Chapter 3
CDS in Unit Disk Graph

Every dance is kind of fever chart, a graph of the heart.
MARTHA GRAHAM

3.1 Motivation and Overview

A unit disk is a disk with diameter one. Denote by $\text{disk}_r(o)$ the disk with center o and radius r. A graph $G = (V, E)$ is called a *unit disk graph* if it can be embedded into the Euclidean plane such that an edge between two nodes u and v exists if and only if $\text{disk}_{0.5}(u) \cap \text{disk}_{0.5}(v) \neq \emptyset$, that is, their Euclidean distance $d(u, v) \leq 1$. The unit disk graph is a mathematical model for wireless sensor networks when all sensors have the same communication radius.

For any node v of a unit disk graph G, the *neighborhood area* of v is the disk $\text{disk}_1(v)$. For any subset V' of nodes, the *neighborhood area* of V' is the union of disks, $\cup_{v \in V'} \text{disk}_1(v)$. For any subgraph H, the *neighborhood area* of H is the union of disks, $\cup_{v \in V(H)} \text{disk}_1(v)$ where $V(H)$ is the node set of subgraph H. Clearly, in a unit disk graph, two nodes u and v are independent if and only if $d(u, v) > 1$. For any two points u and v in the Euclidean plane, if $d(u, v) > 1$, then u and v are also said to be independent.

The boundary of an area Ω is denoted by $\partial\Omega$. Thus, $\partial\text{disk}_r(v) = \text{circle}_r(v)$, which is the circle with radius r and center v.

Clark, Colbourn, and Johnson [24] proved that MIN-CDS in unit disk graphs is still NP-hard. Wan et al. [104] first found that MIN-CDS has polynomial-time constant-approximations. Cheng et al. [22] designed the first PTAS. Since the running time of PTAS is a polynomial of a high degree, which is hard to implement, the design of fast polynomial-time constant-approximation is still an active research topic in the literature [14, 21, 51, 72, 75, 79, 104, 106]. There are many designs using the approach initiated by Wan et al. [104]: At the first stage, construct a maximal independent set. At the second stage, connect the maximal independent set into a CDS.

D.-Z. Du and P.-J. Wan, *Connected Dominating Set; Theory and Applications*, Springer Optimization and Its Applications 77, DOI 10.1007/978-1-4614-5242-3_3,

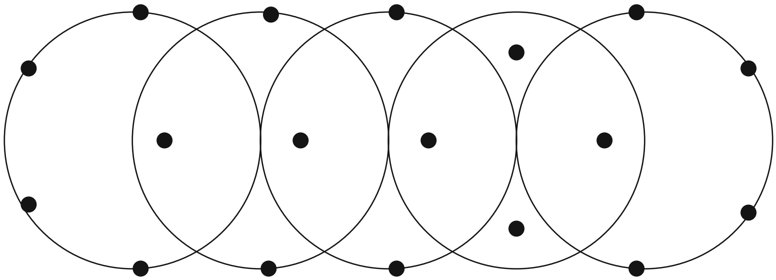

Fig. 3.1 The neighborhood of $n(\geq 3)$ linear points with consecutive distance one may contain $3n+3$ independent points

To analyze such two-stage algorithms, one needs to know what is the maximum size of a maximal independent set (i.e., the size of the maximum independent set) compared with the size of the minimum CDS. The size of the maximum independent set in a graph G is called the independent number, denoted by $\alpha(G)$. The size of the minimum CDS in G is called the connected dominating number, denoted by $\gamma_c(G)$. Wan et al. [106] indicated that there exist some connected unit disk graphs G such that

$$\alpha(G) = 3 \cdot \gamma_c(G) + 3$$

(Fig. 3.1). Many researchers believe that for every connected unit disk graph G

$$\alpha(G) \leq 3 \cdot \gamma_c(G) + 3. \tag{3.1}$$

Many efforts have been made to attack this upper bound. They can be classified into classes based on the difference of basic approaches.

One is based on the study of packing independent points in the neighborhood area of a small subgraph. Wan et al. [104] showed that the neighborhood area of any node can contain at most five independent points (Fig. 3.2) and based on this fact, they showed that for every connected unit disk graph G,

$$\alpha(G) \leq 4 \cdot \gamma_c(G) + 1.$$

Wu et al. [123] showed that the neighborhood area of any edge can contain at most eight independent points (Fig. 3.3), and with this fact, they showed that for every connected unit disk graph G

$$\alpha(G) \leq 3.8 \cdot \gamma_c(G) + 1.2.$$

Along this direction, Wan et al. [106] studied the neighborhood area of a star and proved that for every connected unit disk graphs with at least two nodes

$$\alpha(G) \leq 3\frac{2}{3} \cdot \gamma_c(G) + 1.$$

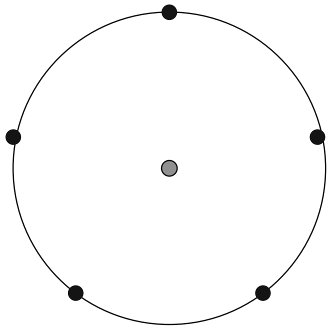

Fig. 3.2 A disk with radius one can contain at most five independent points

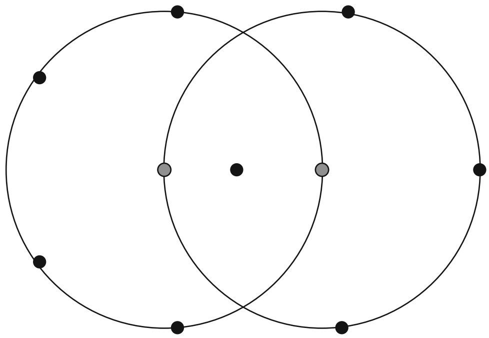

Fig. 3.3 The union of two disks $\text{disk}_1(u) \cap \text{disk}_1(v)$ with $d(u,v) \leq 1$ can contain at most eight independent points

Vahdatpour et al. [103] claimed that they proved (3.1). However, their proof is far from a complete one. An analysis on their proof will be given in Sect. 3.4.

Another approach is to study the total area taken by unattached unit disks in the union of disks of radius 1.5 and with centers at nodes in a CDS. With this approach and Voronoi division, Funke et al. [51] showed that for every connected unit disk graph G

$$\alpha(G) \leq 3.453 \cdot \gamma_c(G) + 8.291.$$

However, in their proof, a key geometric extreme property was used without proof. Therefore, some researchers could not accept this result. Gao et al. [54] gave a detail proof of the geometric extreme property. Li et al. [72] improved approach of Funke et al. and showed that for every connected unit disk graph G,

$$\alpha(G) \leq 3.4306 \cdot \gamma_c(G) + 4.8185.$$

This is the best-known bound so far.

3.2 NP-Hardness and PTAS

In this section, we give a new proof of NP-hardness and a new construction of PTAS for MIN-CDS in unit disk graphs.

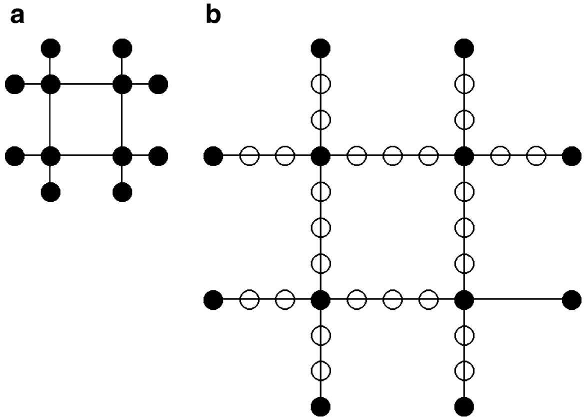

Fig. 3.4 (**a**) A planar graph. (**b**) The constructed graph. The *dark circled* points are candidates of Steiner points and the *light circled* points are terminals

Theorem 3.2.1 (Clark et al. [24]). MIN-CDS *in unit disk graphs is still NP-hard.*

Proof. The following NP-complete problem can be found in [56, 57].

> PLANAR-4-CVC: Given a planar graph $G = (V, E)$ with all vertices of degree at most 4, and a positive integer $k \leq |V|$, determine whether there is a *connected vertex cover* of size k, that is, a subset $V' \subseteq V$ with $|V'| = k$ such that for each edge $\{u, v\} \in E$ at least one of u and v belongs to V' and the subgraph induced by V' is connected.

