure Properties

Throughout this chapter, the compact group G is assumed to be connected. That is a natural assumption to make because a set of two elements of finite order will not even have zhd; just take for U any open subset of G on which the two elements coincide.

Definition 10.2.1. Let $U \subseteq G$. We say that $\mathbf{E} \subseteq \mathbf{\Gamma}$ has U-hd (respectively, U-dhd) if for every $\mu \in M(G)$ (respectively, $\mu \in M_d(G)$) there exists $\nu \in M(U)$ (resp., $\nu \in M_d(U)$) satisfying $\widehat{\mu} = \widehat{\nu}$ on $\mathbf{E}$.

Clearly, $\mathbf{E}$ has zhd (respectively, zdhd) if and only if $\mathbf{E}$ has U-hd (resp., U-dhd) for every non-empty, open set $U \subseteq G$. A translation argument shows that $\mathbf{E}$ has zhd (zdhd) if and only if $\mathbf{E}$ has U-hd (U-dhd) for every e-neighbourhood $U \subseteq G$.

Sets with the U-dhd property can be characterized in an analogous fashion to Kalton's characterization of $I_0(U)$ sets, Theorem 3.2.5. Similar statements characterize U-hd but with $M_d(G)$ and $M_d(U)$ replaced by $M(G)$ and $M(U)$ and $B_d(\mathbf{E})$ replaced by $B(\mathbf{E})$.

Theorem 10.2.2. *Let $U \subseteq G$ be open and $\mathbf{E} \subseteq \mathbf{\Gamma}$. The following are equivalent:*

1. *$\mathbf{E}$ has U-dhd.*
2. *There is a constant N such that for all $\mu \in M_d(G)$ there exists $\nu \in M_d(U)$ with $\|\nu\|_{M(G)} \leq N \|\mu\|_{M(G)}$ and $\widehat{\nu}(\gamma) = \widehat{\mu}(\gamma)$ for all $\gamma \in \mathbf{E}$.*
3. *There is a constant N such that for all $x \in G$ there exists $\nu \in M_d(U)$ with $\|\nu\|_{M(G)} \leq N$ and $\widehat{\nu}(\gamma) = \widehat{\delta_x}(\gamma)$ for all $\gamma \in \mathbf{E}$.*
4. *There exists $0 < \varepsilon < 1$ and constant $N = N(\varepsilon)$ such that for every $\mu \in M_d(G)$ there exists $\nu \in M_d(U)$ with $\|\nu\|_{M(G)} \leq N \|\mu\|_{M(G)}$ and*

$$\|\widehat{\mu} - \widehat{\nu}\|_{B_d(\mathbf{E})} < \varepsilon \|\mu\|_{M(G)}.$$

As in the characterizations of $I_0(U)$ sets, the proof will show that the phrase, "There exists $0 < \varepsilon < 1$", can be replaced by "For every $0 < \varepsilon < 1$".

Proof. (1) $\Rightarrow$ (2) is the closed graph theorem. The implications (2) $\Rightarrow$(3) $\Rightarrow$(4) are clear. (4) $\Rightarrow$(1) is a variation on the standard iteration argument. The details are left as Exercise 10.6.1. □

The proof of the analogous theorem for zdhd is also left as Exercise 10.6.2.

Theorem 10.2.3. *The subset* $\mathbf{E} \subseteq \mathbf{\Gamma}$ *has zdhd if and only if any of the following conditions hold:*

1. *Any one of properties Theorem 10.2.2 (1)–(4) is satisfied for all e-neighbourhoods* $U \subseteq G$.
2. *For every* $x \in G$ *and for every e-neighbourhood* $U \subseteq G$, *there exists* $\nu \in M_d(U)$ *with* $\widehat{\nu}(\gamma) = \widehat{\delta_x}(\gamma)$ *for all* $\gamma \in \mathbf{E}$.
3. *For every e-neighbourhood* $U \subseteq G$ *and* $0 < \varepsilon < 1$ *there exists* $N = N(U, \varepsilon)$ *such that for every* $x \in G$ *there are scalars* $c_n \in \Delta$ *and elements* $u_n \in U$ *such that*

$$\left\| \widehat{\delta_x} - \sum_{n=1}^{N} c_n \widehat{\delta_{u_n}} \right\|_{B_d(\mathbf{E})} < \varepsilon. \tag{10.2.1}$$

Remark 10.2.4. In Theorem 10.2.3(3), δ_x cannot be replaced by arbitrary $\mu \in \text{Ball}(M_d(G))$. Here is why. Suppose $\omega \in \text{Ball}(M(G))$. Let $\mu_\alpha \in \text{Ball}(M_d(G))$ converge weak* to ω, and let $c_{n,\alpha} \in \Delta$ and $u_{n,\alpha} \in U$ have $\|\widehat{\mu_\alpha} - \sum_{n=1}^{N} c_{n,\alpha}\widehat{\delta_{u_{n,\alpha}}}\|_{B(\mathbf{E})} < \varepsilon$ for all α. Letting c_n be a cluster point of $c_{n,\alpha}$ and u_n a cluster point of $u_{n,\alpha}$, we see that $\|\widehat{\omega} - \sum_1^N c_n\widehat{\delta_{u_n}}\|_{B(\mathbf{E})} \leq \varepsilon$. That shows that $\mathbf{E}$ satisfies the conclusion of Lemma 9.4.13. By the completion (Sect. 9.4.2) of the proof of the Ramsey–Wells–Bourgain Theorem 9.4.15, $\mathbf{E}$ is I_0. But there are non-I_0 sets (even non-Sidon sets) that have zdhd; see Sect. 10.4 for several examples.

