Chapter 2
Introduction to Stochastic Processes

2.1 Stochastic Processes

When considering technical, economic, ecological, or other problems, in several cases the quantities $\{X_t,\ t \in \mathcal{T}\}$ being examined can be regarded as a collection of random variables. This collection describes the changes (usually in time and in space) of considered quantities. If the set $\mathcal{T}$ is a subset of the set of real numbers, then the set $\{t \in \mathcal{T}\}$ can be interpreted as time and we can say that the random quantities X_t vary in time. In this case the collection of random variables $\{X_t,\ t \in \mathcal{T}\}$ is called a **stochastic process**. In mathematical modeling of randomly varying quantities in time, one might rely on the highly developed theory of stochastic processes.

Definition 2.1. Let $\mathcal{T} \subset \mathbb{R}$. A **stochastic process** X is defined as a collection $X = \{X_t,\ t \in \mathcal{T}\}$ of indexed random variables X_t, which are given on the same probability space $(\Omega, \mathcal{A}, \mathbf{P}())$.

Depending on the notational complexity of the parameter, we occasionally interchange the notation X_t with $X(t)$.

It is clear that $X_t = X_t(\omega)$ is a function of two variables. For fixed $t \in \mathcal{T}$, X_t is a random variable, and for fixed $\omega \in \Omega$, X_t is a function of the variable $t \in \mathcal{T}$, which is called a **sample path** of the stochastic process.

Depending on the set $\mathcal{T}$, X is called a **discrete-time** stochastic process if the index set $\mathcal{T}$ consists of consecutive integers, for example, $\mathcal{T} = \{0, 1, \ldots\}$ or $\mathcal{T} = \{\ldots, -1, 0, 1, \ldots\}$. Further, X is called a **continuous-time** stochastic process if $\mathcal{T}$ equals an interval of the real line, for example, $\mathcal{T} = [a, b]$, $\mathcal{T} = [0, \infty)$ or $\mathcal{T} = (-\infty, \infty)$.

Note that in the case of discrete time, X is a sequence $\{X_n,\ n \in \mathcal{T}\}$ of random variables, while it determines a random function in the continuous-time case. It should be noted that similarly to the notion of real-valued stochastic processes, we may define complex or vector valued stochastic processes also if X_t take values in a complex plane or in higher-dimensional Euclidean space.

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2.2 Finite-Dimensional Distributions of Stochastic Processes

A stochastic process $\{X_t,\ t \in \mathcal{T}\}$ can be characterized in a statistical sense by its finite-dimensional distributions.

Definition 2.2. The **finite-dimensional distributions** of a stochastic process $\{X_t,\ t \in \mathcal{T}\}$ are defined by the family of all joint distribution functions

$$F_{t_1,\ldots,t_n}(x_1,\ldots,x_n) = \mathbf{P}\,(X_{t_1} < x_1,\ldots,X_{t_n} < x_n),$$

where $n = 1, 2, \ldots$ and $t_1,\ldots,t_n \in \mathcal{T}$.

The family of introduced distribution functions

$$\mathcal{F} = \{F_{t_1,\ldots,t_n},\ \ t_1,\ldots,t_n \in \mathcal{T},\ n = 1, 2, \ldots\}$$

satisfies the following, specified consistency conditions:

(a) For all positive integers n, m and indices $t_1,\ldots,t_{n+m} \in \mathcal{T}$

$$\lim_{x_{n+1}\to\infty} \cdots \lim_{x_{n+m}\to\infty} F_{t_1,\ldots,t_n,t_{n+1},\ldots,t_{n+m}}(x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+m})$$
$$= F_{t_1,\ldots,t_n}(x_1,\ldots,x_n),\ x_1,\ldots,x_n \in R\,.$$

(b) For all permutations $(i_1,\ldots,i_n)$ of the numbers $\{1, 2, \ldots, n\}$

$$F_{s_1,\ldots,s_n}(x_{i_1},\ldots,x_{i_n}) = F_{t_1,\ldots,t_n}(x_1,\ldots,x_n),\ x_1,\ldots,x_n \in R\,,$$

where $s_j = t_{i_j},\ j = 1,\ldots,n$.

Definition 2.3. If the family $\mathcal{F}$ of joint distribution functions defined previously satisfies conditions (a) and (b), then we say that $\mathcal{F}$ satisfies the **consistency conditions**.

The following theorem is a basic one in probability theory and ensures the existence of a stochastic process (in general of a collection of random variables) with given finite-dimensional distribution functions satisfying the consistency conditions.

Theorem 2.4 (Kolmogorov consistency theorem). *Suppose a family of distribution functions* $\mathcal{F} = \{F_{t_1,\ldots,t_n},\ \ t_1,\ldots,t_n \in \mathcal{T},\ n = 1, 2, \ldots\}$ *satisfies the consistency conditions (a) and (b). Then there exists a probability space* $(\Omega, \mathcal{A}, \mathbf{P})$*, and on that a stochastic process* $\{X_t,\ t \in \mathcal{T}\}$*, whose finite-dimensional distributions are identical to* $\mathcal{F}$*.*

For our considerations, it usually suffices to provide the finite-dimensional distribution functions of the stochastic processes, in which case the process is defined in a **weak sense** and it is irrelevant on which probability space it is given. In some instances the behavior of the random path is significant (e.g., continuity in time), which is related to a given probability space $(\Omega, \mathcal{A}, P)$ where the process $\{X_t,\ t \in \mathcal{T}\}$ is defined. In this case the process is given in a **strict sense**.

2.3 Stationary Processes

The class of stochastic processes that show a stationary statistical property in time plays a significant role in practice. Among these processes the most important ones are the stationary processes in strict and weak senses. The main notions are given here for one-dimensional processes, but the notion for high-dimensional processes can be introduced similarly.

Definition 2.5. A process $\{X_t,\ t \in \mathcal{T}\}$ is called **stationary in a strict sense** if the joint distribution functions of random variables

$$(X_{t_1}, \ldots, X_{t_n}) \quad and \quad (X_{t_1+t}, \ldots, X_{t_n+t})$$

are identical for all t, positive integer n, and $t_1, \ldots, t_n \in \mathcal{T}$ satisfying the conditions $t_i + t \in \mathcal{T},\ i = 1, \ldots, n$.

Note that this definition remains valid in the case of vector-valued stochastic processes. Consider a stochastic process X with finite second moment, that is, $\mathbf{E}(X_t^2) < \infty$, for all $t \in \mathcal{T}$. Denote the expected value and covariance functions by

$$\begin{aligned} \mu_X(t) &= \mathbf{E}(X_t),\ t \in \mathcal{T}, \\ R_X(s,t) &= \operatorname{cov}(X_s, X_t) \\ &= \mathbf{E}((X_t - \mu_X(t))(X_s - \mu_X(s))),\ s,t \in \mathcal{T}. \end{aligned}$$

Definition 2.6. A process $\{X_t,\ t \in \mathcal{T}\}$ is called **stationary in a weak sense** if X_t has finite second moment for all $t \in \mathcal{T}$ and the expected value and covariance function satisfy the following relation:

$$\begin{aligned} \mu_X(t) &= \mu_X,\ t \in \mathcal{T}, \\ R_X(s,t) &= R_X(t-s),\ s,t \in \mathcal{T}. \end{aligned}$$

The function R_X is called the **covariance function**.

