Koszul Algebras and Regularity

Aldo Conca, Emanuela De Negri, and Maria Evelina Rossi

1 Introduction

This is a chapter on commutative Koszul algebras and Castelnuovo–Mumford
regularities. Koszul algebras, originally introduced by Priddy [48], are graded
K-algebras R whose residue field K has a linear free resolution as an R-module.
Here linear means that the nonzero entries of the matrices describing the maps in the
resolution have degree 1. For example, over the symmetric algebra $S = \mathrm{Sym}_K(V)$
of a finite dimensional K-vector space V, the residue field K, is resolved by the
Koszul complex which is linear. Similarly, for the exterior algebra $\bigwedge_K V$ the residue
field K is resolved by the Cartan complex which is also linear. In this chapter we
deal mainly with standard graded commutative K-algebras, that is, quotient rings
of the polynomial ring S by homogeneous ideals. The (absolute) Castelnuovo–
Mumford regularity $\mathrm{reg}_S(M)$ is, after Krull dimension and multiplicity, perhaps
the most important invariant of a finitely generated graded S-module M, as it
controls the vanishing of both syzygies and the local cohomology modules of M.
By definition, $\mathrm{reg}_S(M)$ is the least integer r such that the ith syzygy module of
M is generated in degrees $\leq r + i$ for every i. By local duality, $\mathrm{reg}_S(M)$ can be
characterized also as the least number r such that the local cohomology module
$H^i_{\mathbf{m}_S}(M)$ vanishes in degrees $> r - i$ for every i. Analogously when $R = S/I$ is a
standard graded K-algebra and M is a finitely generated graded R-module one can
define the relative Castelnuovo–Mumford regularity as the least integer r such that
the ith syzygy module over R of M is generated in degrees $\leq r + i$ for every i.
The main difference between the relative and the absolute regularity is that over R
most of the resolutions are infinite, that is, there are infinitely many syzygy modules,
and hence it is not at all clear whether $\mathrm{reg}_R(M)$ is finite. Avramov, Eisenbud and

A. Conca (✉) • E. De Negri • M.E. Rossi
Dipartimento di Matematica, Università degli Studi di Genova, Genova, Italy
e-mail: conca@dima.unige.it; denegri@dima.unige.it; rossim@dima.unige.it

Peeva gave in [5, 6] a beautiful characterization of the Koszul property in terms of the relative regularity: R is Koszul iff $\text{reg}_R(M)$ is finite for every M iff $\text{reg}_R(K)$ is finite.

From certain point of views, Koszul algebras behave homologically as polynomial rings. For instance $\text{reg}_R(M)$ can be characterized in terms of regularity of truncated submodules (see Proposition 8). On the other hand, "bad" homological behaviors may occur over Koszul algebras. For instance, modules might have irrational Poincaŕe series over Koszul algebras. Furthermore, Koszul algebras appear quite frequently among the rings that are classically studied in commutative algebra, algebraic geometry and combinatorial commutative algebra. This mixture of similarities and differences with the polynomial ring and their frequent appearance in classical constructions are some of the reasons that make Koszul algebras fascinating, studied and beloved by commutative algebraists and algebraic geometers. In few words, a homological life is worth living in a Koszul algebra. Of course there are other reasons for the popularity of Koszul algebras in the commutative and noncommutative setting, as, for instance, Koszul duality, a phenomenon that generalizes the duality between the symmetric and the exterior algebra (see [13, 14, 50]).

The structure of this chapter is the following. Section 2 contains the characterization, due to Avramov, Eisenbud and Peeva, of Koszul algebras in terms of the finiteness of the regularity of modules (see Theorem 7). It contains also the definition of G-quadratic and LG-quadratic algebras and some fundamental questions concerning the relationships between these notions and the syzygies of Koszul algebras (see Questions 12 and 14).

In Sect. 3 we present three elementary but powerful methods for proving that an algebra is Koszul: the existence of a Gröbner basis of quadrics, the transfer of Koszulness to quotient rings and Koszul filtrations. To illustrate these methods we apply them to Veronese algebras and Veronese modules. We prove that Veronese subalgebras of Koszul algebras are Koszul and that high-enough Veronese subalgebras of any algebra are Koszul. These and related results were proved originally in [3, 11, 12, 25, 32].

Section 4 is devoted to two very strong versions of Koszulness: universally Koszul [21] and absolutely Koszul [43]. An algebra R is universally Koszul if for every ideal $I \subset R$ generated by elements of degree 1 one has $\text{reg}_R(I) = 1$. Given a graded R-module M and $i \in \mathbf{Z}$ one defines $M_{\langle i \rangle}$ as the submodule of M generated by the homogeneous component M_i of degree i of M. The R-module M is componentwise linear if $\text{reg}_R(M_{\langle i \rangle}) = i$ for every i with $M_i \neq 0$. The K-algebra R is absolutely Koszul if any finitely generated graded R-module M has a componentwise linear ith syzygy module for some $i \geq 0$. Two major achievements are the complete characterization of the Cohen–Macaulay domains that are universally Koszul (see [21] or Theorem 4) and the description of two classes of absolutely Koszul algebras (see [43] or Theorem 10). We also present some questions related to these notions, in particular Questions 13 and 14.

In Sect. 5 we discuss some problems regarding the regularity of modules over Koszul algebras. Some are of computational nature, for instance Question 12,

and others are suggested by the analogy with the polynomial ring, for example, Question 9. This section contains also some original results, in particular Proposition 5 and Theorem 11, motivating the questions presented.

Finally Sect. 6 contains a discussion on local variants of the notion of Koszul algebra and the definition of Koszul modules. A local ring $(R, \mathbf{m}, K)$ is called a Koszul ring if the associated graded ring $\mathrm{gr}_{\mathbf{m}}(R)$ is Koszul as a graded K-algebra. The ring R is called Fröberg if its Poincaré series equals to $H_R(-z)^{-1}$, where $H_R(z)$ denotes the Hilbert series of R. Any Koszul ring is Fröberg. The converse holds in the graded setting and is unknown in the local case (see Question 5). Large classes of local rings of almost minimal multiplicity are Koszul. In [41] and [43] a characterization of Koszulness of graded algebras is obtained in terms of the finiteness of the linear defect of the residue field (see Proposition 12). It is an open problem whether the same characterization holds in the local case too (see Question 13).

2   Generalities

Let K be a field and R be a (commutative) standard graded K-algebra, that is, a K-algebra with a decomposition $R = \oplus_{i \in \mathbf{N}} R_i$ (as an Abelian group) such that $R_0 = K$, R_1 is a finite dimensional K-vector space and $R_i R_j = R_{i+j}$ for every $i, j \in \mathbf{N}$. Let S be the symmetric algebra over K of R_1. One has an induced surjection

$$S = \mathrm{Sym}_K(R_1) \to R \tag{1}$$

of standard graded K-algebras. We call Eq. (1) the canonical presentation of R. Hence R is isomorphic (as a standard graded K-algebra) to S/I where I is the kernel of Eq. (1). In particular, I is homogeneous and does not contain elements of degree 1. We say that I defines R. Choosing a K-basis of R_1 the symmetric algebra S gets identified with the polynomial ring $K[x_1, \ldots, x_n]$, with $n = \dim_K R_1$, equipped with its standard graded structure (i.e., $\deg x_i = 1$ for every i). Denote by $\mathbf{m}_R$ the maximal homogeneous ideal of R. We may consider K as a graded R-module via the identification $K = R/\mathbf{m}_R$.

Assumption. With the exception of the last section, K-algebras are always assumed to be standard graded, modules and ideals are graded and finitely generated, and module homomorphisms have degree 0.

For an R-module $M = \oplus_{i \in \mathbf{Z}} M_i$ we denote by $\mathrm{HF}(M, i)$ the Hilbert function of M at i, that is, $\mathrm{HF}(M, i) = \dim_K M_i$ and by $H_M(z) = \sum \dim_K M_i z^i \in \mathbf{Q}[|z|][z^{-1}]$ the associated Hilbert series.

Recall that a minimal graded free resolution of M as an R-module, is a complex of free R-modules

$$\mathbf{F} : \cdots \to F_{i+1} \xrightarrow{\phi_{i+1}} F_i \xrightarrow{\phi_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to 0$$

such that $H_i(\mathbf{F}) = 0$ for $i > 0$ and $H_0(\mathbf{F}) = M$, Image $\phi_{i+1} \subseteq \mathbf{m}_R F_i$ for every i. Such a resolution exists and it is unique up to an isomorphism of complexes, that is why we usually talk of "the" minimal free (graded) resolution of M. By definition, the ith Betti number $\beta_i^R(M)$ of M as an R-module is the rank of F_i. Each F_i is a direct sum of shifted copies of R. The (i, j)th graded Betti number $\beta_{ij}^R(M)$ of M is the number of copies of $R(-j)$ that appear in F_i. By construction one has $\beta_i^R(M) = \dim_K \operatorname{Tor}_i^R(M, K)$ and $\beta_{ij}^R(M) = \dim_K \operatorname{Tor}_i^R(M, K)_j$. The Poincaré series of M is defined as

$$P_M^R(z) = \sum_i \beta_i^R(M) z^i \in \mathbf{Q}[|z|],$$

and its bigraded version is

$$P_M^R(s, z) = \sum_{i,j} \beta_{i,j}^R(M) z^i s^j \in \mathbf{Q}[s][|z|].$$

We set

$$t_i^R(M) = \sup\{j : \beta_{ij}^R(M) \neq 0\}$$

where, by convention, $t_i^R(M) = -\infty$ if $F_i = 0$. By definition, $t_0^R(M)$ is the largest degree of a minimal generator of M. Two important invariants that measure the "growth" of the resolution of M as an R-module are the projective dimension

$$\operatorname{pd}_R(M) = \sup\{i : F_i \neq 0\} = \sup\{i : \beta_{ij}^R(M) \neq 0 \text{ for some } j\}$$

and the Castelnuovo–Mumford regularity

$$\operatorname{reg}_R(M) = \sup\{j - i : \beta_{ij}^R(M) \neq 0\} = \sup\{t_i^R(M) - i : i \in \mathbf{N}\}.$$

We may as well consider M as a module over the polynomial ring S via Eq. (1). The regularity $\operatorname{reg}_S(M)$ of M as an S-module has also a cohomological interpretation via local duality (see , e.g. [15, 31]). Denoting by $H_{\mathbf{m}_S}^i(M)$ the ith local cohomology module with support on the maximal ideal of S one has

$$\operatorname{reg}_S(M) = \max\{j + i : H_{\mathbf{m}_S}^i(M)_j \neq 0\}.$$

Since $H_{\mathbf{m}_R}^i(M) = H_{\mathbf{m}_S}^i(M)$ for every i, nothing changes if on right-hand side of the formula above we replace S with R. So $\operatorname{reg}_S(M)$ is in some sense the "absolute" Castelnuovo–Mumford regularity. Both $\operatorname{pd}_R(M)$ and $\operatorname{reg}_R(M)$ can be infinite.

