Chapter 5
Weighted CDS in Unit Disk Graph

I think the weight really got the best of him today.
JODY PETTY

5.1 Motivation and Overview

It was open for many years whether MINW-CDS in unit disk graphs has a polynomial-time constant-approximation or not. Ambühl et al. [2] discovered the first one. Their solution consists of two stages. At the first stage, they construct a dominating set which is a 72-approximation for the minimum-weight dominating set problem in unit disk graphs as follows.

MINW-DS in Unit Disk Graphs: Given a unit disk graph $G = (V,E)$ with vertex weight $w : V \to R^+$, find a dominating set with minimum total weight.

In the second stage, they connect the dominating set into a CDS with additional cost $12\text{opt}_{\text{WCDS}}$ where opt_{WCDS} is the minimum weight of a CDS. Putting together, they obtained a polynomial-time 94-approximation for MINW-CDS in unit disk graphs.

Huang et al. [66] discovered a new technique on partition, called *double partition*. With the new technique, they obtained a polynomial-time $(6+\varepsilon)$-approximation for MINW-DS in unit disk graphs. Later, the approximation for MINW-DS in unit disk graphs received further improvements, from performance ratio $6+\varepsilon$ to $5+\varepsilon$ by Dai and Yu [27], to $4+\varepsilon$ by Zou et al. [134] and independently by [46], and to 3.63 by Willson et al. [114].

Connecting a weighted dominating set into a weighted CDS is equivalent to solving NODE-WEIGHTED STEINER TREE in unit disk graphs.

In general graphs, it is unlikely for NODE-WEIGHTED STEINER TREE to have a polynomial-time constant-approximation [69]. However, in unit disk graphs, the situation is different. Actually, the work of Ambühl et al. [2] means that there is a polynomial-time 12-approximation for NODE-WEIGHTED STEINER TREE in unit

D.-Z. Du and P.-J. Wan, *Connected Dominating Set; Theory and Applications*, Springer Optimization and Its Applications 77, DOI 10.1007/978-1-4614-5242-3_5,

disk graphs. Huang et al. [66] gave a polynomial-time 4-approximation. Zou et al. [133] constructed a polynomial-time 2.5ρ-approximation for NODE-WEIGHTED STEINER TREE in unit disk graphs provided that there exists a polynomial-time ρ-approximation for the minimum network Steiner tree problem. Recently, the minimum network Steiner tree problem has been found to have a polynomial-time 1.39-approximation [10]. Therefore, the approximation of Zou et al. [133] has performance 3.475. Hence, there exists a polynomial-time 7.105-approximation for MINW-CDS in unit disk graphs.

The following are still open:

Open Problem 5.1.1. *Does* MINW-CDS *in unit disk graphs have a PTAS?*

Open Problem 5.1.2. *Does* MINW-CDS *in unit ball graphs have a polynomial-time constant-approximation?*

5.2 Node-Weighted Steiner Tree

In this section, we introduce the approximation algorithm of Zou et al. [10] for NODE-WEIGHTED STEINER TREE in unit disk graphs.

Their design is motivated from the following property of optimal solutions for NODE-WEIGHTED STEINER TREE in unit disk graphs.

Lemma 5.2.1. *In a unit disk graph, for any set of terminals, there exists an optimal solution T for* NODE-WEIGHTED STEINER TREE *such that every node has degree at most five.*

Proof. Among all optimal trees for NODE-WEIGHTED STEINER TREE, we consider the one with the shortest Euclidean edge length, called the shortest optimal tree. First, note that the shortest optimal tree must have the following properties:

(a1) No two edges cross each other.
(a2) Two edges meet at a node with an angle of at least 60°.
(a3) If two edges meet with an angle of exactly 60°, then they have the same length.

Indeed, if anyone of the above three conditions does not hold, then we can easily find another optimal tree with shorter length.

Now, consider a shortest optimal tree T. By (a2), every node has degree at most six. Suppose T has a node u with degree exactly six, that is, u has six neighbors $v_1, v_2, \ldots, v_6$. By (a2), $\angle v_1uv_2 = \angle v_2uv_3 = \cdots = \angle v_6uv_1 = 60°$. By (a3), $|uv_1| = |uv_2| = \cdots = |uv_6|$. Moreover, v_2 must have degree at most four since replacing (u, v_1) and (u, v_3) by (v_1, v_2) and (v_2, v_3), the result should still be a shortest optimal tree. Now, replace (u, v_1) by (v_1, v_2) and do similar replacement at all nodes with degree six. Then one would obtain a shortest optimal tree with node degree at most five. □

Assign each edge (u,v) with the following weight:

$$w(u,v) = \frac{1}{2}(\chi_P(u)c(u) + \chi_P(v)c(v)),$$

where

$$\chi_P(u) = \begin{cases} 1, & \text{if } u \in P, \\ 0, & \text{otherwise.} \end{cases}$$

Let T^*_{node} be the optimal solution for NODE-WEIGHTED STEINER TREE in unit disk graph G, with the property that every node has degree at most five. Then

$$w(T^*_{\text{node}}) \le 2.5c(T^*_{\text{node}}).$$

Let T^*_{edge} be the minimum edge-weight Steiner tree on terminal set P. Then

$$w(T^*_{\text{edge}}) \le w(T^*_{\text{node}})$$

and from [10], one can compute a 1.39-approximation T for T^*_{edge} in polynomial-time. Therefore

$$w(T) \le 1.39 \cdot w(T^*_{\text{edge}}) \le 3.475 \cdot c(T^*_{\text{node}}).$$

Moreover,

$$c(T) \le w(T)$$

since each Steiner node has degree at least two in T. Therefore,

$$c(T) \le 3.475 \cdot c(T^*_{\text{node}}).$$

Above analysis suggests the following approximation algorithm for NODE-WEIGHTED STEINER TREE for unit disk graphs.

3.475-Approximation
input: unit disk graph $G = (V,E)$ with node weight $c : V \to R^+$
and a terminal set $P \subseteq V$.
compute 1.39-approximation T for the minimum edge-weight Steiner tree on terminal set P in graph G with edge weight
$w(u,v) = \frac{1}{2}(\chi_P(u)c(u) + \chi_P(v)c(v))$ for $(u,v) \in E$;
output T.

Theorem 5.2.2 (Zou et al. [133]). *There exists a polynomial-time 3.475-approximation for* NODE-WEIGHTED STEINER TREE *in unit disk graphs.*

By this theorem, if there exists a polynomial-time τ-approximation for MINW-DS in unit disk graphs, then there exists a polynomial-time $(\tau + 3.475)$-approximation for MINW-CDS in unit disk graphs. The remaining part of this chapter will be contributed to the study of MINW-DS in unit disk graphs.

5.3 Double Partition

The partition is a classical technique to design approximation algorithms [36]. In Sect. 3.2, this technique has been used to design a PTAS for MIN CDS in unit disk graphs. From there, one may see that the approximation performance ratio and the running time have a trade-off. Indeed, the running time of $(1+\varepsilon)$-approximation is $n^{O(1/\varepsilon^2)}$. As the approximation performance ratio $1+\varepsilon$ approaches to 1, the running time $n^{O(1/\varepsilon^2)}$ increases rapidly. Meanwhile, the size of cells, $O(1/\varepsilon)$ in the partition would also increase linearly. Indeed, the design of PTAS is based on the fact that MIN-CDS is polynomial time solvable within any constant-size cell.

When applying the partition to MINW-DS, the trouble one meets is that even for a small constant-size cell, no polynomial-time algorithm for the optimal solution has been obtained so far. Only within a square of edge size at most $\sqrt{2}/2$, a polynomial-time 2-approximation exists. In such a case, the double partition technique can be employed to overcome the trouble.

Initially, the input unit disk graph is put in a square. In the first partition, the square is partitioned into blocks; each block is a small square with edge length $m\sqrt{2}/2$. In the second partition, each block is partitioned into smaller cells with edge length $\sqrt{2}/2$. The advantage of double partition is on the second one. When m is fixed, each block can be seen to contain a constant-number m^2 of cells so that many types of combinations about cells can be enumerated in polynomial-time. In this section, the first partition is introduced under the assumption that there is a ρ-approximation with running time $n^{O(m^2)}$ for the following problem.

> MINW-DS on a Block B: Given a unit disk graph $G=(V,E)$ with a nonnegative node weight $w: V \to R^+$, find the minimum-weight node subset to dominate all nodes lying inside of B.

With the first partition, one shows the following.

Theorem 5.3.1 (Huang et al. [66]). *Suppose there exists a ρ-approximation for* MINW-DS *on a fixed block B with running time $n^{O(m^2)}$. Then for any $\varepsilon > 0$, there exists $(\rho+\varepsilon)$-approximation with computation time $n^{O(1/\varepsilon^2)}$ for* MINW-DS *in unit disk graphs.*

Proof. Choose $m = 12\max(1, \lceil 1/\varepsilon \rceil)$. Put input unit disk graph G into a grid with each block being an $m\mu \times m\mu$ square (Fig. 5.1). All blocks are disjoint. To do so, each block has boundary open on the left and on the top, but close on the right and on the bottom.

For each nonempty block B, compute ρ-approximation for MINW-DS on block B. Unit those ρ-approximation solutions for all nonempty block and denote this union by $A(P)$ for the partition P induced by this grid.

Now, shaft this grid in diagonal direction with distance 4 in each time. This results in $m/4$ partitions $P_1,\dots,P_{m/4}$. Choose $A = A(P_i)$ to be the one with the minimum weight among $A(P_1),\dots,A(P_{m/4})$. Now, one claims that $c(A(P_i)) \le (\rho+\varepsilon)\text{opt}_{\text{WDS}}$ where opt_{WDS} is the total weight of optimal solution for MINW-DS.

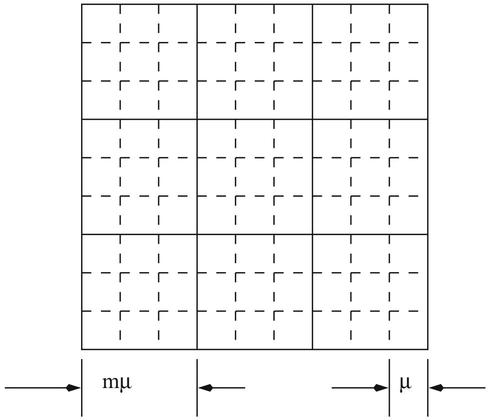


Fig. 5.1 Double partition ($\mu = \sqrt{2}/2$)

Suppose $\mathrm{Opt}_{\mathrm{WDS}}$ is an optimal solution for MINW-DS in unit disk graph G. For each $v \in \mathrm{Opt}_{\mathrm{WDS}}$, the disk $\mathrm{disk}_1(v)$ may intersect more than one blocks of P_i. Let $\zeta_i(v)$ be the number of blocks in partition P_i, intersecting $\mathrm{disk}_1(v)$. Let $O(B) = \{v \in \mathrm{Opt}_{\mathrm{WDS}} \mid \mathrm{disk}_1(v) \cap B \neq \emptyset\}$. Then $O(B)$ is a feasible solution for MINW-DS on block B. Therefore,

$$c(A(P_i)) \leq \sum_{B \in P_i} c(O(B)) = \mathrm{opt}_{\mathrm{WDS}} + \sum_{v \in \mathrm{Opt}_{\mathrm{WDS}}} (\zeta_i(v) - 1)c(v).$$

Note that $\mathrm{disk}_1(v)$ can intersect at most one horizontal cutline and at most one vertical cutline of P_i. Therefore, $\zeta_i(v)$ has only three possible values. $\zeta_i(v) = 1$ if $\mathrm{disk}_1(v)$ does not intersect any cutline of P_i, $\zeta_i(v) = 2$ if $\mathrm{disk}_1(v)$ intersects exactly one cutline of P_i, and $\zeta_i(v) = 4$ if $\mathrm{disk}_1(v)$ intersects two cutlines of P_i, one horizontal cutline and one vertical cutline.