It is clear that if a stochastic process with finite second moment is stationary in a strict sense, then it is stationary in a weak sense also, because the expected value and covariance function depend also on the two-dimensional joint distribution, which is time-invariant if the time shifts. Besides the covariance function $R_X(t)$, the **correlation function** $r_X(t)$ is also used, which is defined as follows:

$$r_X(t) = \frac{1}{R_X(0)} R_X(t) = \frac{1}{\sigma_X^2} R_X(t).$$

2.4 Gaussian Process

In practice, we often encounter stochastic processes whose finite-dimensional distributions are Gaussian. These stochastic processes are called **Gaussian**. In queueing theory Gaussian processes often appear when asymptotic methods are applied.

Note that the expected values and covariances determine the finite-dimensional distributions of the Gaussian process; therefore, it is easy to verify that a Gaussian process is stationary in a strict sense if and only if it is stationary in a weak sense. We also mention here that the discrete-time Gaussian process consists of independent Gaussian random variables if these random variables are uncorrelated.

2.5 Stochastic Process with Independent and Stationary Increments

In several practical modeling problems, stochastic processes have independent and stationary increments. These processes play a significant role both in theory and practice. Among such processes the Wiener and the Poisson processes are defined below.

Definition 2.7. If for any integer $n \geq 1$ and parameters $t_0, \ldots, t_n \in \mathcal{T},\ t_0 < \ldots < t_n$, the increments

$$X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$$

of a stochastic process $X = \{X_t,\ t \in \mathcal{T}\}$ are independent random variables, then X is called a stochastic process with **independent increments**. The process X has **stationary increments** if the distribution of $X_{t+h} - X_t,\ t, t+h \in \mathcal{T}$ does not depend on t.

2.6 Wiener Process

As a special but important case of stochastic processes with independent and stationary increments, we mention here the **Wiener process** (also called **process of Brownian motion**), which gives the mathematical model of diffusion. A process

$X = \{X_t,\ t \in [0, \infty)\}$ is called a Wiener process if the increments of the process are independent and for any positive integer n and $0 \le t_0 < \ldots < t_n$ the joint density function of random variables $X_{t_0}, \ldots, X_{t_n}$ can be given in the form

$$f(x_0, \ldots, x_n; t_0, \ldots, t_n) = (2\pi)^{-n/2} \left[t_0(t_1 - t_0) \ldots (t_n - t_{n-1})\right]^{-1/2} \times \exp\left\{-\frac{1}{2}\left(\frac{x_0^2}{t_0} + \frac{(x_1 - x_0)^2}{t_1 - t_0} + \ldots + \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\right)\right\}.$$

It can be seen from this formula that the Wiener process is Gaussian and the increments

$$X_{t_j} - X_{t_{j-1}},\ j = 1, \ldots, n,$$

are independent Gaussian random variables with expected values 0 and variances $t_j - t_{j-1}$. The expected value function and the covariance function are determined as

$$\mu_X(t) = 0, \quad R_X(s, t) = \min(t, s), \quad t, s \ge 0.$$

2.7 Poisson Process

2.7.1 Definition of Poisson Process

Besides the Wiener process defined above, we discuss in this chapter another important process with independent increments in probability theory, the Poisson process. This process plays a fundamental role not only in the field of queueing theory but in many areas of theoretical and applied sciences, and we will deal with this process later as a Markov arrival process, birth-and-death process, and renewal process. Its significance in probability theory and practice is that it can be used to model different event occurrences in time and space in, for example, queueing systems, physics, insurance, population biology. There are several introductions and equivalent definitions of the Poisson process in the literature according to its different characterizations. First we present the notion in the simple (classical) form and after that in a more general context.

In queueing theory, a frequently used model for the description of the arrival process of costumers is as follows. Assume that costumers arrive at the system one after another at $t_1 < t_2 < \ldots;\ t_n \to \infty$ as $n \to \infty$. The differences in occurrence times, called **interarrival times**, are denoted by

$$X_1 = t_1,\ X_2 = t_2 - t_1, \ldots,\ X_n = t_n - t_{n-1}, \ldots .$$

Define the process $\{N(t),\ t \ge 0\}$ with $N(0) = 0$ and

$$N(t) = \max\{n :\ t_n \le t\} = \max\{n :\ X_1 + \ldots + X_n \le t\},\ t > 0.$$

This process counts the number of customers arriving at the system in the time interval $(0, t]$ and is called the **counting process** for the sequence $t_1 < t_2 < \ldots$. Obviously, the process takes nonnegative integer values only, is nondecreasing, and $N(t) - N(s)$ equals the number of occurrences in the time interval $(s, t]$ for all $0 < s < t$.

In the special case, when $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables with exponential distribution $\text{Exp}(\lambda)$, the increments $N(t) - N(s)$ have a Poisson distribution with the parameter $\lambda(t - s)$. In addition, the counting process $N(t)$ possesses an essential property, that is, it evolves in time **without aftereffects**. This means that the past and current occurrences have no effect on subsequent occurrences. This feature leads to the property of independent increments.

Definition 2.8. We say that the process $N(t)$ is a **Poisson process** with **rate** λ if

1. $N(0) = 0$,
2. $N(t),\ t \geq 0$ is a process with independent increments,
3. The distribution of increments is Poisson with the parameter $\lambda(t - s)$ for all $0 < s < t$.

By definition, the distributions of the increments $N(t + h) - N(t), t \geq 0, h > 0$, do not depend on the moment t; therefore, it is a process with stationary increments and is called a **homogeneous** Poisson process at rate λ. Next, we introduce the Poisson process in a more general setting, and as a special case we have the homogeneous case. After that we will deal with the different characterizations of Poisson processes, which in some cases can serve as a definition of the process. At the end of this chapter, we will introduce the notion of the high-dimensional Poisson process (sometimes called a spatial Poisson process) and give its basic properties.

Let $\{\Lambda(t),\ t \geq 0\}$ be a nonnegative, monotonically nondecreasing, continuous-from-right real-valued function for which $\Lambda(0) = 0$.

Definition 2.9. We say that a stochastic process $\{N(t),\ t \geq 0\}$ taking nonnegative integers is a **Poisson process** if

1. $N(0) = 0$,
2. $N(t)$ is a process with independent increments,
3. The CDFs of the increments $N(t) - N(s)$ are Poisson with the parameter $\Lambda(t) - \Lambda(s)$ for all $0 \leq s \leq t$, that is,

$$\mathbf{P}\left(N(t) - N(s) = k\right) = \frac{(\Lambda(t) - \Lambda(s))^k}{k!} \mathrm{e}^{-(\Lambda(t) - \Lambda(s))},\ k = 0, 1, \ldots .$$

Since for any fixed $t > 0$ the distribution of $N(t) = N(t) - N(0)$ is Poisson with mean $\Lambda(t)$, that is the reason that $N(t)$ is called a Poisson process. We can state that the process $N(t)$ is a monotonically nondecreasing jumping process whose increments $N(t) - N(s),\ 0 \leq s < t$, take nonnegative integers only and the increments have Poisson distributions with the parameter $(\Lambda(t) - \Lambda(s))$. Thus the random variables $N(t),\ t \geq 0$ have Poisson distributions with the parameter $\Lambda(t)$; therefore, the expected value of $N(t)$ is $\mathbf{E}(N(t)) = \Lambda(t),\ t \geq 0$, which is called a **mean value function**.