Example 1. Let $R = K[x]/(x^3)$ and $M = K$. Then $F_{2i} = R(-3i)$ and $F_{2i+1} = R(-3i - 1)$ so that $\operatorname{pd}_R(M) = \infty$ and $\operatorname{reg}_R(M) = \infty$.

Note that, in general, $\operatorname{reg}_R(M)$ is finite if $\operatorname{pd}_R M$ is finite, but, as we will see, not the other way round.

In the study of minimal free resolutions over R, the resolution $\mathbf{K}_R$ of the residue field K as an R-module plays a prominent role. This is because $\operatorname{Tor}_*^R(M, K) =$

$H_*(M \otimes \mathbf{K}_R)$ and hence $\beta_{ij}^R(M) = \dim_K H_i(M \otimes \mathbf{K}_R)_j$. A very important role is played also by the Koszul complex $K(\mathbf{m}_R)$ on a minimal system of generators of the maximal ideal $\mathbf{m}_R$ of R.

When is $\mathrm{pd}_R(M)$ finite for every M? The answer is given by one of the most classical results in commutative algebra: the Auslander–Buchsbaum–Serre Theorem. We present here the graded variant of it that can be seen as a strong version of the Hilbert syzygy theorem.

Theorem 2. *The following conditions are equivalent:*

(1) $\mathrm{pd}_R M$ is finite for every R-module M.
(2) $\mathrm{pd}_R K$ is finite.
(3) R is regular, that is, R is a polynomial ring.

When the conditions hold, then for every M, one has $\mathrm{pd}_R M \leq \mathrm{pd}_R K = \dim R$, and the Koszul complex $K(\mathbf{m}_R)$ resolves K as an R-module, that is, $\mathbf{K}_R \cong K(\mathbf{m}_R)$.

Remark 3. The Koszul complex $K(\mathbf{m}_R)$ has three important features:

(1) It is finite.
(2) It has an algebra structure. Indeed it is a DG-algebra and this has important consequences such as the algebra structure on the Koszul cycles and Koszul homology. See [4] for the definition (and much more) on DG-algebras.
(3) The matrices describing its differentials have nonzero entries only of degree 1.

When R is not a polynomial ring $\mathbf{K}_R$ does not satisfy condition (1) in Remark 3. Can $\mathbf{K}_R$ nevertheless satisfy (2) or (3) of Remark 3?

For (2) the answer is yes: $\mathbf{K}_R$ has always a DG-algebra structure. Indeed a theorem, proved independently by Gulliksen and Schoeller (see [4, 6.3.5]), asserts that $\mathbf{K}_R$ is obtained by the so-called Tate construction. This procedure starts from $K(\mathbf{m}_R)$ and builds $\mathbf{K}_R$ by "adjoining variables to kill homology" while preserving the DG-algebra structure (see [4, 6.3.5]).

Algebras R such that $\mathbf{K}_R$ satisfies condition (3) in Remark 3 in above are called Koszul:

Definition 4. The K-algebra R is Koszul if the matrices describing the differentials of $\mathbf{K}_R$ have nonzero entries only of degree 1, that is, $\mathrm{reg}_R(K) = 0$ or, equivalently, $\beta_{ij}^R(K) = 0$ whenever $i \neq j$.

Koszul algebras were originally introduced by Priddy [48] in his study of homological properties of graded (noncommutative) algebras arising from algebraic topology, leaving the commutative case "for the interested reader". In the recent volume [50] Polishchuk and Positselski present various surprising aspects of Koszulness. We collect below a list of important facts about Koszul commutative algebras. We always refer to the canonical presentation Eq. (1) of R. First we introduce a definition.

Definition 5. We say that R is G-quadratic if its defining ideal I has a Gröbner basis of quadrics with respect to some coordinate system of S_1 and some term order τ on S.

Remark 6. (1) If R is Koszul, then I is generated by quadrics (i.e., homogeneous polynomials of degree 2). Indeed, the condition $\beta_{2j}^R(K) = 0$ for every $j \neq 2$ is equivalent to the fact that I is defined by quadrics. But there are algebras defined by quadrics that are not Koszul. For example, $R = K[x, y, z, t]/I$ with $I = (x^2, y^2, z^2, t^2, xy + zt)$ has $\beta_{34}^R(K) = 5$.

(2) If I is generated by monomials of degree 2 with respect to some coordinate system of S_1, then a simple filtration argument that we reproduce in Sect. 3, (see Theorem 15) shows that R is Koszul in a very strong sense.

(3) If I is generated by a regular sequence of quadrics, then R is Koszul. This follows from a result of Tate [59] asserting that if R is a complete intersection, then $\mathbf{K}_R$ is obtained by $K(\mathbf{m}_R)$ by adding polynomial variables in homological degree 2 to kill $H_1(K(\mathbf{m}_R))$.

(4) If R is G-quadratic, then R is Koszul. This follows from (2) and from the standard deformation argument showing that $\beta_{ij}^R(K) \leq \beta_{ij}^A(K)$ with $A = S/\mathrm{in}_\tau(I)$.

(5) On the other hand there are Koszul algebras that are not G-quadratic. One notes that an ideal defining a G-quadratic algebra must contain quadrics of "low" rank. For instance, if R is Artinian and G-quadratic then its defining ideal must contain the square of a linear form. But most Artinian complete intersection of quadrics do not contain the square of a linear form. For example, $I = (x^2 + yz, y^2 + xz, z^2 + xy) \subset \mathbf{C}[x, y, z]$ is an Artinian complete intersection not containing the square of a linear form. Hence I defines a Koszul and not G-quadratic algebra. See [32] for a general result in this direction.

(6) The Poincaré series $P_K^R(z)$ of K as an R-module can be irrational, see [2]. However, for a Koszul algebra R, one has

$$P_K^R(z) H_R(-z) = 1, \tag{2}$$

and hence $P_K^R(z)$ is rational. Indeed the equality Eq. (2) turns out to be equivalent to the Koszul property of R, [37, 1]. A necessary (but not sufficient) numerical condition for R to be Koszul is that the formal power series $1/H_R(-z)$ has non-negative coefficients (indeed positive unless R is a polynomial ring). Another numerical condition is the following: expand $1/H_R(-z)$ as

$$\frac{\Pi_{h \in 2\mathbf{N}+1}(1 + z^h)^{e_h}}{\Pi_{h \in 2\mathbf{N}+2}(1 - z^h)^{e_h}}$$

with $e_h \in \mathbf{Z}$ (see [4, 7.1.1]). The numbers e_h are the "expected" deviations. If R is Koszul then $e_h \geq 0$ for every h, (indeed $e_h > 0$ for every h unless R is a complete intersection). For example, if $H(z) = 1 + 4z + 5z^2$, then the coefficient

of z^6 in $1/H(-z)$ is negative and the third expected deviation is 0. So for two reasons an algebra with Hilbert series $H(z)$, as the one in (1), cannot be Koszul.

The following characterization of the Koszul property in terms of regularity is formally similar to the Auslander–Buchsbaum–Serre Theorem 2.

Theorem 7 (Avramov–Eisenbud–Peeva). *The following conditions are equivalent:*

(1) $\mathrm{reg}_R(M)$ *is finite for every R-module M.*
(2) $\mathrm{reg}_R(K)$ *is finite.*
(3) *R is Koszul.*

Avramov and Eisenbud proved in [5] that every module has finite regularity over a Koszul algebra. Avramov and Peeva showed in [6] that if $\mathrm{reg}_R(K)$ is finite then it must be 0. Indeed they proved a more general result for graded algebras that are not necessarily standard.

If M is an R-module generated by elements of a given degree, say d, we say that it has a linear resolution over R if $\mathrm{reg}_R(M) = d$. For $q \in \mathbf{Z}$ we set $M_{\langle q \rangle}$ to be the submodule of M generated by M_q and set $M_{\geq q} = \oplus_{i \geq q} M_i$. The module M is said to be componentwise linear over R if $M_{\langle q \rangle}$ has a linear resolution for every q. The (absolute) regularity of a module can be characterized as follows:

$$\mathrm{reg}_S(M) = \min\{q \in \mathbf{Z} : M_{\geq q} \text{ has a linear resolution}\}$$
$$= \min\{q \geq t_0^S(M) : M_{\langle q \rangle} \text{ has a linear resolution}\}$$

One of the motivations of Avramov and Eisenbud in [5] was to establish a similar characterization for the relative regularity over a Koszul algebra. They proved:

Proposition 8. *Let R be a Koszul algebra and M be an R-module. Then:*

$$\mathrm{reg}_R(M) \leq \mathrm{reg}_S(M)$$

and

$$\mathrm{reg}_R(M) = \min\{q \in \mathbf{Z} : M_{\geq q} \text{ has a linear } R\text{−resolution}\}$$
$$= \min\{q \geq t_0^R(M) : M_{\langle q \rangle} \text{ has a linear } R - \text{resolution}\}.$$

Another invariant that measures the growth of the degrees of the syzygies of a module is the slope:

$$\mathrm{slope}_R(M) = \sup\left\{ \frac{t_i^R(M) - t_0^R(M)}{i} : i > 0 \right\}.$$

A useful feature of the slope is that it is finite (no matter if R is Koszul or not). Indeed with respect to the canonical presentation Eq. (1), one has

$$\mathrm{slope}_R(M) \leq \max\{\mathrm{slope}_S(R), \mathrm{slope}_S(M)\}$$

(see [8, 1.2]), and the right-hand side is finite since S is a polynomial ring. Backelin defined in [10] the (Backelin) rate of R to be

$$\mathrm{Rate}(R) = \mathrm{slope}_R(\mathbf{m}_R)$$

as a measure of the failure of the Koszul property. By the very definition, one has $\mathrm{Rate}(R) \geq 1$ and R is Koszul if and only if $\mathrm{Rate}(R) = 1$.

We close the section with a technical lemma:

Lemma 9. *(1) Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of R-modules. Then one has*

$$\begin{aligned}
\mathrm{reg}_R(M_1) &\leq \max\{\mathrm{reg}_R(M_2), \mathrm{reg}_R(M_3) + 1\}, \\
\mathrm{reg}_R(M_2) &\leq \max\{\mathrm{reg}_R(M_1), \mathrm{reg}_R(M_3)\}, \\
\mathrm{reg}_R(M_3) &\leq \max\{\mathrm{reg}_R(M_1) - 1, \mathrm{reg}_R(M_2)\}.
\end{aligned}$$

(2) Let

$$\mathbf{M} : \cdots \to M_i \to \cdots \to M_2 \to M_1 \to M_0 \to 0$$

be a complex of R-modules. Set $H_i = H_i(\mathbf{M})$. Then for every $i \geq 0$ one has

$$t_i^R(H_0) \leq \max\{a_i, b_i\}$$

where $a_i = \max\{t_j^R(M_{i-j}) : j = 0, \ldots, i\}$ and $b_i = \max\{t_j^R(H_{i-j-1}) : j = 0, \ldots, i - 2\}$.