Corollary 10.2.5. *A set with zdhd has zhd.*

Proof. It will be enough to verify that if $\mathbf{E}$ has U-dhd, then $\mathbf{E}$ has $\overline{U}$-hd. Let $\mu \in M(G)$ and choose $\nu_\alpha \in M_d(G)$ with $\|\nu_\alpha\|_{M(G)} \leq \|\mu\|_{M(G)}$ and $\nu_\alpha \to \mu$ weak* in $M(G)$. Since $\mathbf{E}$ has U-dhd, by Theorem 10.2.2(2), there are discrete measures $\sigma_\alpha \in M_d(U)$ and a constant N such that $\widehat{\sigma_\alpha} = \widehat{\nu_\alpha}$ on $\mathbf{E}$ and $\|\sigma_\alpha\|_{M(G)} \leq N\,\|\nu_\alpha\|_{M(G)} \leq N\,\|\mu\|_{M(G)}$. Being norm bounded, the net $\{\sigma_\alpha\}$ has a weak* cluster point $\sigma \in M(G)$. Because the measures σ_α are supported on $\overline{U}$, the same is true of σ and since $\widehat{\nu_\alpha}(\gamma) \to \widehat{\mu}(\gamma)$ for all $\gamma \in \mathbf{\Gamma}$, $\widehat{\sigma} = \widehat{\mu}$ on $\mathbf{E}$. Thus, $\mathbf{E}$ has $\overline{U}$-hd. □

Here are some easy facts about the "size" of a set with zdhd. In particular, Proposition 10.2.6 (1) implies a set with zdhd is not dense in $\overline{\mathbf{\Gamma}}$.

Proposition 10.2.6. *Suppose* $\mathbf{E} \subseteq \mathbf{\Gamma}$ *has zdhd.*

1. *If* $\mathbf{\Lambda}$ *is a non-trivial, closed subgroup of* $\overline{\mathbf{\Gamma}}$*, then* $\overline{\mathbf{E} \cap \mathbf{\Lambda}} \neq \mathbf{\Lambda}$*.*
2. *The interior of* $\overline{\mathbf{E}}$ *in* $\overline{\mathbf{\Gamma}}$ *is empty.*

Proof. (1) Suppose $\mathbf{E} \cap \mathbf{\Lambda}$ is dense in the non-trivial, closed subgroup $\mathbf{\Lambda}$. Let $H \subseteq G$ be the annihilator of $\mathbf{\Lambda}$. Since $\mathbf{\Lambda}$ is non-trivial, its dual group G/H contains a proper open subset UH. Choose $x \in G$ such that the coset $xH \notin UH$. The set UH can also be viewed as an open subset of G. Hence, there is a discrete measure $\nu = \sum c_j \delta_{x_j} \in M_d(UH)$ with $\widehat{\nu} = \widehat{\delta_x}$ on $\mathbf{E}$. Put $\mu = \sum c_j \delta_{x_j H}$. Then μ is a discrete measure on G/Hconcentrated on UH.

If $\gamma \in \mathbf{\Lambda}$, then, since $\gamma(H) = 1$,

$$\widehat{\mu}(\gamma) = \sum c_j \widehat{\delta_{x_j H}}(\gamma) = \widehat{\nu}(\gamma) = \widehat{\delta_x}(\gamma) = \widehat{\delta_{xH}}(\gamma).$$

Thus, the Fourier–Stieltjes transforms of μ and δ_{xH} agree on $\mathbf{\Lambda}$. But $xH \notin UH$, so this is not possible.

(2) Suppose $\overline{\mathbf{E}}$ contains a non-empty, open set in $\overline{\mathbf{\Gamma}}$. There is no loss of generality in assuming this open set is a neighbourhood of the identity since translates of sets with property zdhd also have zdhd. Thus, $\overline{\mathbf{E}}$ contains a set of the form $\{\gamma : |\gamma(x_j) - 1| < \varepsilon \text{ for } j = 1, \ldots, J\}$. In particular, $\overline{\mathbf{E}}$ will contain $H^{\perp}$, where $H = \langle\{x_1, \ldots, x_J\}\rangle$. By the first part of the proposition, this subgroup must be trivial and so H must be dense in G.

Consider the map $T : \overline{\mathbf{\Gamma}} \to \mathbb{T}^J$ given by $T(\gamma) = (\gamma(x_1), \ldots, \gamma(x_J))$. The map T is clearly continuous and, since H is dense, T is $1-1$. Thus, $T : \overline{\mathbf{\Gamma}} \to T(\overline{\mathbf{\Gamma}})$ is a homeomorphism. This shows $\overline{\mathbf{\Gamma}}$ is homeomorphic to a compact subgroup of $\mathbb{T}^J$, and so G is countable. But there are no countably infinite, compact abelian groups (Exercise C.4.17 (1)). □

Remark 10.2.7. It is known that, unlike I_0 sets, a set $\mathbf{E}$ with zdhd can cluster at a continuous character; see Corollary 10.4.5, but it is unknown if $\overline{\mathbf{E}}$ is a U_0 set **[P 12]**, for example.

Proposition 10.2.8. *A subset* $\mathbf{E} \subseteq \mathbb{Z}$ *which has zhd cannot contain arbitrarily long arithmetic progressions of fixed step length.*

Proof. Since translation and dilation do not affect the property zhd, or even the zhd constants, there is no loss of generality in assuming $\mathbf{E}$ contains arbitrarily long arithmetic progressions of step length 1.

Let $U \subseteq \mathbb{T}$ be a non-empty, open subset that is not dense and choose $x \notin \overline{U}$. Let N be the U-hd constant of $\mathbf{E}$, that is, for every $\mu \in M(G)$ there is some $\nu \in M(U)$ such that $\|\nu\|_{M(G)} \leq N \|\mu\|_{M(G)}$ and $\widehat{\nu} = \widehat{\mu}$ on $\mathbf{E}$. By translating $\mathbf{E}$ repeatedly, we can obtain measures $\nu_j \in M_d(U)$ such that $\widehat{\nu_j} = \widehat{\delta_x}$ on $[-j, j]$ and $\|\nu_j\|_{M(G)} \leq N$. Let ν be a weak* limit. This measure is supported on $\overline{U}$ and its transform agrees with δ_x on all of $\mathbb{Z}$. But this is impossible because $x \notin \overline{U}$. □

10.3 Union Results

It is unknown if the union of two sets with zdhd has zdhd [**P 11**]. In this section it will be shown that a finite union of zdhd sets has property zdhd under additional assumptions. In particular, we will prove that a finite union of I_0 sets, although not necessarily I_0, has zdhd.