Moreover, two vertical cutlines, possibly from two different partitions, have distance $2\sqrt{2} > 2$. Thus, over all partitions, each disk $\mathrm{disk}_1(v)$ for $v \in \mathrm{Opt}_{\mathrm{WDS}}$ can intersect at most one vertical cutline and similarly at most one horizontal cutline. This means that for every $v \in \mathrm{Opt}_{\mathrm{WDS}}$,

$$\sum_{i=1}^{m/4} (\zeta_i(v) - 1) \leq 3.$$

Therefore,

$$\begin{aligned} c(A) &= \min_{1 \leq i \leq m/4} c(A(P_i)) \\ &\leq \frac{1}{m/4} \sum_{i=1}^{m/4} \sum_{B \in P_i} c(O(B)) \end{aligned}$$

$$= \text{opt}_{\text{WDS}} + \frac{4}{m} \sum_{v \in \text{Opt}_{\text{WDS}}} \sum_{i=1}^{m/4} (\zeta_i(v) - 1)c(v)$$
$$\leq \text{opt}_{\text{WDS}} + \frac{12}{m} \text{opt}_{\text{WDS}}$$
$$\leq (1+\varepsilon)\text{opt}_{\text{WDS}}.$$

□

5.4 Cell Decomposition

In the second partition, each block is partitioned into $\frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2}$ cells also by a grid. To have all cells nonoverlapping, assume that each cell has open boundary on the right and on the top, and close on the left and on the bottom. This section is contributed to study the following problem.

MINW-DS on a Cell e: Given a unit disk graph $G = (V, E)$ with a nonnegative node weight $w : V \to R^+$, find the minimum-weight node subset to dominate all nodes lying inside of e.

The main duty of this section is to prove the following result.

Lemma 5.4.1. *There is a polynomial-time 2-approximation for* MINW-DS *in a cell e.*

The proof of this lemma is based on a decomposition of nodes in the cell e into two parts which form two polynomial-time solvable subproblems. This decomposition stems from the property of optimal solution for MINW-DS in the cell e.

Suppose $\text{Opt}(e)$ is an optimal solution for MINW-DS in a cell e. If $\text{Opt}(e)$ contains a node v lying in e, then $\text{Opt}(e) = \{v\}$ and $c(v) = \min_{u \in e} c(u)$ because any node in e is able to dominate every point of e. The difficult part of characterizing $\text{Opt}(e)$ is in the case that $\text{Opt}(e)$ does not contain any node in e.

To deal with this case, let A, B, C, D be four vertices of e and divide outside of e into eight areas NE (northeastern), NC (north-central), NW (northwestern), ME (middle-east), MW (middle-west), SE (southeastern), SC (south-central), and SW (southwestern) as shown in Fig. 5.2.

Let $V(e)$ be the set of nodes lying in the cell e. $V(e)$ will be decomposed into two parts $V(e) = V_1 \cup V_2$ ($V_1 \cap V_2 = \emptyset$) such that all points in V_1 can be dominated by nodes in $\text{Opt}_1(e) = \text{Opt}(e) \cap (N \cup S)$ where $N = NE \cup NC \cup NW$ and $S = SE \cup SC \cup SW$, and V_2 can be dominated by nodes in $\text{Opt}_2(e) = \text{Opt}(e) \cap (E \cup W)$ where $E = NE \cup ME \cup SE$ and $W = NW \cup MW \cup SW$.

Next, the existence of such a partition of $V(e)$ for $\text{Opt}(e)$ would be proved through presentation of two lemmas.

For any vertex $p \in V(e)$, let $\angle p$ be a right angle at p such that two edges intersect horizontal line AB each at an angle of $\pi/4$. Let $\Delta_{\text{south}}(p)$ denote the part of e lying inside of $\angle p$. Similarly, we can define $\Delta_{\text{north}}(p)$, $\Delta_{\text{east}}(p)$ and $\Delta_{\text{west}}(p)$ as shown in Fig. 5.3.

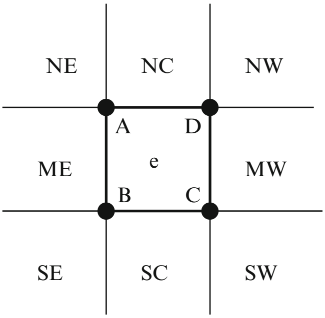


Fig. 5.2 Outside of e is divided into eight areas

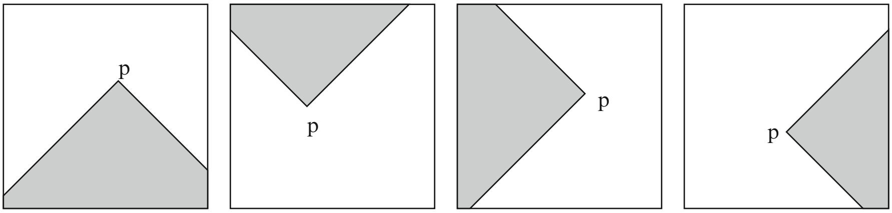


Fig. 5.3 $\Delta_{\text{south}}(p)$, $\Delta_{\text{north}}(p)$, $\Delta_{\text{east}}(p)$ and $\Delta_{\text{west}}(p)$

Lemma 5.4.2. *If p is dominated by a vertex u in area SC then every point in $\Delta_{\text{south}}(p)$ can be dominated by u. The similar statement holds for ME and $\Delta_{\text{east}}(p)$, MW and $\Delta_{\text{west}}(p)$, and NC and $\Delta_{\text{north}}(p)$.*

Proof. Note that $\Delta_{\text{south}}(p)$ is a convex polygon. It is sufficient to show that the distance from u to every vertex of $\Delta_{\text{south}}(p)$ is at most one.

Suppose v is a vertex of $\Delta_{\text{south}}(p)$ on BC (Fig. 5.4). Draw a line L' perpendicular to pv and equally divide pv. If u is below L', then $d(u,v) \leq d(u,p) \leq 1$. If u is above L', then $\angle uvp < \pi/2$ and hence $\angle uvC < 3\pi/4$. Hence, $d(u,v) < \mu/\cos\pi/4 = 1$.

A similar argument can be applied in the case that the vertex v of $\Delta_{\text{south}}(p)$ is on DA or on AB. □

Consider two nodes $p, p' \in V(e)$. Suppose p is on the left of p'. Extend the left edge of $\angle p$ and the right edge of $\angle p'$ to intersect at point p''. Define $\Delta_{\text{south}}(p,p')$ to be the part of e lying inside of $\angle p''$ (Fig. 5.5). Similarly, we can define $\Delta_{\text{north}}(p.p')$.

Lemma 5.4.3. *Let K be a set of nodes which dominates $V(e)$. Suppose $p, p' \in V(e)$ are dominated by some nodes in $K \cap SC$, but neither p nor p' is dominated by any node in $K \cap (ME \cup MW)$. Then every node in $\Delta_{\text{south}}(p,p')$ can be dominated by node in $K \cap (N \cup S)$ where $N = NE \cup NC \cup NW$ and $S = SE \cup SC \cup SW$.*

Proof. By Lemma 5.4.2, it suffices to consider a node u lying in $\Delta_{\text{south}}(p,p') \setminus (\Delta_{\text{south}}(p) \cup \Delta_{\text{south}}(p'))$. For contradiction, suppose u is dominated by a node v in $K \cap (ME \cup MW)$. If $v \in ME$, then $\Delta_{\text{east}}(v)$ contains p and by Lemma 5.4.2, p is dominated by v, a contradiction. A similar contradiction can result from $v \in MW$. □

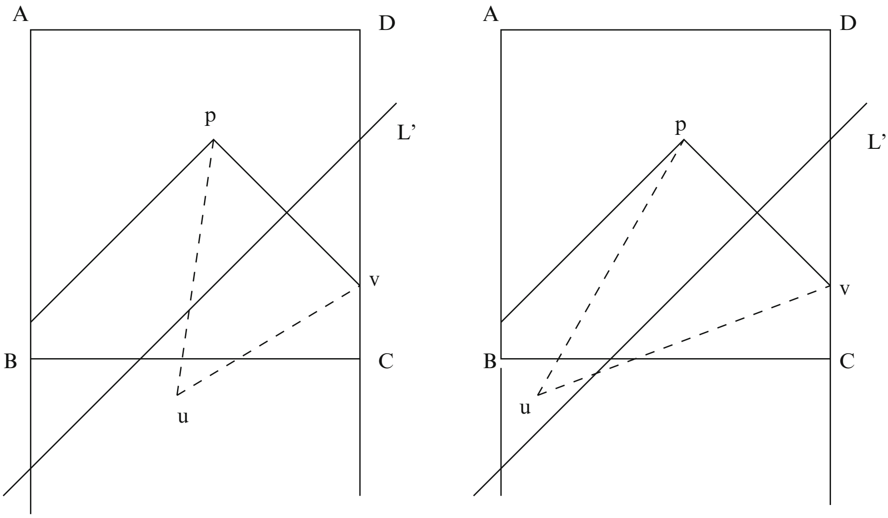
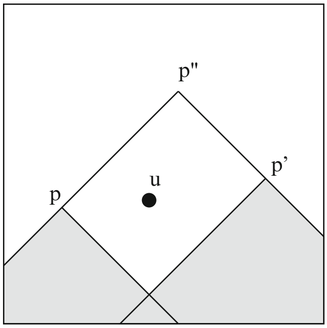


Fig. 5.4 The proof of Lemma 5.4.2

Fig. 5.5 $\Delta_{\text{south}}(p, p')$



Now, it is ready to give a property of $\text{Opt}(e)$ in case that $\text{Opt}(e) \cap V(e) = \emptyset$.