Consider a graph $G = (V, E)$ and a positive integer k, which is an instance of this problem. We construct a unit disk graph as follows. First, note that we can embed G into the plane so that all edges consist of horizontal and vertical segments of lengths being an integer at least 4, so that every two edges meet at an angle of 90° or 180°. Add new vertices on the interior of each edge in G to divide the edge into a path of many edges, each of length exactly one. Denote by W the set of all such new vertices. (See Fig. 3.4. New vertices are light circled points.)

Now, consider a horizontal path $(u, w_1, \ldots, w_h, v)$ obtained from an edge (u, v). Choose a constant $0 < c < 0.5(\sqrt{2} - 1)$. For each new vertex w_i, add another new vertex w_i' such that $d(w_i, w_i') = c$ and w_i' is above w_i if i is odd and below w_i if i is even (Fig. 3.5). This placement of w_i' implies that w_i' can connect to only w_i. Similarly, we can deal with path obtained from vertical edges. Denote by G' the constructed graph. Then every CDS C of G' must contain w_i. In fact, in order to dominate w_i', C must contain either w_i or w_i'. If C contains w_i', then w_i' has to connect other vertices in C through w_i. Therefore, we must have w_i in C.

Now, it is easy to see the following facts:

1. W is a dominating set of G'.
2. C is a connected vertex-cover of G if and only if $C \cup W$ is a CDS of G'.

Therefore, G has a connected vertex-cover of size at most k if and only if G' has a CDS of size at most $|W| + k$. □

Next, we give a new construction of a PTAS for MIN-CDS in unit disk graphs.

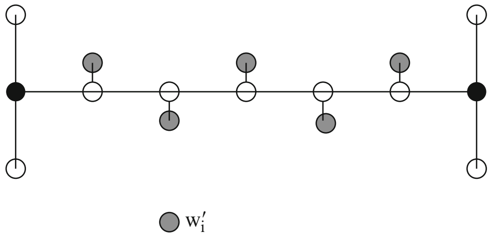

Fig. 3.5 Add w_i''s

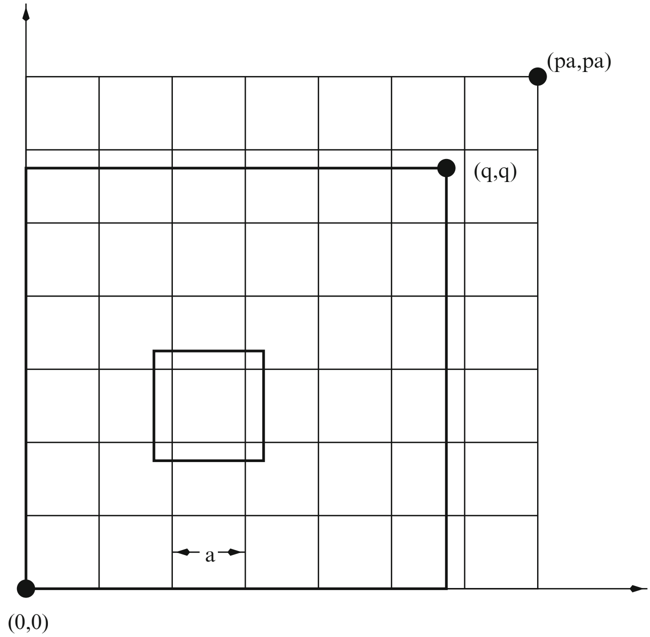

Fig. 3.6 Grid $P(0)$

Initially, we put input connected unit disk graph $G = (V,E)$ in the interior of the square $[0,q] \times [0,q]$ and construct a grid $P(0)$ as shown in Fig. 3.6. $P(0)$ divides the square $[0,pa] \times [0,pa]$ into p^2 cells where $a = 8k$ for a positive integer k and $p = 1 + \lceil q/a \rceil$. Each cell e is a $a \times a$ square, including its left boundary and its lower boundary, so that all cells are disjoint and their union covers the interior of the square $[0,q] \times [0,q]$.

For each cell e, let $C(e)$ be the closed area bounded by the $(a+4) \times (a+4)$ square with the same center as that of e, called the *central area* of cell e. Let $CB(e)$ be the interior of the $(a+8) \times (a+8)$ square with the same center as that of e. Denote by $B(e)$ obtained from $CB(e)$ by removing e, called the *boundary area* of cell e (Fig. 3.7).

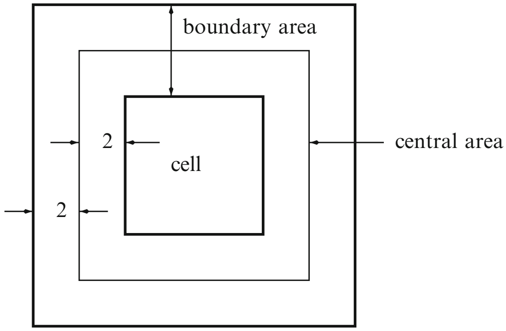

Fig. 3.7 Central area $C(e)$ and Boundary area $B(e)$

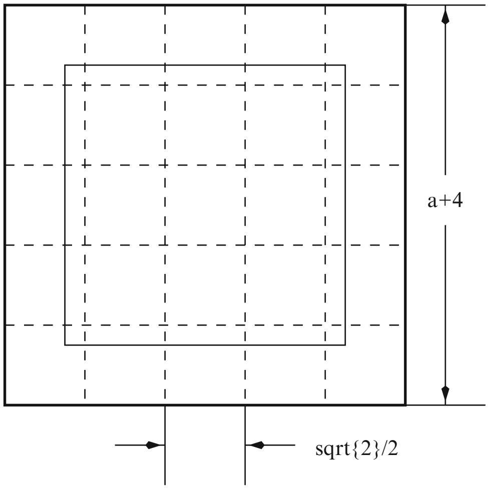

Fig. 3.8 Partition of central area $C(e)$

For each cell e, we study the following problem.

LOCAL(e): Find the minimum subset D of vertices in $V \cap CB(e)$ such that (a) D dominates all nodes in $V \cap C(e)$, and (b) for any connected component F of subgraph $G[V \cap C(e)]$, $G[D]$ contains a connected component dominating F.

Lemma 3.2.2. *The minimum solution of* LOCAL(e) *problem can be computed in time* $n_e^{O(a^2)}$ *where* $n_e = |V \cap CB(e)|$.

Proof. Partition $C(e)$ into $\lceil (a+4)\sqrt{2} \rceil^2$ small squares with edge length at most $\sqrt{2}/2$ (Fig. 3.8). Then for each closed small square s, if $V \cap s \neq \emptyset$, then choose one from $V \cap s$, which would dominate all vertices in $V \cap s$. All chosen vertices form a set D dominating $V \cap C(e)$ and $|D| \leq \lceil (a+4)\sqrt{2} \rceil^2$.

Now, consider each connected component F of $G[V \cap C(e)]$. If D does not contain a connected component dominating F, then we may add at most $2|F \cap D|$ vertices to connect all vertices $F \cap D$ into one connected component. This means that LOCAL(e) has a feasible solution of size at most $3\lceil (a+4)\sqrt{2} \rceil^2$. Therefore, we can find the optimal solution for LOCAL(e) in time $n_e^{3\lceil (a+4)\sqrt{2} \rceil^2} = n_e^{O(a^2)}$. □

Let D_e denote the minimum solution for LOCAL(e). Define $D(0) = \cup_{e \in P(0)} D_e$ where $e \in P(0)$ means that e is over all cells in partition $P(0)$.

Lemma 3.2.3. *$D(0)$ contains a CDS for G, which can be computed in time $n^{O(a^2)}$.*

Proof. Consider two adjacent cells e and e'. Let F be a connected component of $G[V \cap C(e)]$ dominated by a connected component D_e^F of $G[D \cap CB(e)]$. Let F' be a connected component of $G[V \cap C(e')]$ dominated by a connected component $D_e^{F'}$ of $G[D \cap CB(e')]$. Suppose $F \cup F'$ is connected. We claim that $G[D_e^F \cup D_e^{F'}]$ is also connected.

To show the claim, we first note that $C(e) \cap C(e')$ is a strip with width 4. Since $F \cup F'$ is connected, there is a vertex x of $F \cup F'$ in $C(e) \cap C(e')$. x must belong to $F \cap F'$. Let $y \in D_e^F$ dominate x and $y' \in D_e^{F'}$ dominate x. We next consider two cases.

Case 1. $y \in C(e')$ or $y' \in C(e)$. If $y \in C(e')$, then y must belong to F' and hence y is dominated by $D_e^{F'}$. Thus, $G[D_e^F \cup D_e^{F'}]$ is connected.