10.3.1 Unions of Zdhd Sets with Disjoint Closures

We begin by showing that a finite union of zdhd sets with disjoint closures has zdhd.

Proposition 10.3.1. *Suppose* $\mathbf{E},\mathbf{F} \subseteq \mathbf{\Gamma}$ *have* V*-dhd for some symmetric e-neighbourhood* V *and that* $\overline{\mathbf{E}} \cap \overline{\mathbf{F}}$ *is empty. Then* $\mathbf{E} \cup \mathbf{F}$ *has* V^6*-dhd.*

Proof. We claim it will be enough to find $\nu \in M_d(V^5)$ with $\widehat{\nu} = 1$ on $\mathbf{E}$ and $\widehat{\nu} = 0$ on $\mathbf{F}$. To see this, note that for each $\mu \in M_d(G)$ there exist $\omega_{\mathbf{E}}, \omega_{\mathbf{F}} \in M_d(V)$ such that $\widehat{\omega_{\mathbf{E}}} = \widehat{\mu}$ on $\mathbf{E}$ and $\widehat{\omega_{\mathbf{F}}} = \widehat{\mu}$ on $\mathbf{F}$. Set $\omega = \nu * \omega_{\mathbf{E}} + (1-\nu) * \omega_{\mathbf{F}}$. We have $\omega \in M_d(V^6)$ and $\widehat{\omega} = \widehat{\mu}$ on $\mathbf{E} \cup \mathbf{F}$.

We turn to finding ν. That is done exactly as in the proof of Proposition 5.2.2, up to the point at which the $I_0(V)$ property for $\mathbf{F}$ is called upon. In the notation of the proof of Proposition 5.2.2, for each $\gamma \in \overline{\mathbf{E}}$ there is a measure $\tau_1 \in M_d(V)$, with $\widehat{\tau_1} \geq 1/2$ on $\overline{\mathbf{F}}$ and $\widehat{\tau_1}(\gamma) = 0$. In the present context, we call upon Gel'fand's Theorem C.1.12. Applying that theorem to $A(\overline{\mathbf{F}})$ and $\widehat{\tau_1}$, we see that there exists $\widehat{\tau_0} \in A(\overline{\mathbf{F}})$ such that $\widehat{\tau_0} = 1/\widehat{\tau_1}$ on $\mathbf{F}$. We may assume $\tau_0 \in M_d(G)$. Because $\mathbf{F}$ has V-dhd, there exists $\tau \in M_d(V)$ with $\widehat{\tau} = \widehat{\tau_0}$ on $\mathbf{F}$. Then $\omega_\gamma = (1 - \tau_1 * \tau) * (1 - \widetilde{\tau_1} * \widetilde{\tau})$ has $\widehat{\omega_\gamma}(\gamma) = 1$, $\widehat{\omega_\gamma} = 0$ on $\mathbf{F}$ and $\widehat{\omega_\gamma} \geq 0$ everywhere. Also, $\omega_\gamma \in M_d(V^4)$.

By the compactness of $\overline{\mathbf{E}}$, there are $\gamma_1, \ldots, \gamma_M$ such that $\tau_1' := \sum_1^M \omega_{\gamma_m}$ has $\widehat{\tau_1'} \geq 1/2$ on $\overline{\mathbf{E}}$ (and 0 on $\mathbf{F}$). Again, by Gel'fand's theorem, there exists $\tau_0' \in M_d(G)$ such that $\widehat{\tau_0'} = 1/\widehat{\tau_1'}$ on $\mathbf{E}$. Because $\mathbf{E}$ has V-dhd, there exists $\tau' \in M_d(V)$ such that $\widehat{\tau'} = \widehat{\tau_0'}$ on $\mathbf{E}$. Then $\nu = \tau_1' * \tau'$ has $\widehat{\nu} = 1$ on $\mathbf{E}$ and $\widehat{\nu} = 0$ on $\mathbf{F}$. Also, $\nu \in M_d(V^5)$. □

Corollary 10.3.2. *If* $\mathbf{E},\mathbf{F}$ *have zdhd and* $\overline{\mathbf{E}} \cap \overline{\mathbf{F}}$ *is empty, then* $\mathbf{E} \cup \mathbf{F}$ *has zdhd.*

Corollary 10.3.3. *If* $\mathbf{E}$ *has zdhd and* $\mathbf{F}$ *is finite, then* $\mathbf{E} \cup \mathbf{F}$ *has zdhd.*

Proof. There is no loss in assuming $\mathbf{F} = \{\gamma\}$. If μ and ν are discrete measures whose Fourier transforms agree on $\mathbf{E}$, then by continuity $\widehat{\mu} = \widehat{\nu}$ on $\overline{\mathbf{E}}$. Thus, if $\gamma \in \overline{\mathbf{E}}$, then $\mathbf{E} \cup \{\gamma\}$ has zdhd. Otherwise, $\overline{\mathbf{E}} \cap \overline{\mathbf{F}}$ is empty and we may apply the previous corollary. □

10.3.2 A Finite Union of I_0 Sets Has Zdhd

The goal of this section is to prove the theorem stated in its title. The proof will require the notion of a Helson set (Remark 3.5.5). Recall that the Helson constant (p. 165) of a closed set $\mathbf{S} \subset \overline{\mathbf{\Gamma}}$ is the infimum of the numbers C such that $\|f\|_{B_d(\mathbf{S})} \le C\|f\|_\infty$ for all $f \in B_d(\mathbf{S})$.

Helson sets are relevant here because, as was observed in Remark 3.5.5, the closure of an I_0 set is a Helson set. Like Sidon sets, a finite union of Helson sets is Helson. This deep result, stated below, will be used in what follows.

Theorem 10.3.4 (Varopoulos's union theorem).
The union of two Helson sets is Helson.

Here is our union theorem for I_0 sets.