Lemma 5.4.4. *Let* $\text{Opt}(e)$ *be an optimal solution for* MINW-DS *in the cell* e*. Suppose* $\text{Opt}(e) \cap V(e) = \emptyset$*. Then there exist four nodes* $p, p', q, q' \in V(e)$ *such that* $V_1(e) = V(e) \cap (\Delta_{\text{south}}(p, p') \cup \Delta_{\text{north}}(q, q'))$ *is dominated by* $\text{Opt}_1(e) = \text{Opt}(e) \cap (N \cup S)$ *and* $V_2(e) = V(e) - V_1(e)$ *is dominated by* $\text{Opt}_2(e) = \text{Opt}(e) \cap (E \cup W)$.

Proof. Let V_S be the set of nodes in $V(e)$, each of which can be dominated by a node in SC but not dominated by any node in $ME \cup MW$. Let p be the node in $V(S)$ such that the left edge of $\Delta(p)$ is on the leftmost position among all left edges of $\Delta(v)$ for $v \in V_S$. Let p' be the node in V_S such that the right edge of $\Delta(p')$ is on the rightmost position among all right edges of $\Delta_{\text{south}}(v)$ for $v \in V_S$. Clearly, $\Delta_{\text{south}}(p, p')$ has the following properties:

(p1) Every node in $\Delta_{\text{south}}(p, p')$ can be dominated by $\text{Opt}_1(e)$.
(p2) $V_S \subset \Delta_{\text{south}}(p, p')$.

Similarly, let V_N be the set of nodes in $V(e)$, each of which can be dominated by a node in SC but not dominated by any node in $ME \cup MW$. One can find nodes $q, q' \in V_N$ to meet the following requirement.

(q1) Every node in $\Delta_{\text{north}}(q, q')$ can be dominated by $\text{Opt}_1(e)$.
(q2) $V_N \subset \Delta_{\text{north}}(p, p')$.

It follows from (p1) and (q1) that $V_1(e)$ is dominated by $\text{Opt}_1(e)$. If follows from (p2) and (q2) that $V_2(e)$ is dominated by $\text{Opt}_2(e)$. □

Based on Lemma 5.4.4, one can design a 2-approximation for MINW-DS in the cell e as follows.

2-Approximation for MINW-DS in a cell e
input a weighted unit disk graph G and a cell e.
$u \leftarrow \text{argmin}_{v \in V(e)} c(v)$;
$V^+(e) \leftarrow \{v \in V \mid \text{disk}_1(v) \cap e \neq \emptyset\}$;
$V_1^+ \leftarrow V^+(e) \cap (N \cup S)$;
$V_2^+ \leftarrow V^+(e) \cap (E \cup W)$;
$A \leftarrow \{u\}$;
for every $\{p, p', q, q'\} \subset V(e)$
do begin
$V_1 \leftarrow V(e) \cap (\Delta_{\text{south}}(p, p') \cup \Delta_{\text{north}}(q, q'))$;
$V_2 \leftarrow V(e) \setminus V_1$;
find the minimum-weight subset O_1 of V_1^+, dominating V_1;
find the minimum weight subset O_2 of V_2^+, dominating V_2;
if $c(A) > c(O_1 \cup O_2)$
$A \leftarrow O_1 \cup O_2$;
end-for;
output $A(e) = A$.

Clearly, if $\text{Opt}(e) \cap V(e) \neq \emptyset$, then $c(A) = c(\text{Opt}(e))$. If $\text{Opt}(e) \cap V(e) = \emptyset$, then for $\{p, p', q, q'\}$ in Lemma 5.4.4, one has

$$c(O_1) \leq c(\text{Opt}_1(e)), c(O_2) \leq c(\text{Opt}_2(e)).$$

Therefore,

$$c(A(e)) \leq c(O_1 \cup O_2) \leq c(\text{Opt}_1(e)) + c(\text{Opt}_2(e)) \leq 2c(\text{Opt}(e)).$$

The next section will show that O_1 and O_2 can be computed in polynomial-time. Therefore, the following holds.

Lemma 5.4.5. *There is a polynomial-time 2-approximation for* MINW-DS *in the cell e.*

The following result can be easily obtained based on Lemma 5.4.5.

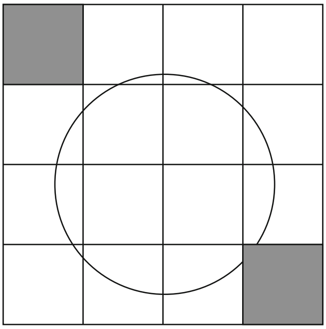


Fig. 5.6 For two cells at ends of a diagonal, at most one has its interior intersecting a disk of radius one

Theorem 5.4.6. *There is a polynomial-time 28-approximation for* MINW-DS *in any block B.*

Proof. Suppose for each node v, $\text{disk}_1(v)$ intersects at most α cells e. Then

$$c(\cup_{e\in B}A(e)) \leq \sum_{e\in B} c(A(e)) \leq \sum_{e\in B} 2c(\text{Opt}(e)) \leq 2\alpha c(\text{Opt}(b))$$

where $\text{Opt}(B)$ is the optimal solution for MINW-DS in the block B.

Note that $\text{disk}_1(v)$ can intersect at most four horizontal strips and at most four vertical strips, hence at most 16 cells. Furthermore, consider two cells at two ends of a diagonal. Only one of them has its interiors intersecting a disk with radius one (Fig. 5.6). Thus, $\alpha \leq 14$. This means that $\cup_{e\in B}A(e)$ is a 28-approximation for MINW-DS in the block B. □

5.5 6-Approximation

Why and O_1 and O_2 be computed within polynomial-time in the **2-Approximation** for MINW-DS in a cell e? This section first answers this question. To do so, it suffices to study the following problem.

> MINW-SENSOR-COVER with Targets in a Strip: Consider a set P of targets lying inside a horizontal strip and a set $\mathcal{D}$ of disks with radius one and centers lying either above or below the strip (Fig. 5.7). Assume every target in P is covered by at least one disk in D. Given disks with a nonnegative weight $c : \mathcal{D} \to R^+$, find the minimum total weight subset of disks covering all targets.

Let

$$\mathcal{D}^+ = \{D \in \mathcal{D} \mid \text{ the center of } D \text{lies above the strip}\}$$

and

$$\mathcal{D}^- = \{D \in \mathcal{D} \mid \text{ the center of } D \text{ lies below the strip}\}.$$

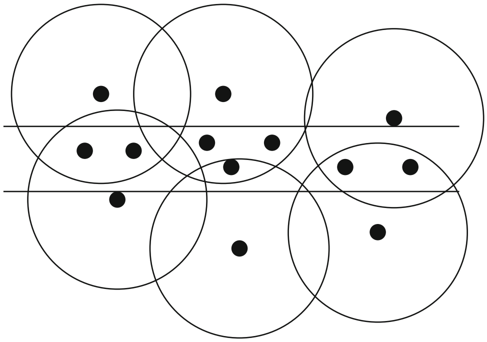


Fig. 5.7 Sensor Cover with targets in a strip

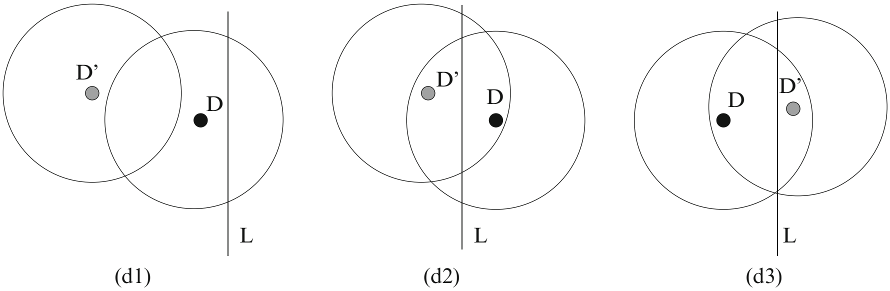


Fig. 5.8 D' is controlled by D at line L

Consider a disk $D \in \mathcal{D}^+$ intersecting a vertical line L. A disk $D' \in \mathcal{D}^+$ is said to be *controlled* by D at L, denoted by $D' \prec_L^+ D$, if one of the following holds (Fig. 5.8):

(d1) D' does not intersect L.
(d2) The lower endpoint of $D' \cap L$ is higher than the lower endpoint of $D \cap L$.
(d3) The lower endpoint of $D' \cap L$ is identical to the lower endpoint of $D \cap L$. But, the center of D' is on the right of the center of D.

Similarly, let $D \in \mathcal{D}^-$ intersect a line L. Then a disk $D' \in \mathcal{D}^-$ is said to be *controlled* by D, denoted by $D' \prec_L^-$, if one of the following holds:

(d1) D' does not intersect L.
(d2) The upper endpoint of $D' \cap L$ is lower than the upper endpoint of $D \cap L$.
(d3) The upper endpoint of $D' \cap L$ is identical to the upper endpoint of $D \cap L$. But, the center of D' is on the right of the center of D.

The following are important properties of controlledness.

Lemma 5.5.1. *Let $D, D', D'' \in \mathcal{D}^+$ and L a vertical line. If $D'' \prec_L^+ D'$ and $D' \prec_L^+ D$, then $D'' \prec_L^+ D$. Similarly, for $D, D', D'' \in \mathcal{D}^-$, if $D'' \prec_L^- D'$ and $D' \prec_L^- D$, then $D'' \prec_L^- D$.*

Proof. It follows immediately from the definition of controlledness. □

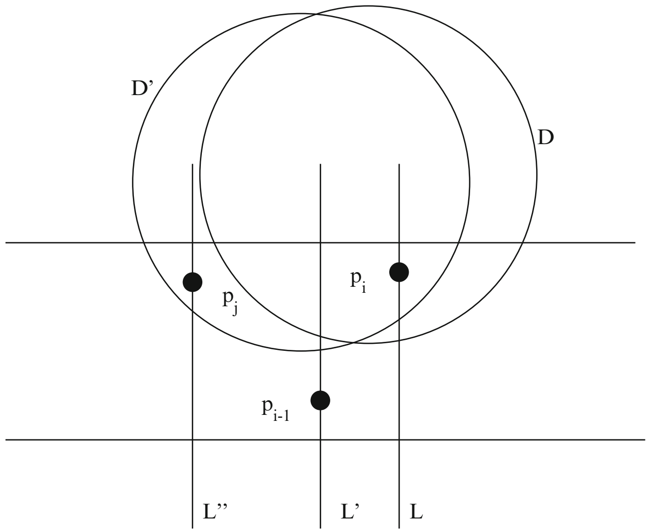


Fig. 5.9 The proof of Lemma 5.5.1

Lemma 5.5.2. *Let D and D' be two disks and L a vertical line. Suppose $D \cap L \neq \emptyset$. Then*

(e1) If $D, D' \in \mathcal{D}^+$, then either $D' \prec_L^+ D$ or $D \prec_L^+ D'$;
(e2) If $D, D' \in \mathcal{D}^-$, then either $D' \prec_L^- D$ or $D \prec_L^- D'$.

