Chapter 9
CDS in Disk-Intersection Graphs

I don't like to hurt people, I really
don't like it at all. But in order to get a red light
at the intersection, you sometimes have to have an accident.
JACK ANDERSON

9.1 Motivation and Overview

Consider a finite set V of nodes in the plane and a radius function $r : V \to \mathbb{R}^+$. The *disk-intersection graph* (DIG) of V with the radius function r, denoted by $G_r(V)$, is the undirected graph on V in which u and v are adjacent if and only if the disk centered at u of radius $r(u)$ and the disk centered at v of radius $r(v)$ intersect, or equivalently,

$$\|uv\| \leq r(u) + r(v).$$

If $r(v) = 1/2$ for all $v \in V$, then $G_r(V)$ is exactly the unit disk graph (UDG) of V. Thus, the class of UDGs is a subclass of the class of DIGs. Hence, MIN-DS and MIN-CDS restricted to DIGs are also NP-hard. However, the approximation algorithms for MIN-DS and MIN-CDS restricted to UDGs cannot be directly extended to those for MIN-DS and MIN-CDS restricted to DIGs.

In this chapter, we present a simple local-search approximation algorithm for MIN-DS of DIGs, which yields a polynomial time approximation scheme (PTAS) for MIN-DS of DIGs [59]. In addition, we show that for any fixed $\varepsilon > 0$, there is a polynomial $(3+\varepsilon)$-approximation algorithm for MIN-CDS of DIGs. The rest of this chapter is organized as follows. In Sect. 9.2, we introduce the Voronoi diagram and Voronoi dual of a set of disks and their geometric properties. In Sect. 9.3, we describe a local-search approximation algorithm for MIN-DS of DIGs and show that it yields a PTAS. In Sect. 9.4, we present a two-stage approximation algorithm for MIN-CDS of DIGs.

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9.2 Voronoi Diagram and Dual of Disks

A pair of disk disks are said to be *geometrically redundant* if one is contained in the other. A set of four disks form a *degenerate quadruple* if there is a circle which is either externally tangent to all of them (see Fig. 9.1a) or internally tangent to all of them (see Fig. 9.1b).

Let $\mathcal{D}$ be a finite set of disks in which no pair of disk are geometrically redundant and no quadruple of disk are degenerate. Then the centers of the disk in $\mathcal{D}$ are all distinct. Let V be the set of centers of the disk in $\mathcal{D}$. For $v \in V$, we use $D(v)$ to denote the disk in $\mathcal{D}$ centered at v and $\rho(v)$ to denote the radius of the disk $D(v)$. The *shifted distance* from a point p and a node $v \in V$ is defined to be

$$\ell(p,v) = \|pv\| - \rho(v)$$

For a point p and a node $v \in V$, denote

$$\ell(p,v) = \|pv\| - \rho(v)$$

In other words, $|\ell(p,v)|$ is the Euclidean distance from p to the boundary of the disk $D(v)$, and $\ell(p,v)$ is positive (respectively, negative) if p is outside (respectively, inside) $D(v)$. Figure 9.2 illustrates the shifted distances. Clearly, for each point p and any two nodes u and v in V, if $\ell(p,u) \leq \ell(p,v)$ and $p \in D(v)$, then $p \in D(u)$ as well. For each $v \in V$, the set of points p in the plane satisfying that

$$\ell(p,v) = \min_{u \in V} \ell(p,u)$$

is referred to the *Voronoi cell* of $D(v)$. The lemma below shows that the Voronoi cell of $D(v)$ is nonempty and is star-shaped with respect to v.