Theorem 10.3.5. *If* $\mathbf{E}$, $\mathbf{F} \subseteq \mathbf{\Gamma}$ *are* I_0 *sets, then* $\mathbf{E} \cup \mathbf{F}$ *has zdhd.*

One can immediately deduce that there are non-I0 sets with zdhd (such as Example 1.5.2). We will see later (Proposition 10.4.4) that there are non-Sidon sets that have zdhd.

First we prove a technical lemma.

Lemma 10.3.6. *Suppose* $\mathbf{E}, \mathbf{F} \subseteq \mathbf{\Gamma}$ *are* I_0 *sets. Let* $V \subseteq G$ *be an e-neighbourhood. Then there is an open set* $\boldsymbol{\Omega} \subseteq \overline{\mathbf{\Gamma}}$, *containing* $\overline{\mathbf{E}} \cap \overline{\mathbf{F}}$, *such that* $(\mathbf{E} \cup \mathbf{F}) \cap \boldsymbol{\Omega}$ *has* V*-dhd.*

Proof (of Lemma 10.3.6). The characterization of V-dhd given in Theorem 10.2.2 (4) implies that it will suffice to show that there is an open set $\boldsymbol{\Omega} \supseteq \overline{\mathbf{E}} \cap \overline{\mathbf{F}}$ and constant N such that for every $\mu \in M_d(G)$ with $\|\mu\|_{M(G)} \le 1$ there exists $\nu \in M_d(V)$ such that $\|\nu\|_{M(G)} \le N$ and $\|\widehat{\mu} - \widehat{\nu}\|_{B_d((\mathbf{E}\cup\mathbf{F})\cap\boldsymbol{\Omega})} \le 1/2$.

Choose an e-neighbourhood $W \subseteq G$ such that $W^2 \subseteq V$. By compactness, there are finitely many points $x_k, k = 1, \ldots, K$, such that $\bigcup_{k=1}^K x_k W = G$. Being I_0, the set $\mathbf{E}$ has zdhd, and hence there are measures $\nu_k \in M_d(W)$ and a constant $C_{\mathbf{E}}$ such that $\|\nu_k\|_{M(G)} \le C_{\mathbf{E}}$ and $\widehat{\nu_k} = \widehat{\delta_{x_k}}$ on $\mathbf{E}$. Continuity implies that this equality continues to hold on $\overline{\mathbf{E}}$.

The sets $\mathbf{E}, \mathbf{F}$ are both Helson, and therefore so is their union. Let C_0 be the Helson constant of $\overline{\mathbf{E}} \cup \overline{\mathbf{F}}$. Put

$$\boldsymbol{\Omega} = \bigcap_{k=1}^{K} \{\gamma \in \overline{\mathbf{\Gamma}} : \left|\widehat{\nu_k}(\gamma) - \widehat{\delta_{x_k}}(\gamma)\right| < 1/(4C_0)\} \subseteq \overline{\mathbf{\Gamma}}.$$

The set $\boldsymbol{\Omega}$ is open as each of the sets $\{\gamma : \left|\widehat{\nu_k}(\gamma) - \widehat{\delta_{x_k}}(\gamma)\right| < 1/(4C_0)\}$ is open in $\overline{\mathbf{\Gamma}}$. Since $\widehat{\nu_k} = \widehat{\delta_{x_k}}$ on $\overline{\mathbf{E}}$, $\boldsymbol{\Omega}$ contains all of $\overline{\mathbf{E}}$.

Let μ be any discrete measure on G with $\|\mu\|_{M(G)} \leq 1$. Then μ can be written as

$$\mu = \sum_{j,k} c_{j,k}\delta_{x_k w_{j,k}} = \sum_{k=1}^{K}\sum_{j=1}^{\infty} c_{j,k}\delta_{x_k} * \delta_{w_{j,k}},$$

where $w_{j,k} \in W$ and $\sum_{j,k} |c_{j,k}| = \|\mu\|_{M(G)} \leq 1$.

Consider the measure $\nu = \sum_{j,k} c_{j,k}\nu_k * \delta_{w_{j,k}} \in M_d(W^2) \subseteq M_d(V)$. This measure satisfies $\|\nu\|_{M(G)} \leq \sum_{j,k} |c_{j,k}|\, \|\nu_k\|_{M(G)} \leq C_{\mathbf{E}}$. Furthermore, the definition of $\boldsymbol{\Omega}$ ensures that for each $\gamma \in \overline{\boldsymbol{\Omega}}$,

$$\begin{aligned}|\widehat{\nu}(\gamma) - \widehat{\mu}(\gamma)| &= \Big|\sum_{j,k} c_{j,k}(\widehat{\nu_k}(\gamma) - \widehat{\delta_{x_k}}(\gamma))\widehat{\delta_{w_{j,k}}}(\gamma)\Big| \\ &\leq \sum_{j,k} |c_{j,k}|\, |\widehat{\nu_k}(\gamma) - \widehat{\delta_{x_k}}(\gamma)| \leq \frac{1}{4C_0}\,\|\mu\| \leq \frac{1}{4C_0}.\end{aligned}$$

Thus, the function $\widehat{\nu} - \widehat{\mu}$, viewed as an element of $C((\overline{\mathbf{E}} \cup \overline{\mathbf{F}}) \cap \overline{\boldsymbol{\Omega}})$, has norm at most $1/(4C_0)$. Since $\overline{\mathbf{E}} \cup \overline{\mathbf{F}}$ is Helson, it follows that there is a measure $\sigma \in M_d(G)$ such that $\widehat{\sigma} = \widehat{\nu} - \widehat{\mu}$ on $(\overline{\mathbf{E}} \cup \overline{\mathbf{F}}) \cap \overline{\boldsymbol{\Omega}}$ and $\|\sigma\|_{M(G)} \leq 2C_0/(4C_0) = 1/2$. Therefore, $\|\widehat{\nu} - \widehat{\mu}\|_{B_d((\mathbf{E}\cup\mathbf{F})\cap\boldsymbol{\Omega})} \leq \|\sigma\|_{M(G)} \leq 1/2$, which proves that ν has the desired properties. □

Proof (of Theorem 10.3.5). Let $U \subseteq G$ be an e-neighbourhood and let $V \subseteq G$ be an e-neighbourhood with $V^5 \subseteq U$. As in Lemma 10.3.6, choose $\boldsymbol{\Omega} \subseteq \overline{\boldsymbol{\Gamma}}$, an open set containing $\overline{\mathbf{E}} \cap \overline{\mathbf{F}}$, such that $(\mathbf{E} \cup \mathbf{F}) \cap \boldsymbol{\Omega}$ has V-dhd. The regularity of the topology of $\overline{\boldsymbol{\Gamma}}$ implies there is an open set $\boldsymbol{\Omega}_1 \supseteq \overline{\mathbf{E}} \cap \overline{\mathbf{F}}$, such that $\overline{\boldsymbol{\Omega}_1} \subseteq \boldsymbol{\Omega}$.