Case 2. $y \notin C(e')$ and $y' \notin C(e)$. In this case, path (y, x, y') passes through $C(e) \cap C(e')$. However, $d(y, y') \leq 2$. Hence, it is impossible for this case to occur.

The truth of our claim implies that for any connected component F of $G[C(e) \cup C(e')]$, $D_e \cup D_{e'}$ has a connected component dominating F. Putting cells together into a horizontal trip and then putting all horizontal strips together into $P(0)$, we would obtain a property of $D(0)$ that for every connected component of G, $D(0)$ has a connected component to dominate it. Since G is connected, $D(0)$ contains a CDS.

Note that each vertex may appear in $CB(e)$ for at most four cells e. Therefore, by Lemma 3.2.2, $D(0)$ can be computed in time

$$\sum_{e \in P(0)} n_e^{O(a^2)} \leq (4n)^{O(a^2)} = n^{O(a^2)},$$

where $n = |V|$. □

To estimate $|D(0)|$, we consider a minimum solution D^* of MIN-CDS. Let $PB(0) = \cup_{e \in P(0)} B(e)$.

Lemma 3.2.4. *Let $PB(0) = \cup_{e \in P(0)} B(e)$. Then $|D(0)| \leq |D^*| + 24|D^* \cap PB(0)|$.*

Proof. For each cell e, we modify $D^* \cap CB(e)$ into a feasible solution of LOCAL(e) as follows. Consider a connected component F of $G[V \cap C(e)]$. Suppose F is dominated by k connected components $C_1, C_2, \ldots, C_k$ of $G[D^* \cap CB(e)]$ $(k \geq 2)$ and they are connected outside of $CB(e)$. Then every C_i has a vertex lying in $CB(e) - C(e)$. Since F is connected, there exist C_i and C_j $(i \neq j)$ such that C_i and C_j can be connected together by adding two new vertices. We can charge these two vertices to the one vertex of C_i, lying in $CB(e) - C(e)$. Moreover, each vertex x in $D^* \cap (CB(e) - C(e))$ can dominate at most three connected components of $G[V \cap C(e)]$ (this is because each connected component F_i contributes a vertex u_i in a half disk with center at x and radius one, and $d(u_i, u_j) > 1$ for $i \neq j$, which

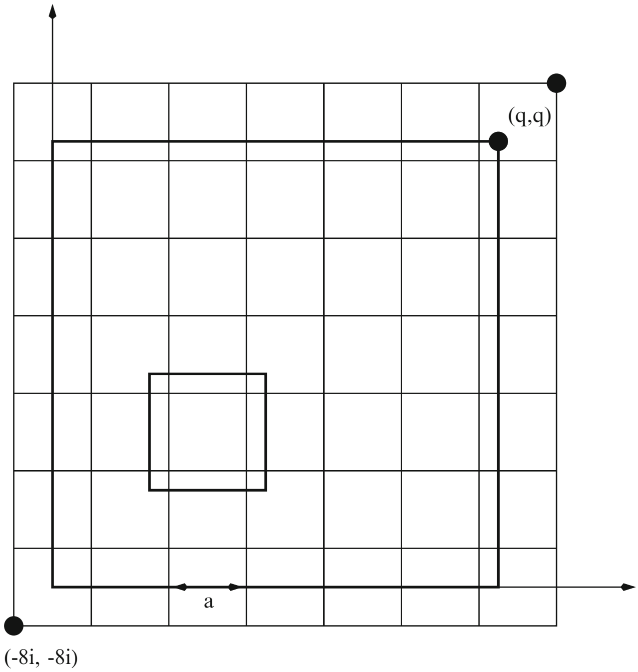

Fig. 3.9 Grid $P(i)$

implies that $\angle u_i x u_j > 60°$). Hence x can be charged at most six times. Let D'_e be obtained from $D^* \cap CB(e)$ by above modification. Then

$$\begin{aligned}|D'_e| &\le |D^* \cap CB(e)| + 6|D^* \cap (CB(e) - C(e))| \\ &\le |D^* \cap e| + 6|D^* \cap B(e)|.\end{aligned}$$

Now, note that each vertex can appear in $B(e)$ for at most four cells e and $|D_e| \le |D'_e|$ where D_e is a minimum solution of LOCAL(e). Thus, we have

$$\begin{aligned}|D(0)| &\le \sum_{e \in P(0)} |D'_e| \\ &\le \sum_{e \in P(0)} (|D^* \cap e| + 6|D^* \cap B(e)|) \\ &\le |D^*| + 24|D^* \cap PB(0)|.\end{aligned}$$ □

Now, we shift partition $P(0)$ to $P(i)$ as shown in Fig. 3.9 such that the left and lower corner of the grid is moved to point $(-8i, -8i)$. For each $P(i)$, we can compute a feasible solution $D(i)$ in the same way as computing $D(0)$ for $P(0)$. Then we have

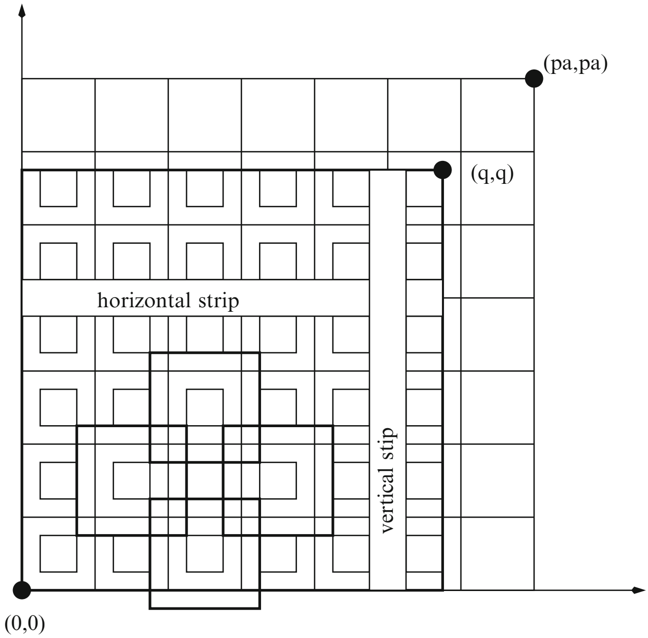

Fig. 3.10 Horizontal and vertical strips

(a) $D(i)$ is a CDS.
(b) $D(i)$ can be computed in time $n^{O(a^2)}$.
(c) $|D(i)| \leq |D^*| + 24|D^* \cap PB(i)|$.

From (c), we can obtain the following

Lemma 3.2.5. *For* $k = a/8$, $|D(0) + |D(1)| + \cdots + |D(k-1)| \leq (k+48)|D^*|$.

Proof. Note that $PB(i)$ consists of a group of horizontal strips and a group of vertical strips (Fig. 3.10). All horizontal strips in $PB(0) \cup PB(1) \cup \cdots \cup PB(k-1)$ are disjoint and all vertical strips in $PB(0) \cup PB(1) \cup \cdots \cup PB(k-1)$ are also disjoint. Therefore,

$$\sum_{i=0}^{k-1} |D^* \cap PB(i)| \leq 2|D^*|.$$

Hence,

$$\sum_{i=0}^{k-1} |D(i)| \leq (k+48)|D^*|.$$ □

Set $k = \lceil 1/(8\varepsilon) \rceil$ and run the following algorithm.

Algorithm PTAS
input a unit disk graph G;
  Compute $D(0), D(1), \ldots, D(k-1)$;
  Choose i^*, $0 \le i^* \le k-1$ such that
    $|D(i^*)| = \min(|D(0)|, |D(1)|, \ldots, |D(k-1)|)$;
output $D(i^*)$.

Theorem 3.2.6 (Cheng et al. [22]). *Algorithm PTAS produces an approximation solution for* MIN-CDS *with size*

$$|D(i^*)| \le (1+\varepsilon)|D^*|$$

and runs in time $n^{O(1/\varepsilon^2)}$.

Proof. It follows from Lemmas 3.2.3 and 3.2.5. □

3.3 Two-Stage Algorithm

Although MIN-CDS in unit disk graph has a PTAS, the running time is a polynomial with very high degree and hence not able to be implemented for a real world problem. Therefore, one still wants to find faster approximations with small constant performance ratio. So far, all approximation algorithms of this type are designed in the same manner: First, construct a maximal independent set and then connected it into a CDS. Here, one notes that every maximal independent set is a dominating set.

To save the spending at the second stage, one usually constructs a maximal independent set in the following way:

Algorithm MIS
input a connected graph;
Color a node in black, its neighbors in grey and all other nodes in white;
while a white node exists **do**
    choose a white node x with a grey neighbor and
    color x in black and its white neighbors in grey;
output the set of black nodes.