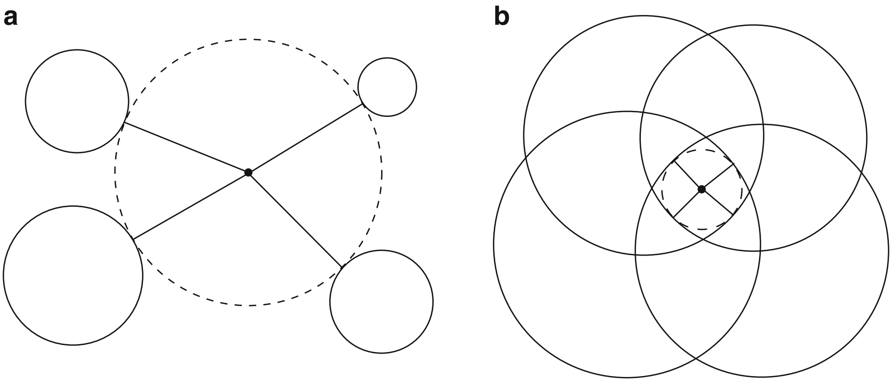

Fig. 9.1 Degenerate quadruples

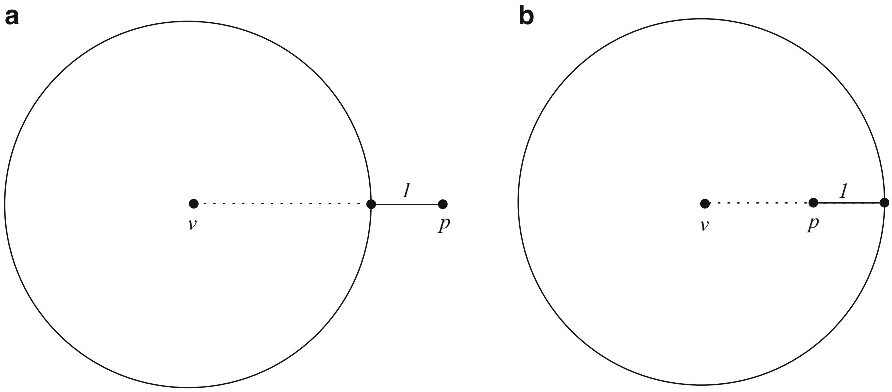

Fig. 9.2 The shifted distance

Lemma 9.2.1. *Consider any $v \in V$.*

1. *v lies in only the Voronoi cell of $D(v)$.*
2. *For any point p in the cell of v, each point in the interior of the line segment vp lies in only the Voronoi cell of $D(v)$.*

Proof. (1) For any $u \in V \setminus \{v\}$,

$$\ell(v,u) - \ell(v,v) = \|vu\| - (\rho(u) - \rho(v)) > 0,$$

where the last inequality follows from the fact that $D(u)$ and $D(v)$ are not geometrically redundant. Thus, the first part of the lemma holds.

(2) Consider any point q in the interior of the line segment vp and any $u \in V \setminus \{v\}$. We have

$$\begin{aligned}
\ell(q,v) &= \|qv\| - \rho(v) \\
&= \|pv\| - \|pq\| - \rho(v) \\
&= \ell(p,v) - \|pq\| \\
&\leq \ell(p,u) - \|pq\| \\
&= \|pu\| - \|pq\| - \rho(u) \\
&\leq \|qu\| - \rho(u) \\
&= \ell(q,u).
\end{aligned}$$

We further claim that $\ell(q,v) \neq \ell(q,u)$. Assume to the contrary that the claim does not hold. Then,

$$\|pu\| - \|pq\| = \|qu\|$$

and

$$\ell(p,v) = \ell(p,u).$$

So, q lies in the line segment pu. By symmetry, we assume that v also lies in the segment pu. Then,

$$\begin{aligned}\|uv\| &= \|pu\| - \|pv\| \\ &= (\ell(p,u) + \rho(u)) - (\ell(p,v) + \rho(v)) \\ &= \rho(u) - \rho(v).\end{aligned}$$