Because $\mathbf{E}$ has zdhd, there is a measure $\mu_{\mathbf{E}} \in M_d(V)$ such that $\widehat{\mu_{\mathbf{E}}} = 0$ on $\overline{\mathbf{E}} \setminus \boldsymbol{\Omega}_1$ and $\widehat{\mu_{\mathbf{E}}} = 1$ on the disjoint closed set $\overline{\mathbf{E}} \cap \overline{\mathbf{F}}$. Similarly, there exists a measure $\mu_{\mathbf{F}} \in M_d(V)$ such that $\widehat{\mu_{\mathbf{F}}} = 0$ on $\overline{\mathbf{F}} \setminus \boldsymbol{\Omega}_1$ and $\widehat{\mu_{\mathbf{F}}} = 1$ on the disjoint set $\overline{\mathbf{E}} \cap \overline{\mathbf{F}}$. Put $\sigma = \mu_{\mathbf{E}} * \mu_{\mathbf{F}} \in M_d(V^2)$. Then $\widehat{\sigma} = 0$ on $(\overline{\mathbf{E}} \cup \overline{\mathbf{F}}) \setminus \boldsymbol{\Omega}_1$ and $\widehat{\sigma} = 1$ on $\overline{\mathbf{E}} \cap \overline{\mathbf{F}}$. Let

$$\boldsymbol{\Omega}_2 = \{\gamma \in \overline{\mathbf{E}} \cup \overline{\mathbf{F}} : |\widehat{\sigma}(\gamma)| > 1/2\}.$$

The choice of σ ensures that $\overline{\mathbf{E}} \cap \overline{\mathbf{F}} \subseteq \boldsymbol{\Omega}_2 \subseteq \overline{\boldsymbol{\Omega}_2} \subseteq \boldsymbol{\Omega}_1$.

Since $\widehat{\sigma}|_{\boldsymbol{\Omega}_2}$ is bounded away from 0, an application of Gel'fand's theorem (as in Proposition 10.3.1) implies there exists $\sigma_1 \in M_d(G)$ such that $\widehat{\sigma} \cdot \widehat{\sigma_1} = 1$ on $\boldsymbol{\Omega}_2$. Since $(\mathbf{E} \cup \mathbf{F}) \cap \boldsymbol{\Omega}$ has V-dhd, the same is true for its subset $(\mathbf{E} \cup \mathbf{F}) \cap \boldsymbol{\Omega}_2$. Hence, we can choose $\sigma_2 \in M_d(V)$ such that $\widehat{\sigma_2} = \widehat{\sigma_1}$ on $(\mathbf{E} \cup \mathbf{F}) \cap \boldsymbol{\Omega}_2$.

Because the closed sets $\overline{\boldsymbol{\Omega}_2}$ and $\boldsymbol{\Omega}_1^c$ are disjoint, there is a discrete measure ν such that $\widehat{\nu} = 1$ on $\boldsymbol{\Omega}_2$, $\widehat{\nu} = 0$ on $\boldsymbol{\Omega}_1$ and $0 \leq \widehat{\nu} \leq 1$. Again, because $(\mathbf{E} \cup \mathbf{F}) \cap \boldsymbol{\Omega}$ has V-dhd, there exists $\nu_1 \in M_d(V)$ such that $\widehat{\nu}_1 = \widehat{\nu}$ on $(\mathbf{E} \cup \mathbf{F}) \cap \boldsymbol{\Omega}$. Finally, put

$$\mu = \sigma * \sigma_2 * \nu_1 \in M_d(V^4).$$

By construction, $\widehat{\mu} = 1$ on $(\mathbf{E} \cup \mathbf{F}) \cap \boldsymbol{\Omega}_2$ and $\widehat{\mu} = 0$ on $(\mathbf{E} \cup \mathbf{F}) \setminus \boldsymbol{\Omega}_1$.

Let $\rho \in M_d(G)$ be given. To prove that $\mathbf{E} \cup \mathbf{F}$ has U-dhd it will be enough to prove there exists $\rho_0 \in M_d(V^5)$ such that $\widehat{\rho_0} = \widehat{\rho}$ on $\mathbf{E} \cup \mathbf{F}$.

The closed sets $\overline{\mathbf{E}} \setminus \boldsymbol{\Omega}_2$ and $\overline{\mathbf{F}} \setminus \boldsymbol{\Omega}_2$ are disjoint since $\overline{\mathbf{E}} \cap \overline{\mathbf{F}} \subseteq \boldsymbol{\Omega}_2$. By Corollary 10.3.2, $(\mathbf{E} \setminus \boldsymbol{\Omega}_2) \cup (\mathbf{F} \setminus \boldsymbol{\Omega}_2) = (\mathbf{E} \cup \mathbf{F}) \setminus \boldsymbol{\Omega}_2$ has zdhd and this ensures there is a discrete measure $\rho_1 \in M_d(V)$ such that $\widehat{\rho_1} = \widehat{\rho}$ on $(\mathbf{E} \cup \mathbf{F}) \setminus \boldsymbol{\Omega}_2$. Since $(\mathbf{E} \cup \mathbf{F}) \cap \boldsymbol{\Omega}$ has V-dhd, there is also a measure $\rho_2 \in M_d(V)$ such that $\widehat{\rho_2} = \widehat{\rho}$ on $(\mathbf{E} \cup \mathbf{F}) \cap \boldsymbol{\Omega}$. Set

$$\rho_0 = \rho_1 * (\delta_e - \mu) + \rho_2 * \mu \in M_d(V^5).$$

We claim this measure interpolates $\widehat{\rho}$ on $\mathbf{E} \cup \mathbf{F}$. To see this, observe the following:

- On $(\mathbf{E} \cup \mathbf{F}) \cap (\boldsymbol{\Omega}_1 \setminus \boldsymbol{\Omega}_2)$, $\widehat{\rho_1} = \widehat{\rho} = \widehat{\rho_2}$; hence, $\widehat{\rho_0} = \widehat{\rho_1}(1 - \widehat{\mu}) + \widehat{\rho_2}\widehat{\mu} = \widehat{\rho}$.
- On $(\mathbf{E} \cup \mathbf{F}) \setminus \boldsymbol{\Omega}_1$, $\widehat{\rho_1} = \widehat{\rho}$ and $\widehat{\mu} = 0$; thus, $\widehat{\rho_0} = \widehat{\rho_1} = \widehat{\rho}$.
- Finally, on $(\mathbf{E} \cup \mathbf{F}) \cap \boldsymbol{\Omega}_2$, $\widehat{\rho_2} = \widehat{\rho}$ and $\widehat{\mu} = 1$, and so $\widehat{\rho_0} = \widehat{\rho_2} = \widehat{\rho}$.

These observations demonstrate that $\widehat{\rho_0} = \widehat{\rho}$ on $\mathbf{E} \cup \mathbf{F}$, as claimed. □

10.3.3 Other Union Results for Zdhd Sets

The union of two zdhd sets can also be shown to have zdhd if their closures have finite intersection, though we know of no non-trivial instances of this. This is a consequence of a more general result which relies on the property of spectral synthesis (p. 213).

Theorem 10.3.7. *Suppose* $\mathbf{E}$, $\mathbf{F} \subseteq \boldsymbol{\Gamma}$ *have zdhd and* $\overline{\mathbf{E}} \cap \overline{\mathbf{F}}$ *obeys spectral synthesis. Then* $\mathbf{E} \cup \mathbf{F}$ *has zdhd.*

Corollary 10.3.8. *If* $\mathbf{E}, \mathbf{F}$ *have zdhd and* $\overline{\mathbf{E}} \cap \overline{\mathbf{F}}$ *is finite, then* $\mathbf{E} \cup \mathbf{F}$ *has zdhd.*

Proof. This follows since finite sets have spectral synthesis (see p. 213). □

The theorem is deduced from a technical lemma similar to Lemma 10.3.6. We remark that the definition of zdhd extends to subsets of $\overline{\boldsymbol{\Gamma}}$ in the obvious fashion and the characterization results, Theorems 10.2.2 and 10.2.3, continue to hold in this setting.

Lemma 10.3.9. *Suppose* $\mathbf{E}$ *has zdhd,* $\overline{\mathbf{E}} \cap \overline{\mathbf{F}}$ *obeys spectral synthesis and* $V \subseteq G$ *is an e-neighbourhood. Then there is an open set* $\boldsymbol{\Omega} \supseteq \overline{\mathbf{E}} \cap \overline{\mathbf{F}}$ *such that* $\boldsymbol{\Omega}$ *has* V*-dhd.*

Proof (of Lemma 10.3.9). We begin in the same manner as the proof of Lemma 10.3.6. Let $W^2 \subseteq V$ be open and assume $\bigcup_{k=1}^{K} x_k W = G$. Use the zdhd property of $\mathbf{E}$ to obtain a constant C and measures $\nu_k \in M_d(W)$

such that $\|\nu_k\|_{M(G)} \le C$ and $\widehat{\nu_k} = \widehat{\delta_{x_k}}$ on $\mathbf{E}$ for $k = 1, \ldots, K$. Since $\overline{\mathbf{E}} \cap \overline{\mathbf{F}}$ obeys spectral synthesis, there is a neighbourhood $\boldsymbol{\Omega}$ of $\overline{\mathbf{E}} \cap \overline{\mathbf{F}}$ such that $\|\widehat{\nu_k} - \widehat{\delta_{x_k}}\|_{B_d(\boldsymbol{\Omega})} < 1/(2K)$ for each $k = 1, \ldots, K$ (see Lemma C.1.14). This inequality is enough to prove that if $\mu = \sum_{j,k} c_{j,k}\delta_{x_k w_{j,k}}$ is any norm one, discrete measure, with $w_{j,k} \in W$, then $\nu = \sum_{j,k} c_{j,k}\nu_k * \delta_{w_{j,k}} \in M_d(V)$ has measure norm at most C and satisfies $\|\widehat{\mu} - \widehat{\nu}\|_{B_d(\boldsymbol{\Omega})} \le 1/2$. □

Proof (of Theorem 10.3.7). The proof is like that of Theorem 10.3.5 but without the intersections with $\mathbf{E} \cup \mathbf{F}$ and using Lemma 10.3.9. □

10.4 Examples and Applications

Proposition 10.4.1. *Let $\mathbf{E} \subseteq \mathbf{\Gamma}$ and suppose the cluster points of $\mathbf{E}$ are contained in the intersection of cosets of the form $\gamma_n H_n^{\perp} \subseteq \overline{\mathbf{\Gamma}}$, where H_n are finite subgroups of G and $\bigcup_{n=1}^{\infty} H_n$ is dense in G. Then $\mathbf{E}$ has zdhd.*

Proof. Fix an e-neighbourhood $U \subseteq G$ and let V be a symmetric e-neighbourhood such that $V^{12} \subseteq U$. Assume $G = \bigcup_{k=1}^{K} x_k V$. Since $\bigcup_n H_n$ is dense in G, for each k, there are elements $h_k \in x_k V \cap (\bigcup_n H_n)$, say $h_k = x_k v_k \in H_{n_k}$ with $v_k \in V$. Because V is symmetric,

$$\bigcup_{k=1}^{K} h_k V^2 = \bigcup_{k=1}^{K} x_k v_k V^2 \supseteq \bigcup_{k=1}^{K} x_k V = G.$$