The maximal independent set constructed as above has the following property.

Lemma 3.3.1 (AoA Property). *Every subset of the maximal independent set constructed as above is within distance two from its complement.*

In the second stage, consider the constructed maximal independent set as a set of terminals and then find the minimum number of Steiner nodes (added nodes) to interconnect all terminal. This means to solve the following problem.

ST-MSP-IN-UDG: Given a unit disk graph $G = (V,E)$ and a node subset $P \subseteq V$ with AoA Property, find a node subset S with the minimum cardinality, such that $G[P \cup S]$ is connected. (Nodes in P are called *terminals* while nodes in S are called *Steiner nodes*.)

This is an NP-hard problem with many approximation solutions. Any one of them can play the role in the second stage. The following is an example, a greedy approximation.

For any subset C of nodes, let $p(C)$ denote the number of connected components of $G[C]$. Denote $\Delta_x p(C) = p(C \cup \{x\}) - p(C)$. Suppose a maximal independent set D with AoA property is already constructed.

Greedy Connection
input a dominating set D;
$C \leftarrow D$;
while $p(C) \geq 2$ **do**
    choose a node x to maximize $-\Delta_x p(C)$ and
    $C \leftarrow C \cup \{x\}$;
output C.

The following theorem states the performance of this approximation.

Theorem 3.3.2 (Zou et al. [132]). *Suppose G is a graph with $\alpha(G) \leq a \cdot \gamma_c(G) + b$ and D is a maximal independent set with AoA property. Then the CDS produced by* ***Greedy Connection*** *has size at most*

$$(a + 2 + \ln(a-1))\gamma_c(G) + b + \lfloor b \rfloor - 1.$$

Proof. Suppose $x_1, \ldots, x_g$ are selected in turn by the greedy algorithm. Let $\{y_1, \ldots, y_{\gamma_c(G)}\}$ be a minimum CDS and for any i, $\{y_1, \ldots, y_i\}$ induces a connected subgraph. Denote $C_0 = D$, $C_{i+1} = C_i \cup \{x_{i+1}\}$ and $C_j^* = \{y_1, \ldots, y_j\}$. Then

$$-\Delta_{y_j} p(C_i \cup C_{j-1}^*) + \Delta_{y_j} p(C_i) \leq 1.$$

So, $-\Delta_{x_{i+1}} p(C_i) \geq -\Delta_{y_j} p(C_i)$ for all $1 \leq j \leq \gamma_c(G)$. Thus,

$$\begin{aligned} -\Delta_{x_{i+1}} p(C_i) &\geq \frac{-\sum_{j=1}^{\gamma_c(G)} \Delta_{y_j} p(C_i)}{\gamma_c(G)} \\ &\geq \frac{-\gamma_c(G) + 1 - \sum_{j=1}^{\gamma_c(G)} \Delta_{y_j} p(C_i \cup C_{j-1}^*)}{\gamma_c(G)} \\ &= \frac{-\gamma_c(G) + 1 - p(C_i \cup C^*) + p(C_i)}{\gamma_c(G)} \\ &= \frac{\gamma_c(G) + p(C_i)}{\gamma_c(G)}, \end{aligned}$$

that is,

$$-p(C_{i+1}) \geq -p(C_i) + \frac{-\gamma_c(G) + p(C_i)}{\gamma_c(G)}.$$

Denote $a_i = -\gamma_c(G) - b + p(C_i)$. Then

$$a_{i+1} \leq a_i \left(1 - \frac{1}{\gamma_c(G)}\right).$$

Thus,

$$a_i \leq a_0 \left(1 - \frac{1}{\gamma_c(G)}\right)^i \leq a_0 \mathrm{e}^{-\mathrm{i}/\gamma_c(G)}.$$

First, assume the existence of i, $0 \leq i < g$ such that

$$a_{i+1} < \gamma_c(G) \leq a_i.$$

Then $g \leq i + 2\gamma_c(G) - 1 + \lfloor b \rfloor$ and

$$\gamma_c(G) \leq a_0 e^{-i/\gamma_c(G)}.$$

Hence,

$$i \leq \gamma_c(G) \ln(a_0/\gamma_c(G)).$$

Moreover,

$$a_0/\gamma_c(G) = (-\gamma_c(G) - b + |D|)/\gamma_c(G) \leq a - 1.$$

Therefore,

$$|D| + g \leq (a + 2 + \ln(a-1))\gamma_c(G) + b + \lfloor b \rfloor - 1.$$

Now, consider the case that there is no i such that $a_{i+1} < \gamma_c(G) \leq a_i$. Note that $a_g = -\gamma_c(G) - b + 1 < \gamma_c(G)$. Thus, it must have $a_0 < \gamma_c(G)$. This implies that $g \leq 2\gamma_c(G) - 2 + \lfloor b \rfloor$. Thus,

$$|D| + g \leq (a+2)\gamma_c(G) + b + \lfloor b \rfloor - 2.$$ □

There is a better analysis found by Wan et al. [106]. They found some geometric properties of this approximation and gave a better performance ratio by taking this advantage.

The best-known approximation for ST-MSP-IN-UDG is given by Min et al. [79] as follows.

Algorithm MHHW:
input a maximal independent set with AoA property.

Color all its nodes in black and others in gray. In the following, we will change some gray nodes to black in certain rules. A *black component* is a connected component of the subgraph induced by black nodes.

Stage 1 **while** there exists a grey node adjacent to at least three
black components **do**
change its color from gray to black;
end-while;
Stage 2 **while** there exists a grey node adjacent to at least two
black components **do**
change its color from gray to black;
end-while;
output all black nodes.

They showed the following.

Theorem 3.3.3 (Min et al. [79]). *In Algorithm MHHW, the number of gray nodes changed their color to black is at most $3 \cdot \gamma_c(G)$ where G is input unit disk graph.*

3.4 Independent Number (I)

Two points u and v are *independent* if $d(u,v) > 1$. To establish the upper bound of independent number $\alpha(G)$ for unit disk graphs G, one way is to study packing independent points in the neighborhood area of the minimum CDS.

The following result is first proved by Wan et al. [104].

Lemma 3.4.1 (Wan et al. [104]). *A disk D with radius one can contain at most five independent points.*

Proof. Let o be the center of D. Suppose $u_1, u_2, \ldots, u_k$ are all independent points in D, in counterclockwise ordering. Then we must have $\angle u_1 x u_2 > 60°$, $\angle u_2 x u_3 > 60°, \ldots, \angle u_k x u_1 > 60°$ since $d(x,u_i) \leq 1$ and $d(u_1,u_2) > 1, d(u_2,u_3) > 1, \ldots, d(u_k,u_1) > 1$. Therefore, $k \cdot 60° < 360°$. Hence, $k \leq 5$. □

With Lemma 3.4.1, Wan et al. [104] proved the following

Theorem 3.4.2 (Wan et al. [104]). *Let $\alpha(G)$ and $\gamma_c(G)$ be the independent number and the connected dominating number of unit disk graph G, respectively. Then*

$$\alpha(G) \leq 4 \cdot \gamma_c(G) + 1.$$

Proof. The proof is by induction on $\gamma_c(G)$. If $\gamma_c(G) = 1$, then the inequality follows immediately from Lemma 3.4.1. In general, suppose $\gamma_c(G) = n > 1$. Choose a node x in the minimum CDS C such that $C - \{x\}$ is still connected. This can be done by choosing x as a leaf of spanning tree of $G[C]$. By induction hypothesis, there are at most $4(n-1)+1$ independent points lying in $\mathcal{A} = \cup_{y \in C - \{x\}} \text{disk}_1(y)$.

Let $z \in C - \{x\}$ be adjacent to x. Suppose $w_1, \ldots, w_k$ are independent points in $\text{disk}_1(x) \setminus \mathcal{A}$. Note that every point in $\text{disk}_1(x) \setminus \mathcal{A}$ is independent from z. Thus, $z, w_1, \ldots, w_k$ are independent in $\text{disk}_1(x)$. By Lemma 3.4.1, $k \leq 4$. Therefore, there exist at most $4(n-1)+1+4 = 4n+1$ independent points lying in $\mathcal{A} \cup \text{disk}_1(x)$. □

Wu et al. [123] showed a result on packing independent points in two disks.

Lemma 3.4.3 (Wu et al. [123]). *Let u and v be two points with distance at most one. Then $\text{disk}_1(u) \cup \text{disk}_1(v)$ can contain at most eight independent points.*

Proof. For contradiction, suppose there exists an independent set I of at least nine points lying in $\text{disk}_1(u) \cup \text{disk}_1(v)$. One claims that the intersection $A = \text{disk}_1(u) \cap \text{disk}_1(v)$ contains at most one point in I.