This means that $D(v)$ is internally tangent to $D(u)$, which is a contradiction. Thus, our claim holds. Therefore,

$$\ell(q,v) < \ell(q,u).$$

So, the second part of the lemma holds. □

Clearly, the boundary of the Voronoi cell of each disk in $\mathcal{D}$ is a concatenation of parts of hyperbolic curves and/or lines. The Voronoi cells of all disks in $\mathcal{D}$ induce a decomposition of the plane, which is known as the *Voronoi diagram* of $\mathcal{D}$. Since $\mathcal{D}$ contains no degenerate quadruple, no point belongs to Voronoi cells of more than three disks in $\mathcal{D}$. A vertex of the Voronoi diagram of $\mathcal{D}$ is an point which belongs to the Voronoi cells of three disks in V. The *Voronoi dual* of $\mathcal{D}$ is a graph on V in which two nodes u and v are adjacent if and only if the Voronoi cells of $D(u)$ and $D(v)$ share a common point. It is a planar graph as shown in the lemma below.

Lemma 9.2.2. *The Voronoi dual of $\mathcal{D}$ is a planar graph.*

Proof. Consider any edge $e = uv$ of the Voronoi dual of $\mathcal{D}$. Let p_e be an arbitrary common point shared by the Voronoi cells of $D(u)$ and $D(v)$ which is not a vertex of the Voronoi diagram of $\mathcal{D}$. The poly-segment up_ev, which is the concatenation of the two line segments up_e and vp_e, is referred to as the geometric embedding of e in the plane. We show that the geometric embeddings of any two edges e and e' do not cross each other (i.e., have no common interior point). Assume to the contrary that they have a common interior point q. We consider in two cases.

Case 1: e and e' have no common endpoint. Let $e = uv$ and $e' = u'v'$. By Lemma 9.2.1, any interior point of the poly-segment up_ev other than p_e either lies only in the Voronoi cell of $D(u)$ or only in the Voronoi cell of $D(v)$, and hence cannot lie in poly-segment $u'p_{e'}v'$. Thus, q must be the point p_e. Similarly, q must be the point $p_{e'}$. However, $q = p_e = p_{e'}$ would imply that $\{u, v, u', v'\}$ is a degenerate quadruple, which is a contradiction.

Case 2: e and e' have one common endpoint. Let $e = uv$ and $e' = u'v$. By Lemma 9.2.1, any interior point of the line segment up_e lies only in the Voronoi cell of $D(u)$, and hence cannot lie in poly-segment $u'p_{e'}v$. Thus, q must lie in the line segment vp_e. Similarly, q must lie in the line segment $vp_{e'}$. However, $p_e \neq p_{e'}$ for otherwise, p_e would be a vertex of the Voronoi diagram of $\mathcal{D}$, which contradicts to the selection of p_e. Thus, the two line segments vp_e and $vp_{e'}$ only meet at v. So, $q = v$, which is a contradiction.

In either case, we have reached a contradiction. So, the geometric embeddings of any two edges e and e' do not have a cross each other. Therefore, the lemma holds. □

9.3 Local Search for MIN-DS

In this section, we present a local-search algorithm for MIN-DS. Suppose that each node in V has a unique ID for tie-breaking. A node $v \in V$ is said to be *redundant* if there exists a node $u \in V$ satisfying that either v only dominates a proper subset of nodes dominated by u, or v dominates exactly the same set of nodes but has a larger ID than u. Let V^* denote the set of nonredundant nodes in V. Clearly, V^* still contains a minimum DS. Let B be a DS contained in V^*. A set $U \subseteq B$ is said to be a *loose* subset of B if there is a subset U' of V^* such that $|U'| < |U|$ and $(B \setminus U) \cup U'$ is still a DS, and to be a *tight* subset of B otherwise. B is said to be *k-tight* if every subset $U \subseteq B$ with $|U| \leq k$ is tight. Intuitively, for sufficiently large k the size of a k-tight DS is close to the domination number γ, which is the size of a minimum DS. Technically, we relate a k-tight DS with a minimum DS using the following planar expansion theorem established in [83].

Theorem 9.3.1. *There are two fixed positive constants c and K such that for any planar bipartite graph $H = (X, Y; E)$ satisfying that $|X| \geq 2$ and for every subset $Y' \subseteq Y$ of size at most $k \geq K$, $|N_H(Y')| \geq |Y'|$, we have*

$$|Y| \leq (1 + c/\sqrt{k})\,|X|.$$

With the help of the above theorem, we shall prove the following relation between the size of k-tight DS and the domination number γ.