We first check that $\mathbf{\Lambda} = \bigcap_{k=1}^{K} \gamma_{n_k} H_{n_k}^{\perp}$ has V^2-dhd by verifying Theorem 10.2.2 (3). Let $x \in G$. Then $x = h_k v$ for some $k = 1, \ldots, K$ and $v \in V^2$. Fix $\gamma_0 \in \gamma_{n_k} H_{n_k}^{\perp}$ and let σ be the discrete, norm one measure $\sigma = \gamma_0(h_k)\delta_{v^{-1}} \in M_d(V^2)$. If $\gamma \in \mathbf{\Lambda}$, then $\gamma \in \gamma_{n_k} H_{n_k}^{\perp}$, so $\gamma(h_k) = \gamma_{n_k}(h_k) = \gamma_0(h_k)$. Thus, $\widehat{\sigma}(\gamma) = \gamma_0(h_k)\gamma(v) = \gamma(h_k v) = \gamma(x) = \widehat{\delta_{x^{-1}}}(\gamma)$, showing that Theorem 10.2.2 (3) is satisfied with $N = 1$.

Since H_n is a finite group, $\mathbf{\Lambda}$ is open. By assumption, all the cluster points of $\mathbf{E}$ belong to $\mathbf{\Lambda}$. Thus, $\mathbf{E} \setminus \mathbf{\Lambda}$ must be finite. Since $\mathbf{\Lambda}$ is also closed, $\mathbf{E}$ has V^{12}-dhd by Proposition 10.3.1. □

Definition 10.4.2. Sets $\mathbf{E} \subseteq \mathbf{\Gamma}$ which satisfy the hypotheses of the proposition will be said to have *strong zdhd*.

Example 10.4.3. Hadamard sets of the form $\{r^n\}_{n=1}^{\infty}$, with integer $r \ge 2$, are strong zdhd. Just take H_n to be the subgroup of the r^nth roots of unity. Similarly, the set $\{k \cdot 100^{j!} : k = 1, \ldots, j, 1 \le j < \infty\}$ is another example of a strong zdhd set, and this set is not Sidon since it contains arbitrarily long arithmetic progressions.

The next proposition gives another example of a non-Sidon, zdhd set.

Proposition 10.4.4. *If* $\mathbf{E} = \{r^n\}_{n=1}^{\infty} \subset \mathbb{Z}$ *with* $r \geq 3$ *an integer, then* $\mathbf{E}+\mathbf{E}$ *and* $\mathbf{E}-\mathbf{E}$ *have zdhd.*

Proof. Let H_n be the subgroup consisting of the r^nth roots of unity and let $\boldsymbol{\Lambda} = \bigcap_n H_n^{\perp} \subseteq \overline{\mathbb{Z}}$. Fix a non-empty, open set $U \subseteq \mathbb{T}$ and choose $h_1, \ldots, h_K \in \bigcup_{n=1}^{\infty} H_n$ such that $\bigcup_{k=1}^{K}(h_k + U) = \mathbb{T}$.

Being a compact subgroup, $\boldsymbol{\Lambda}$ is a set of spectral synthesis (p. 213) and, since $\widehat{\delta_{h_k}} = 1$ on $\boldsymbol{\Lambda}$, it follows that there is a neighbourhood $\boldsymbol{\Omega} \subseteq \overline{\mathbb{Z}}$ containing $\boldsymbol{\Lambda}$ such that $\left\|\widehat{\delta_{h_k}} - \widehat{\delta}_1\right\|_{B_d(\boldsymbol{\Omega})} < 1$ for each $k = 1, \ldots, K$. By an argument similar to the proof that Theorem 10.2.3(3) implies zdhd (see Exercise 10.6.3), we conclude that $\boldsymbol{\Omega}$ has U-dhd.

Notice that $(\overline{\mathbf{E}+\mathbf{E}}) \smallsetminus \boldsymbol{\Omega}$ consists of a finite number of elements of $\mathbf{E}+\mathbf{E}$, plus a finite number of sets of the form $r^n + \overline{\mathbf{E}}$. Each of these finitely many sets has zdhd and their closures in $\overline{\mathbb{Z}}$ are pairwise disjoint (see Exercise 10.6.6). Therefore, their union has zdhd by Proposition 10.3.1.

The argument is similar for $\mathbf{E}-\mathbf{E}$. □

Corollary 10.4.5. *A zdhd set can cluster at a continuous character.*

Proof. $\mathbf{1}$ is a cluster point of $\mathbf{E}\cdot\mathbf{E}^{-1}$ whenever $\mathbf{E}$ is an infinite set. □

Remark 10.4.6. Being Hadamard, the $\mathbf{E}$ of Proposition 10.4.4 is $I_0(U)$ with bounded length (Remark 3.2.15 (i)). In [30] it is shown that if $\mathbf{E} = \{n_j\}$ is any Hadamard set with ratio at least 3, then for each k the set $\{n_{j_1} \pm \cdots \pm n_{j_k} : j_1 < \cdots < j_k\}$ has zhd. It is unknown, in general, if $\mathbf{E}\cdot\mathbf{E}^{\pm 1}$ has zdhd whenever $\mathbf{E}$ has bounded length (or bounded constants) and if all such "sums" of Hadamard sets have zdhd [**P 11**].

10.4.1 The Hadamard Gap Theorem for Sets with Zhd

We began this book by introducing Hadamard sets and giving examples of some of the unusual properties possessed by power series and trigonometric series with frequencies supported on a Hadamard set. Throughout the book, we have seen various generalizations these properties. To conclude, we give a short proof of a generalization of the classical Hadamard gap Theorem 1.2.2 for sets with property zhd.

Proposition 10.4.7. *Suppose* $\{n_j\}_{j=1}^{\infty}$ *is an increasing sequence of positive integers and has property zhd. The function* $f(z) = \sum_{j=1}^{\infty} c_j z^{n_j}$ *cannot be analytically continued, at any point, across the circle of convergence.*

Remark 10.4.8. This proves the classical Hadamard gap theorem since Hadamard sets, being Sidon, have property zhd.