Proof. It follows immediately from the definition of controlledness. □

Lemma 5.5.3. *Let L and L' be two vertical lines such that L lies on the right of L'. Then the following holds:*

(L1) Let $D, D \in \mathcal{D}^+$. If $D' \prec_L^+ D$ and $D \prec_{L'}^+ D'$, then $D \cap Q(L') \subseteq D' \cap Q(L')$ where $Q(L')$ is the closed lower-left quarter-plane bounded by the upper boundary of the strip and L'.
(L2) Let $D, D \in \mathcal{D}^-$. If $D' \prec_L^- D$ and $D \prec_{L'}^- D'$, then $D \cap Q'(L') \subseteq D' \cap Q'(L')$ where $Q'(L')$ is the closed upper-left quarter-plane bounded by the lower boundary of the strip and L'.

Proof. For contradiction, suppose (L1) is not true, that is, there exists a point $p \in (D \cap Q(L')) \setminus (D' \cap Q(L'))$. Let L'' be the vertical line passing through p. Then $D' \prec_{L''}^+ D$. Then L'' is on the left of L'. Let A be the lower endpoint of $D \cap L$ and A'' the lower endpoint of $D \cap L''$. Then $D \cap L' \neq \emptyset$. Let A' be the lower endpoint of $D \cap L'$ and B' the lower endpoint of $D' \cap L'$. Then A' is above B'. They both should lie below the line AA'' (Fig. 5.9). Therefore, $D' \cap AA''$ is located inside the segment $[A, A'']$. Let E and F be two endpoints of $D' \cap AA''$. Clearly, $\angle EAF < \angle A''A'A$ and $|EF| \leq |AA''|$. Therefore,

$$\text{radius}(D') = \frac{|EF|}{2\sin\angle EB'F} < \frac{|AA''|}{2\sin\angle A''A'A} = \text{radius}(D).$$

a contradiction. □

Now, it is ready to show the following.

Theorem 5.5.4 (Ambühl et al. [2]). MINW-SENSOR-COVER *with targets in a strip can be solved in* $O(m^4 n)$ *where* $n = |P|$ *and* $m = |\mathcal{D}|$.

Proof. First, assume every disk in $\mathcal{D}$ has a positive weight since disks with zero weight can be removed together with targets covered by them at the beginning.

Let $p_1, .., p_n$ be all points in P in the ordering from left to right. Call as an *upper disk* (*lower disk*) for any disk with center above (below) the strip. A dynamic programming will be employed to find the optimal solution. For simplicity of description, assume the two boundaries of the strip are also two disks with infinite radius and weight zero. They do not cover any point in P. The upper bound is an upper disk, and the lower bound is a lower disk. Note that these two disks do not belong to $\mathcal{D}$. But, the relations $\prec_L^+$ and $\prec_L^+$ can be extended to all upper disks and all lower disks, respectively.

For an upper disks D and a lower disk D' with $D \cup D'$ covering p_i, define by $T_i(D, D')$ the one with the minimum total weight among disk subsets $\mathcal{D}'$ satisfying the following conditions:

1. $\mathcal{D}'$ covers $p_1, \ldots, p_i$.
2. $D, D' \in \mathcal{D}'$.
3. Let L_i be the vertical line passing through p_i. Then D controls every upper disk in $\mathcal{D}'$ at L_i, and D' controls every lower disk in $\mathcal{D}'$ at L_i.

Since two boundaries of the strip have zero weight and cover nothing, for simplicity of the discussion, one assume that they cannot appear in $T_i(D, D') - \{D, D'\}$. In other word, they can play only the role of D or D'.

Let $c(T_i(D, D'))$ be the total weight of disks in $T_i(D, D')$. One claims that the following recursion holds.

$$c(T_i(D, D')) = \min_{D_1, D_2} \{c(T_{i-1}(D_1, D_2)) + c(\{D, D'\} \setminus \{D_1, D_2\})\}, \tag{5.1}$$

where upper disk D_1 and lower disk D_2 are over all possible pairs satisfying conditions:

(c1) $D_1 \cup D_2$ covers p_{i-1}.
(c2) Let L_i be the vertical line passing through p_i. Then $D_1 \prec_{L_i}^+ D$ and $D_2 \prec_{L_i}^- D'$.

To show this claim, at the first choose D_1 to be the upper disk in $T_i(D, D')$ which controls every upper disk in $T_i(D, D')$ at L_{i-1}. By Lemma 5.5.1, such a choice must exist. Similarly, one can choose D_2 to the lower disk in $T_i(D, D')$ which controls every lower disk in $T_i(D,D')$. By Lemma 5.5.3, $D \cap Q(L_{i-1}) \subseteq D_1 \cap Q(L_{i-1})$ and $D' \cap Q' \subseteq D_2 \cap Q'$. Therefore, $(T_i(D,D') - \{D,D'\}) \cup \{D_1,D_2\}$ covers $p_1,\ldots,p_{i-1}$. Hence,

$$c(T_i(D,D')) - c(\{D,D'\} \setminus \{D_1,D_2\}) \geq c(T_{i-1}(D_1,D_2)),$$

that is,

$$c(T_i(D,D')) \geq \min_{D_1,D_2} (c(T_{i-1}(D_1,D_2)) + c(\{D,D'\} \setminus \{D_1,D_2\})).$$

On the other hand, for any pair $\{D_1,D_2\}$ satisfying (c1) and (c2), $T_{i-1}(D_1,D_2) \cup \{D,D'\}$ covers $p_1,\ldots,p_i$. Moreover, for any upper disk $\hat{D}$ in $T_{i-1}(D_1,D_2)$, one must have $\hat{D} \prec^+_{L_i} D$. Indeed, for contradiction, suppose $D \prec^+_{L_i} \hat{D}$. Then $\hat{D} \neq D_1$ and hence $\hat{D} \in \mathcal{D}$ has a positive weight. By Lemma 5.5.1, $D_1 \prec^+_{L_i} \hat{D}$. Note that by the definition of $T_{i-1}(D_1,D_2)$, $\hat{D} \prec^+_{L_{i-1}} D_1$. By Lemma 5.5.3, then $\hat{D} \cap Q(L_{i-1}) \subseteq D_1 \cap Q(L_{i-1})$, which means that $\hat{D}$ can be deleted from $T_{i-1}(D_1,D_2)$. This contradicts the minimality of $T_{i-1}(D_1,D_2)$.

Similarly, for any lower disk $\hat{D}$ in $T_{i-1}(D_1,D_2)$, one must have $\hat{D} \prec^-_{L_i} D'$. Therefore,

$$c(T_i(D,D')) \leq T_{i-1}(D_1,D_2) + c(\{D,D'\} \setminus \{D_1,D_2\})$$

for any pair $\{D_1,D_2\}$ satisfying (c1) and (c2). Therefore,

$$c(T_i(D,D')) \leq \min_{D_1,D_2} (T_{i-1}(D_1,D_2) + c(\{D,D'\} \setminus \{D_1,D_2\}))$$

for $\{D_1,D_2\}$ over all pairs satisfying (c1) and (c2). Hence, (5.1) holds.

This recursion suggests a dynamic program for computing all $T_i(D,D')$. There are $O(nm^2)$ $T_i(D,D')$'s and each needs to be computed recursively in time $O(m^2)$. Therefore, this dynamic program runs in time $O(nm^4)$. Finally, the minimum weight of subset of disks covering all targets can be computed from $\min_{D,D'} c(T_n(D,D'))$, which requires $O(m^2)$ time. □

By Theorem 5.5.4, O_1 and O_2 in **2-Approximation** for MinW-DS in a cell e can be computed in polynomial time. Hence, a polynomial-time 28-approximation has been obtained for MinW-DS in a block B.

However, an idea motivated from Theorem 5.5.4 may give a big improvement. That is to combine $V_1(e)$ along a horizontal strip and combine $V_2(e)$ along a vertical strip. With such an idea, the approximation performance ratio can be reduced from 28 to 6.

6-Approximation for MinW-DS in a block B.

input a unit disk graph $G=(V,E)$ and a block B.

Let C be the set of cells e in B with $V(e) = V \cap e \neq \emptyset$. Let $H_1,\ldots,H_m$ be horizontal strips and $Y_1,\ldots,Y_m$ vertical strips of B.

Let $C' \subseteq C$. For each cell $e \in C'$, choose a vertex $v_e \in V(e)$ and let $U = \{v_e \mid e \in C'\}$. For every subset C' and every U, compute a vertex subset $A(C',U)$ in the following way:

Step 1. Let $Z = V \cap (\cup_{v\in U} \text{disk}_1(v))$. For every $e \in C - C'$, update $V(e) \leftarrow V(e) \setminus Z$.
Step 2. For every cell $e \in C - C'$ and for every choice of $\{p, p', q, q'\} \subseteq V(e)$, let $V_1(e) = V(e) \cap (\Delta_{\text{south}}(p, p') \cup \Delta_{\text{north}}(q, q'))$ and $V_2(e) = V(e) - V_1(e)$.

Step 2.1. For each horizontal strip H_i, compute a minimum weight subset $\text{Opt}(H_i)$ of disks with centers lying outside H_i to dominate $(\cup_{e\in H_i\cap(C-C')} V_1(e))$.
Step 2.2. For each vertical strip Y, compute a minimum weight subset $\text{Opt}(Y_i)$ of disks with centers lying outside Y_i to dominate $(\cup_{e\in Y_i\cap(C-C')} V_2(e))$.
Step 2.3 Compute $O = (\cup_{i=}^m \text{Opt}(H_i)) \cup (\cup_{i=1}^m \text{Opt}(Y_i))$.
Step 2.4 Compute O^* to minimize the total weight $c(O)$ over all possible combinations of $\{p.p', q, q'\}$ for all $e \in C - C'$.

Step 3. Set $A(C',U) = O^* \cup U$.

Finally, compute an $A^* = A(C',U)$ to minimize the total weight $c(A(C',U)$ for C' over all subsets of C and U over all choices of v_e for all $e \in C'$.
output A^*.

Theorem 5.5.5 (Huang et al. [66]). *There exists a 6-approximation for* MINW-DS *in a block B, with running time $n^{O(m^2)}$ where n is the number of nodes v such that* $\text{disk}_1(v) \cap B \neq \emptyset$.

Proof. First, let us estimate the time for computing A^*. There are $O(2^{m^2})$ possible subsets of C, $n^{O(m^2)}$ possible choices of U and $O(n^{4m^2})$ possible combinations of $\{p, p', q, q'\}$ for all cells in $C - C'$. For each combination, computing all $\text{Opt}(H_i)$ and all $\text{Opt}(Y_i)$ needs time $O(n^5)$. Therefore, total computation time is $n^{O(m^2)}$.