We also note that using the property of independent increments, the joint distribution of the random variables $N(t_1), \ldots, N(t_n)$ can be derived for all positive integers n and all $0 < t_1 < \ldots < t_n$ without difficulty because for any integers $0 \leq k_1 \leq \ldots \leq k_n$ we get

$$\begin{aligned}
&\mathbf{P}(N(t_1) = k_1, \ldots, N(t_n) = k_n) \\
&= \mathbf{P}(N(t_1) = k_1, N(t_2) - N(t_1) = k_2 - k_1, \ldots, N(t_n) - N(t_{n-1}) = k_n - k_{n-1}) \\
&= \frac{(\Lambda(t_1))^{k_1}}{k_1!} \mathrm{e}^{-\Lambda(t_1)} \prod_{i=2}^{n} \frac{(\Lambda(t_i) - \Lambda(t_{i-1}))^{k_i - k_{i-1}}}{(k_i - k_{i-1})!} \mathrm{e}^{-(\Lambda(t_i) - \Lambda(t_{i-1}))}.
\end{aligned}$$

Since the mean value function $\Lambda(t) = \mathbf{E}(N(t))$ is monotonically nondecreasing, the set of discontinuity points $\{\tau_n\}$ of $\Lambda(t)$ is finite or countably infinite. It can happen that the set of discontinuity points $\{\tau_n\}$ has more than one convergence point, and in this case we cannot give the points of $\{\tau_n\}$ as an ordered sequence $\tau_1 < \tau_2 < \ldots$. Define the jumps of the function $\Lambda(t)$ at discontinuity points τ_n as follows:

$$\lambda_n = \Lambda(\tau_n + 0) - \Lambda(\tau_n - 0) = \Lambda(\tau_n) - \Lambda(\tau_n - 0).$$

By definition, the increments of a Poisson process are independent; thus it is easy to check that the following decomposition exists:

$$N(t) = N_r(t) + N_s(t),$$

where $N_r(t)$ and $N_s(t)$ are independent Poisson processes with mean value functions

$$\Lambda_r(t) = \Lambda(t) - \sum_{\tau_n < t} \lambda_n \text{ and } \Lambda_s(t) = \sum_{\tau_n < t} \lambda_n.$$

The **regular** part $N_r(t)$ of $N(t)$ has jumps equal to 1 only, whose mean value function $\Lambda_r(t)$ is continuous. Thus we can state that the process $N_r(t)$ is continuous in probability, that is, for any point t, $0 \leq t < \infty$, the relation

$$\lim_{s \to 0} \mathbf{P}(N_r(t + s) - N_r(t) > 0) = \lim_{s \to 0} \mathbf{P}(N_r(t + s) - N_r(t) \geq 1) = 0$$

is true. The second part $N_s(t)$ of $N(t)$ is called a **singular** Poisson process because it can have jumps only in discrete points $\{\tau_n\}$. Then

$$\mathbf{P}(N_s(\tau_n) - N_s(\tau_n - 0)) = k) = \frac{\lambda_n^k}{k!} \mathrm{e}^{-\lambda_n}, k = 0, 1, 2, \ldots .$$

Definition 2.10. If the mean value function $\Lambda(t)$ of a Poisson process $\{N(t),\ t \geq 0\}$ is differentiable with the derivative $\lambda(s),\ s \geq 0$ satisfying $\Lambda(t) = \int_0^t \lambda(s)\mathrm{d}s$, then the function $\lambda(s)$ is called a **rate** (or **intensity**) **function** of the process.

In accordance with our first definition (2.8), we say that the Poisson process $N(t)$ is **homogeneous** with the rate λ if the rate function is a constant $\lambda(t) = \lambda,\ t \geq 0$. In this case, $\Lambda(t) = \lambda t,\ t \geq 0$ is satisfied; consequently, the distributions of all increments $N(t) - N(s),\ 0 \leq s < t$ are Poisson with the parameter $\lambda(t-s)$ and $\mathbf{E}\,(N(t) - N(s)) = \lambda(t-s)$. This shows that the average number of occurrences is proportional to the length of the corresponding interval and the constant of proportionality is λ. These circumstances justify the name of the rate λ.

If the rate can vary with time, that is, the rate function does not equal a constant, the Poisson process is called **inhomogeneous**.

2.7.2 Construction of Poisson Process

The construction of Poisson processes plays an essential role both from a theoretical and a practical point of view. In particular, it is essential in simulation methods. The Poisson process $N(t)$ and the sequence of the random jumping points $t_1 < t_2 < \ldots$ of the process uniquely determine each other. This fact provides an opportunity to give another definition of the Poisson process on the real number line. We prove that the following two constructions of Poisson processes are valid (see, for example, pp. 117–118 in [85]).

Theorem 2.11 (*Construction I*). *Let $X_1, X_2, \ldots$ be independent and identically distributed random variables whose common CDF is exponential with parameter* 1. *Define*

$$M(t) = \sum_{m=1}^{\infty} \mathcal{I}_{\{X_1+\ldots+X_m \leq t\}},\ t \geq 0. \tag{2.1}$$

Then the process $M(t)$ is a homogeneous Poisson process with an intensity rate equal to 1.

Theorem 2.12 (*Construction II*). *Let $U_1, U_2, \ldots$ be a sequence of independent and identically distributed random variables having common uniform distribution on the interval $(0,T)$, and let N be a random variable independent of U_i with a Poisson distribution with the parameter λT. Define*

$$N(t) = \sum_{m=1}^{N} \mathcal{I}_{\{U_m \leq t\}},\ 0 \leq t \leq T. \tag{2.2}$$

Then $N(t)$ is a homogeneous Poisson process on the interval $[0,T]$ at rate λ.

We begin with the proof of Construction II. Then, using this result, we verify Construction I.

Proof (Construction II). Let K be a positive integer and $t_1, \ldots, t_K$ positive constants such that $t_0 = 0 < t_1 < t_2 < \ldots < t_K = T$. Since, by Eq. (2.2), $N(T) = N$ and $N(t) = \sum_{m=1}^{N} \mathcal{I}_{\{U_m \le t\}}$, the increments of $N(t)$ on the intervals $(t_{k-1}, t_k],\ k = 1, \ldots, K$, can be given in the form

$$N(t_k) - N(t_{k-1}) = \sum_{n=1}^{N} \mathcal{I}_{\{t_{k-1} < U_n \le t_k\}},\ k = 1, \ldots, K.$$

Determine the joint characteristic function of the increments $N(t_k) - N(t_{k-1})$. Let $s_k \in \mathbb{R},\ k = 1, \ldots, K$, be arbitrary; then

$$\begin{aligned}
\varphi(s_1, \ldots, s_K) &= \mathbf{E}\left(\exp\left\{\sum_{k=1}^{K} i s_k (N(t_k) - N(t_{k-1}))\right\}\right) \\
&= \mathbf{P}(N = 0) + \sum_{n=1}^{\infty} \mathbf{E}\left(\exp\left\{\sum_{k=1}^{K} i s_k (N(t_k) - N(t_{k-1}))\right\} \,\middle|\, N = n\right) \mathbf{P}(N = n) \\
&= \mathrm{e}^{-\lambda T} + \sum_{n=1}^{\infty} \mathbf{E}\left(\exp\left\{\sum_{k=1}^{K} i s_k \sum_{\ell=1}^{n} \mathcal{I}_{\{t_{k-1} < U_\ell \le t_k\}}\right\}\right) \mathbf{P}(N = n) \\
&= \mathrm{e}^{-\lambda T} + \sum_{n=1}^{\infty} \prod_{\ell=1}^{n} \mathbf{E}\left(\exp\left\{\sum_{k=1}^{K} i s_k \mathcal{I}_{\{t_{k-1} < U_\ell \le t_k\}}\right\}\right) \frac{(\lambda T)^n}{n!} \mathrm{e}^{-\lambda T} \\
&= \mathrm{e}^{-\lambda T} \sum_{n=0}^{\infty} \left(\sum_{k=1}^{K} \frac{t_k - t_{k-1}}{T} \mathrm{e}^{i s_k}\right)^n \frac{(\lambda T)^n}{n!} = \mathrm{e}^{-\lambda T} \exp\left\{\sum_{k=1}^{K} \mathrm{e}^{i s_k} \lambda (t_k - t_{k-1})\right\},
\end{aligned}$$

and using the relation $T = t_K - t_0 = \sum_{k=1}^{K} (t_k - t_{k-1})$ we get

$$\varphi(s_1, \ldots, s_K) = \prod_{k=1}^{K} \exp\left\{\lambda (t_k - t_{k-1})(\mathrm{e}^{i s_k} - 1)\right\}.$$

Since the characteristic function $\varphi(s_1, \ldots, s_K)$ derived here is equal to the joint characteristic function of independent random variables having a Poisson distribution with the parameters $\lambda(t_k - t_{k-1}),\ k = 1, \ldots, K$, the proof is complete. □

For the proof of Construction I we need the following well-known lemma of probability theory.