Indeed, suppose A contains k vertices in I. By Lemma 3.4.1, $\text{disk}_1(u) - A$ contains at most $5-k$ points in I and $\text{disk}_1(v) - A$ contains at most $5-k$ points in I. Thus, $\text{disk}_1(u) \cup \text{disk}_1(v)$ contains at most $10-k$ point in I. Hence, $10-k \geq 9$, that is, $k \leq 1$.

In Lemma 3.4.4, one shows that $\text{disk}_1(u) \cup \text{disk}_1(v) - A$ contains at most seven independent points. Therefore, $\text{disk}_1(u) \cup \text{disk}_1(v)$ contains at most eight independent points, a contradiction. □

Lemma 3.4.4. *Let u and v be two points with distance at most one. Then $\text{disk}_1(u) \triangle \text{disk}_1(v)$ can contain at most seven independent points where*

$$\text{disk}_1(u) \triangle \text{disk}_1(v) = (\text{disk}_1(u) \setminus \text{disk}_1(v)) \cup (\text{disk}_1(v) \setminus \text{disk}_1(u)).$$

Proof. By Lemma 3.4.1, $\text{disk}_1(u) \setminus \text{disk}_1(v)$ contains at most four independent points and $\text{disk}_1(v) \setminus \text{disk}_1(u)$ contains at most four independent points. For contraction, suppose $\text{disk}_1(u) \triangle \text{disk}_1(v)$ contains eight independent points. Then $\text{disk}_1(u) \setminus \text{disk}_1(v)$ contains exactly four independent points a_1, a_2, a_3, a_4 and $\text{disk}_1(v) \setminus \text{disk}_1(u)$ contains exactly four independent points a_5, a_6, a_7, a_8. Assume $a_1, \ldots, a_4$ lie counter-clockwisely in $\text{disk}_1(u)$ and $a_5, \ldots, a_8$ lie counter-clockwisely in $\text{disk}_1(v)$. Denote by ub_i the radius passing through a_i for $i = 2, \ldots, 4$ and by vb_i the radius passing through a_i for $i = 5, \ldots, 8$. Using arcs with radius one, draw four arc-triangles ub_2c_2, ub_3c_3, vb_6c_6, and vb_7c_7 as shown in Fig. 3.11. Their boundaries intersect the boundary of $\text{disk}_1(u) \cap \text{disk}_1(v)$ at d_2, d_3, d_6, d_7, respectively. Note that none of a_1, a_4, a_5, a_8 can lie in the four arc-triangles ub_2c_2, ub_3c_3, vb_6c_6, and vb_7c_7. Therefore, a_1, a_4, a_5, a_8 must lie in the four small dark areas xc_2d_2, yc_3d_3, yc_6d_6 and xc_7d_7, respectively, as shown in Fig. 3.11.

Next, one find a contradiction by proving the fact that there exist two small dark areas too close to contain two independent vertices.

To show this fact, note that $\angle b_2ub_3 > 60°$ and $\angle c_2ub_2 = \angle b_3uc_3 = 60°$. Hence, $\angle c_2uc_3 > 180°$ and $\angle c_3uc_2 < 180°$ (note that $\angle c_3uc_2$ is the one obtained by moving c_3u counterclockwisely to c_2u). Similarly, $\angle c_7vc_6 < 180°$. Therefore $\angle uc_2c_7 + \angle c_2c_7v + \angle vc_6c_3 + \angle c_6c_3u > 360°$. This means that either $\angle uc_2c_7 + \angle c_2c_7v > 180°$

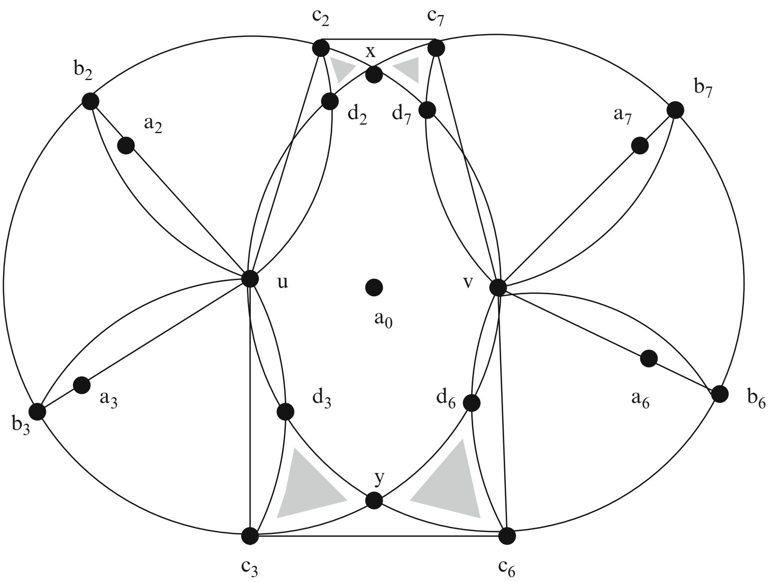

Fig. 3.11 Four small *dark* areas

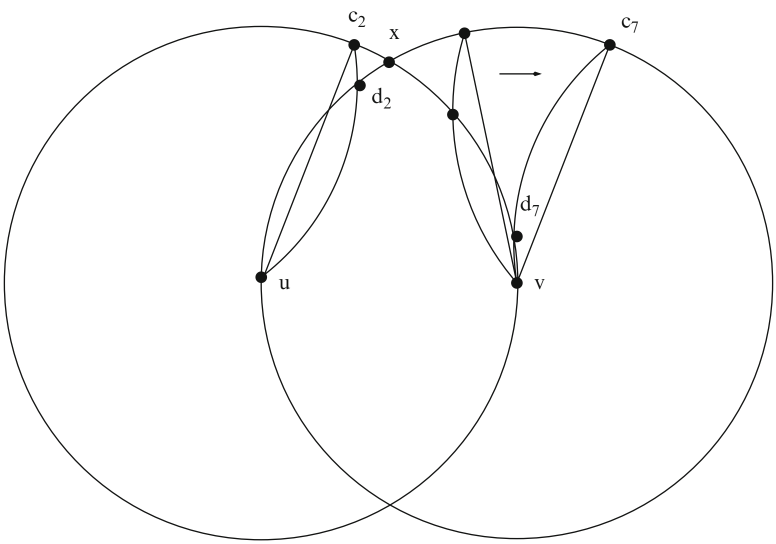

Fig. 3.12 Turn unit arc-triangle vb_7c_7 until $vc_7 \parallel uc_2$

or $\angle vc_6c_3 + \angle c_6c_3u > 180°$. Without loss of generality, assume the former occurs (Fig. 3.12). Next, it will be showed that dark areas xc_2d_2 and xc_7d_7 cannot contain two independent points.

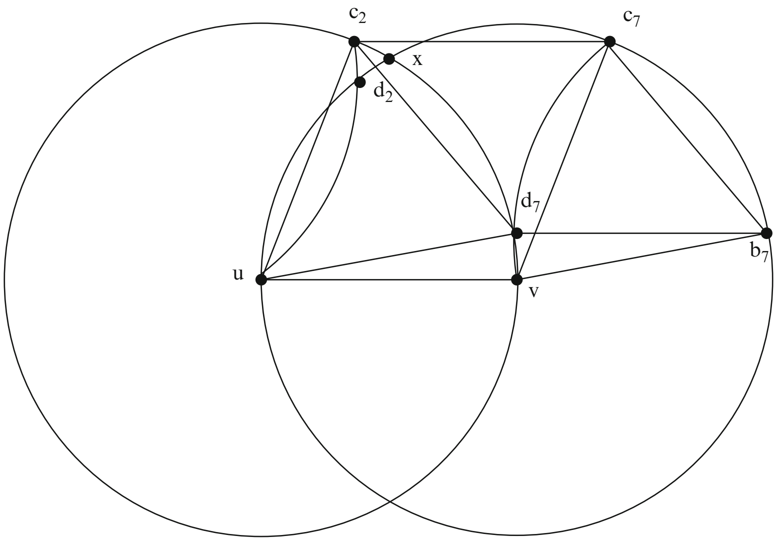

Fig. 3.13 Move u until $|uv| = 1$

To do so, at the first, the area xc_7d_7 is enlarged by turning the arc-triangle vb_7c_7 around v until vc_7 is parallel to uc_2. At this limit position, quadrilateral c_2uvc_7 becomes a parallelogram so that $|c_2c_7| = |uv| \leq 1$. It follows that the distance between two points in areas xc_2d_2 and xc_7d_7 cannot exceed $\max(|c_2c_7|, |c_2d_7|, |d_2c_7|, |d_2d_7|)$. Moreover, it can be proved that $|d_2d_7| \leq \max(|c_2d_7|, |d_2c_7|)$. In fact, note that $\angle c_7d_7d_2 + \angle d_7d_2c_2 > 360°$. Thus, either $\angle c_7d_7d_2 > 90°$ or $\angle d_7d_2c_2 > 90°$. Therefore, either $|d_2c_7 > |d_2d_7|$ or $c_2d_7 > |d_2d_7|$.