Next, estimate the performance ratio. Let Opt be an optimal solution for MINW-DS in the block B. Set $C' = \{e \in C \mid e \cap \text{Opt} \neq \emptyset\}$. For each $e \in C'$, choose a node $v_e \in \text{Opt} \cap e$. Set $U = \{v_e \mid e \in C'\}$ and $Z = V \cap (\cup_{v\in U} \text{disk}_1(v))$. Update $V(e)$ for all $e \in C - C'$ by $V(e) \leftarrow V(e) \setminus Z$. For each $e \in C - C'$, by Lemma 5.4.4, there exists a set $\{p, p', q, q'\}$ of at most four nodes in $V(e)$ such that $V_1(e) = V(e) \cap (\Delta_{\text{south}}(p, p') \cup \Delta_{\text{north}}(q, q'))$ is dominated by $\text{Opt}_1(e) = \text{Opt}(e) \cap (S \cup N)$ and $V_2(e) = V(e) - V_1(e)$ is dominated by $\text{Opt}_2(e) = \text{Opt}(e) \cap (E \cup W)$ where

$$\text{Opt}(e) = \{v \in V - U \mid e \cap \text{disk}_1(v) \neq \emptyset\}.$$

Note that $\cup_{e\in H_i\cap(C-C_i)} \text{Opt}_1(e)$ is a feasible solution for the minimization problem solved at Step 2.1. Therefore,

$$c(\text{Opt}(H_i)) \leq c(\cup_{e\in H_i\cap(C-C_i)} \text{Opt}_1(e)).$$

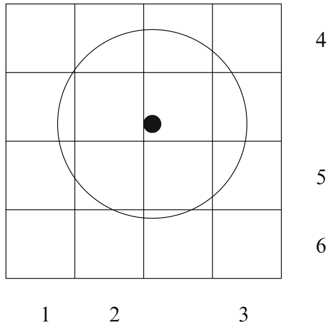


Fig. 5.10 Each $\text{disk}_1(v)$ intersects at most six strips not containing v

Similarly,

$$c(\text{Opt}(Y_i)) \leq c(\cup_{e \in Y_i \cap (C - C_i)} \text{Opt}_2(e)).$$

Therefore

$$\begin{aligned} c(O) &\leq \sum_{i=1}^{m} c(\cup_{e \in H_i \cap (C - C_i)} \text{Opt}_2(e)) \\ &\quad + \sum_{i=1}^{m} c(\cup_{e \in Y_i \cap (C - C_i)} \text{Opt}_2(e)) \\ &\leq 6 \cdot c(\text{Opt} - U) \end{aligned}$$

since each disk $\text{disk}_i(v)$ can intersect at most six strips which do not contain v (Fig. 5.10). Hence,

$$\begin{aligned} c(A^*) &\leq c(A(C', U)) = c(O^*) + c(U) \\ &\leq c(O) + c(U) \\ &\leq 6 \cdot c(\text{Opt} - U) + c(U) \\ &\leq 6 \cdot c(\text{Opt}). \end{aligned}$$

□

5.6 4-Approximation

Zou et al. [134] studied a generalization of MinW-Sensor-Cover with targets in a strip.

MINW-CHROMATIC-DISK-COVER: Consider m parallel horizontal strips $H_1, \ldots, H_m$ as shown in Fig. 5.11. To have all strips disjoint, assume that each strip has open boundary on the top and close boundary on the bottom. Given a set $\mathcal{R}$ of red disks with radius one, a set $\mathcal{B}$ of blue disk with radius one, a positive weight function $c : \mathcal{R} \cup \mathcal{B} \to R^+$, and a set P of targets points lying in those strips, find the minimum-weight subset of red disks and blue disks such that every target in strip H_i is covered by a chromatic disk, that is, by either a red disk with center lying above H_i or a blue disk lying below H_i.

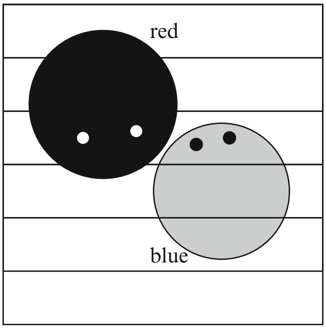


Fig. 5.11 Chromatic disk cover

When $m = 1$, MINW-CHROMATIC-DISK-COVER is exactly MINW-SENSOR-COVER with targets in a strip.

Theorem 5.6.1 (Zou et al. [134]). MINW-CHROMATIC-DISK-COVER *can be solved in time* $O(nd^{4m})$ *where* $n = |P|$ *and* $d = |\mathcal{R} \cup \mathcal{B}|$.

Proof. Denote $\mathcal{R}_i = \mathcal{R} \cap H_{i-1}$ for $i = 2, \ldots, m$ and $\mathcal{R}_1 = \{\text{disk}_1(v) \in \mathcal{R} \mid v \text{ lies above } H_1\}$. Denote $\mathcal{B}_i = \mathcal{B} \cap H_{i+1}$ for $i = 1, \ldots, m-1$ and $\mathcal{B}_m = \{\text{disk}_1(v) \in \mathcal{B} \mid v \text{ lies below } H_m\}$.

Let $\mathcal{R}_i^+$ be obtained from $\mathcal{R}$ by putting a dumming disk with radius infinite, which is the lower half plane bounded by the upper boundary of H_i. Let $\mathcal{B}_i^+$ be obtained from $\mathcal{B}_i$ by putting a dumming disk with radius infinite, which is the upper half plane bounded by the lower bound of H_i. Consider a $2m$-dimensional vectors $\mathbf{D}$ in

$$\mathcal{S} = \mathcal{R}_1^+ \times \cdots \times \mathcal{R}_m^+ \times \mathcal{B}_1^+ \times \cdots \times \mathcal{B}_m^+.$$

For simplicity, $\mathbf{D}$ is also used to denote the set of components of $\mathbf{D}$. Denote by D_i the ith component of $\mathbf{D}$. Then one has $D_i \in \mathcal{R}_i^+$ and $D_{m+i} \in \mathcal{B}_i^+$ for $1 \le i \le m$. Consider $\mathbf{D}, \mathbf{D}' \in \mathcal{S}$. For any vertical line L, $\mathbf{D}'$ is said to be controled by $\mathbf{D}$, written as $\mathbf{D}' \prec_L \mathbf{D}$, if for $1 \le i \le m$, $D'_i \prec_L^+ D_i$ and $D'_{m+i} \prec_L^- D - m + i$.

Let $L_1, .., L_k$ be all vertical lines passing through target points in P in the ordering from left to right. Let P_i be the subset of targets in P lying on L_i or on the left of L_i. For any $\mathbf{D} \in \mathcal{S}$, if $\mathbf{D}$ does not cover $P_j - P_{j-1}$, then define $T_j(\mathbf{D}) = \text{nil}$ if there does not exist a disk subset $\mathcal{D}'$ satisfying the following conditions:

- $\mathcal{D}'$ is a chromatic disk cover for P_j.
- $\mathbf{D} \subseteq \mathcal{D}'$.
- For any disk $D' \in \mathcal{D}' \cap \mathcal{R}_i$, $D' \prec_{L_j}^+ D_i$.
- For any disk $D' \in \mathcal{D}' \cap \mathcal{B}_i$, $D' \prec_{L_j}^- D_{m+i}$.

If such a disk subset $\mathcal{D}'$ exists, then define $T_j(\mathcal{D})$ to be the one with the minimum total weight.

Since all dumming disks have zero weight and cover nothing, for simplicity of the discussion, one assumes that they cannot appear in $T_j(\mathbf{D}) \setminus \mathbf{D}$.

Now, one claims that for $\mathbf{D}$ covering $P_j - P_{j-1}$, the following recursion holds.

$$c(T_j(\mathbf{D})) = \min_{\mathbf{D}' \prec_{L_j} \mathbf{D}} \{c(T_{j-1}(\mathbf{D}')) + c(\mathbf{D} \setminus \mathbf{D}')\}. \tag{5.2}$$

To show this claim, one first chooses D'_i to be the disk in $T_j(\mathcal{D}) \cap \mathcal{R}_j^+$ which controls every disk in $T_j(\mathbf{D}) \cap \mathcal{R}_j^+$ at L_{j-1}. By Lemma 5.5.1, such a choice must exists. Similarly, one can choose D'_{m+i} to the disk in $T_j(\mathbf{D})$ which controls every disk in $T_j(\mathbf{D}) \cap \mathcal{B}_i^+$ at line L_{j-1}. Define $\mathbf{D}' = (D'_i, 1 \le i \le 2m)$.

By Lemma 5.5.3, $D_i \cap Q_i(L_{j-1}) \subseteq D'_i \cap Q_i(L_{j-1})$ for $1 \le i \le m$ where $Q_i(L_j)$ is the close lower-left quarter-plane bounded by L_{j-1} and the upper boundary of H_i. Similarly, $D_{m+i} \cap Q'_i(L_{j-1}) \subseteq D'_{m+i} \cap Q'_i(L_{j-1})$ for $1 \le i \le m$. This means that if $T_j(\mathbf{D}) \ne$ nil, then $(T_j(\mathbf{D}) - (\mathbf{D} \setminus \mathbf{D}')$ is a chromatic disk cover for P_{j-1}. Hence,

$$c(T_j(\mathbf{D})) - c(\mathbf{D} \setminus \mathbf{D}') \ge c(T_{j-1}(\mathbf{D}')),$$

that is,

$$c(T_j(\mathbf{D})) \ge \min_{\mathbf{D}' \prec_{L_j} \mathbf{D}} (c(T_{j-1}(\mathbf{D}')) + c(\mathbf{D} \setminus \mathbf{D}')). \tag{5.3}$$

If $T_j(\mathbf{D}) =$ nil, then $c(T_j(\mathbf{B})) = \infty$ and hence (5.3) holds trivially.

On the other hand, for any $\mathbf{D}' \prec \mathbf{D}$, if $T_{j-1}(\mathbf{D}') =$ nil, then $c(T_{j-1}(\mathbf{D}') = \infty > c(T_j(\mathbf{D}))$. Next, assume that $T_{j-1}(\mathbf{D}') \ne$ nil. Then $T_{j-1}(\mathbf{D}') \cup \mathbf{D}$ is a chromatic cover of P_j.

Moreover, for any disk $\hat{D}$ in $T_{j-1}(\mathbf{D}')$, one must have $\hat{D} \prec_{L_j}^+ D_i$ if $\hat{D} \in \mathcal{R}_i^+$ and $\hat{D} \prec_{L_j}^- D_{m+i}$ if $\hat{D} \in \mathcal{B}_i^+$. In fact, for contradiction, suppose $\hat{D} \in \mathcal{R}_i^+$ and $\hat{D}$ is not controlled by D_i. Thus, $\hat{D} \notin \mathbf{D}'$ and hence $c(\hat{D}) > 0$. Moreover, by Lemma 5.5.2, $D_i \prec_{L_j}^+ \hat{D}$. By Lemma 5.5.1, $D'_i \prec_{L_j}^+ \hat{D}$. By Lemma 5.5.3, $\hat{D} \cap Q_i(L_{j-1}) \subseteq D'_i \cap Q_i(L_{j-1})$. This means that $\hat{D}$ can be deleted from $T_{j-1}(\mathcal{D}')$, contradicting the minimality of $T_{j-1}(\mathbf{D}')$. Similarly, it is also impossible that $\hat{D} \in \mathcal{B}_i^+$ and $\hat{D}$ is not controlled by D_{m+i}.