Lemma 2.13. *Let T be a positive constant and k be a positive integer. Let $U_1, \dots, U_k$ be independent and identically distributed random variables having a common uniform distribution on the interval $(0, T)$. Define by $U_{1k} \le \dots \le U_{kk}$ the ordered random variables $U_1, \dots, U_k$. Then the joint PDF of random variables $U_{1k}, \dots, U_{kk}$ is*

$$f_{U_{1k},\dots,U_{kk}}(t_1, \dots, t_k) = \begin{cases} \frac{k!}{T^k}, \text{ if } 0 < t_1 \le t_2 \le \dots \le t_k < T, \\ 0, \qquad \text{otherwise.} \end{cases}$$

Proof. Since $U_{1k} \le \dots \le U_{kk}$, it is enough to determine the joint PDF of random variables $U_{1k}, \dots, U_{kk}$ on the set

$$\mathcal{K} = \{(t_1, \dots, t_k) : \ 0 \le t_1 \le \dots \le t_k < T\}.$$

Under the assumptions of the lemma, the random variables $U_1, \dots, U_k$ are independent and uniformly distributed on the interval $(0, T)$; thus for every permutation $i_1, \dots, i_k$ of the numbers $1, 2, \dots, k$ (the number of all permutations is equal to $k!$)

$$\begin{aligned} &\mathbf{P}\left(U_{i_1} \le \dots \le U_{i_k}, U_{i_1} \le t_1, \dots, U_{i_k} \le t_k\right) \\ &\quad = \mathbf{P}\left(U_1 \le \dots \le U_k, U_1 \le t_1, \dots, U_k \le t_k\right), \end{aligned}$$

then

$$\begin{aligned} F_{U_{1k},\dots,U_{kk}}(t_1, \dots, t_k) &= \mathbf{P}\left(U_{1k} \le t_1, \dots, U_{kk} \le t_k\right) \\ &= k!\mathbf{P}\left(U_1 \le \dots \le U_k, U_1 \le t_1, \dots, U_k \le t_k\right) \\ &= k! \int_0^{t_1} \dots \int_0^{t_k} \frac{1}{T^k} \mathcal{I}_{\{u_1 \le \dots \le u_k\}} \mathrm{d}u_k \dots \mathrm{d}u_1 \\ &= \frac{k!}{T^k} \int_0^{t_1} \int_{u_1}^{t_2} \dots \int_{u_{k-1}}^{t_k} \mathrm{d}u_k \dots \mathrm{d}u_1. \end{aligned}$$

From this we immediately have

$$f_{U_{1k},\dots,U_{kk}}(t_1, \dots, t_k) = \frac{k!}{T^k}, \quad (t_1, \dots, t_k) \in \mathcal{K},$$

which completes the proof. □

Proof (Construction I). We verify that for any $T > 0$ the process $M(t),\ 0 \le t \le T$ is a homogeneous Poisson process with rate λ. By Construction II, $N(T) = N$, where the distribution of random variable N is Poisson with the parameter λT. From Eq. (2.2) it follows that the process $N(t),\ 0 \le t < T$, can be rewritten in the form

$$N(t) = \sum_{m=1}^{N} \mathcal{I}_{\{U_m \le t\}} = \sum_{n=1}^{N} \mathcal{I}_{\{U_{nN} \le t\}},$$

where for every $k \ge 1$ and under the condition $N(T) = k$ the random variables $U_1, \dots U_k$ are independent and uniformly distributed on the interval $(0, T)$ and $U_{1k} \le U_{2k} \le \dots \le U_{kk}$ are the ordered random variables $U_1, \dots U_k$. Note that we used these properties only to determine the joint characteristic function of the increments. Define

$$T_n = X_1 + \dots + X_n,\ \ n = 1, 2, \dots,$$

where, by assumption, $X_1, X_2, \dots$ are independent and identically distributed random variables with a common exponential CDF of parameter 1. Then, using the relation (2.1), for any $0 \le t \le T$,

$$M(t) = \sum_{n=1}^{\infty} \mathcal{I}_{\{T_n \le t\}} = \begin{cases} \sum\limits_{n=1}^{M(T)} \mathcal{I}_{\{T_n \le t\}}, & \text{if } T \ge T_1, \\ 0, & \text{if } T < T_1. \end{cases}$$

By the previous note it is enough to prove that

(a) The random variable $M(T)$ has a Poisson CDF with the parameter λT;
(b) For every positive integer k and under the condition $M(T) = k$, the joint CDF of the random variables $T_1, \dots, T_n$ are identical with the CDF of the random variables $U_{1k}, \dots, U_{kk}$.

(a) First we prove that for any positive t the CDF of the random variable $M(t)$ is Poisson with the parameter (λT). Since the common CDF of independent and identically distributed random variables X_i is exponential with the parameter λ, the random variable T_n has a gamma(n, λ) distribution whose PDF (see the description of gamma distribution in Sect. 1.2.2.) is

$$f_{T_n}(x) = \begin{cases} \frac{\lambda^n}{\Gamma(n)} x^{n-1} \mathrm{e}^{-\lambda x}, & \text{if } x > 0, \\ 0, & \text{if } x \le 0. \end{cases}$$

From the exponential distribution of the first arrival we have

$$\mathbf{P}\left(M(t) = 0\right) = \mathbf{P}\left(X_1 > t\right) = \mathrm{e}^{-\lambda t}.$$

Using the theorem of the total expected value, for every positive integer k we obtain

$$\begin{aligned}
\mathbf{P}(M(t)=k) &= \mathbf{P}(X_1+\ldots+X_k \le t < X_1+\ldots+X_{k+1}) \\
&= \mathbf{P}(T_k \le t < T_k + X_{k+1}) \\
&= \int_0^t \mathbf{P}(T_k \le t < T_k + X_{k+1} \mid T_k = z)\frac{\lambda^k}{\Gamma(k)} z^{k-1}\mathrm{e}^{-\lambda z}\mathrm{d}z \\
&= \int_0^t \mathbf{P}(t - z < X_{k+1})\frac{\lambda^k}{\Gamma(k)} z^{k-1}\mathrm{e}^{-\lambda z}\mathrm{d}z \\
&= \int_0^t \mathrm{e}^{-\lambda(t-z)}\frac{\lambda^k}{\Gamma(k)} z^{k-1}\mathrm{e}^{-\lambda z}\mathrm{d}z \\
&= \frac{\lambda^k}{\Gamma(k)}\mathrm{e}^{-\lambda t}\int_0^t z^{k-1}\mathrm{d}z = \frac{(\lambda t)^k}{\Gamma(k)k}\mathrm{e}^{-\lambda t} = \frac{(\lambda t)^k}{k!}\mathrm{e}^{-\lambda t};
\end{aligned}$$

thus the random variable $M(t),\ t \ge 0$ has a Poisson distribution with the parameter λt.