Moreover one has

$$\mathrm{reg}_R(H_0) \leq \max\{a, b\}$$

where $a = \sup\{\mathrm{reg}_R(M_j) - j : j \geq 0\}$ and $b = \sup\{\mathrm{reg}_R(H_j) - (j + 1) : j \geq 1\}$.

Proof. (1) follows immediately by considering the long exact sequence obtained by applying $\mathrm{Tor}(K, -)$. For (2) one breaks the complex into short exact sequences and proves by induction on i the inequality for $t_i^R(H_0)$. Then one deduces the second inequality by translating the first into a statement about regularities.   □

We collect below some problems about the Koszul property and the existence of Gröbner bases of quadrics. Let us recall the following.

Definition 10. A K-algebra R is LG-quadratic if there exists a G-quadratic algebra A and a regular sequence of linear forms $y_1, \ldots, y_c$ such that $R \simeq A/(y_1, \ldots, y_c)$.

We have the following implications:

$$\text{G-quadratic} \implies \text{LG-quadratic} \implies \text{Koszul} \implies \text{quadratic} \qquad (3)$$

As discussed in Remark 6 the third implication in Eq. (3) is strict. The following remark, due to Caviglia, in connection with Remark 6(5) shows that also the first implication in Eq. (3) is strict.

Remark 11. Any complete intersection R of quadrics is LG-quadratic.
Say $R = K[x_1, \ldots, x_n]/(q_1, \ldots, q_m)$ then set

$$A = R[y_1, \ldots, y_m]/(y_1^2 + q_1, \ldots, y_m^2 + q_m)$$

and note that A is G-quadratic for obvious reasons and $y_1, \ldots, y_m$ is a regular sequence in A by codimension considerations.

But we do not know an example of a Koszul algebra that is not LG-quadratic. So we ask:

Question 12. Is any Koszul algebra LG-quadratic?

Our feeling is that the answer should be negative. But how can we exclude that a Koszul algebra is LG-quadratic? One can look at the h-vector (i.e., the numerator of the Hilbert series) since it is invariant under Gröbner deformation and modifications as the one involved in the definition of LG-quadratic. Alternatively one can look at syzygies over the polynomial ring because they can only grow under such operations. These observations lead to a new question:

Question 13. Is the h-vector of any Koszul algebra R the h-vector of an algebra defined by quadratic monomials? And, if yes, does there exist an algebra A with quadratic monomial relations, h-vector equal to that of R and satisfying $\beta_{ij}^S(R) \leq \beta_{ij}^T(A)$ for every i and j? Here S and T denote the polynomial rings canonically projecting onto R and A.

A negative answer to Question 13 would imply a negative answer to Question 12. Note that any h-vector of an algebra defined by quadratic monomials is also the h-vector of an algebra defined by square-free quadratic monomials (by using the polarization process). The simplicial complexes associated to square-free quadratic monomial ideals are called flag. There has been a lot of activity concerning combinatorial properties and characterizations of h-vectors and f-vectors of flag simplicial complexes, see [28] for recent results and for a survey of what is known and conjectured. Here we just mention that Frohmader has proved in [39] a conjecture of Kalai asserting that the f-vectors of flag simplicial complexes are f-vectors of balanced simplicial complexes.

Regarding the inequality for Betti numbers in Question 13, LG-quadratic algebras R satisfy the following restrictions:

1. $t_i^S(R) \leq 2i$
2. $t_i^S(R) < 2i$ if $t_{i-1}^S(R) < 2(i-1)$
3. $t_i^S(R) < 2i$ if $i > \dim S - \dim R$
4. $\beta_i^S(R) \leq \binom{\beta_1^S(R)}{i}$

deduced from the deformation to the (non-minimal) Taylor resolution of quadratic monomial ideals (see for instance [47, 4.3.2]). As shown in [8] the same restrictions are satisfied by any Koszul algebra, with the exception of possibly (4). So we ask:

Question 14. Let R be a Koszul algebra quotient of the polynomial ring S. Is it true that $\beta_i^S(R) \leq \binom{\beta_1^S(R)}{i}$?

It can be very difficult to decide whether a given Koszul algebra is G-quadratic. In the 1990s, Peeva and Sturmfels asked whether the coordinate ring

$$PV = K[x^3, x^2y, x^2z, xy^2, xz^2, y^3, y^2z, yz^2, z^3]$$

of the pinched Veronese is Koszul. For about a decade this was a benchmark example for testing new techniques for proving Koszulness. In 2009 Caviglia [18] gave the first proof of the Koszulness of PV. Recently a new one has been presented in [19] that applies also to a larger family of rings including all the general projections to $\mathbf{P}^8$ of the Veronese surface in $\mathbf{P}^9$. The problem remains to decide whether:

Question 15. Is PV G-quadratic?

The answer is negative if one considers the toric coordinates only (as it can be checked by computing the associated Gröbner fan using CaTS [1]), but unknown in general. There are plenty of quadratic monomial ideals defining algebras with the Hilbert function of PV and larger Betti numbers.

The algebra PV is generated by all monomials in $n = 3$ variables of degree $d = 3$ that are supported on at most $s = 2$ variables. By varying the indices n, d, s one gets a family of pinched Veronese algebras $PV(n, d, s)$, and it is natural to ask:

Question 16. For which values of n, d, s is $PV(n, d, s)$ quadratic or Koszul?

Not all of them are quadratic, for instance, $PV(4, 5, 2)$ is not. Questions as Question 16 are very common in the literature: in a family of algebras one asks which ones are quadratic or Koszul or if quadratic and Koszul are equivalent properties for the algebras in the family. For example, in [26, 6.10] the authors ask:

Question 17. Let R be a quadratic Gorenstein algebra with Hilbert series $1 + nz + nz^2 + z^3$. Is R Koszul?

For $n = 3$ the answer is obvious as R must be a complete intersection of quadrics and for $n = 4$ the answer is positive by [26, 6.15]. See Theorem 12 for results concerning this family of algebras.

3   How to Prove that an Algebra is Koszul?

To prove that an algebra is Koszul is usually a difficult task. There are examples, due to Roos, showing that a sort of Murphy's law (anything that can possibly go wrong, does) holds in this context. Indeed there exists a family of quadratic algebras $R(a)$ depending on an integer $a > 1$ such that the Hilbert series of $R(a)$ is $1 + 6z + 8z^2$ for every a. Moreover K has a linear resolution for a steps and a nonlinear syzygy

in homological position $a + 1$ (see [51]). So there is no statement of the kind: if R is an algebra with Hilbert series H then there is a number N depending on H, such that if the resolution of K over R is linear for N steps, it will be linear forever.

The goal of this section is to present some techniques to prove that an algebra is Koszul (without pretending they are the most powerful or interesting). For the sake of illustration we will apply these techniques to discuss the Koszul properties of Veronese algebras and modules. The material we present is taken from various sources (see [3, 8, 10–12, 16, 24–27, 32, 42, 58]).

3.1 Gröbner Basis of Quadrics

The simplest way to prove that an algebra is Koszul is to show that it is G-quadratic. A weak point of this prospective is that Gröbner bases refer to a system of coordinates and a term order. As said earlier, not all the Koszul algebras are G-quadratic. On the other hand many of the classical constructions in commutative algebra and algebraic geometry lead to algebras that have a privileged, say, natural, system of coordinates. For instance, the coordinate ring of the Grassmannian comes equipped with the Plücker coordinates. Toric varieties come with their toric coordinates. So one looks for a Gröbner basis of quadrics with respect to the natural system of coordinates. It turns out that many of the classical algebras (Grassmannian, Veronese, Segre, etc..) do have Gröbner bases of quadrics in the natural system of coordinates. Here we treat in details the Veronese case:

Theorem 1. *Let $S = K[x_1, \ldots, x_n]$ and $c \in \mathbf{N}$. Then the Veronese subring $S^{(c)} = \oplus_{j \in \mathbf{N}} S_{jc}$ is defined by a Gröbner basis of quadrics.*

Proof. For $j \in \mathbf{N}$ denote by M_j the set of monomials of degree j of S. Consider $T_c = \mathrm{Sym}_K(S_c) = K[t_m : m \in M_c]$ and the surjective map $\Phi : T_c \to S^{(c)}$ of K-algebras with $\Phi(t_m) = m$ for every $m \in M_c$. For every monomial m we set $\max(m) = \max\{i : x_i | m\}$ and $\min(m) = \min\{i : x_i | m\}$. Furthermore for monomials $m_1, m_2 \in M_c$ we set $m_1 \prec m_2$ if $\max(m_1) \leq \min(m_2)$. Clearly $\prec$ is a transitive (but not reflexive) relation. We say that $m_1, m_2 \in M_c$ are incomparable if $m_1 \not\prec m_2$ and $m_2 \not\prec m_1$ and that are comparable otherwise. For a pair of incomparable elements $m_1, m_2 \in M_c$, let $m_3, m_4 \in M_c$ be the uniquely determined elements in M_c such that $m_1 m_2 = m_3 m_4$ and $m_3 \prec m_4$. Set

$$F(m_1, m_2) = t_{m_1} t_{m_2} - t_{m_3} t_{m_4}.$$

By construction $F(m_1, m_2) \in \mathrm{Ker}\Phi$ and we claim that the set of the $F(m_1, m_2)$'s is a Gröbner basis of $\mathrm{Ker}\Phi$ with respect to any term order τ of T_c such that $\mathrm{in}_\tau(F(m_1, m_2)) = t_{m_1} t_{m_2}$. Such a term order exists: order the $t_m's$ totally as follows:

$$t_u \geq t_v \text{ iff } u \geq v \text{ lexicographically}$$

and then consider the degree reverse lexicographic term order associated to that total order. Such a term order has the required property as it is easy to see. It remains to prove that the $F(m_1, m_2)$'s form a Gröbner basis of $\text{Ker}\Phi$. Set

$$U = (t_{m_1} t_{m_2} : m_1, m_2 \in M_c \text{ are incomparable})$$

By construction we have $U \subset \text{in}_\tau(\text{Ker}\Phi)$ and we have to prove equality. We do it by checking that the two associated quotients have the same Hilbert function. The inequality $\text{HF}(T_c/\text{Ker}\Phi, i) \leq \text{HF}(T_c/U, i)$ follows from the inclusion of the ideals. For the other note that

$$\text{HF}(T_c/\text{in}_\tau(\text{Ker}\Phi), i) = \text{HF}(T_c/\text{Ker}\Phi, i) = \text{HF}(S^{(c)}, i) = \#M_{ic}$$

The key observations are:

1. A monomial in the t's, say $t_{m_1} \cdots t_{m_i}$, is not in U if (after a permutation) $m_1 \prec m_2 \prec \cdots \prec m_i$.
2. Every monomial $m \in M_{ic}$ has a uniquely determined decomposition $m = m_1 \cdots m_i$ with $m_1 \prec m_2 \prec \cdots \prec m_i$.