From above argument, one can see that $T_{j-1}(\mathbf{D}') \cup \mathbf{D}$ satisfies all conditions for above $\mathcal{D}'$. Therefore,

$$\begin{aligned} c(T_j(\mathbf{D})) &\le c(T_{j-1}(\mathbf{D}' \cup \mathbf{D})) \\ &= c(T_{j-1}(\mathbf{D}')) + c(\mathbf{D} \setminus \mathbf{D}') \end{aligned}$$

for all $\mathbf{D}' \prec_{L_j} \mathbf{D}$. This completes the proof of (5.2).

The recursion (5.2) suggests a dynamic program for computing all $T_j(\mathbf{D})$. There are $O(nd^{2m})$ $T_j(\mathbf{D})$'s, and each needs to be computed recursively in time $O(d^{2m})$. Therefore, this dynamic program runs in time $O(nd^{4m})$. Finally, the minimum weight of subset of disks covering all targets can be computed from $\min_{\mathbf{D} \in \mathcal{S}} c(T_k(\mathbf{D}))$, which requires $O(d^{2m})$ time. □

4-Approximation for MINW-DS in a block B.

input a unit disk graph $G = (V, E)$ and a block B.

Let C be the set of cells e in B with $V(e) = V \cap e \neq \emptyset$. Let $H_1, \ldots, H_m$ be horizontal strips and $Y_1, \ldots, Y_m$ vertical strips of B.

Let $C' \subseteq C$. For each cell $e \in C'$, choose a vertex $v_e \in V(e)$ and let $U = \{v_e \mid e \in C'\}$. For every subset C' and every U, compute a vertex subset $A(C', U)$ in the following way:

Step 1. Let $Z = V \cap (\cup_{v \in U} \text{disk}_1(v))$. For every $e \in C - C'$, update $V(e) \leftarrow V(e) \setminus Z$.

Step 2. For every cell $e \in C - C'$ and for every choice of $\{p, p', q, q'\} \subseteq V(e)$, let $V_1(e) = V(e) \cap (\Delta_{\text{south}}(p, p') \cup \Delta_{\text{north}}(q, q'))$ and $V_2(e) = V(e) - V_1(e)$.

Step 2.1. Compute an optimal solution $\text{Opt}(H)$ for MINW-CHROMATIC-DISK-COVER with horizontal strips $H_1, \ldots, H_m$, target set $P = (\cup_{e \in (C - C')} V_1(e))$, red disk set $\mathcal{R} = \{(\text{disk}_1(v), \text{red}) \mid v \in V - U\}$ and blue disk set $\mathcal{B} = \{(\text{disk}_1(v), \text{blue}) \mid v \in V - U\}$.

Step 2.2. Compute an optimal solution $\text{Opt}(Y)$ for MINW-CHROMATIC-DISK-COVER with vertical strips $Y_1, \ldots, Y_m$, target set $P = (\cup_{e \in (C - C')} V_2(e))$, red disk set $\mathcal{R} = \{(\text{disk}_1(v), \text{red}) \mid v \in V - U\}$ and blue disk set $\mathcal{B} = \{(\text{disk}_1(v), \text{blue}) \mid v \in V - U\}$. (Note: Each target is required to be covered by either a red disk from the left or a blue disk from the right.)

Step 2.3 Compute $O = \text{Opt}(H) \cup \text{Opt}(Y)$.

Step 2.4 Compute O^* to minimize the total weight $c(O)$ over all possible combinations of $\{p.p', q, q'\}$ for all $e \in C - C'$.

Step 3. Set $A(C', U) = O^* \cup U$.

Finally, compute an $A^* = A(C', U)$ to minimize the total weight $c(A(C', U))$ for C' over all subsets of C and U over all choices of v_e for all $e \in C'$.

output A^*.

Theorem 5.6.2 (Zou et al. [134]). *There exists a 4-approximation for* MINW-DS *in a block B, with running time $n^{O(m^2)}$ where n is the number of nodes v such that* $\text{disk}_1(v) \cap B \neq \emptyset$.

Proof. First, let us estimate the time for computing A^*. There are $O(2^{m^2})$ possible subsets of C, $n^{O(m^2)}$ possible choices of U and $O(n^{4m^2})$ possible combinations of $\{p, p', q, q'\}$ for all cells in $C - C'$. For each combination, computing all $\text{Opt}(H)$ and all $\text{Opt}(Y)$ needs time $O(n^{4m+1})$. Therefore, total computation time is $n^{O(m^2)}$.

Next, estimate the performance ratio. Let Opt be an optimal solution for MINW-DS in the block B. Set $C' = \{e \in C \mid e \cap \text{Opt} \neq \emptyset\}$. For each $e \in C'$, choose a node $v_e \in \text{Opt} \cap e$. Set $U = \{v_e \mid e \in C'\}$ and $Z = V \cap (\cup_{v \in U} \text{disk}_1(v))$. Update $V(e)$ for all $e \in C - C'$ by $V(e) \leftarrow V(e) \setminus Z$. For each $e \in C - C'$, by Lemma 5.4.4, there exists a set $\{p, p', q, q'\}$ of at most four nodes in $V(e)$ such that $V_1(e) = V(e) \cap (\Delta_{\text{south}}(p, p') \cup$

$\Delta_{\text{north}}(q,q'))$ is dominated by $\text{Opt}_1(e) = \text{Opt}(e) \cap (S \cup N)$ and $V_2(e) = V(e) - V_1(e)$ is dominated by $\text{Opt}_2(e) = \text{Opt}(e) \cap (E \cup W)$ where

$$\text{Opt}(e) = \{v \in V - U \mid e \cap \text{disk}_1(v) \neq \emptyset\}.$$

Let

$$\mathcal{D}_1(e) = \{(\text{disk}_1(v), \text{red}) \mid v \in \text{Opt}_1(v) \cap N\} \cup \{(\text{disk}_1, \text{blue}) \mid v \in \text{Opt}_1(v) \cap S\}.$$

Then $\cup_{e \in (C - C_i)} \mathcal{D}_1(e)$ is a feasible solution for the minimization problem solved at Step 2.1. Therefore,

$$c(\text{Opt}(H)) \leq c(\cup_{e \in (C - C_i)} \mathcal{D}_1(e)).$$

Similarly, let

$$\mathcal{D}_2(e) = \{(\text{disk}_1(v), \text{red}) \mid v \in \text{Opt}_2(v) \cap E\} \cup \{(\text{disk}_1, \text{blue}) \mid v \in \text{Opt}_2(v) \cap W\}.$$

Then $\cup_{e \in (C - C_i)} \mathcal{D}_2(e)$ is a feasible solution for the minimization problem solved at Step 2.2. Therefore,

$$c(\text{Opt}(Y) \leq c(\cup_{e \in (C - C_i)} \mathcal{D}_2(e)).$$

Therefore

$$\begin{aligned} c(O) &\leq c(\cup_{e \in (C - C_i)} \mathcal{D}_1(e)) \\ &\quad + c(\cup_{e \in (C - C_i)} \mathcal{D}_2(e)) \\ &\leq 4 \cdot c(\text{Opt} - U) \end{aligned}$$

since each disk $\text{disk}_i(v)$ can involve feasible solutions for at most two minimization problems and at each feasible solution $\text{disk}_1(v)$ has at most two copies, one in red and one in blue. Hence,

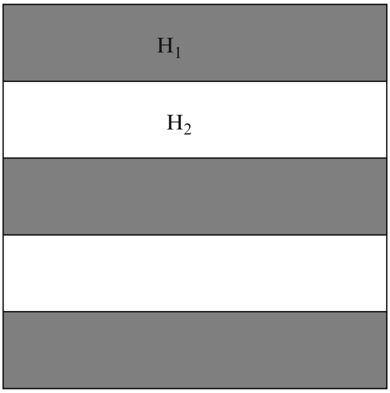
$$\begin{aligned} c(A^*) &\leq c(A(C', U)) = c(O^*) + c(U) \\ &\leq c(O) + c(U) \\ &\leq 4 \cdot c(\text{Opt} - U) + c(U) \\ &\leq 4 \cdot c(\text{Opt}). \end{aligned}$$

□

5.7 3.63-Approximation

Erlebach and Mihalak [46] studied another generalization of MINW-SENSOR-COVER with targets in a strip.

Fig. 5.12 Multi-strips



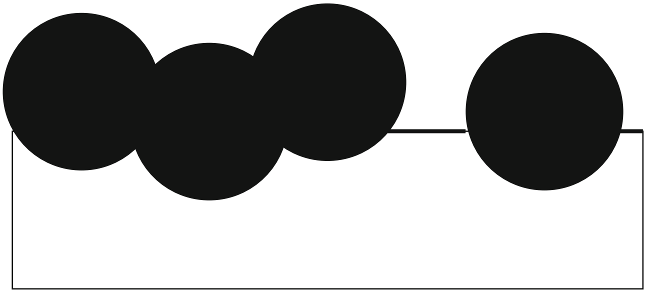


Fig. 5.13 The upper envelope

MINW-SENSOR-COVER with Targets in Multi-Strips: Consider m parallel horizontal strips $H_1, \ldots, H_m$ in a block as shown in Fig. 5.12. To have all strips disjoint, assume that each strip has open boundary on the top and close boundary on the bottom. Given a set $\mathcal{D}$ of n disks with radius one and a positive weight function $c : \mathcal{D} \to R^+$, and a set P of k targets points lying in strips $H_1 \cup H_3 \cup \cdot \cup H_{2\lceil m/2 \rceil - 1}$, find the minimum weight subset of disks such that every target in strip H_i is covered by a disk with center lying outside of H_i.

Erlebach and Mihalak [46] transformed this problem to a shortest path problem. To explain this transformation, the first is to study an optimal solution Opt for MINW-SENSOR-COVER with targets in multi-strips.

Consider a strip H_i for some odd i, $1 \le i \le m$. A disk in Opt is called an *upper disk* with respect to H_i if its center lies above H_i. For simplicity of discussion, the half plane above H_i is also considered as an upper (dumming) disk with respect to H_i. All upper disks with respect to H_i form an *upper area* of H_i. The boundary of this area is called the *upper envelope* of H_i (Fig. 5.13). Similarly, one may define lower disks, the lower dumming disks the lower area and the lower envelope of H_i.