(b) Let T be a fixed positive number and let $U_1,\ldots U_k$ be independent random variables uniformly distributed on the interval $(0,1)$. Denote by $U_{1k} \le \ldots \le U_{kk}$ the ordered random variables $U_1,\ldots U_k$. Now we verify that for every positive integer k the joint CDF of random variables $T_1,\ldots,T_k$ under the condition $M(T) = k$ is identical with the joint CDF of the ordered random variables $U_{1k},\ldots,U_{kk}$ (see Theorem 2.3 of Ch. 4. in [48]).

For any positive numbers $t_1,\ldots,t_k$, the joint conditional CDF of random variables $T_1,\ldots,T_k$ given $M(t) = k$ can be written in the form

$$\mathbf{P}(T_1 \le t_1,\ldots,T_k \le t_k \mid M(T)=k) = \frac{\mathbf{P}(T_1 \le t_1,\ldots,T_k \le t_k, M(T)=k)}{\mathbf{P}(M(T)=k)}.$$

By the result proved in part (a), the denominator has the form

$$\mathbf{P}(M(T)=k) = \frac{(\lambda T)^k}{k!}\mathrm{e}^{-\lambda T},\ k=0,1,\ldots,$$

while the numerator can be written as follows:

$$
\begin{aligned}
&\mathbf{P}\left(T_1 \le t_1, \ldots, T_k \le t_k, M(T) = k\right) \\
&\quad = \mathbf{P}\left(X_1 \le t_1, X_1 + X_2 \le t_2, \ldots, X_1 + \ldots + X_k \le t_k, X_1 + \ldots + X_{k+1} > T\right) \\
&\quad = \int\limits_0^{t_1} \int\limits_0^{t_2-u_1} \int\limits_0^{t_3-(u_1+u_2)} \ldots \int\limits_0^{t_k-(u_1+\ldots+u_{k-1})} \int\limits_{T-(u_1+\ldots+u_k)}^{\infty} \prod_{i=1}^{k+1} \left(\lambda \mathrm{e}^{-\lambda u_i}\right) \mathrm{d}u_{k+1} \ldots \mathrm{d}u_1 \\
&\quad = \lambda^k \int\limits_0^{t_1} \int\limits_0^{t_2-u_1} \int\limits_0^{t_3-(u_1+u_2)} \ldots \int\limits_0^{t_k-(u_1+\ldots+u_{k-1})} \mathrm{e}^{-\lambda(u_1+\ldots+u_k)} \mathrm{e}^{-\lambda(T-u_1+\ldots+u_k)} \mathrm{d}u_{k+1} \ldots \mathrm{d}u_1 \\
&\quad = \lambda^k \mathrm{e}^{-\lambda T} \int\limits_0^{t_1} \int\limits_0^{t_2-u_1} \int\limits_0^{t_3-(u_1+u_2)} \ldots \int\limits_0^{t_k-(u_1+\ldots+u_{k-1})} \mathrm{d}u_k \ldots \mathrm{d}u_1 .
\end{aligned}
$$

Setting $v_1 = u_1, v_2 = u_1 + u_2, \ldots, v_k = u_1 + \ldots + u_k$, the last integral takes the form

$$
\frac{(\lambda T)^k}{k!} \mathrm{e}^{-\lambda T} \frac{k!}{T^k} \int\limits_0^{t_1} \int\limits_{v_1}^{t_2} \int\limits_{v_2}^{t_3} \ldots \int\limits_{v_{k-1}}^{t_k} \mathrm{d}v_k \ldots \mathrm{d}v_1 ,
$$

thus

$$
\mathbf{P}\left(T_1 \le t_1, \ldots, T_k \le t_k \,|M(T) = k\right) = \frac{k!}{T^k} \int\limits_0^{t_1} \int\limits_{v_1}^{t_2} \int\limits_{v_2}^{t_3} \ldots \int\limits_{v_{k-1}}^{t_k} \mathrm{d}v_k \ldots \mathrm{d}v_1 .
$$

From this we get that the joint conditional PDF of random variables $T_1, \ldots, T_k$ given $M(T) = k$ equals the constant value $\frac{k!}{T^k}$, which, by the preceding lemma, is identical with the joint PDF of random variables $U_{1k}, \ldots, U_{kk}$. Using the proof of Construction II, we obtain that Construction I has the result of a homogeneous Poisson process at rate λ on the interval $(0, T]$, and at the same time on the whole interval $(0, \infty)$, because T was chosen arbitrarily. □

2.7.3 *Basic Properties of a Homogeneous Poisson Process*

Let $N(t),\ t \ge 0$ be a homogeneous Poisson process with a rate λ. We enumerate below the main properties of $N(t)$.

(a) For any $t \ge 0$ the CDF of $N(t)$ is Poisson with the parameter λt, that is,

$$
\mathbf{P}\left(N(t) = k\right) = \frac{(\lambda t)^k}{k!} \mathrm{e}^{-\lambda t},\ k = 0, 1, \ldots .
$$

(b) The increments of $N(t) - N(s)$, $0 \le s < t$, are independent and have a Poisson distribution with the parameter $\lambda(t-s)$.
(c) The sum of two independent homogeneous Poisson processes $N_1(t;\lambda_1)$ and $N_2(t;\lambda_2)$ at rates λ_1 and λ_2, respectively, is a homogeneous Poisson process with a rate $(\lambda_1 + \lambda_2)$.
(d) Given $0 < t < T < \infty$, a positive integer N_0 and an integer k satisfy the inequality $0 \le k \le N_0$. The conditional CDF of the random variable $N(t)$ given $N(T) = N_0$ is binomial with the parameters $(N_0, 1/T)$.

Proof.

$$\begin{aligned}
\mathbf{P}\left(N(t) = k \mid N(T) = N_0\right) &= \frac{\mathbf{P}\left(N(t) = k, N(T) = N_0\right)}{\mathbf{P}\left(N(T) = N_0\right)} \\
&= \frac{\mathbf{P}\left(N(t) = k, N(T) - N(t) = N_0 - k\right)}{\mathbf{P}\left(N(T) = N_0\right)} \\
&= \frac{(\lambda t)^k}{k!} \mathrm{e}^{-\lambda t} \frac{(\lambda(T-t))^{N_0 - k}}{(N_0 - k)!} \mathrm{e}^{-\lambda(T-t)} \left(\frac{(\lambda T)^{N_0}}{N_0!} \mathrm{e}^{-\lambda T} \right)^{-1} \\
&= \binom{N_0}{k} \left(\frac{t}{T} \right)^k \left(1 - \frac{t}{T} \right)^{N_0 - k}.
\end{aligned}$$

□

(e) The following asymptotic relations are valid as $h \to +0$:

$$\begin{aligned}
&\mathbf{P}\left(N(h) = 0\right) = 1 - \lambda h + o(h), \\
&\mathbf{P}\left(N(h) = 1\right) = \lambda h + o(h), \\
&\mathbf{P}\left(N(h) \ge 2\right) = o(h). \qquad \text{(orderliness)}
\end{aligned}$$

Lemma 2.14. *For every nonnegative integer m the inequality*

$$\left| \mathrm{e}^x - \sum_{k=0}^{m} \frac{x^k}{k!} \right| < \frac{|x|^{m+1}}{(m+1)!} \mathrm{e}^{|x|} = o(|x|^m),\ x \to 0,$$

holds.