This implies that

$$\text{HF}(T_c/U, i) \leq \#M_{ic},$$

proving the desired assertion. □

3.2   Transfer of Koszulness

Let A be a K-algebra A and $B = A/I$ a quotient of it. Assume one of the two algebras is Koszul. What do we need to know about the relationship between A and B to conclude that the other algebra is Koszul too? Here is an answer:

Theorem 2. *Let A be a K-algebra and B be a quotient of A.*

(1) If $\text{reg}_A(B) \leq 1$ and A is Koszul, then B is Koszul.
(2) If $\text{reg}_A(B)$ is finite and B is Koszul, then A is Koszul.

The theorem is a corollary of the following:

Proposition 3. *Let A be a K-algebra and B a quotient algebra of A. Let M be a B-module. Then:*

(1) $\text{reg}_A(M) \leq \text{reg}_B(M) + \text{reg}_A(B)$.
(2) If $\text{reg}_A(B) \leq 1$ then $\text{reg}_B(M) \leq \text{reg}_A(M)$.

Proof. One applies Lemma 9(2) to the minimal free resolution $\mathbf{F}$ of M as a B-module and one has:

$$\text{reg}_A(M) \leq \sup\{\text{reg}_A(F_j) - j : j \geq 0\}.$$

Since $\operatorname{reg}_A(F_j) = \operatorname{reg}_A(B) + t_j^B(M)$, we can conclude that (1) holds.

For (2) it is enough to prove that the inequality

$$t_i^B(M) - i \leq \max\{t_j^A(M) - j : j = 0, \dots, i\}$$

holds for every i. We argue by induction on i; the case $i = 0$ is obvious because $t_0^A(M) = t_0^B(M)$. Assume $i > 0$ and take a minimal presentation of M as a B-module

$$0 \to N \to F \to M \to 0$$

where F is B-free. Since $t_i^B(M) = t_{i-1}^B(N)$, by induction we have:

$$t_i^B(M) - i = t_{i-1}^B(N) - i \leq \max\{t_j^A(N) - j - 1 : j = 0, \dots, i - 1\}$$

Since $t_j^A(N) \leq \max\{t_j^A(F), t_{j+1}^A(M)\}$ and $t_j^A(F) = t_j^A(B) + t_0^A(M) \leq j + 1 + t_0^A(M)$ we may conclude that the desired inequality holds. $\qquad\square$

Proof of Theorem 2. (1) Applying Proposition 3(1) with M equal to K one has that $\operatorname{reg}_A(K) \leq \operatorname{reg}_A(B)$ which is finite by assumption. It follows then from Theorem 7 that A is Koszul. For (2) one applies Proposition 3(2) with $M = K$, and one gets $\operatorname{reg}_B(K) \leq \operatorname{reg}_A(K)$ which is 0 by assumption; hence $\operatorname{reg}_B(K) = 0$ as required. $\quad\square$

Lemma 4. *Let R be Koszul algebra and M be an R-module. Then*

$$\operatorname{reg}_R(\mathbf{m}_R M) \leq \operatorname{reg}_R(M) + 1.$$

In particular, $\operatorname{reg}_R(\mathbf{m}_R^u) = u$, (unless $\mathbf{m}_R^u = 0$) that is, $\mathbf{m}_R^u$ has a linear resolution for every $u \in \mathbf{N}$.

Proof. Apply Lemma 9 to the short exact sequence

$$0 \to \mathbf{m}_R M \to M \to M/\mathbf{m}_R M \to 0$$

and use the fact that $M/\mathbf{m}_R M$ is a direct sum of copies of K shifted at most by $-t_0^R(M)$. $\qquad\square$

We apply now Theorem 2 to prove that the Veronese subrings of a Koszul algebra are Koszul.

Let $c \in \mathbf{N}$ and $R^{(c)} = \oplus_{j \in \mathbf{Z}} R_{jc}$ be the cth Veronese subalgebra of R. Similarly one defines $M^{(c)}$ for every R-module M. The formation of the cth Veronese submodule is an exact functor from the category of R-modules to the category of graded $R^{(c)}$-modules (recall that, by convention, modules are graded and maps are homogeneous of degree 0). For $u = 0, \dots, c - 1$ consider the Veronese submodules $V_u = \oplus_{j \in \mathbf{Z}} R_{jc+u}$. Note that V_u is an $R^{(c)}$-module generated in degree 0 and that for $a \in \mathbf{Z}$ one has

$$R(-a)^{(c)} = V_u(-\lceil a/c \rceil)$$

where $u = 0$ if $a \equiv 0 \bmod(c)$ and $u = c - r$ if $a \equiv r \bmod(c)$ and $0 < r < c$.

Theorem 5. *Let R be Koszul. Then $R^{(c)}$ is Koszul and $\operatorname{reg}_{R^{(c)}}(V_u) = 0$ for every $u = 0, \ldots, c - 1$.*

Proof. Set $A = R^{(c)}$. First we prove that $\operatorname{reg}_A(V_u) = 0$ for every $u = 0, \ldots, c - 1$. To this end, we prove by induction on i that $t_i^A(V_u) \leq i$ for every i. The case $i = 0$ is obvious. So assume $i > 0$. Let $M = \mathbf{m}_R^u(u)$. By Lemma 4 and by construction we have $\operatorname{reg}_R(M) = 0$ and $M^{(c)} = V_u$. Consider the minimal free resolution $\mathbf{F}$ of M over R and apply the functor $-^{(c)}$. We get a complex $\mathbf{G} = \mathbf{F}^{(c)}$ of A-modules such that $H_0(\mathbf{G}) = V_u$, $H_j(\mathbf{G}) = 0$ for $j > 0$ and $G_j = F_j^{(c)}$ is a direct sum of copies of $R(-j)^{(c)}$. Applying Lemma 9 we get $t_i^A(V_u) \leq \max\{t_{i-j}^A(G_j) : j = 0, \ldots, i\}$. Since G_0 is A-free we have $t_i^A(G_0) = -\infty$. For $j > 0$ we have $R(-j)^{(c)} = V_w(-\lceil j/c \rceil)$ for some number w with $0 \leq w < c$. Hence, by induction, $t_{i-j}^A(G_j) \leq i - j + \lceil j/c \rceil \leq i$. Summing up,

$$t_i^A(V_u) \leq \max\{i - j + \lceil j/c \rceil : j = 1, \ldots, i\} = i.$$

In order to prove that A is Koszul we consider the minimal free resolution $\mathbf{F}$ of K over R and apply $-^{(c)}$. We get a complex $\mathbf{G} = \mathbf{F}^{(c)}$ of A-modules such that $H_0(\mathbf{G}) = K$, $H_j(\mathbf{G}) = 0$ for $j > 0$ and $G_j = F_j^{(c)}$ is a direct sum of copies of $V_u(-\lceil j/c \rceil)$. Hence $\operatorname{reg}_A(\mathbf{G}_j) = \lceil j/c \rceil$ and applying Lemma 9 we obtain

$$\operatorname{reg}_A(K) \leq \sup\{\lceil j/c \rceil - j : j \geq 0\} = 0.$$
$\square$

We also have:

Theorem 6. *Let R be a K-algebra, then the Veronese subalgebra $R^{(c)}$ is Koszul for $c \gg 0$. More precisely, if $R = A/I$ with A Koszul, then $R^{(c)}$ is Koszul for every $c \geq \sup\{t_i^A(R)/(1 + i) : i \geq 0\}$.*

Proof. Let $\mathbf{F}$ be the minimal free resolution of R as an A-module. Set $B = A^{(c)}$ and note that B is Koszul because of Theorem 5. Then $\mathbf{G} = \mathbf{F}^{(c)}$ is a complex of B-modules such that $H_0(\mathbf{G}) = R^{(c)}$, $H_j(\mathbf{G}) = 0$ for $j > 0$. Furthermore $G_i = F_i^{(c)}$ is a direct sum of shifted copies of the Veronese submodules V_u. Using Theorem 5 we get the bound $\operatorname{reg}_B(G_i) \leq \lceil t_i^A(R)/c \rceil$. Applying Lemma 9 we get

$$\operatorname{reg}_B(R^{(c)}) \leq \sup\{\lceil t_i^A(R)/c \rceil - i : i \geq 0\}.$$

Hence, for $c \geq \sup\{t_i^A(R)/(1+i) : i \geq 0\}$ one has $\operatorname{reg}_B(R^{(c)}) \leq 1$ and we conclude from Theorem 2 that $R^{(c)}$ is Koszul.
$\square$

Remark 7. (1) Note that the number $\sup\{t_i^A(R)/(1 + i) : i \geq 0\}$ in Theorem 6 is finite. For instance it is less than or equal to $(\operatorname{reg}_A(R) + 1)/2$ which is finite because $\operatorname{reg}_A(R)$ is finite. Note however that $\sup\{t_i^A(R)/(1+i) : i \geq 0\}$ can be much smaller than $(\operatorname{reg}_A(R) + 1)/2$; for instance if $R = A/I$ with I generated

by a regular sequence of r elements of degree d, then $t_i^A(R) = id$ so that $\operatorname{reg}_A(R) = r(d-1)$ while $\sup\{t_i^A(R)/(1+i) : i \geq 0\} = dr/(r+1)$.

(2) In particular, if we take the canonical presentation $R = S/I$ Eq. (1), then we have that $R^{(c)}$ is Koszul if $c \geq \sup\{t_i^S(R)/(1+i) : i \geq 0\}$. In [32, 2] it is proved that if $c \geq (\operatorname{reg}_S(R) + 1)/2$, then $R^{(c)}$ is even G-quadratic. See [57] for other interesting results in this direction.

(3) Backelin proved in [10] that $R^{(c)}$ is Koszul if $c \geq \operatorname{Rate}(R)$.

(4) The proof of Theorem 6 shows also that $\operatorname{reg}_{A^{(c)}}(R^{(c)}) = 0$ if $c \geq \operatorname{slope}_A(R)$.