Proof. Suppose that f could be analytically continued at z_0 on the circle of convergence. There is no loss of generality in assuming the circle of convergence has radius 1 and that $z_0 = 1$. Then f can be continued to be analytic in the open set $U_\varepsilon = \{z \in \mathbb{C} : |z - 1| < \varepsilon\}$ for some $\varepsilon > 0$.

Let $t \in [0, 2\pi]$. Since $\{n_j\}$ has zhd, it is possible to obtain a measure ν on $\mathbb{T}$, concentrated on $\mathbb{T} \cap U_{\varepsilon/3}$ (an open subset of $\mathbb{T}$), such that $\widehat{\nu}(n_j) = \widehat{\delta_t}(n_j)$ for all j.

The function $g(z) = \int f(\mathrm{e}^{-\mathrm{i}\theta} z) \mathrm{d}\nu(\theta)$ is analytic in the interior of the unit disk, as well as in the set $U_{\varepsilon/3}$. Since

$$\int \mathrm{e}^{-\mathrm{i}\theta n_j} \mathrm{d}\nu(\theta) = \widehat{\nu}(n_j) = \mathrm{e}^{\mathrm{i}t n_j},$$

the Taylor coefficients of g are the same as those of the function $z \mapsto f(\mathrm{e}^{\mathrm{i}t} z)$. Thus, f has an analytic continuation to $\{z \in \mathbb{C} : \left|z - \mathrm{e}^{\mathrm{i}t}\right| < \varepsilon/3\}$. Since t was arbitrary, f can be continued to $\{z : |z| < 1 + \varepsilon/3\}$, which contradicts the assumption that 1 is the radius of convergence. □

10.5 Remarks and Credits

The term “zero harmonic density” seems to have first appeared in Déchamps-Gondim’s 1976 note [28]. In [29], she gave the proof that if $\{\gamma_j\}$ is dissociate, then $\{\pm\gamma_j \pm \gamma_k : 1 \leq j < k < \infty\}$ has zhd. Déchamps and Selles in 1996 [30] gave a lengthy construction to establish that the sets $\{n_{j_1} \pm \cdots \pm n_{j_k} : j_1 < \cdots < j_k\}$, where $\{n_j\}$ is Hadamard with ratio at least 3, have zhd. Lust [125] showed that if $\mathbf{E} \subseteq \mathbb{Z}$ is such that for each $k \geq 1$, $C_{k\mathbf{E}}$ (where $k\mathbf{E} = \mathbf{E} + \cdots + \mathbf{E}$ k times) does not contain a subspace isomorphic to c_0, then $\mathbf{E}$ has zhd.

The concept of zero discrete harmonic density was introduced in [58] and most of the results of this chapter can be found there. The property zdhd with bounded constants, defined analogously to that of $I_0(U)$ with bounded constants, was also studied in that paper. It is shown there that a set which has zdhd with bounded constants has at most one cluster point in $\mathbf{\Gamma}$ and that $M_0(\overline{\mathbf{E}}) = \{0\}$ if $\mathbf{E}$ is a finite union of sets having zdhd with bounded constants. Like Sidon sets, a set which contains the sum of two disjoint infinite sets cannot be a finite union of sets with zdhd with bounded constants. Thus, $\{3^n\} + \{3^n\}$ is a zdhd set that is not a finite union of sets with zdhd with bounded constants. The proof of Proposition 10.4.1 actually shows that strong zdhd sets have zdhd with bounded constants.

Theorem 10.3.4 was proved by Varopoulos for compact Helson sets, one of which was metrizable [188, 189]. It was extended to the non-metrizable (but still compact) case by Lust [124] and then to the general case by Saeki [170]. Other proofs can be found in [56, Chapter 2], [86, 120]. See also [177], which improves on the original constants.

The connection between zero harmonic density and zero density for subsets of $\mathbb{Z}$ (meaning $\limsup |\mathbf{E} \cap [-N, N]|/(2N+1) = 0$) is unclear **[P 13]**. It is only known that not all sets of zero density have zero harmonic density. The set $\{100^{j!} + k : k \le j\}$ is such an example since it contains arbitrarily long arithmetic progressions of fixed step length (see Proposition 10.2.8).

10.6 Exercises

Exercise 10.6.1. Prove Theorem 10.2.2 (4) $\Rightarrow$ (1).

Exercise 10.6.2. Prove Theorem 10.2.3.

Exercise 10.6.3. Let $V \subseteq G$ be open and $\varepsilon < 1$. Show that if there exist finitely many points $\{x_j\}_{j=1}^{N} \subseteq G$ with $\bigcup_{j=1}^{N} x_j V = G$ and $\nu_j \in M_d(V)$ with $\left\| \widehat{\nu_j} - \widehat{\delta_{x_j}} \right\|_{B_d(E)} < 1$, then $\mathbf{E} \subseteq \mathbf{\Gamma}$ has V^2-dhd.

Exercise 10.6.4. Suppose that $U \subseteq [-\pi/k, \pi/k) \subseteq \mathbb{T}$. Show that $\mathbf{E} \subseteq \mathbb{Z}$ is kU-dhd with constant N (as in Theorem 10.2.2 (2)) if and only if $k\mathbf{E}$ is U-dhd with constant N.

Exercise 10.6.5. Show that if $\mathbf{E}$ has U-dhd for some proper open subset $U \subseteq G$, then $\mathbf{E}$ is not dense in $\mathbf{\Gamma}$.

Exercise 10.6.6. Suppose $\mathbf{E} = \{r^n\}_{n=1}^{\infty}$ with $r \ge 3$ an integer. Show that the sets $r^n + \mathbf{E}$ and $r^m + \mathbf{E}$ for $m \ne n$ have disjoint closures.

Exercise 10.6.7. One can define the notion of *zdhd with bounded constants* analogously to that of I_0 sets with bounded constants.

1. Prove that a set which has zdhd with bounded constants has at most one cluster point in $\mathbf{\Gamma}$.
2. Prove that a set with strong zdhd has zdhd with bounded constants.

Exercise 10.6.8. Prove that the union of a set with strong zdhd and a set with zdhd has zdhd.