Proof. The assertion of the lemma follows from the nth-order Taylor approximation to e^x with the Lagrange form of the remainder term (see Sect. 7.7 of [4]), but one can obtain it by simple computations. Using the Taylor expansion

$$\mathrm{e}^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

of the function e^x, which implies that

$$\left| e^x - \sum_{k=0}^{m} \frac{x^k}{k!} \right| = \left| \sum_{k=m+1}^{\infty} \frac{x^k}{k!} \right| \leq \frac{|x|^{m+1}}{(m+1)!} \sum_{k=0}^{\infty} \frac{(m+1)!}{(m+1+k)!} |x|^k <$$

$$< \frac{|x|^{m+1}}{(m+1)!} \sum_{k=0}^{\infty} \frac{|x|^k}{k!} = \frac{|x|^{m+1}}{(m+1)!} e^{|x|} = o(|x|^m),\ x \to 0.$$

□

Proof of Property (e). From the preceding lemma we have as $h \to +0$

$$\mathbf{P}(N(h) = 0) = e^{-\lambda h} = 1 - \lambda h + o(h),$$

$$\mathbf{P}(N(h) = 1) = \frac{\lambda h}{1!} e^{-\lambda h} = \lambda h (1 - \lambda h + o(h)) = \lambda h + o(h),$$

$$\mathbf{P}(N(h) \geq 2) = 1 - \left(e^{-\lambda h} + \frac{(\lambda h)^1}{1!} e^{-\lambda h} \right) = -\left(e^{-\lambda h} - 1 + \frac{(\lambda h)^1}{1!} e^{-\lambda h} \right) = o(h).$$

□

(f) Given that exactly one event of a homogeneous Poisson process $[N(t),\ t \geq 0]$ has occurred during the interval $(0, t]$, the time of occurrence of this event is uniformly distributed over $(0, t]$.

Proof of Property (f). Denote by λ the rate of the process $N(t)$. Immediate application of the conditional probability gives for all $0 < x < t$

$$\begin{aligned}
\mathbf{P}(X_1 \leq x | N(t) = 1) &= \frac{\mathbf{P}(X_1 \leq x,\ N(t) = 1)}{\mathbf{P}(N(t) = 1)} \\
&= \frac{\mathbf{P}(N(x) = 1, N(t) - N(x) = 0)}{\mathbf{P}(N(t) = 1)} \\
&= \frac{\mathbf{P}(N(x) = 1)\mathbf{P}(N(t - x) = 0)}{\mathbf{P}(N(t) = 1)} \\
&= \left(\frac{(\lambda x)^1}{1!} e^{-\lambda x} \frac{[\lambda(t - x)]^0}{0!} e^{-\lambda(t-x)} \right) \left(\frac{(\lambda t)^1}{1!} e^{-\lambda t} \right)^{-1} = \frac{x}{t}.
\end{aligned}$$

□

(g) **Strong Markov property**. Let $\{N(t),\ t \geq 0\}$ be a homogeneous Poisson process with the rate λ, and assume that $N(t)$ is $\mathcal{A}_t$ measurable for all $t \geq 0$, where $\mathcal{A}_t \subset \mathcal{A},\ t \geq 0$, is a monotonically increasing family of σ-algebras. Let τ be a random variable such that the condition $\{\tau \leq t\} \in \mathcal{A}_t$ holds for all $t \geq 0$. This type of random variable is called a **Markov point** with respect to the family of σ-algebra $\mathcal{A}_t, t \geq 0$. For example, the constant $\tau = t$ and the so-called **first hitting time**, $\tau_k = \sup\{s : N(s) < k\}$, where k is a positive integer, are Markov points. Denote

$$N_\tau(t) = N(t+\tau) - N(\tau),\ t \geq 0.$$

Then the process $N_\tau(t),\ t \geq 0$, is a homogeneous Poisson process with the rate λ, which does not depend on the Markov point τ or on the process $\{N(t),\ 0 \leq t \leq \tau\}$.

(h) **Random deletion (filtering) of a Poisson process**. Let $N(t),\ t \geq 0$ be a homogeneous Poisson process with intensity $\lambda > 0$. Let us suppose that we delete points in the process $N(t)$ independently with probability $(1-p)$, where $0 < p < 1$ is a fixed number. Then the new process $M(t),\ t \geq 0$, determined by the undeleted points of $N(t)$ constitutes a homogeneous Poisson process with intensity $p\lambda$.

Proof of the Property (h). Let us represent the Poisson process $N(t)$ in the form

$$N(t) = \sum_{k=1}^{\infty} \mathcal{I}_{\{t_k \leq t\}},\ \ t \geq 0,$$

where $t_k = X_1 + \ldots + X_k,\ k = 1, 2, \ldots$ and $X_1, X_2, \ldots$ are independent exponentially distributed random variables with the parameter λ. The random deletion in the process $N(t)$ can be realized with the help of a sequence of independent and identically distributed random variables $I_1, I_2, \ldots,$ which do not depend on the process $N(t),\ t \geq 0$ and have a distribution $\mathbf{P}(I_k = 1) = p$, $\mathbf{P}(I_k = 0) = 1 - p$. The deletion of a point t_k in the process $N(t)$ happens only in the case $I_k = 0$. Let $T_0 = 0$, and denote by $0 < T_1 < T_2 < \ldots$ the sequence of remaining points. Thus the new process can be given in the form

$$M(t) = \sum_{k=1}^{\infty} \mathcal{I}_{\{T_k \leq t\}} = \sum_{k=1}^{\infty} \mathcal{I}_{\{t_k \leq t,\ I_k = 1\}},\ \ t \geq 0.$$

Using the property of the process $N(t)$ and the random sequence $I_k,\ k \geq 1$, it is clear that the sequence of random variables $Y_k = T_k - T_{k-1},\ k = 1, 2, \ldots$, are independent and identically distributed; therefore, it is enough to prove that they have an exponential distribution with the parameter $p\lambda$, i.e., $\mathbf{P}(Y_k < y) = 1 - \mathrm{e}^{p\lambda y}$.

The sequence of the remaining points T_k can be given in the form $T_k = t_{n_k},\ k = 1, 2, \ldots,$ where the random variables n_k are defined as follows:

$$n_1 = \min\{j : \ j \geq 1,\ I_j = 1\},$$

$$n_k = \min\{j : \ j > n_{k-1},\ I_j = 1\},\ k \geq 2.$$

Let us compute the distribution of the random variable

$$Y_1 = T_1 = X_1 + \ldots + X_{n_1}.$$

By the use of the formula of total probability, we obtain

$$\begin{aligned}\mathbf{P}(Y_1 < y) &= \mathbf{P}(X_1 + \ldots + X_{n_1} < y) \\ &= \sum_{k=1}^{\infty} \mathbf{P}(X_1 + \ldots + X_{n_1} < y | n_1 = k)\mathbf{P}(n_1 = k) \\ &= \sum_{k=1}^{\infty} \mathbf{P}(X_1 + \ldots + X_k < y)\mathbf{P}(n_1 = k).\end{aligned}$$