Theorem 9.3.2. *Let c and K be the two fixed constants in Theorem 9.3.1. Then, for any k-tight DS $B \subseteq V^*$ with $k \geq \max\{K, 2\}$,*

$$|B| \leq \left(1 + c/\sqrt{k}\right)\gamma.$$

Theorem 9.3.2 suggests a local-search algorithm for MIN-DS, referred to as *k-Local Search* (***k*-LS**), where k is a positive integer parameter at least two. It computes a k-tight cover $B \subseteq V'$ in two phases:

- *Preprocessing Phase*: Compute the set V^* of nonredundant nodes in V, and then compute a cover $B \subseteq V^*$ by the well-known greedy algorithm for Minimum Set Cover.

- *Replacement Phase*: While B is not k-tight, find a subset U of B with size at most k and a subset U' of V^* with size at most $|U|-1$ satisfying that $(B\setminus U)\cup U'$ is still a DS; replace B by $(B\setminus U)\cup U'$. Finally, we output B.

By Theorem 9.3.2, the algorithm k-**LS** has an approximation ratio at most $1+O\left(1/\sqrt{k}\right)$ when $k\geq K$. Its running time is dominated by the second phase. Let $m=|V^*|$. Then, the second phase consists of $O(m)$ iterations. In each iteration, the search for the subset U and its replacement U' takes at most

$$O\left(m^k\right)\cdot O\left(m^{k-1}\right)=O\left(m^{2k-1}\right)$$

time. So, the total running time is

$$O(m)\cdot O\left(m^{2k-1}\right)=O\left(m^{2k}\right).$$

This means that the algorithm k-**LS** is a PTAS.

We move on to the proof of 9.3.2. Consider a minimum DS O contained in V^*. Theorem 9.3.2 holds trivially if $|B|=|O|$. So, we assume that $|B|>|O|$. Then,

$$|B\setminus O|>|O\setminus B|.$$

In addition, $|O\setminus B|\geq k$ for otherwise, we can choose a subset of $|O\setminus B|+1$ nodes from $B\setminus O$ and replace them by $O\setminus B$ to get a smaller DS, which contradicts to the fact that B is k-tight. Let T be the set of nodes in V not dominated by $O\cap B$. Then, each node in T is dominated by some node in $B\setminus O$ and by some node in $O\setminus B$. In addition, we have the following stronger property.

Lemma 9.3.3. *There is a planar bipartite graph H on $O\setminus B$ and $B\setminus O$ satisfying the following* "locality condition"*: For each $t\in T$, there are two adjacent nodes in H both of which dominate t.*

Let H be the planar bipartite graph satisfying the property in the above lemma. We claim that for any $U\subseteq B\setminus O$, $(B\setminus U)\cup N_H(U)$ is still a DS. Indeed, consider any $t\in V$. If t is dominated by $B\setminus U$, then it is also dominated by $(B\setminus U)\cup N_H(U)$. If t is not dominated by $B\setminus U$, then t is only dominated by nodes in U and hence $t\in T$. By Lemma 9.3.3, there exist two adjacent nodes $u\in B\setminus O$ and $v\in O\setminus B$ both of which dominate t. Then, we must have $u\in U$ and hence $v\in N_H(U)$. Thus, t is still dominated by $(B\setminus U)\cup N_H(U)$. So, the claim holds.

Now, consider any $U\subseteq B\setminus O$ with $|U|\leq k$. Then $|N_H(U)|\geq|U|$, for otherwise $(B\setminus U)\cup N_H(U)$ is a DS smaller than B, which contradicts to the fact that B is k-tight. By Theorem 9.3.1, we have

$$|B\setminus O|\leq(1+c/\sqrt{k})\,|O\setminus B|$$

and hence

$$|B|\leq(1+c/\sqrt{k})\,|O|.$$

So, Theorem 9.3.2 holds.