Now, to complete the proof of claimed fact, it suffices to prove that $|c_2d_7| \leq 1$ and $|d_2c_7| \leq 1$. To see $|c_2d_7| \leq 1$, make $|c_2d_7|$ longer by moving v away from u until $|uv| = 1$ (Fig. 3.13). At this limit position, $|uv| = |vb_7| = |b_7d_7| = |d_7u| = 1$. Therefore, uvb_7d_7 is a parallelogram. It follows that $|d_7b_7| = |c_2c_7| = 1$ and d_7b_7 is parallel to uv and hence parallel to c_2c_7. Thus, $c_2d_7b_7c_7$ is a parallelogram. Therefore, $|c_2d_7| = |c_7b_7| = 1$. Similarly, one can show $|d_2c_7| \leq 1$. □

With Lemma 3.4.3, Wu et al. [123] established the following

Theorem 3.4.5 (Wu et al. [123]). *Let $\alpha(G)$ and $\gamma_c(G)$ be the independent number and the connected dominating number of unit disk graph G, respectively. Then*

$$\alpha(G) \leq 3.8 \cdot \gamma_c(G) + 1.2.$$

Proof. First, one proves the following two lemmas about unit disk graph and general graphs.

Lemma 3.4.6. *In any unit disk graph, there exists a minimum spanning tree such that every vertex has degree at most five.*

Proof. First, note that in any minimum spanning tree, each vertex u has degree at most six. In fact, for contradiction, suppose u has degree more than six. Then there are two edges uv and uv' such that $\angle vuv' < 60°$. It follows that $|vv'| < |uv|$ or $|vv'| < |uv'|$. Replacing uv (in the former case) or uv' (in the latter case) by vv' would result in a shorter spanning tree, a contradiction. A similar argument can also proved that if a vertex u has degree six, then all edges at u have the same length and all angles at u equal $60°$.

Suppose T is a minimum spanning tree with the minimum number of vertices with degree six. For contradiction, suppose T has a vertex u with degree six. Then, every angle at u equals $60°$ and all edges incident to u have the same length. Consider a vertex v adjacent to u. Then u has two edges uw and ux such that $\angle wuv = \angle vux = 60°$ and $|uv| = |uw| = |ux|$. Thus, $|vw| = |uw|$ and $|vx| = |ux|$. Replacing uw and ux by vw and vx results in still a minimum spanning tree. But, v gets two more edges. This means that v has degree at most four in T. Thus, replacing uv by vw in T would result in a minimum spanning tree in which u has degree five and v has degree at most five, so that the number of vertices with degree six is reduced by one, a contradiction. □

Lemma 3.4.7. *Every tree T with at least three vertices has a non-leaf vertex adjacent to at most one non-leaf vertex.*

Proof. Let T' be the subtree obtained from T by removal of all leaves. Since T has at least three vertices, T' contains at least one vertex. If T' contains only one vertex, then it meets our requirement. If T' contains more than one vertex, then every leaf of T' is a non-leaf vertex of T satisfying the condition stated in the lemma. □

Now, it is ready to prove Theorem 3.4.5. Let H be a subgraph induced by a minimum CDS in the given unit disk graph G. Then H is also a unit disk subgraph. By Lemma 3.4.6, H has a minimum spanning tree T such that every vertex has degree at most five. Let $|T|$ denote the number of vertices in T. It will be proved by induction on $|T|$ that there exist at most $3.8|T| + 1.2$ independent vertices in the neighbor area of T. For $|T| = 1$ or 2, this is true by Lemmas 3.4.1 and 3.4.3. Next, assume $|T| \geq 3$. By Lemma 3.4.7, T contains a non-leaf vertex v adjacent to at most one non-leaf vertex. Let u be the non-leaf neighbor of v if it exists, or a leaf neighbor of v, otherwise. Let $x_1, \ldots, x_k$ $(k \leq 4)$ be other neighbors of v. Note that by Lemma 3.4.1, each $\text{disk}_1(x_i)$ for $1 \leq i \leq k-1$ contains at most four independent points which are also independent from v, and by Lemma 3.4.3, $\text{disk}_1(v) \cup \text{disk}_1(x_k)$ contains at most seven independent points which are also independent from u. Moreover, by the induction hypothesis, the neighbor area of $T - \{v, x_1, .., x_k\}$ contains at most $3.8(|T| - k - 1) + 1.2$ independent vertices. Therefore, the neighbor area of T contains at most

$$3.8(|T| - k - 1) + 1.2 + 7 + 4(k-1) = 3.8|T| + 1.2 + 0.2(k-4) \leq 3.8|T| + 1.2$$

independent vertices. Note that $|T| = mcds$. This completes the proof of Theorem 3.4.5. □

Wan, Wang and Yao [106] found an idea to prove a better bound based on the study on packing independent points in the neighborhood area of a star.

First, they note that every tree can be partitioned into nontrivial stars. (A star is trivial if it contains only one node.)

Lemma 3.4.8. *For any tree T, its node set has a partition $V(T) = (V_1, \ldots, V_k)$ such that for every part V_i, $T[V_i]$ is a star with at least two nodes.*

Proof. Choose any node r of T as a root and consider T as a rooted tree. Then one can compute such a partition as follows.

$V \leftarrow V(T)$;
$i \leftarrow 0$;
while $V \neq \emptyset$ **do begin**
  $i \leftarrow i+1$;
  choose a leaf u at lowest level and find its parent node v;
  let V' be the set of v and its all children;
  if $|V - V'| > 1$
   then $V_i \leftarrow V'$
   else $V_i \leftarrow V$
  $V \leftarrow V - V_i$;
end-while
output $(V_1, \ldots, V_i)$.

□

Wan et al. [106] then found tight upper bound for the number of independent points lying in the neighborhood area of a star.

Lemma 3.4.9. *The neighborhood area of a star with n nodes can contain at most ϕ_n independent points where*

$$\phi_n = \begin{cases} 3n+2 & \text{if } n \le 2, \\ 3n+3 & \text{if } n \le 5, \\ 21 & \text{if } 6 \le n. \end{cases}$$

Consider a star S. Let o be the center of S. Then the neighborhood area of S is contained in $\text{disk}_2(o)$. By Zassenhaus–Groemer–Oler inequality (in Section 3.6), $\text{disk}_2(o)$ can contain at most 21 independent points. This means that Lemma 3.4.9 holds for $n \ge 6$. By Lemmas 3.4.1 and 3.4.3, Lemma 3.4.9 holds for $n \le 2$. For $n = 3$, suppose $V(S) = \{v_1, v_2, v_3\}$. By Lemma 3.4.3, $\text{disk}_1(v_1) \cup \text{disk}_1(v_2)$ can contain at most eight independent points and by Lemma 3.4.1, $\text{disk}_1(v_3) \setminus (\text{disk}_1(v_1) \cup \text{disk}_1(v_2))$ can contain at most four independent points. Thus, $\text{disk}_1(v_1) \cup \text{disk}_1(v_2) \cup \text{disk}_1(v_3)$ can contain at most twelve independent points. For $n = 4, 5$, the proof is given by a tedious geometric argument and the interested reader may see the original paper [106] for detail.