Lemma 5.7.1. *An upper disk* $\mathrm{disk}_1(v)$ *with respect to* H_i *can appear in the upper envelope of* H_i *at most once. If an upper disk* $\mathrm{disk}_1(u)$ *is on the left of another upper disk* $\mathrm{disk}_1(v)$ *on the upper envelope of* H_i*, then the center* u *is on the left of the center* v*.*

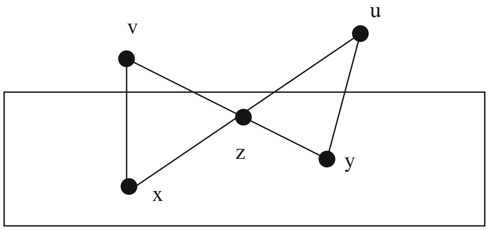


Fig. 5.14 The proof of Lemma 5.7.1

Similarly, a lower disk $\text{disk}_1(v)$ *with respect to* H_i *can appear in the lower envelope of* H_i *at most once. If a lower disk* $\text{disk}_1(u)$ *is on the left of another lower disk* $\text{disk}_1(v)$ *on the lower envelope of* H_i*, then the center* u *is on the left of the center* v*.*

Proof. For contradiction, suppose the upper disk $\text{disk}_1(u)$ is on the left of the upper disk $\text{disk}_1(v)$ on the upper envelope of strip H_i, but the center u is on the right of the center v. Choose a point x from $\text{circle}_1(u)$ appearing on the upper envelope and a point y from $\text{circle}_1(v)$ appearing on the upper envelope. Then segments ux and vy intersect, say at z. Since x and y appear in the upper envelope, one must have $|ux| < |vx|$ and $|vy| < |uy|$. Therefore, $|ux| + |vy| < |vx| + |uy|$. However, $|vx| < |vz| + |zx|$ and $|uy| < |uz| + |zy|$. Hence $|vx| + |uy| < |vz| + |zx| + |uz| + |zy| = |ux| + |vy|$, a contradiction. Therefore, the second sentence is true (Fig.5.14).

The first sentence is a corollary of the second sentence. In fact, if $\text{disk}_1(v)$ appears twice on the upper envelope, then between two appearances, there must exist another disk $\text{disk}_1(u)$ appearing. This means that v is on the left of u and also on the right of u, a contradiction. □

A *corner* of the upper envelope of H_i is an intersection point of two upper disks on the envelope. A *corner* of the lower envelope of H_i is an intersection point of two lower disks on the envelope. A *sweep line* L_i for H_i is a vertical line that starts from a position on the left of all disks in $\mathcal{D}$ and moves to right until a position on the right of all disks in $\mathcal{D}$. L_i's movement is discrete. Each intermediate position of L_i must pass through a corner on either the upper envelope or the lower envelope. Each of such positions is denoted by a quadruple $(d_1, d_2; d_3, d_4)$ with either $d_1 = d_2$ or $d_3 = d_4$ where d_1, d_2 are upper disks and d_3, d_4 are lower disks. If $d_1 \neq d_2$, then L_i passes through the intersection point of d_1 and d_2 on the upper envelope. If $d_3 \neq d_4$, then L_i passes through the intersection point of d_3 and d_4 on the lower envelope. For the initial and the end position of L_i, $d_1 = d_2$ is the upper dumming disk and $d_3 = d_4$ is the lower dumming disk.

A *move* of L_i is from its current position $(d_1, d_2; d_3, d_4)$ to an adjacent position on the right. If $d_1 = d_2$, this adjacent position is either $(d_2, d; d_4, d_4)$ or $(d_1, d_1; d_4, d)$. In this case, one says that the disk d_3 leaves L_i and the disk d enters. If $d_3 = d_4$, then the right adjacent position is either $(d_2, d; d_4, d_4)$ or $(d_2, d_2; d_4, d)$. In this case, one says that the disk d_1 leaves and the disk d enters (Fig. 5.15).

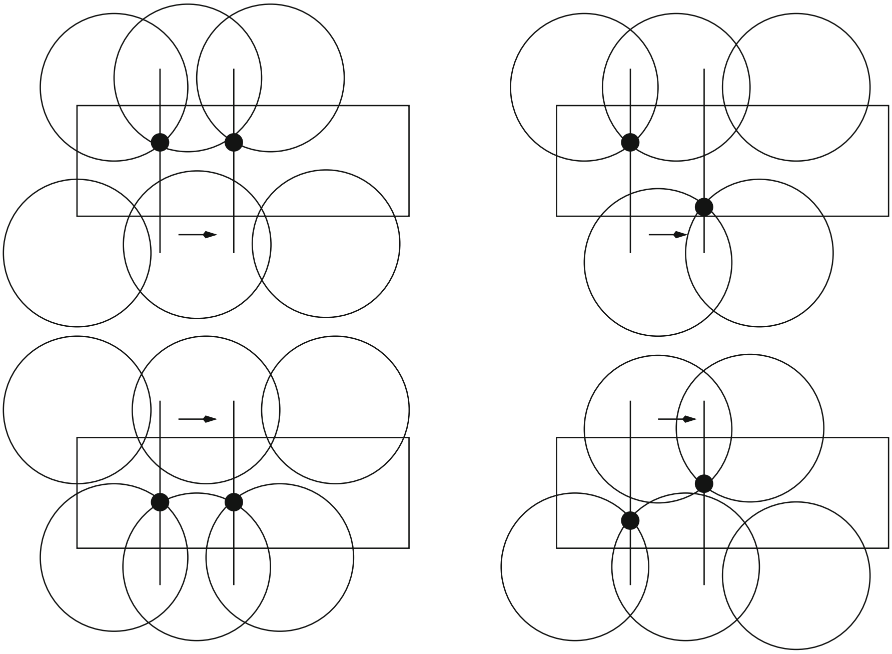


Fig. 5.15 Four possibilities for a sweep line to move to right adjacent position

Note that every disk in Opt must appear in the upper or lower envelope for some strip H_i and the weight of Opt equals the total weight of disks appearing on envelopes. The sweep line is used to calculate this total weight so that one can turn the sweeping process into a shortest path of a graph. There is one trouble if the sweep line is doing counting individually. Each disk may appear in the lower envelope of strip H_i and also in the upper envelope of H_{i+2}. If every sweep line does individual counting, then the total weight is not exactly $c(\text{Opt})$. Therefore, all sweep line may be required to do a combined action and to employ a technique for avoiding the double counting.

A *configuration* of sweep lines consists of positions of sweep lines L_1, L_3, ..., $L_{2\lceil m/2\rceil -1}$ at a moment. A *legal move* from a configuration A to another configuration B consists of exactly one move of one sweep line which is required to satisfy the following constraint:

(m1) In this move, if a lower disk d leaves line L_i, then the disk d should be already passed by L_{i+2}. Otherwise, L_i has to wait for L_{i+2} to pass the disk d.

(m2) In this move, if an upper disk d leaves line L_i, then the disk d should be already passed by L_{i-2}. Otherwise, L_i has to wait for L_{i-2} to pass the disk d.

Here, by a disk passed by a sweep line at position $(d_1, d_2; d_3, d_4)$, one means that if d is an upper disk, then the center of d is not on the right of the center of d_2; if d is a lower disk, then the center of d is not on the right of the center of d_4.

Why can this constraint eliminate the double counting? It is because one can set counting rule as follows: The counting is performed only on a newly entered disk in each move. This means that when a disk d enters a configuration in a move, the weight of d is counted if the configuration does not contain d, and the weight of d is not counted if the configuration already contains d.

Can this constraint stop the moving of configuration? The answer is no because if the moving stops, then every sweep line is waiting for another line to pass a disk. Then there are two cases.

Case 1. There are two sweep lines L_i and L_{i+2} such that L_i waits for L_{i+2} to pass a disk d and L_{i+2} waits for L_i to pass another disk d'. In this case, L_i should be in position $(d_1, d_2; d, d_4)$ and L_{i+2} should be in position $(d', d'_2; d'_3, d'_4)$. Since L_i waits for L_{i+2} to pass d, the center of d is on the right of the center of d'_2 and hence by Lemma 5.7.1, the center of d is on the right of the center of d'. Similarly, the center of d' should be on the right of the center of d, a contradiction.
Case 2. Case 1 does not occur. If L_i waits for L_{i+2}, then L_{i+2} waits for L_{i+4}, etc. However, since the number of strips if finite, this process cannot go forever, a contradiction. If L_i waits for L_{i-2}, then L_{i-2} waits for L_{i-4}. This process cannot go forever, neither.

Next, an auxiliary graph $G(\text{Opt})$ can be constructed to turn the moving of configuration into a shortest path. All possible configurations form all vertices. There exists an arc (A, B) from a configuration A to another configuration B if and only if B can be reached from A through a legal move. The start vertex s is the configuration consisting of all sweep lines on the left of all disks in $\mathcal{D}$. The target vertex is the configuration consisting of all sweep lines on the right of all disks in $\mathcal{D}$.

Now, it is ready to show the following.

Theorem 5.7.2 (Eriksson et al. [45]). MinW-Sensor-Cover *with targets in multi-strips can be solved in time $O(n^{3(m+1)})$.*

Proof. Use $\mathcal{D}$ instead of Opt to construct the sweep line positions, configurations, legal move of configurations, and graph $G(\mathcal{D})$ by following the same rules in the construction of graph $G(\text{Opt})$, except an additional requirement for a sweep line move: during the move, targets between two positions should be covered by two envelope. Then $G(\mathcal{D})$ contains $G(\text{Opt})$ as a subgraph, and the shortest path from configuration s to configuration t would give an optimal solution for Sensor Cover with targets in multi-strips. Since each sweep line position is determined by three disks, each sweep line has at most $O(n^3)$ positions. Hence, the number of configurations is at most $O(n^{3m/2})$ for even m and $O(n^{3(m+1)/2})$ for odd m. Therefore, computing the shortest path in $G(\mathcal{D})$ takes time $O(n^{3m})$ for even m and $O(n^{3(m+1)})$ for odd m. □

Based on Theorem 5.7.2, Willson et al. [114] constructed a polynomial-time 3.63-approximation for MinW-DS in a block B. Their main idea is motivated from the following observation. To construct an approximation solution, MinW-DS in a block B is divided into four problems, two on horizontal strips and two on vertical strips. Consider the two on horizontal strips. One is on strips $H_1, H_3, \ldots$.

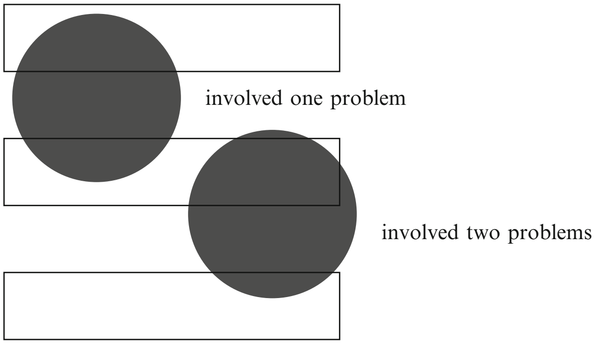


Fig. 5.16 $\text{disk}_1(v)$ involves only one problem if v is nearly at the central of H_i

The other one is on strips $H_2, H_4, \ldots$. Suppose $\text{disk}_1(v)$ with $v \in H_i$. Then $\text{disk}_1(v)$ will involve only one problem which is on strips H_{i-1} and H_{i+1} if v lies nearly at the central of H_i. This means that in average, $\text{disk}_1(v)$ involves less than two problems. How to take advantage of this average estimation? Shifting the partition on the block B is a traditional technique (Fig. 5.16).