In the remaining of this section, we prove Lemma 9.3.3. Let B' (respectively, O') be a replication of $B \setminus O$ (respectively, $O \setminus B$), and let $V' = O' \cup B'$. Each replication $v \in V'$ also has a radius $r(v)$ equal to the radius of the original node in V it is replicated from. For each $v \in V'$, define

$$\bar{r}(v) = \min\{\|uv\| - r(u) : \|uv\| > r(u) + r(v), u \in V\}.$$

Clearly, $\bar{r}(v) > r(v)$, and if we increase the radius of v to any value below $\bar{r}(v)$, the set of nodes in V dominated by v remains the same. A function ρ on V' is said to be *domination-preserving* if $r(v) \leq \rho(v) < \bar{r}(v)$ for each $v \in V'$. For each domination-preserving function ρ, we use $\mathcal{D}_\rho$ to denote the collection of disks centered at v of radius $\rho(v)$ for all $v \in V'$.

Lemma 9.3.4. *There exists a domination-preserving function ρ on V' such that $\mathcal{D}_\rho$ contains no degenerate quadruple.*

Proof. We prove the lemma by contradiction. Assume the lemma is not true. Let ρ be the "fewest counterexample", in other words, $\mathcal{D}_\rho$ contains the least number of degenerate quadruples. Suppose that the disk centered at $u \in V'$ is contained in at least one quadruple in $\mathcal{D}_\rho$. We show that we can change the radius of u to some value in $[r(u), \bar{r}(u))$ such that the disk of u is not involved in any degenerate quadruple. Consider any triple disks D_1, D_2, D_3 in $\mathcal{D}_\rho$ which can potentially form a degenerate quadruple with some disk centered at u. Let v_i be the center of D_i for $1 \leq i \leq 3$. For each circle which is either externally tangent to the triple or internally tangent to the triple, its center q must satisfy the equalities

$$\|qv_1\| - \rho(v_1) = \|qv_2\| - \rho(v_2) = \|qv_3\| - \rho(v_3).$$

So, q lies in a branch of a hyperbola with two foci v_1 and v_2 (which can be degenerated to the perpendicular bisector of v_1v_2), and similarly, q also lies in a branch of a hyperbola with two foci v_1 and v_3 (which can be degenerated to the perpendicular bisector of v_1v_3). Since these two branches may have at most 4 intersection points, q can take at most 4 positions. Thus, for a disk centered at u to form a degenerate quadruple with D_1, D_2, and D_3, its radius can be of at most 4 values, each of which is referred to as a *forbidden* radius of u. As the number of triples of disks in $\mathcal{D}_\rho$ which can potentially form a degenerate quadruple with some disk centered at u is at most $\binom{|V'|-1}{3}$, the total number of forbidden radii of u is at most $4\binom{|V'|-1}{3}$. Now consider the radius function ρ' on V' satisfying that $\rho'(u)$ takes some value in $[r(u), \bar{r}(u))$ other than the forbidden radii of u, and $\rho'(v) = \rho(v)$ for each $v \neq u$. Then, ρ' is still domination-preserving but $\mathcal{D}_{\rho'}$ contains strictly fewer degenerate quadruples. This contradicts to the choice of ρ. Therefore, the lemma holds. □

Now, we fix a domination-preserving function ρ on V' such that $\mathcal{D}_\rho$ contains no degenerate quadruple. For each node $v \in V'$, let $D(v)$ denote the disk centered at v of radius $\rho(v)$. We claim that any pair of disks in $\mathcal{D}_\rho$ are geometrically nonredundant.

Indeed, assume to the contrary that there exist two nodes in u and v such that $D(u) \subseteq D(v)$. Since ρ is domination-preserving, all nodes in V dominated by u are also dominated by v, which is a contradiction. Thus, our claim holds. Let H be the graph obtained from the Voronoi dual of $\mathcal{D}_\rho$ by removing all edges between two nodes in O' and all edges between two nodes in B'. By Lemma 9.2.2, H is a planar bipartite graph on O' and B'.