Theorem 3.4.6 (Wan et al. [106]). *For any connected unit disk graph G with at least two nodes,*

$$\alpha(G) \leq \frac{11}{3} \cdot \gamma_c(G) + 1.$$

Proof. Let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a nontrivial star partition of a spanning tree of the minimum CDS of G such that for any $1 \leq i \leq k$, $S_1 \cup \cdots \cup S_i$ is connected. Let I be the maximum independent set of G. For any subgraph H, denote by $I(H)$ the intersection of I and the neighborhood area of H. The proof is by a mathematical induction on k. By Lemma 3.4.9,

$$|I(S_k)| \leq \frac{11}{3} \cdot |S_k| + 1.$$

Since $\cup_{i=1}^{k} S_i$ is connected, S_k must have a node v lying in the neighborhood area of $\cup_{i=1}^{k-1} S_i$ and v is independent to any point in $I(\cup_{i=1}^{k-1} S_i) \setminus I(S_k)$. By the induction hypothesis,

$$|I(\cup_{i=1}^{k-1} S_i) \setminus I(S_k)| + 1 \leq \frac{11}{3} \cdot |\cup_{i=1}^{k-1} S_i| + 1,$$

that is,

$$|I(\cup_{i=1}^{k-1} S_i) \setminus I(S_k)| \leq \frac{11}{3} \cdot |\cup_{i=1}^{k-1} S_i|.$$

Therefore,

$$\begin{aligned} |I(\cup_{i=1}^{k} S_i)| &= |I(\cup_{i=1}^{k-1} S_i) \setminus I(S_k)| + |I(S_k)| \\ &\leq \frac{11}{3} \cdot |\cup_{i=1}^{k} S_i| + 1. \end{aligned}$$ □

Vahdatpour et al. [103] claimed that they proved that for any connected unit disk graph G,

$$\alpha(G) \leq 3\gamma_c(G) + 3.$$

If their proof is correct, then this is the best possible result. Wan et al. [106] have showed that for some unit disk graph G

$$\alpha(G) = \gamma_c(G) + 3.$$

Unfortunately, the proof of Vahdatpour et al. is far from a complete one. In the following, we give an analysis on their proof and indicate what important parts their proof miss. First, note that their proof use a mathematical induction on the number of vertices in the minimum CDS based on two important lemmas.

Let T be a spanning tree of the minimum CDS. For any node v, denote $N(v) = \text{disk}_1(v)$. Let U be any set of independent points lying in the neighborhood area $Q(T)$ of T. Assume $v_1, v_2, \ldots, v_T$ is an arbitrary traversal of T. For any i, $2 \leq i \leq |T|$, consider $U_i = N(v_i) \cap U - \cup_{j=1}^{i-1} N(v_j)$ be the subset of nodes in U that are adjacent to v_i but not to any of $v_1, v_2, \ldots, v_{i-1}$. We will call U_i the semi-exclusive neighboring set of node v_i." The first lemma is as follows:

Lemma 3.4.10. *For two distinct vertices v_i and v_j of T with $|U_i| = |U_j| = 4$, there exists a node v_k on the path between v_i and v_j such that $|U_k| \leq 2$.*

Now, consider a leaf v_j. There are several cases.

Case 1. $|U_j| \leq 3$. Then we apply the induction hypothesis on $T \setminus v_j$ and finish the induction proof.

Case 2. $|U_j| = 4$ and $|U_i| \leq 3$ for every $i \neq j$. In this case, we immediately have $|U| \leq 3|T| + 3$.

Case 3. $|U_j| = 4$ and there is $i \neq j$ such that $|U_i| = 4$. By Lemma 3.4.10, there exists a vertex v_k on the path between v_i and v_j such that $|U_k| \leq 2$.

Subcase 3.1. Path $P = (v_j, \ldots, v_k)$ does not contain a fork vertex (i.e., a vertex with degree at least three). In this subcase, we can apply the induction hypothesis to $T \setminus P$ and finish the induction proof.

Subcase 3.2. Path $P = (v_j, \ldots, v_k)$ contains some fork vertices. In this subcase, $T \setminus P$ is not connected and hence not a tree so that we cannot apply induction hypothesis to $T \setminus P$. This is a complicated subcase. However, Vahdatpour et al. [103] did not give sufficient argument to deal with it. Indeed, they provided the second lemma to handle this subcase. However, (1) the second lemma is not sufficient to handle this subcase and (2) the proof of the second lemma is far from a complete one.

Therefore, it is still an open problem whether the inequality (3.1) holds or not.

3.5 Independent Number (II)

Funke et al. [51] initiated another idea to establish the upper bound of the independent number $\alpha(G)$ for unit disk graph G. The idea is based on the fact that all $\text{disk}_{0.5}(v)$ for v over nodes in the maximum independent set are disjoint and they all lie in the union Ω of $\text{disk}_{1.5}(x)$ for x over all nodes in the minimum CDS. The following result follows immediately from this fact.

Theorem 3.5.1 (Funke et al. [51]). *Let $\alpha(G)$ and $\gamma_c(G)$ be the independent number and the connected dominating number of unit disk graph G, respectively. Then*

$$\alpha(G) \leq 3.748\gamma_c(G) + 5.252.$$

Proof. Define the dominating area of a vertex x to be the disk $\text{disk}_{1.5}(x)$. Then two adjacent nodes have at least $\frac{9}{2} \arccos \frac{1}{3} - \sqrt{2}$ area in common. Thus, the union Ω of dominating areas of a minimum CDS can have at most

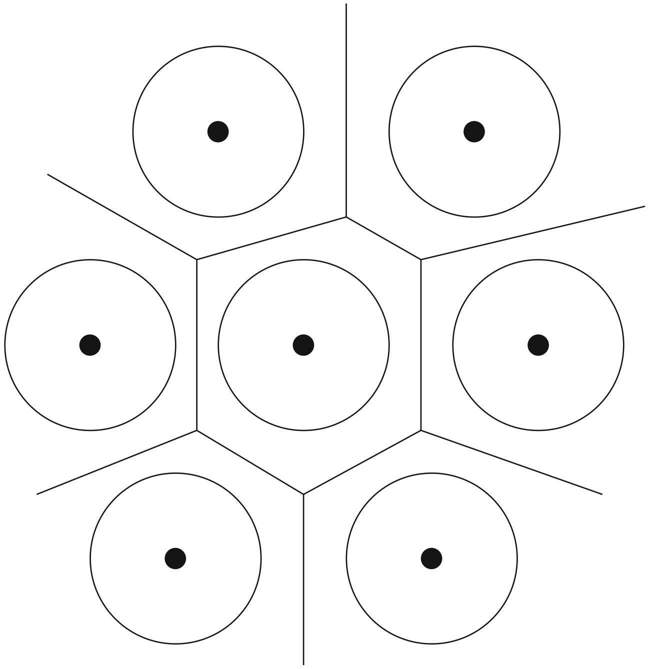

Fig. 3.14 Voronoi division

$$(\gamma_c(G)-1)\left(\frac{9}{2}\arccos\frac{1}{3}-\sqrt{2}\right)+\pi 1.5^2$$

area. For every node v in a maximal independent set, draw a disk $\text{disk}_{0.5}(v)$. All such disks are disjoint and lie in the adjacent area of the maximum independent set. Therefore, the size of a maximal independent set $\alpha(G)$ is at most

$$\frac{(\gamma_c(G)-1)(\frac{9}{2}\arccos\frac{1}{3}-\sqrt{2})+\pi 1.5^2}{0.25\pi} \leq 3.748\gamma_c(G)+5.252. \qquad \square$$

To improve this approach, Funke et al. further introduced Voronoi division (Fig. 3.14) of the maximum independent set. Denote by $\text{voro}(v)$ the Voronoi cell of node v. Since $\text{disk}_{0.5}(v) \subset \text{voro}(v)$, the area of $\text{voro}(v)$ would be bigger that the area of $\text{disk}_{0.5}(v)$. In fact, they claimed that the area of $\text{voro}(v)$ is at least $\sqrt{3}/2$ and $\text{voro}(v) \cap \Omega$ has area at least 0.8525. This fact has been verified by Gao et al. [54]. Therefore, Funde et al. [51] established the following.

Theorem 3.5.2 (Funke et al. [51]). *Let $\alpha(G)$ and $\gamma_c(G)$ be the independent number and the connected dominating number of unit disk graph G, respectively. Then*

$$\alpha(G) \leq 3.453\gamma_c(G)+4.839.$$

Proof. Similar to the proof of Theorem 3.5.1.

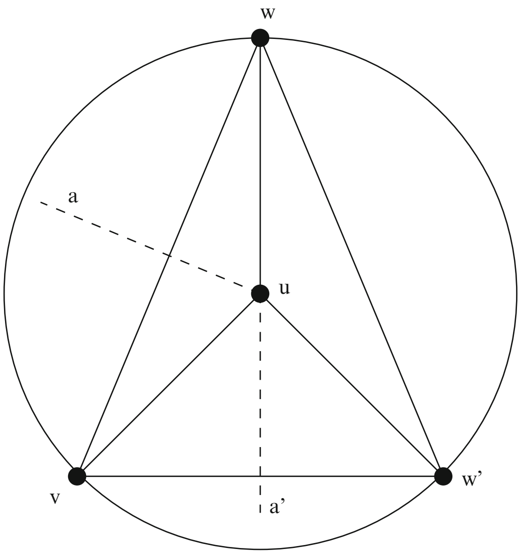

Fig. 3.15 The proof of Lemma 3.5.3

$$\alpha(G) \leq \frac{(\gamma_c(G)-1)(\frac{9}{2}\arccos\frac{1}{3}-\sqrt{2})+\pi 1.5^2}{0.8525} \leq 3.453\gamma_c(G)+4.839. \qquad \square$$

Li et al. [72] found two ideas to make an improvement. The first idea is based on the following facts.