3.3 Filtrations

Another tool for proving that an algebra is Koszul is a "divide and conquer" strategy that can be formulated in various technical forms, depending on the goal one has in mind. We choose the following:

Definition 8. A Koszul filtration of a K-algebra R is a set $\mathscr{F}$ of ideals of R such that:

(1) Every ideal $I \in \mathscr{F}$ is generated by elements of degree 1.
(2) The zero ideal 0 and the maximal ideal $\mathbf{m}_R$ are in $\mathscr{F}$.
(3) For every $I \in \mathscr{F}$, $I \neq 0$, there exists $J \in \mathscr{F}$ such that $J \subset I$, I/J is cyclic and $\operatorname{Ann}(I/J) = J : I \in \mathscr{F}$.

By the very definition a Koszul filtration must contain a complete flag of R_1, that is, an increasing sequence $I_0 = 0 \subset I_1 \subset \cdots \subset I_{n-1} \subset I_n = \mathbf{m}_R$ such that I_i is minimally generated by i elements of degree 1. The case where $\mathscr{F}$ consists of just a single flag deserves a name:

Definition 9. A Gröbner flag for R is a Koszul filtration that consists of a single complete flag of R_1. In other words, $\mathscr{F} = \{I_0 = 0 \subset I_1 \subset \cdots \subset I_{n-1} \subset I_n = \mathbf{m}_R\}$ with $I_{i-1} : I_i \in \mathscr{F}$ for every i.

One has:

Lemma 10. *Let $\mathscr{F}$ be a Koszul filtration for R. Then one has:*

(1) $\operatorname{reg}_R(R/I) = 0$ *and R/I is Koszul for every $I \in \mathscr{F}$.*
(2) *R is Koszul.*
(3) *If $\mathscr{F}$ is a Gröbner flag, then R is G-quadratic.*

Proof. (1) and (2): One easily proves by induction on i and on the number of generators of I that $t_i^R(R/I) \leq i$ for every i and $I \in \mathscr{F}$. This implies that R is Koszul (take $I = \mathbf{m}_R$) and that $\operatorname{reg}_R(R/I) = 0$, hence R/I is Koszul by Theorem 2.

(3) We just sketch the argument: let $x_1, \ldots, x_n$ be a basis for the flag, that is, $\mathscr{F} = \{I_0 = 0 \subset I_1 \subset \cdots \subset I_{n-1} \subset I_n = \mathbf{m}_R\}$ and $I_i = (x_1, \ldots, x_i)$ for every i.

For every i there exists $j_i \geq i$ such that $(x_1, \ldots, x_i) : x_{i+1} = (x_1, \ldots, x_{j_i})$. For every $i < h \leq j_i$ the assertion $x_h x_{i+1} \in (x_1, \ldots, x_i)$ is turned into a quadratic equation in the defining ideal of R. The claim is that these quadratic equations form a Gröbner basis with respect to a term order that selects $x_h x_{i+1}$ as leading monomial. To prove the claim one shows that the identified monomials define an algebra, call it A, whose Hilbert function equals that of R. This is done by showing that the numbers $j_1, \ldots, j_n$ associated to the flag of R determine the Hilbert function of R and then by showing that also A has a Gröbner flag with associated numbers $j_1, \ldots, j_n$. $\square$

There are Koszul algebras without Koszul filtrations and G-quadratic algebras without Gröbner flags see the examples given in [26, pp. 100 and 101]. Families of algebras having Koszul filtrations or Gröbner flags are described in [26]. For instance, it is proved that the coordinate ring of a set of at most $2n$ points in $\mathbf{P}^n$ in general linear position has a Gröbner flag, and that the general Gorenstein Artinian algebra with socle in degree 3 has a Koszul filtration. The results for points in [26] generalize results of [27, 44] and are generalized in [49]. Filtrations of more general type are used in [24] to control the Backelin rate of coordinate rings of sets of points in the projective space.

The following notion is very natural for algebras with privileged coordinate systems (e.g. in the toric case).

Definition 11. An algebra R is said to be strongly Koszul if there exists a basis X of R_1 such that for every $Y \subset X$ and for every $x \in X \setminus Y$ there exists $Z \subseteq X$ such that $(Y) : x = (Z)$.

Our definition of strongly Koszul is slightly different than the one given in [42]. In [42] it is assumed that the basis X of R_1 is totally ordered, and in the definition one adds the requirement that x is larger than every element in Y.

Remark 12. If R is strongly Koszul with respect to a basis X of R_1, then the set $\{(Y) : Y \subseteq X\}$ is obviously a Koszul filtration.

We have:

Theorem 13. *Let $R = S/I$ with $S = K[x_1, \ldots, x_n]$ and $I \subset S$ an ideal generated by monomials of degrees $\leq d$. Then $R^{(c)}$ is strongly Koszul for every $c \geq d - 1$.*

Proof. In the proof we use the following basic facts:

Fact (1): If $m_1, \ldots, m_t, m$ are monomials of S, then $(m_1, \ldots, m_t) :_S m$ is generated by the monomials $m_i / \gcd(m_i, m)$ for $i = 1, \ldots, t$.

Fact (2): If T is an algebra and $A = T^{(c)}$, then for every ideal $I \subset A$ and $f \in A$ one has $IT \cap A = I$ and $(IT :_T f) \cap A = I :_A f$.

The first is an elementary and well-know, property of monomials; the second holds true because A is a direct summand of T.

Let $A = R^{(c)}$. Let X be the set of the residue classes in R of the monomials of degree c that are not in I. Clearly X is a basis of A_1. Let $Y \subset X$ and $z \in X \setminus Y$,

say $Y = \{\bar{m}_1, \ldots, \bar{m}_v\}$ and $z = \bar{m}$. We have to compute $(Y) :_A z$. To this end let us consider $J = (I + (m_1, \ldots, m_v)) :_S m$ and note that $J = I + H$ with H a monomial ideal generated in degrees $\leq c$. Then $(Y) :_A z = (\bar{m} : m \in H \setminus I$ is a monomial of degree c). $\qquad\square$

Let us single out two interesting special cases:

Theorem 14. *Let* $S = K[x_1, \ldots, x_n]$. *Then* $S^{(c)}$ *is strongly Koszul for every* $c \in \mathbf{N}$.

Theorem 15. *Let* $S = K[x_1, \ldots, x_n]$ *and let* $I \subset S$ *be an ideal generated by monomials of degree* 2. *Then* S/I *is strongly Koszul.*

Given a Koszul filtration $\mathscr{F}$ for an algebra R we may also look at modules having a filtration compatible with $\mathscr{F}$. This leads us to the following:

Definition 16. Let R be an algebra with a Koszul filtration $\mathscr{F}$. Let M be an R-module. We say that M has linear quotients with respect to $\mathscr{F}$ if M is minimally generated by elements $m_1, \ldots, m_v$ such that $\langle m_1, \ldots, m_{i-1} \rangle :_R m_i \in \mathscr{F}$ for $i = 1, \ldots, v$.

One easily deduces:

Lemma 17. *Let* R *be an algebra with a Koszul filtration* $\mathscr{F}$ *and* M *an* R-module *with linear quotients with respect to* $\mathscr{F}$. *Then* $\mathrm{reg}_R(M) = t_0^R(M)$.

As an example we have:

Proposition 18. *Let* $S = K[x_1, \ldots, x_n]$ *and* I *be a monomial ideal generated in degree* $\leq d$. *Consider* $R = S/I$ *and the Veronese ring* $R^{(c)}$ *equipped with the Koszul filtration described in the proof of Theorem* 13. *For every* $u = 0, \ldots, c - 1$ *the Veronese module* $V_u = \oplus_j R_{u+jc}$ *has linear quotients with respect to* $\mathscr{F}$.

The proof is easy, again, based on Fact (1) in the proof of Theorem 13. In particular, this gives another proof of the fact that the Veronese modules V_u have a linear $R^{(c)}$-resolution.

The results and the proofs presented for Veronese rings and Veronese modules have their analogous in the multigraded setting (see [25]). For later applications we mention explicitly one case.

Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ with $\mathbf{Z}^2$-grading induced by the assignment $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. For every $c = (c_1, c_2)$ we look at the diagonal subalgebra $S_\Delta = \oplus_{a \in \Delta} S_a$ where $\Delta = \{ic : i \in \mathbf{Z}\}$. The algebra S_Δ is nothing but the Segre product of the c_1th Veronese ring of $K[x_1, \ldots, x_n]$ and the c_2th Veronese ring of $K[y_1, \ldots, y_m]$. We have:

Proposition 19. *For every* $(a, b) \in \mathbf{Z}^2$ *the* S_Δ-submodule *of* S *generated by* $S_{(a,b)}$ *has a linear resolution.*

4   Absolutely and Universally

We have discussed in the previous sections some notions, such as being G-quadratic, strongly Koszul, having a Koszul filtration or a Gröbner flag that imply Koszulness. In this section we discuss two variants of the Koszul property: universally Koszul and absolutely Koszul.

4.1   Universally Koszul

When looking for a Koszul filtration, among the many families of ideals of linear forms one can take the set of all ideals of linear forms. This leads to the following definition:

Definition 1. Let R be a K-algebra and set

$$\mathscr{L}(R) = \{I \subset R : I \text{ ideal generated by elements of degree } 1\}.$$

We say that R is universally Koszul if the following equivalent conditions hold:

(1) $\mathscr{L}(R)$ is a Koszul filtration of R.
(2) $\operatorname{reg}_R(R/I) = 0$ for every $I \in \mathscr{L}(R)$.
(3) For every $I \in \mathscr{L}(R)$ and $x \in R_1 \setminus I$, one has $I : x \in \mathscr{L}(R)$.