3.63-Approximation for MINW-DS in a block.
input a unit disk graph $G = (V, E)$ and a block B.
Put the block B at the position with $(0,0)$ as its left-lower corner. Let $P(a,b)$ be a grid with cell size $\mu \times \mu$ and the left-lower corner $(-am\mu/q, -bm\mu/q)$ such that the block B is covered where $\mu = \frac{\sqrt{2}}{2}$.
$A \leftarrow nil$ $(c(nil) = \infty)$;
for $a = 0$ **to** $q-1$ **do**
  for $b = 0$ **to** $q-1$ **do begin**
    compute $A(a,b)$ with procedure $A(a,b)$;
    if $c(A) > c(A(a,b))$
      then $A \leftarrow A(a,b)$;
  end-for;
output A.

Procedure $A(a,b)$
input a unit disk graph $G = (V, E)$ and a block B. Use grid $P(a,b)$ partition the block B into cells.

Let C be the set of cells e in B with $V(e) = V \cap e \neq \emptyset$. Let $H_1, \ldots, H_m$ be horizontal strips and $Y_1, \ldots, Y_m$ vertical strips of B. Define

$$\begin{aligned} H_{\text{odd}} &= H_1 \cup H_3 \cup \cdots \cup H_{2\lceil m/2 \rceil - 1}, \\ H_{\text{even}} &= H_2 \cup H_4 \cup \cdots \cup H_{2\lfloor m/2 \rfloor}, \\ Y_{\text{odd}} &= Y_1 \cup Y_3 \cup \cdots \cup Y_{2\lceil m/2 \rceil - 1}, \\ Y_{\text{even}} &= Y_2 \cup Y_4 \cup \cdots \cup Y_{2\lfloor m/2 \rfloor}. \end{aligned}$$

Let $C' \subseteq C$. For each cell $e \in C'$, choose a vertex $v_e \in V(e)$ and let $U = \{v_e \mid e \in C'\}$. For every subset C' and every U, compute a vertex subset $A(C',U)$ in the following way:

Step 1. Let $Z = V \cap (\cup_{v \in U} \text{disk}_1(v))$. For every $e \in C - C'$, update $V(e) \leftarrow V(e) \setminus Z$.
Step 2. For every cell $e \in C - C'$ and for every choice of $\{p, p', q, q'\} \subseteq V(e)$, let $V_1(e) = V(e) \cap (\Delta_{\text{south}}(p,p') \cup \Delta_{\text{north}}(q,q'))$ and $V_2(e) = V(e) - V_1(e)$.

Step 2.1. Compute an optimal solution $\text{Opt}(H_{\text{odd}})$ for SENSOR-COVER with targets in multi-strips and with horizontal strips H_{odd}, target set $P = (\cup_{e \in (C-C') \cap H_{\text{odd}}} V_1(e))$, disk set $\mathcal{D} = \{\text{disk}_1(v) \mid v \in V - U\}$.
Step 2.2. Compute an optimal solution $\text{Opt}(H_{\text{even}})$ for SENSOR-COVER with targets in multi-strips and with horizontal strips H_{even}, target set $P = (\cup_{e \in (C-C') \cap H_{\text{even}}} V_1(e))$, disk set $\mathcal{D} = \{\text{disk}_1(v) \mid v \in V - U\}$.
Step 2.3. Compute an optimal solution $\text{Opt}(Y_{\text{odd}})$ for SENSOR-COVER with targets in multi-strips and with vertical strips Y_{odd}, target set $P = (\cup_{e \in (C-C') \cap Y_{\text{odd}}} V_2(e))$, disk set $\mathcal{D} = \{\text{disk}_1(v) \mid v \in V - U\}$.
Step 2.4. Compute an optimal solution $\text{Opt}(Y_{\text{even}})$ for SENSOR-COVER with targets in multi-strips with vertical strips Y_{even}, target set $P = (\cup_{e \in (C-C') \cap Y_{\text{even}}} V_2(e))$, disk set $\mathcal{D} = \{\text{disk}_1(v) \mid v \in V - U\}$.
Step 2.5 Compute $O = \text{Opt}(H_{\text{odd}}) \cup \text{Opt}(H_{\text{even}}) \cup \text{Opt}(Y_{\text{odd}}) \cup \text{Opt}(Y_{\text{even}})$
Step 2.6 Compute O^* to minimize the total weight $c(O)$ over all possible combinations of $\{p.p', q, q'\}$ for all $e \in C - C'$.

Step 3. Set $A(C',U) = O^* \cup U$.

Finally, compute an $A(a,b) = A(C',U)$ to minimize the total weight $c(A(C',U)$ for C' over all subsets of C and U over all choices of v_e for all $e \in C'$.

output $A(a,b)$.

Theorem 5.7.3 (Willson et al. [114]). *There exists a 3.63-approximation for* MINW-DS *in a block B, with running time $n^{O(m^2)}$ where n is the number of nodes v such that* $\text{disk}_1(v) \cap B \neq \emptyset$.

Proof. First, let us estimate the time for computing $A(a,b)$. There are $O(2^{m^2})$ possible subsets of C, $n^{O(m^2)}$ possible choices of U and $O(n^{4m^2})$ possible combinations of $\{p, p', q, q'\}$ for all cells in $C - C'$. For each combination, computing $\text{Opt}(H_{\text{odd}})$, $\text{Opt}(H_{\text{even}})$, $\text{Opt}(Y_{\text{odd}})$ and $\text{Opt}(Y_{\text{even}})$ needs time $O(n^{4m+1})$. Therefore, total computation time is $n^{O(m^2)}$.

Next, estimate the performance ratio. Let Opt be an optimal solution for MINW-DS in the block B. Set $C' = \{e \in C \mid e \cap \text{Opt} \neq \emptyset\}$. For each $e \in C'$, choose a node $v_e \in \text{Opt} \cap e$. Set $U = \{v_e \mid e \in C'\}$ and $Z = V \cap (\cup_{v \in U} \text{disk}_1(v))$. Update $V(e)$ for all $e \in C - C'$ by $V(e) \leftarrow V(e) \setminus Z$. For each $e \in C - C'$, by Lemma 5.4.4, there exists a set $\{p, p', q, q'\}$ of at most four nodes in $V(e)$ such that $V_1(e) = V(e) \cap (\Delta_{\text{south}}(p,p') \cup$

$\Delta_{\text{north}}(q,q'))$ is dominated by $\text{Opt}_1(e) = \text{Opt}(e) \cap (S \cup N)$ and $V_2(e) = V(e) - V_1(e)$ is dominated by $\text{Opt}_2(e) = \text{Opt}(e) \cap (E \cup W)$ where

$$\text{Opt}(e) = \{v \in V - U \mid e \cap \text{disk}_1(v) \neq \emptyset\}.$$

Then that $\cup_{e \in (C-C_i) \cap H_{\text{odd}}} \text{Opt}_1(e)$ is a feasible solution for the minimization problem solved at Step 2.1. Therefore,

$$c(\text{Opt}(H_{\text{odd}})) \leq c(\cup_{e \in (C-C_i) \cap H_{\text{odd}}} \text{Opt}_1(e)). \tag{5.4}$$

Similarly,

$$c(\text{Opt}(H_{\text{even}})) \leq c(\cup_{e \in (C-C_i) \cap H_{\text{even}}} \text{Opt}_1(e)). \tag{5.5}$$

$$c(\text{Opt}(Y_{\text{odd}})) \leq c(\cup_{e \in (C-C_i) \cap Y_{\text{odd}}} \text{Opt}_2(e)). \tag{5.6}$$

$$c(\text{Opt}(H_{\text{odd}})) \leq c(\cup_{e \in (C-C_i) \cap Y_{\text{even}}} \text{Opt}_2(e)). \tag{5.7}$$

For every $v \in \text{Opt}$, define

$$\tau_1(v;a,b) = \begin{cases} 1 \text{ if disk}_1(v) \text{ intersets three horizontal strips,} \\ 2 \text{ otherwise,} \end{cases}$$

and

$$\tau_2(v;a,b) = \begin{cases} 1 \text{ if disk}_1(v) \text{ intersets three vertical strips,} \\ 2 \text{ otherwise.} \end{cases}$$

Then $\text{disk}_1(v)$ involves $\tau_1(v;a,b)$ of two equations (5.4) and (5.5), and $\tau_2(v;a,b)$ of two equations (5.6) and (5.7), Therefore

$$\begin{aligned} c(O) &\leq c(\cup_{e \in (C-C_i) \cap H_{\text{odd}}} \text{Opt}_1(e)) + c(\cup_{e \in (C-C_i) \cap H_{\text{even}}} \text{Opt}_1(e)) \\ &\quad + c(\cup_{e \in (C-C_i) \cap Y_{\text{odd}}} \text{Opt}_2(e)) + c(\cup_{e \in (C-C_i) \cap Y_{\text{even}}} \text{Opt}_2(e)) \\ &\leq \sum_{v \in \text{Opt}-U} c(v)(\tau_1(v;a,b) + \tau_2(v:a,b)). \end{aligned}$$

Hence,

$$\begin{aligned} c(A(a,b)) &\leq c(A(C',U)) = c(O^*) + c(U) \\ &\leq c(O) + c(U) \\ &\leq \sum_{v \in \text{Opt}} c(v)(\tau_1(v;a,b) + \tau_2(v:a,b)). \end{aligned}$$

For any $v \in \text{Opt}$, note that for any fixed b, there exist at least $\lfloor \frac{3\mu-2}{\mu/q} \rfloor$ values of a such that $\tau_1(v;a,b) = 1$, and for any fixed a, there exist at least $\lfloor \frac{3\mu-2}{\mu/q} \rfloor$ values of b such that $\tau_2(v;a,b) = 1$. Therefore,

$$\begin{aligned} A &\leq \frac{1}{q^2} \sum_{a=0}^{q-1} \sum_{b=0}^{q-1} \sum_{v \in \mathrm{Opt}} c(v)(\tau_1(v;a,b) + \tau_2(v;a,b)) \\ &\leq \left(4 - 2\frac{\lfloor (3-2\sqrt{2})q \rfloor}{q}\right) \cdot c(\mathrm{Opt}). \end{aligned}$$

As $q \to \infty$, $\frac{\lfloor (3-2\sqrt{2})q \rfloor}{q}$ goes to $3 - 2\sqrt{2}$. Since $4 - 2(3 - 2\sqrt{2}) < 3.63$, there exists a fixed q such that $(4 - 2\frac{\lfloor (3-2\sqrt{2})q \rfloor}{q}) < 3.63$. □