The sum $X_1 + \ldots + X_k$ of independent exponentially distributed random variables X_i has a gamma distribution with the density function

$$f(y; k, \lambda) = \frac{\lambda^k}{(k-1)!} y^{k-1} \mathrm{e}^{-\lambda y}, \ y > 0,$$

whereas, on the other hand, the random variable n_1 has a geometric distribution with the parameter p, i.e.,

$$\mathbf{P}(n_1 = k) = (1-p)^{k-1} p;$$

therefore, we get

$$\sum_{k=1}^{\infty} \mathbf{P}(X_1 + \ldots + X_k < y)\mathbf{P}(n_1 = k) = \sum_{k=1}^{\infty} \int_0^y \frac{\lambda^k}{(k-1)!} x^{k-1} \mathrm{e}^{-\lambda x} (1-p)^{k-1} p \mathrm{d}x$$

$$= \lambda p \int_0^y \left(\sum_{k=0}^{\infty} \frac{[(1-p)\lambda x]^k}{k!} \right) \mathrm{e}^{-\lambda x} \mathrm{d}x = \lambda p \int_0^y \mathrm{e}^{(1-p)\lambda x} \mathrm{e}^{-\lambda x} \mathrm{d}x$$

$$= \lambda p \int_0^y \mathrm{e}^{-p\lambda x} \mathrm{d}x = 1 - \mathrm{e}^{-p\lambda y}.$$

□

(1) **Modeling an inhomogeneous Poisson process**. Let $\{\Lambda(t),\ t \geq 0\}$ be a nonnegative, monotonically nondecreasing, continuous-from-left function such that $\Lambda(0) = 0$. Let $N(t),\ t \geq 0$, be a homogeneous Poisson process with rate 1. Then the process defined by the equation

$$N_{\Lambda}(t) = N(\Lambda(t)),\ t \geq 0,$$

is a Poisson process with mean value function $\Lambda(t),\ t \geq 0$.

Proof of Property (i). Obviously, $N(\Lambda(0)) = N(0) = 0$, and the increments of the process $N_\Lambda(t)$ are independent and the CDF of the increments are Poissonian, because for any $0 \leq s \leq t$ the CDF of the increment $N_\Lambda(t) - N_\Lambda(s)$ is Poisson with the parameter $\Lambda(t) - \Lambda(s)$,

$$\mathbf{P}(N_\Lambda(t) - N_\Lambda(s) = k) = \mathbf{P}(N(\Lambda(t)) - N(\Lambda(s)))$$
$$= \frac{(\Lambda(t) - \Lambda(s))^k}{k!} \mathrm{e}^{-(\Lambda(t)-\Lambda(s))},\ k = 0, 1, \ldots .$$

□

2.7.4 Higher-Dimensional Poisson Process

The Poisson process can be defined, in higher dimensions, as a model of random points in space. To do this, we first concentrate on the process on the real number line, from the aspect of a possible generalization.

Let $\{N(t),\ t \geq 0\}$ be a Poisson process on a probability space $(\Omega, \mathcal{A}, P)$. Assume that it has a rate function $\lambda(t),\ t \geq 0$; thus, the mean value function has the form

$$\Lambda(t) = \int_0^t \lambda(s)\mathrm{d}s,\ t \geq 0,$$

where the function $\lambda(t)$ is nonnegative and locally integrable function. Denote by $t_1, t_2, \ldots$ the sequence of the random jumping points of $N(t)$. Since the mean value function is continuous, the jumps of $N(t)$ are exactly 1; moreover, the process $N(t)$ and the random points $\Pi = \{t_1, t_2, \ldots\}$ determine uniquely each other. If we can characterize the countable set Π of random points $\{t_1, t_2, \ldots\}$, then at the same time we can give a new definition of the Poisson process $N(t)$.

Denote by $\mathcal{B}_+ = \mathcal{B}(\mathbb{R}_+)$ the Borel σ-algebra of the half line $\mathbb{R}_+ = [0, \infty)$, i.e., the minimal σ-algebra that consists of all open intervals of $\mathbb{R}_+$. Let $B_i = (a_i, b_i],\ i = 1, \ldots, n$, be nonoverlapping intervals of $\mathbb{R}_+$; then obviously $B_i \in \mathcal{B}_+$. Introduce the random variables

$$\Pi(B_i) = \#\{\Pi \cap B_i\} = \#\left\{t_j : t_j \in B_i\right\},\ i = 1, \ldots, n,$$

where $\#\{\cdot\}$ means the number of elements of a set; then

$$\Pi(B_i) = N(b_i) - N(a_i).$$

By the use of the properties of Poisson processes, the following statements hold:

(1) The random variables $\Pi(B_i)$ are independent because the increments of the process $N(t)$ are independent.
(2) The CDF of $\Pi(B_i)$ is Poisson with the parameter $\Lambda(B_i)$, i.e.,

$$\mathbf{P}\left(\Pi(B_i) = k\right) = \frac{(\Lambda(B_i))^k}{k!}\mathrm{e}^{-\Lambda(B_i)},$$

where $\Lambda(B_i) = \int_{B_i} \lambda(s)\mathrm{d}s,\ 1 \le i \le n$.

Observe that by the definition of random variables $\Pi(B_i)$, it is unimportant whether or not the set of random points $\Pi = \{t_i\}$ is ordered and $\Pi(B_i)$ is determined by the number of points t_i only, which is included in the interval $(a_i, b_i]$. This circumstance is important because we want to define the Poisson processes on higher-dimensional spaces, which do not constitute an ordered set, contrary to the one-dimensional case.

More generally, let $B_i \in \mathcal{B}(\mathbb{R}_+),\ 1 \le i \le n$, be disjoint Borel sets and denote $\Pi(B_i) = \#\{\Pi \cap B_i\}$. It can be checked that $\Pi(B_i)$ are random variables defined by the random points $\Pi = \{t_1, t_2, \ldots\}$ and they satisfy properties (1) and (2). On this basis, the Poisson process can be defined in higher-dimensional Euclidean spaces and, in general, in metric spaces also (see Chap. 2. of [54]).

Consider the d-dimensional Euclidean space $S = \mathbb{R}^d$ and denote by $\mathcal{B}(S)$ the Borel σ-algebra of the subset of S. We will define the Poisson process Π as a random set function satisfying properties (1) and (2). Let $\Pi : \Omega \to \mathcal{S}$ be a random point set in $\mathcal{S}$, where $\mathcal{S}$ denotes the set of all subsets of S consisting of countable points. Then the quantities $\Pi(A) = \#\{\Pi \cap A\}$ define random variables for all $A \in \mathcal{B}(S)$.

Definition 2.15. We say that Π is a Poisson process on the space S if $\Pi \in \mathcal{S}$ is a random countable set of points in S and the following conditions are satisfied:

(1) The random variables $\Pi(A_i) = \#\{\Pi \cap A_i\}$ are independent for all disjoint sets $A_1, \ldots, A_n \in \mathcal{B}(S)$.
(2) For any $A \in \mathcal{B}(S)$ the CDF of random variables $\Pi(A)$ are Poisson with the parameter $\Lambda(A)$, where $0 \le \Lambda(A) \le \infty$.
The function $\Lambda(A),\ A \in \mathcal{B}(S)$ is called a **mean measure** of a Poisson process (see [54], p. 14).

Properties:

1. Since the random variable $\Pi(A)$ has a Poisson distribution with the parameter $\Lambda(A)$, then $\mathbf{E}\left(\Pi(A)\right) = \Lambda(A)$ and $\mathbf{D}^2\left(\Pi(A)\right) = \Lambda(A)$.
2. If $\Lambda(A)$ is finite, then the random variable $\Pi(A)$ is finite with probability 1, and if $\Lambda(A) = \infty$, then the number of elements of the random point set $\Pi \cap A$ is countably infinite with probability 1.
3. For any disjoint sets $A_1, A_2, \ldots \in \mathcal{B}(S)$,

$$\Pi(A) = \sum_{i=1}^{\infty} \Pi(A_i) \quad \text{and} \quad \Lambda(A) = \sum_{i=1}^{\infty} \Lambda(A_i),$$

where $A = \cup_{i=1}^{\infty} A_i$. The last relation means that the mean measure $\Lambda(B),\ B \in \mathcal{B}(S)$ satisfies the conditions of a measure, i.e., it is a nonnegative, σ-additive set function on the measurable space $(S, \mathcal{B}(S))$, which justifies the name of Λ.