That the three conditions are indeed equivalent is easy to see (see [21, 1.4]). In [21, 2.4] it is proved that:

Theorem 2. *Let $S = K[x_1, \ldots, x_n]$ and $m \in \mathbf{N}$. If $m \leq n/2$, then a generic space of quadrics of codimension m in the vector space of quadrics defines a universally Koszul algebra.*

One should compare the result above with the following:

Theorem 3. *Let $S = K[x_1, \ldots, x_n]$ and $m \in \mathbf{N}$.*

(1) *A generic space of quadrics of codimension m defines a Koszul algebra if $m \leq n^2/4$.*
(2) *A generic space of quadrics of codimension m defines an algebra with a Gröbner flag if $m \leq n - 1$.*

For (1) see [27, 3.4], for (2) [20, 10]. Fröberg and Löfwall proved in [38] that, apart from spaces of quadrics of codimension $\leq n^2/4$, the only generic spaces of quadrics defining Koszul algebras are the complete intersections. Returning to universally Koszul algebras, in [21] it is also proved that:

Theorem 4. *Let R be a Cohen–Macaulay domain K-algebra with K algebraically closed of characteristic 0. Then R is universally Koszul if and only if R is a polynomial extension of one of the following algebras:*

(1) *The coordinate ring of a quadric hypersurface.*
(2) *The coordinate ring of a rational normal curve, that is, $K[x, y]^{(c)}$ for some c.*
(3) *The coordinate ring of a rational normal scroll of type (c, c), that is, the Segre product of $K[x, y]^{(c)}$ with $K[s, t]$.*
(4) *The coordinate ring of the Veronese surface in $\mathbf{P}^5$, that is, $K[x, y, z]^{(2)}$.*

4.2 Absolutely Koszul

Let $\mathbf{F}_M^R$ be the minimal free resolution of a graded module M over R. One defines a $\mathbf{m}_R$-filtration on $\mathbf{F}_M^R$ whose associated graded complex $\mathrm{lin}(\mathbf{F}_M^R)$ has, in the graded case, a very elementary description. The complex $\mathrm{lin}(\mathbf{F}_M^R)$ is obtained from $\mathbf{F}_M$ by replacing with 0 all entries of degree > 1 in the matrices representing the homomorphisms. In the local case the definition of $\mathrm{lin}(\mathbf{F}_M^R)$ is more complicated (see Sect. 6 for details). One defines

$$\mathrm{ld}_R(M) = \sup\{i : H_i(\mathrm{lin}(\mathbf{F}_M^R)) \neq 0\}. \qquad (4)$$

Denote by $\Omega_i^R(M)$ the ith syzygy module of a module M over R. It is proved in Römer PhD thesis and also in [43] that:

Proposition 5. *Assume R is Koszul. Then:*

(1) *M is componentwise linear iff $\mathrm{ld}_R(M) = 0$.*
(2) *$\mathrm{ld}_R(M) = \inf\{i : \Omega_i(M) \text{ is componentwise linear}\}$.*
(3) *If $\Omega_i^R(M)$ is componentwise linear then $\Omega_{i+1}^R(M)$ is componentwise linear.*

Iyengar and Römer introduced in [43] the following notion:

Definition 6. A K-algebra R is said to be absolutely Koszul if $\mathrm{ld}_R(M)$ is finite for every module M.

It is shown in [41] that:

Proposition 7. *If $\mathrm{ld}_R(M)$ is finite, then $\mathrm{reg}_R(M)$ is finite as well. Furthermore the Poincaré series $P_M(z)$ of M is rational and its "denominator" only depends on R.*

One obtains the following characterization of the Koszul property:

Corollary 8. *Let R be a K-algebra. Then R is Koszul if and only if $\mathrm{ld}_R(K)$ is finite. In particular, if R is absolutely Koszul then R is Koszul.*

On the other hand there are Koszul algebras that are not absolutely Koszul.

Example 9. The algebra

$$R = K[x_1, x_2, x_3, y_1, y_2, y_3]/(x_1, x_2, x_3)^2 + (y_1, y_2, y_3)^2$$

is Koszul but not absolutely Koszul because there are R-modules with non-rational Poincaré series. This and other examples of "bad" Koszul algebras are discussed by Roos in [52].

One also has [41, 5.10].

Theorem 10. *Let $R = S/I$ with $S = K[x_1, \ldots, x_n]$. Then R is absolutely Koszul if either R is a complete intersection of quadrics or $\text{reg}_S(R) = 1$.*

There is however an important difference between the two cases [41, 6.2, 6.7]:

Remark 11. When $\text{reg}_S(R) = 1$ one has $\text{ld}_R(M) \leq 2 \dim R$ for every M and even $\text{ld}_R(M) \leq \dim R$ is furthermore R is Cohen–Macaulay. But when R if a complete intersection of quadrics of codimension > 1 (or more generally when R is Gorenstein of with socle in degree > 1) one has $\sup_M \text{ld}_R(M) = \infty$.

Another important contribution is the following:

Theorem 12. *Let R be a Gorenstein Artinian algebra with Hilbert function $1 + nz + nz^2 + z^3$ and $n > 2$. Then:*

(1) *If there exist $x, y \in R_1$ such that $0 : x = (y)$ and $0 : y = (x)$ (an exact pair of zero-divisors in the terminology of [40]), then R has a Koszul filtration and it is absolutely Koszul.*
(2) *If R is generic then it has an exact pair of zero-divisors.*

See [26, 2.13,6.3] for the statement on Koszul filtration in (1) and for (2) and see [40, 3.3] for the absolutely Koszulness.

What are the relationships between the properties discussed in this and the earlier sections? Here are some questions:

Question 13. (1) Strongly Koszul $\Rightarrow$ G-quadratic?
(2) Universally Koszul $\Rightarrow$ G-quadratic?
(3) Universally Koszul $\Rightarrow$ absolutely Koszul?

Question 13 (1) is mentioned in [42, p. 166] in the toric setting. Another interesting question is:

Question 14. What is the behavior of absolutely Koszul algebras under standard algebra operations (e.g. forming Veronese subalgebras or Segre and fiber products)?

The same question for universally Koszul algebras is discussed in [21] and for strongly Koszul in [42]. Note however that in [42] the authors deal mainly with toric algebras and their toric coordinates. Universally Koszul algebras with monomial relations have been classified in [22]. We may ask:

Question 15. Is it possible to classify absolutely Koszul algebras defined by monomials?

5 Regularity and Koszulness

We list in this section some facts and some questions that we like concerning Koszul algebras and regularity. We observe the following.

Remark 1. Regularity over the polynomial ring S behaves quite well with respect to products of ideals and modules:

(1) $\operatorname{reg}_S(I^u M)$ is a linear function in u for large u (see [29, 45, 60]).
(2) $\operatorname{reg}_S(IM) \leq \operatorname{reg}_S(M) + \operatorname{reg}_S(I)$ (does not hold in general but it) holds provided $\dim S/I \leq 1$, [23].
(3) More generally,

$$\operatorname{reg}_S(\operatorname{Tor}_i^S(N, M)) \leq \operatorname{reg}_S(M) + \operatorname{reg}_S(N) + i$$

provided the Krull dimension of $\operatorname{Tor}_1^S(N, M)$ is ≤ 1, [17, 33].
(4) $\operatorname{reg}_S(I_1 \cdots I_d) = d$ for ideals I_i generated in degree 1, [23]

where M, N are S-modules and I, I_i are ideals of S.

What happens if we replace in Remark 1 the polynomial ring S with a Koszul algebra R and consider regularity over R? Trung and Wang proved in [60] that $\operatorname{reg}_S(I^u M)$ is asymptotically a linear function in u when I is an ideal of R and M is a R-module. If R is Koszul, $\operatorname{reg}_R(I^u M) \leq \operatorname{reg}_S(I^u M)$, and hence $\operatorname{reg}_R(I^u M)$ is bounded above by a linear function in u.

Question 2. Let R be a Koszul algebra $I \subset R$ an ideal and M an R-module. Is $\operatorname{reg}_R(I^u M)$ a linear function in u for large u?

The following examples show that statements (2) and (3) in Remark 1 do not hold over Koszul algebras.

Example 3. Let $R = \mathbf{Q}[x, y, z, t]/(x^2+y^2, z^2+t^2)$. With $I = (x, z)$ and $J = (y, t)$ one has $\operatorname{reg}_R(I) = 1$, $\operatorname{reg}_R(J) = 1$ because x, z and y, t are regular sequences on R, $\dim R/I = 0$ and $\operatorname{reg}_R(IJ) = 3$.

Example 4. Let $R = K[x, y]/(x^2 + y^2)$. Let $M = R/(x)$ and $N = R/(y)$ and note that $\operatorname{reg}_R(M) = 0$, $\operatorname{reg}_R(N) = 0$ because x and y are non-zero divisors in R while $\operatorname{Tor}_1^R(M, N) = H_1(x, y, R) = K(-2)$.

Nevertheless statements (2), (3) of Remark 1 might hold for special type of ideals/modules over special type of Koszul algebras. For example, one has:

Proposition 5. *Let R be a Cohen–Macaulay K-algebra with $\operatorname{reg}_S(R) = 1$, let I be an ideal generated in degree 1 such that $\dim R/I \leq 1$ and M an R-module. Then $\operatorname{reg}_R(IM) \leq \operatorname{reg}_R(M) + 1$. In particular, $\operatorname{reg}_R(I) = 1$.*

Proof. We may assume K is infinite. The short exact sequence

$$0 \to IM \to M \to M/IM \to 0$$

implies that $\text{reg}_R(IM) \leq \max\{\text{reg}_R(M), \text{reg}_R(M/IM) + 1\}$. It is therefore enough to prove that $\text{reg}_R(M/IM) \leq \text{reg}_R(M)$. Then let $J \subset I$ be an ideal generated by a maximal regular sequence of elements of degree 1 and set $A = R/J$. Since $\text{reg}_R(A) = 0$ and since M/IM is an A-module, by virtue of Proposition 3, we have $\text{reg}_R(M/IM) \leq \text{reg}_A(M/IM)$. By construction, A is Cohen–Macaulay of dimension ≤ 1 and has regularity 1 over the polynomial ring projecting onto it. So, by Lemma 14 we have $\text{reg}_A(M/IM) = \max\{t_0^A(M/IM), t_1^A(M/IM) - 1\}$. Summing up, since $t_0^A(M/IM) = t_0^R(M)$, it is enough to prove $t_1^A(M/IM) \leq \text{reg}_R(M) + 1$. Now we look at

$$0 \to IM/JM \to M/JM \to M/IM \to 0$$

that gives $t_1^A(M/IM) \leq \max\{t_1^A(M/JM), t_0^A(IM/JM)\}$. Being $t_0^A(IM/JM) \leq t_0^R(M) + 1 \leq \text{reg}_R(M) + 1$, it remains to prove that $t_1^A(M/JM) \leq \text{reg}_R(M) + 1$, and this follows from the right exactness of the tensor product. $\square$

The following example shows that the assumption $\dim R/I \leq 1$ in Proposition 5 is essential.

Example 6. The algebra $R = K[x, y, z, t]/(xy, yz, zt)$ is Cohen–Macaulay of dimension 2 and $\text{reg}_S(R) = 1$. The ideal $I = (y - z)$ has $\text{reg}_R(I) = 2$ and $\dim R/I = 2$.