Next, we show that H satisfies the locality condition: For each $t \in T$, there are two adjacent nodes in H both of which dominate t. Clearly, t is dominated by a node $v \in V'$ if and only if $\ell(t,v) \leq \rho(t)$ where $\ell(t,v) = \|tv\| - \rho(v)$ is the shifted distance from t to v. Thus, if $\ell(t,u) \leq \ell(t,v)$ for some two nodes u and v in V' and t is dominated by v, then t is also dominated by u as well. We consider two cases:

Case 1: t lies in the Voronoi cell of $D(u)$ for some $u \in O'$. Then, u must dominate t as t is dominated by O'. Let v be a node in B' to which t has the smallest shifted distance. Then, v must also dominate t, as t is dominated by B'. If u and v are adjacent, then the locality condition holds trivially. So, we assume that u and v are nonadjacent. Then, t lies outside the Voronoi cell of $D(v)$. We walk from t to v along the straight line segment tv. During this walk, we may cross some Voronoi cells of the disks in $\mathcal{D}_\rho$, and at some point before reaching v we will enter the Voronoi cell of $D(v)$ the first time. Let x be the point at which we first enter the Voronoi cell of $D(v)$. We must enter this cell from another cell, and we assume this cell the Voronoi cell of $D(w)$. Then, $\ell(t,w) \leq \ell(t,v)$ as

$$\begin{aligned}
\ell(t,w) &= \|tw\| - \rho(w) \\
&\leq \|tx\| + \|xw\| - \rho(w) \\
&= \|tx\| + \ell(x,w) \\
&= \|tx\| + \ell(x,v) \\
&= \|tx\| + \|xv\| - \rho(v) \\
&= \|tv\| - \rho(v) \\
&= \ell(t,v).
\end{aligned}$$

We further claim that $\ell(t,w) < \ell(t,v)$. Indeed, assume to the contrary that $\ell(t,w) = \ell(t,v)$. Then, we must have $\|tw\| = \|tx\| + \|xw\|$, in other words, w lies in the ray tv. As $\ell(t,w) = \ell(t,v)$, either $D(v) \subseteq D(w)$ or $D(w) \subseteq D(v)$, which is a contradiction. Therefore, our claim is true. By the choice of v, $w \in O'$ and w is adjacent to v. In addition, w dominates t since $\ell(t,w) < \ell(t,v)$ and v dominates t. Thus, the locality condition is satisfied.

Case 2: t lies in the Voronoi cell of $D(u)$ for some $u \in B'$. The proof is the same as in Case 1 is thus omitted.

Since ρ is domination-preserving and B' (respectively, O') be a replication of $B \setminus O$ (respectively, $O \setminus B$), Lemma 9.3.3 holds.

9.4 A Two-Staged Algorithm for MIN-CDS

In this section, we present a two-staged approximation algorithm for MIN-CDS of DIGs. The first stage applies the local-search algorithm k**-LS** presented in the previous section to compute a DS B. The second stage compute a set C of connectors such that $B \cup C$ is a CDS as follows. Initially C is empty. Repeat the following iteration until $B \cup C$ is connected. First we find a pair of closest connected components of $G_r(B \cup C)$ and compute a shortest (in terms the number of hops) path P between them. Then, we all internal nodes in P to C.

Clearly, the number of iterations executed in the second stage is at most $|B|-1$. In addition, it is easy to show that at most two nodes are added to C in each iteration. Thus, We claim that $|C| \leq 2(|B|-1)$. So, $|B \cup C| \leq 3|B|-2$. By Theorem 9.3.2,

$$|B| = \left(1 + O\left(1/\sqrt{k}\right)\right)\gamma.$$

Therefore,

$$|B \cup C| = \left(3 + O\left(1/\sqrt{k}\right)\right)\gamma.$$

Since γ is no more than the connected domination number γ_c, the two-staged approximation algorithm has an approximation bound $3 + O\left(1/\sqrt{k}\right)$.