Like the one-dimensional case, when the Poisson process has a rate function, it is an important class of Poisson processes for which there exists a nonnegative locally integrable function λ with the property

$$\Lambda(B) = \int_B \lambda(s)\mathrm{d}s,\ B \in \mathcal{B}(S)$$

(here the integral is defined with respect to the Lebesgue measure ds). Then the mean measure Λ is **nonatomic**, that is, there is no point $s_0 \in \mathcal{B}(S)$ such that $\Lambda(\{s_0\}) > 0$.

4. By the use of properties 1 and 3, it is easy to obtain the relation

$$\mathbf{D}^2\left(\Pi(A)\right) = \sum_{i=1}^{\infty} \mathbf{D}^2\left(\Pi(A_i)\right) = \sum_{i=1}^{\infty} \Lambda(A_i) = \Lambda(A).$$

5. For any $B, C \in \mathcal{B}(S)$,

$$\mathrm{cov}(\Pi(B), \Pi(C)) = \Lambda(B \cap C).$$

Proof. Since $\Pi(B) = \Pi(B \cap C) + \Pi(B \backslash C)$ and $\Pi(C) = \Pi(B \cap C) + \Pi(C \backslash B)$, where the sets $A \cap C$, $A \backslash C$ and $C \backslash A$ are disjoint, the $\Pi(A \cap C)$, $\Pi(A \backslash C)$, and $\Pi(C \backslash A)$ are independent random variables, and thus

$$\begin{aligned} \mathrm{cov}(\Pi(A), \Pi(C)) &= \mathrm{cov}(\Pi(A \cap C), \Pi(A \cap C)) \\ &= \mathbf{D}^2\left(\Pi(A \cap C)\right) = \Lambda(A \cap C). \end{aligned}$$

□

6. For any (not necessarily disjoint) sets $A_1, \ldots, A_n \in \mathcal{B}(S)$ the joint distribution of random variables $\Pi(A_1), \ldots, \Pi(A_n)$ is uniquely determined by the mean measure Λ.

Proof. Denote the set of the 2^n pairwise disjoint sets by

$$\mathcal{C} = \{C = B_1 \cap \ldots \cap B_n,\ \text{where } B_i \text{ means the set either } A_i, \text{ or } \bar{A_i}\};$$

then the random variables $\Pi(C)$ are independent and have a Poisson distribution with the parameter $\Lambda(C)$. Consequently, the random variables $\Pi(A_1), \ldots, \Pi(A_n)$

can be given as a sum from a 2^n number of independent random variables $\Pi(C),\ C \in \mathcal{C}$, having a Poisson distribution with the parameter $\Lambda(C)$; therefore, the joint distribution of random variables $\Pi(A_i)$ is uniquely determined by $\Pi(C),\ C \in \mathcal{C}$, and the mean measure Λ. □

Comment 2.16. *Let $S = \mathbb{R}^d$, and assume*

$$\Lambda(A) = \int_A \lambda(x)\mathrm{d}x,\ A \in \mathcal{B}(S),$$

where $\lambda(x)$ is a nonnegative and locally integrable function and $dx = dx_{1...}dx_n$. If $|A|$ denotes the n-dimensional (Lebesgue) measure of a set A and the function $\lambda(x)$ is continuous at a point $x_0 \in S$, then

$$\Lambda(A) \sim \lambda(x_0)\,|A|$$

if the set A is included in a small neighborhood of the point x_0.

The Poisson process Π is called ***homogeneous*** *if $\lambda(x) = \lambda$ for a positive constant λ. In this case for any $A \in \mathcal{B}(S)$ the inequality $\Lambda(A) = \lambda\,|A|$ holds.*

The following three theorems state general assertions on the Poisson processes defined in higher-dimensional spaces (see Chap. 2 of [54]).

Theorem 2.17 (*Existence theorem). If the mean measure Λ is nonatomic on the space S and it is σ-finite, i.e., it can be expressed in the form*

$$\Lambda = \sum_{i=1}^{\infty} \Lambda_i,\ \ \textit{where}\ \ \Lambda_i(S) < \infty,$$

then there exists a Poisson process Π on the space S and has mean measure Λ.

Theorem 2.18 (*Superposition theorem). If $\Pi_i, i = 1, 2, \ldots$, is a sequence of independent Poisson processes with mean measure $\Lambda_1, \Lambda_2, \ldots$ on the space S, then the superposition $\Pi = \cup_{i=1}^{\infty}\Pi_i$ is a Poisson process with mean measure $\Lambda = \sum_{i=1}^{\infty}\Lambda_i$.*

Theorem 2.19 (*Restriction theorem). Let Π be a Poisson process on the space S with mean measure Λ. Then for any $S_0 \in \mathcal{B}(S)$ the process*

$$\Pi_0 = \Pi \cap S_0$$

can be defined as a Poisson process on S with mean measure

$$\Lambda_0(A) = \Lambda(A \cap S_0).$$

The process Π_0 can be interpreted as a Poisson process on the space S_0 with mean measure Λ_0, where Λ_0 is called the restriction of mean measure Λ to S_0.

2.8 Exercises

Exercise 2.1. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with finite absolute moment $\mathbf{E}(|X_1|) < \infty$. Let N be a random variable taking positive integer numbers and independent of the random variable $(X_i, i = 1, 2, \ldots)$. Prove that

(a) $\mathbf{E}(X_1 + \ldots + X_N) = \mathbf{E}(X_1)\,\mathbf{E}(N)$,
(b) $\mathbf{D}^2(X_1 + \ldots + X_N) = \mathbf{D}^2(X_1) + (\mathbf{E}(X_1))^2(\mathbf{E}(N))^2$
(Wald identities or Wald lemma).

Exercise 2.2. Let $X_0, X_1, \ldots$ be independent random variables with joint distribution $\mathbf{P}(X_i = 1) = \mathbf{P}(X_i = -1) = \frac{1}{2}$.

Define $Z_0 = 0$, $Z_k = Z_{k-1} + X_k$, $k = 0, 1, \ldots$. Determine the expectation and covariance function of the process $(Z_k, k = 1, 2, \ldots)$ (random walk on the integer numbers).

Let a and b be real numbers, $|b| < 1$. Denote $W_0 = aX_0$, $W_k = bW_{k-1} + X_k$, $k = 1, 2, \ldots$ [here the process $(W_k, k = 0, 1, \ldots)$ constitutes a first-degree autoregressive process with the initial value aX_0 and with the innovation process $(X_k, k = 1, 2, \ldots)$]. If we fix the value b, for which value of a will the process W_k be stationary in a weak sense?

Exercise 2.3. Let a and b be real numbers, and let U be a random variable uniformly distributed on the interval $(0, 2\pi)$. Denote $X_t = a\cos(bt + U)$, $-\infty < t < \infty$. Prove that the random cosine process $(X_t, -\infty < t < \infty)$ is stationary.

Exercise 2.4. Let $N(t), T \geq 0$ be a homogeneous Poisson process with intensity λ.

(a) Determine the covariance and correlation functions of $N(t)$.
(b) Determine the conditional expectation $\mathbf{E}(N(t + s) \mid N(t))$.