Example 3 shows also that statement (4) of Remark 1 does not hold over a Koszul algebra even if we assume that each I_i is an ideal of regularity 1 and of finite projective dimension. Statement (4) of Remark 1 might be true if one assumes that the ideals I_i belongs to a Koszul filtration. We give a couple of examples in this direction:

Proposition 7. *Let $S = K[x_1, \ldots, x_n]$, $R = S/I$ with I generated by monomials of degree 2. Let $X = \{\bar{x}_1, \ldots, \bar{x}_n\}$ and $\mathscr{F} = \{(Y) : Y \subset X\}$. Let $I_1, \ldots, I_d \in \mathscr{F}$. Then $\text{reg}_R(I_1 \cdots I_d) = d$ unless $I_1 \cdots I_d = 0$.*

Proof. First we observe the following. Let $m_1, \ldots, m_t$ be monomials of degree d and $J = (m_1, \ldots, m_t)$. Assume that they have linear quotients (in S), that is, $(m_1, \ldots, m_{i-1}) :_S m_i$ is generated by variables for every i. Fact (1) in the proof of Theorem 13 implies that JR has linear quotients with respect to the Koszul filtration $\mathscr{F}$ of R. By Lemma 17 we have that $\text{reg}_R(JR) = d$ (unless $JR = 0$). Now the desired result follows because products of ideals of variables have linear quotients in S by [23, 5.4]. $\square$

Example 4.3 in [23] shows that the inequality $\text{reg}_R(IM) \leq \text{reg}_R(M) + \text{reg}_R(I)$ does not even hold over a K-algebra R with a Koszul filtration $\mathscr{F}$, $I \in \mathscr{F}$ and M an R-module with linear quotient with respect to $\mathscr{F}$. The following are natural questions:

Question 8. Let R be an algebra with a Koszul filtration $\mathscr{F}$. Is it true that $\text{reg}_R(I_1 \cdots I_d) = d$ for every $I_1, \ldots, I_d \in \mathscr{F}$ whenever the product is non-zero?

In view of the analogy with statement (4) of Remark 1 the following special case deserves attention:

Question 9. Let R be a universally Koszul algebra. Is it true that $\operatorname{reg}_R(I_1 \cdots I_d) = d$ for every $I_1, \ldots, I_d$ ideals of R generated in degree 1 (whenever the product is non-zero)?

Remark 10. In a universally Koszul algebra a product of elements of degree 1 has a linear annihilator. This can be easily shown by induction on the number of factors. Hence, the answer to Question 9 is positive if each I_i is principal.

We are able to answer Question 8 in the following cases:

Theorem 11. *Products of ideals of linear forms have linear resolutions over the following rings:*

(1) R *is Cohen–Macaulay with* $\dim R \leq 1$ *and* $\operatorname{reg}_S(R) = 1$.
(2) $R = K[x, y, z]/(q)$ *with* $\deg q = 2$.
(3) $R = K[x, y]^{(c)}$ *with* $c \in \mathbf{N}_{>0}$.
(4) $R = K[x, y, z]^{(2)}$.
(5) $R = K[x, y] * K[s, t]$ *(∗ denotes the Segre product).*

Proof. The rings in the list are Cohen–Macaulay with $\operatorname{reg}_S(R) = 1$. Let $I_1, \ldots, I_d$ be ideals generated by linear forms. We prove by induction on d that $\operatorname{reg}_R(I_1 \cdots I_d) = d$. The case $d = 1$ follows because the rings in the list are universally Koszul. If for one of the I_i we have $\dim R/I_i \leq 1$ then we may use Proposition 5 and conclude by induction. Hence we may assume $\dim R/I_i \geq 2$ for every i. For the ring (1) and (3) (which is a 2-dimensional domain) we are done. In the case (2), the only case left is when the I_i are principal. But then we may conclude by virtue of Remark 10. In cases (3) and (4) we have that $\dim R/I_i = 2$ for each i, that is, $\operatorname{height} I_i = 1$. If one of the I_i is principal, then we are done by induction (because the R is a domain). Denote by A either $K[x, y, z]$ in case (3) or $K[x, y, s, t]$ in case (4). Since R is a direct summand of A we have $IA \cap R = I$ for every ideal I of R. It follows that $\operatorname{height}(I_i A) = 1$ for every i and hence there exist non-units $f_i \in A$ such that $I_i A \subset (f_i)$. In case (3) we have that each f_i must have degree 1 in A and $I_i A = (f_1) J_1$ with J_1 an ideal generated by linear forms of A. Hence $I_1 I_2 = (f_1 f_2) H$ where $H = J_1 J_2$ is an ideal generated by linear forms of R. Hence we are done because one of the factor is principal. In case (4) we have that each f_i is either a linear form in x, y or a linear form in s, t. If one of the f_i's is a linear form in x, y and another one is a linear form in s, t we can proceed as in the case (3). So we are left with the case that every f_i is a linear form in, say, x and y and $I_i = (f_i) J_i$ with J_i generated by linear forms in z, t. Since none of the I_i is principal we have that $J_i = (z, t)$ for every i. Hence $I_1 \cdots I_d$ is generated by $(\prod_{i=1}^{d} f_i)(z, t)^d$ and it isomorphic to the R-submodule of A generated by its component of degree $(0, d)$. That such a module has a linear resolution over R follows from Proposition 19. $\square$

We state now a very basic question of computational nature.

Question 12. Let R be a Koszul algebra and M an R-module. How does one compute $\mathrm{reg}_R(M)$? Can one do it algorithmically?

Few comments concerning Question 12. We assume to be able to compute syzygies over R and so to be able to compute the first steps of the resolution of a R-module M. Let $S \to R$ the canonical presentation Eq. (1) of R. We know that $\mathrm{reg}_R(M) \leq \mathrm{reg}_S(M)$ and $\mathrm{reg}_S(M)$ can be computed algorithmically because $\mathrm{pd}_S(M)$ is finite. A special but already interesting case of Question 12 is:

Question 13. Let R be a Koszul algebra and M an R-module generated in degree 0, with $M_1 \neq 0$ and $M_i = 0$ for $i > 1$. Can one decide algorithmically whether $\mathrm{reg}_R(M) = 0$ or $\mathrm{reg}_R(M) = 1$?

Set
$$r_R(M) = \min\{i \in \mathbf{N} : t_i^R(M) - i = \mathrm{reg}_R(M)\}.$$

So $r_R(M)$ is the first homological position where the regularity of M is attained. If one knows $r_R(M)$ or a upper bound $r \geq r_R(M)$ for it, then one can compute $\mathrm{reg}_R(M)$ by computing the first r steps of the resolution of M. Note that

$$r_R(M) \leq \mathrm{ld}_R(M)$$

because $\mathrm{reg}_R(N) = t_0^R(N)$ if N is componentwise linear. One has:

Lemma 14. *Let R be a K-algebra with $\mathrm{reg}_S(R) = 1$. Then $r_R(M) \leq 2\dim R$ for every M, that is, the regularity of any R-module is attained within the first $2\dim R$ steps of the resolution. If furthermore R is Cohen–Macaulay, $r_R(M) \leq \dim R - \mathrm{depth}M$.*

The first assertion follows from Remark 11; the second is proved by a simple induction on $\mathrm{depth}M$.

Note that the ith syzygy module of M cannot have a free summand if $i > \dim R$ by [30, 0.1] and so
$$t_{j+1}^R(M) > t_j^R(M) \text{ if } j > \dim R.$$

Unfortunately there is no hope to get a bound for $r_R(M)$ just in terms of invariants of R for general Koszul algebras. The argument of [41, 6.7] that shows that if R is a Gorenstein algebra with socle degree > 1 then $\sup \mathrm{ld}_R(M) = \infty$ shows also that $\sup_M r_R(M) = \infty$. For instance, over $R = K[x, y]/(x^2, y^2)$ let M_n be the dual of the nth syzygy module $\Omega_n^R(K)$ shifted by n. One has that M_n is generated be in degree 0, $\mathrm{reg}_R(M_n) = 1$ and $r_R(M_n) = n$. On the other hand, the number of generators of M_n is n. So we ask:

Question 15. Let R be a Koszul algebra. Can one bound $r_R(M)$ in terms of invariants of R and "computable" invariants of M such as its Hilbert series or its Betti numbers over S?

The questions above make sense also over special families of Koszul rings. For instance, there has been a lot of activity to understand resolutions of modules over

short rings, that is, rings with $\mathbf{m}_R^3 = 0$ or $\mathbf{m}_R^4 = 0$, both in the graded and local case (see [7, 9, 40]). It would be very interesting to answer Questions 12, 13, and 15 for short rings.

6 Local Variants

This section is concerned with "Koszul-like" behaviors of local rings and their modules.

Assumption: From now on, when not explicitly said, R is assumed to be a local or graded ring with maximal ideal $\mathbf{m}$ and residue field $K = R/\mathbf{m}$. Moreover all modules and ideals are finitely generated, and homogeneous in the graded case.

We define the associated graded ring to R with respect to the $\mathbf{m}$-adic filtration

$$G = \mathrm{gr}_{\mathbf{m}}(R) = \oplus_{i \geq 0} \mathbf{m}^i / \mathbf{m}^{i+1}.$$

The Hilbert series and the Poincaré series of R are

$$H_R(z) = H_G(z) = \sum_{i \geq 0} \dim(\mathbf{m}^i/\mathbf{m}^{i+1})z^i \quad \text{and} \quad P_R(z) = \sum_{i \geq 0} \dim \mathrm{Tor}_i^R(K, K)z^i.$$

6.1 Koszul Rings

Following Fröberg [35] we extend the definition of Koszul ring to the local case as follows:

Definition 1. The ring R is Koszul if its associated graded ring G is a Koszul algebra (in the graded sense), that is, R is Koszul if K has a linear resolution as a G-module.

As it is said in Remark 6 (6) in the graded setting the Koszul property holds equivalent to the following relation between the Poincaré series of K and the Hilbert series of R:

$$P_K^R(z)H_R(-z) = 1. \tag{5}$$

The following definition is due to Fitzgerald [34]:

Definition 2. The ring R is Fröberg if the relation Eq. (5) is verified.

We want to explain why every Koszul ring is Fröberg. To this end we need to introduce few definitions.

Let A be a regular local ring with maximal ideal $\mathbf{m}_A$ and let I be an ideal of A such that $I \subseteq \mathbf{m}_A^2$. Set $R = A/I$. Then $G \simeq S/I^*$ where S is the polynomial ring

and I^* is the homogeneous ideal generated by the initial forms f^* of the elements $f \in I$.

Definition 3. (1) A subset $\{f_1, \ldots, f_t\}$ of I is a standard basis of I if $I^* = (f_1^*, \ldots, f_t^*)$;
(2) The ideal I is d-isomultiple if I^* is generated in degree d.

If $\{f_1, \ldots, f_t\}$ is a standard basis of I, then $I = (f_1, \ldots, f_t)$. See [53] for more details on d-isomultiple ideals. Notice that by Remark 6 (1) we have

$$R \quad \text{Koszul} \quad \Longrightarrow \quad I \text{ is 2-isomultiple}.$$

Obviously the converse does not hold true because a quadratic K-algebra is not necessarily Koszul.

We now explore the connection between Koszul and Fröberg rings. By definition $H_R(z) = H_G(z)$, and

$$P_K^R(z) \le P_K^G(z)$$

(see, e.g. [36, 4]). Conditions are known under which $\beta_i^R(K) = \beta_i^G(K)$, for instance, this happens if

$$t_i^G(K) = \max\{j : \beta_{ij}^G(K) \ne 0\} \le \min\{j : \beta_{i+1\,j}^G(K) \ne 0\} \text{ for every } i; \quad (6)$$

see [36, 4]. This is the case if K has a linear resolution as a G-module. Hence,

Proposition 4. *If R is Koszul, then R is Fröberg.*

Proof. By definition, $H_R(z) = H_G(z)$. If R is a Koszul ring, then G is Koszul, in particular $P_K^G(z)H_G(-z) = 1$. The result follows because the graded resolution of K as a G-module is linear and hence Eq. (6) and therefore $P_K^R(z) = P_K^G(z)$. $\quad\square$

Since in the graded case R is Fröberg iff it is Koszul, it is natural to ask the following question.

Question 5. Is a Fröberg (local) ring Koszul?

We give a positive answer to this question for a special class of rings. If f is a non-zero element of R, denote by $v(f) = v$ the valuation of f, that is the largest integer such that $f \in \mathbf{m}^v$.

Proposition 6. *Let I be an ideal generated by a regular sequence in a regular local ring A. The following facts are equivalent:*

(1) A/I *is Koszul.*
(2) A/I *is Fröberg.*
(3) I *is 2-isomultiple.*

Proof. By Proposition 4 we know (1) implies (2). We prove that (2) implies (3). Let $I = (f_1, \ldots, f_r)$ with $v(f_i) = v_i \ge 2$. By [59] we have $P_K^{A/I}(z) = (1 + z)^n/(1 - z^2)^r$. Since A/I is Fröberg, one has that $H_{A/I}(z) = (1 - z^2)^r/(1 - z)^n$,

in particular the multiplicity of A/I is 2^r. From [53, 1.8], it follows that $v_i = 2$ for every $i = 1, \ldots, r$ and $f_1^*, \ldots, f_r^*$ is a regular sequence in G. Hence $I^* = (f_1^*, \ldots, f_r^*)$, so I is 2-isomultiple. If we assume (3), then G is a graded quadratic complete intersection, hence $P_K^G(z) H_G(-z) = 1$ and since G is graded this implies that G is Koszul. □

Next example is interesting to better understand what happens in case the regular sequence is not 2-isomultiple.

Example 7. Consider $I_s = (x^2 - y^s, xy) \subset A = K[[x, y]]$ where s is an integer ≥ 2. Then, as we have seen before, $P_K^{A/I}(z) = (1 + z)^2/(1 - z^2)^2$ and it does not depend on s. On the contrary the Hilbert series depends on s, precisely $H_{A/I}(z) = 1 + 2z + \sum_{i=2}^{s} z^i$. It follows that A/I is Koszul (hence Fröberg) if and only if $s = 2$ if and only if I is 2-isomultiple. In fact if $s > 2$, then $I_s^* = (x^2, xy, y^{s+1})$ is not quadratic.

In the following we denote by $e(M)$ the multiplicity (or degree) of an R-module M and by $\mu(M)$ its minimal number of generators. Let R be a Cohen–Macaulay ring. Abhyankar proved that $e(R) \geq h + 1$ and $h = \mu(\mathbf{m}) - \dim R$ is the so-called embedding codimension. If equality holds R is said to be of minimal multiplicity.

Proposition 8. *Let R be a Cohen–Macaulay ring of multiplicity e and Cohen–Macaulay type τ. If one of the following conditions holds:*

(1) $e = h + 1$
(2) $e = h + 2$ *and* $\tau < h$

then R is a Koszul ring.

Proof. In both cases the associated graded ring is Cohen–Macaulay and quadratic (see [53, 3.3, 3.10]). We may assume that the residue field is infinite; hence there exist $x_1^*, \ldots, x_d^*$ filter regular sequence in G and it is enough to prove that $G/(x_1^*, \ldots, x_d^*) \simeq \mathrm{gr}_{\mathbf{m}/(x_1, \ldots, x_d)}(R/(x_1, \ldots, x_d))$ is Koszul (see, e.g. [43, 2.13]). Hence the problem is reduced to an Artinian quadratic K-algebra with $\mu(\mathbf{m}) = h > 1$ and $\dim_K \mathbf{m}^2 \leq 1$, and the result follows (see [34] or [20]). □

Remark 9. (1) There are Cohen–Macaulay rings R with $e = h + 2$ and $\tau = h$ whose associated graded ring G is not quadratic, hence not Koszul. For example this is the case if $R = k[[t^5, t^6, t^{13}, t^{14}]]$, where $e = h + 2 = 3 + 2 = 5$ and $\tau = 3$.

(2) Let R be Artinian of multiplicity $e = h + 3$. Then R is stretched if its Hilbert function is $1 + hz + z^2 + z^3$, and short if its Hilbert function is $1 + hz + 2z^2$ (for details see [54]). For example, if R is Gorenstein, then R is stretched. Sally classified, up to analytic isomorphism, the Artinian local rings which are stretched in terms of the multiplicity and the Cohen–Macaulay type. As a consequence one verifies that if R is stretched of multiplicity $\geq h + 3$, then I^* is never quadratic, hence R is never Koszul. If R is short, then R is Koszul if and only if G is quadratic. In fact, by a result in Backelin's PhD thesis (see also [20]), if $\dim_K G_2 = 2$ and G is quadratic, then G is Koszul, so R is Koszul.

6.2 Koszul Modules and Linear Defect

Koszul modules have been introduced in [41]. Let us recall the definition.

Consider $(\mathbf{F}_M^R, \delta)$ a minimal free resolution of M as an R-module. The property $\delta(\mathbf{F}_M^R) \subseteq \mathbf{m}\mathbf{F}_M^R$ (the minimality) allows us to form for every $j \geq 0$ a complex

$$\lin_j(\mathbf{F}_M^R): \quad 0 \to \frac{F_j}{\mathbf{m}F_j} \to \cdots \to \frac{\mathbf{m}^{j-i} F_i}{\mathbf{m}^{j-i+1}F_i} \to \cdots \to \frac{\mathbf{m}^j F_0}{\mathbf{m}^{j+1}F_0} \to 0$$

of K-vector spaces. Denoting $\lin(\mathbf{F}_M^R) = \oplus_{j\geq 0}\lin_j(\mathbf{F}_M^R)$, one has that $\lin(\mathbf{F}_M^R)$ is a complex of free graded modules over $G = \gr_{\mathbf{m}}(R)$ whose ith free module is

$$\oplus_j \mathbf{m}^{j-i} F_i / \mathbf{m}^{j-i+1} F_i = \gr_{\mathbf{m}}(F_i)(-i) = G(-i) \otimes_K F_i/\mathbf{m}F_i.$$

By construction the differentials can be described by matrices of linear forms.

Accordingly with the definition given by Herzog and Iyengar in [41]:

Definition 10. M is a Koszul module if $H_i(\lin_j(\mathbf{F}_M^R)) = 0$ for every $i > 0$ and $j \geq 0$, that is, $H_i(\lin(\mathbf{F}_M^R)) = 0$ for every $i > 0$.

Remark 11. Notice that, if R is graded, the K-algebras G and R are naturally isomorphic. In particular $\lin(\mathbf{F}_M^R)$ coincides with the complex already defined in Sect. 4.2. This is why $\lin(\mathbf{F}_M^R)$ is called the linear part of $\mathbf{F}_M^R$.

As in the graded case (see Eq. (4)), one defines the linear defect of M over R:

$$\ld_R(M) = \sup\{i : H_i(\lin(\mathbf{F}_M^R)) \neq 0\}. \tag{7}$$

The linear defect gives a measure of how far is $\lin(\mathbf{F}_M^R)$ from being a resolution of $\gr_{\mathbf{m}}(M) = \oplus_{j\geq 0}\mathbf{m}^j M/\mathbf{m}^{j+1}M$. By the uniqueness of minimal free resolution, up to isomorphism of complexes, one has that $\ld_R(M)$ does not depend on $\mathbf{F}_M^R$, but only on the module M. When $\ld_R(M) < \infty$, we say that in the minimal free resolution the linear part predominates.

Koszul modules have appeared previously in the literature under the name "modules with linear resolution" in [56] and "weakly Koszul" in [46].

By definition R is Koszul as an R-module because it is R-free. But, accordingly with Definition 1, R is a Koszul ring if and only if K is a Koszul R-module. We have:

R is a Koszul ring $\iff$ K is a Koszul R-module $\iff$ K is a Koszul G-module.

If R is a graded K-algebra, Corollary 8 in particular says that R is a Koszul ring if K has finite linear defect or equivalently K is a Koszul module. By [41, 1.13] and [43, 3.4] one gets the following result, that is, the analogous of Theorem 7, with the regularity replaced by the linear defect.

Proposition 12. *Let R be a graded K-algebra. The following facts are equivalent:*

(1) *R is Koszul.*
(2) $\mathrm{ld}_R(K) = 0$.
(3) $\mathrm{ld}_R(K) < \infty$.
(4) *There exists a Koszul Cohen–Macaulay R-module M with $\mu(M) = e(M)$.*
(5) *Every Cohen–Macaulay R-module M with $\mu(M) = e(M)$ is Koszul.*

In [43] the modules verifying $\mu(M) = e(M)$ are named modules of minimal degree. When R itself is Cohen–Macaulay, the maximal Cohen–Macaulay modules of minimal degree are precisely the so-called Ulrich modules. Cohen–Macaulay modules of minimal degree exist over any local ring, for example, the residue field is one.

The following question appears in [41, 1.14].

Question 13. Let R be a local ring. If $\mathrm{ld}_R(K) < \infty$, then is $\mathrm{ld}_R(K) = 0$?

To answer Question 13 one has to compare $\mathrm{lin}(\mathbf{F}_K^R)$ and $\mathrm{lin}(\mathbf{F}_K^G)$. From a minimal free resolution of K as a G-module we can build up a free resolution (not necessarily minimal) of K as an R-module. In some cases the process for getting the minimal free resolution is under control via special cancellations (see [55, 3.1]), but in general it is a difficult task.

We may define absolutely Koszul local rings exactly as in the graded case. A positive answer to Question 13 would give a positive answer to the following:

Question 14. Let R be an absolutely Koszul local ring. Is R Koszul?

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