Lemma 3.5.3. *Every vertex u of Voronoi cell* $\text{voro}(v)$ *lies outside the disk* $\text{disk}_{1/\sqrt{3}}(v)$.

Proof. Suppose ua and ua' are two edges of $\text{voro}(v)$ at vertex u. Let w be the symmetric point of v with respect to line ua and w' the symmetric point of v with respect to line ua' (Fig. 3.15). Then by the construction of Voronoi division, it can be seen that v, w and w' are independent and they are on circle $\text{circle}_{d(v,u)}(u)$. Note that one of angles $\angle vuw, \angle wuw', \angle w'uv$ is at most 120°. This means one of $d(v,w), d(w,w'), d(w',v)$ is at most $\sqrt{3} \cdot d(v,u)$, that is,

$$1 < \sqrt{3}d(v,u) \text{ or } d(v,u) > 1/\sqrt{3}. \qquad \square$$

Lemma 3.5.4. *Let P be a polygon inscribed in the circle* $\text{circle}_{1/\sqrt{3}}(v)$ *such that* $\text{disk}_{0.5}(v) \subset P$. *Then*

$$\text{area}(P) \geq \sqrt{3}/2,$$
$$\text{area}(P \cap \text{disk}_{1.5}(s)) \geq \sigma = 0.85505328..,$$

for $\text{disk}_{0.5}(v) \subset \text{disk}_{1.5}(s)$.

By Lemma 3.5.3, for each v in the maximum independent set, one can construct a polygon P_v which is inscribed in circle circle$_{1/\sqrt{3}}(v)$ and disk$_{0.5}(v) \subset P_v \subset$ voro(v). By Lemma 3.5.4, the area of $P_v \cap \Omega \geq \sigma$ and hence $\alpha(G) \leq \text{area}(\Omega)/\sigma$.

The second idea is motivated from an observation on Lemma 3.5.4. Lemma 3.5.4 indicates that the maximum independent set I can be partitioned into two parts

$$I_1 = \{v \mid \text{voro}(v) \cap \text{disk}_{1/\sqrt{3}}(v) \subseteq \Omega\},$$
$$I_2 = \{v \mid \text{voro}(v) \cap \text{disk}_{1/\sqrt{3}}(v) \not\subseteq \Omega\}.$$

For $v \in I_1$, area$(\text{voro}(v) \cap \Omega) \geq \sqrt{3}/2$ and for $v \in I_2$, area$(\text{voro}(v) \cap \Omega) \geq \sigma$. If $|I_2|$ can be upper-bounded in some way, the upper bound for $\alpha(G)$ could be improved. In fact, since

$$\text{area}(\Omega) \geq \frac{\sqrt{3}}{2} \cdot |I_1| + \sigma \cdot |I_2|,$$

one has

$$|I| \leq \frac{\text{area}(\Omega)}{\frac{\sqrt{3}}{2}} + \left(1 - \frac{\sigma}{\frac{\sqrt{3}}{2}}\right) \cdot |I_2|.$$

Li et al. successfully established an upper bound for $|I_2|$ as follows.

Lemma 3.5.5. *Let C be a minimum CDS of unit disk graph $G = V, E)$. Define $\Omega' = \cup_{x \in C}\text{disk}_{1.5-1/\sqrt{3}}(x)$. Then the boundary length of Ω is at most*

$$2\left(3 - \frac{2}{\sqrt{3}}\right)\left((\gamma_c(G) - 1)\arcsin\frac{1}{3 - \frac{2}{\sqrt{3}}} + \frac{\pi}{2}\right)$$

and at least $2(1 - 1/\sqrt{3})|I_2|$.

With this lemma, they showed the following best-known upper bound for $\alpha(G)$.

Theorem 3.5.6 (Li et al. [72]). *Let $\alpha(G)$ and $\gamma_c(G)$ be the independent number and the connected dominating number of unit disk graph G, respectively. Then*

$$\alpha(G) \leq 3.4305176\gamma_c(G) + 4.8184688.$$

3.6 Zassenhaus–Groemer–Oler Inequality

Suppose a compact convex region C contains centers of n non-overlapping unit disks. Then

$$n \leq \frac{2}{\sqrt{3}}A(C) + \frac{1}{2}P(C) + 1,$$

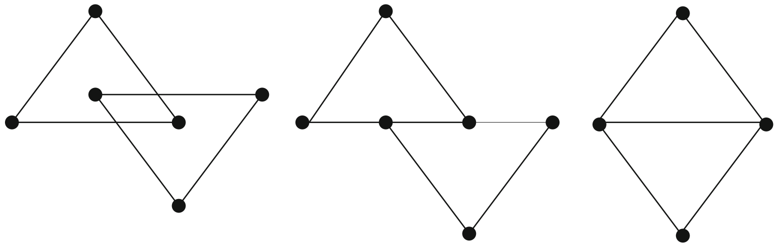

Fig. 3.16 Three families of simplexes

where $A(C)$ is the area of C and $P(C)$ is its perimeter. This inequality is conjectured by Zassenhaus in 1947 (see [125]) and proved independently by Groemer [61] and Oler [85].

This inequality has been used in the proof of Lemma 3.4.9. Indeed, the application of this inequality has been found in many places for analysis of approximation algorithms for optimization problems in unit disk graphs; especially it will be used in later chapters. Therefore, we introduce it here.

There are several proofs of this inequality [50, 61, 78, 85]. The following was given by Folkman and Graham [50] with an extension to two-dimensional simplicial complex.

A zero-dimensional simplex is a point. A one-dimensional simplex is a straight line segment. A two-dimensional simplex is a triangle. In general, a *simplex* is a polytope with minimum number of vertices among all polytopes with certain dimension. For example, a tetrahedron is a three-dimensional polytope with minimum number of vertices and hence a three-dimensional simplex.

Any simplex is the convex hull of its vertices. The convex hull of any subset of vertices in a simplex S is also a simplex, which is called a *face* of simplex S. A family Δ of simplexes is called a *simplicial complex* if it satisfies the following two conditions:

(a) For $S \in \Delta$, every face of S is in Δ.
(b) For $S, S' \in \Delta$, $S \cap S'$ is a face for both S and S'.

From (a) and (b), it is easy to see the following holds:

(c) For $S, S' \in \Delta$, $S \cap S'$ is also a simplex in Δ.

In Fig. 3.16, there are three families of simplexes. While the first two are not simplicial complexes, the last one is.

For any simplex A, $|A|$ denotes the number of vertices in A. Thus, $|A|-1$ is the dimension of A. The *Euler characteristic* of a simplicial complex Δ is defined by

$$\chi(\Delta) = \sum_{A \in \Delta, A \neq \emptyset} (-1)^{|A|-1} = \sum_{A \in \Delta} (-1)^{|A|-1} + 1.$$

Let $m(A)$ denote the area of A for two-dimensional simplex A and the length of A for one-dimensional simplex A. For one-dimensional simplex A, let $\varepsilon(A,\Delta)$ denote the number of two-dimensional simplex in Δ having A as a face. For simplicial complex Δ in the Euclidean plane, it is easy to see from (b) that $\varepsilon(A,\Delta) \leq 2$. When $\varepsilon(A,\Delta) = 1$, A is on the boundary of the union of simplexes in Δ. When $\varepsilon(A,\Delta) = 2$, A is in the interior of the union of simplexes in Δ. Now, a proper definition is given for the area $A(\Delta)$ and the perimeter $P(\Delta)$ of a simplicial complex Δ in the Euclidean plane.

$$A(\Delta) = \sum_{A\in\Delta,|A|=3} m(A)$$

and

$$P(\Delta) = \sum_{A\in\Delta,|A|=2} (2-\varepsilon(A,\Delta))m(A).$$

The inequality of Folkman and Graham [50] is as follows:

Theorem 3.6.1 (Folkman–Graham [50]). *Let Δ be a simplicial complex in the Euclidean plane. Suppose for any two distinct vertices x and y in Δ, $d(x,y) \geq 1$. Then*

$$|\Delta| \leq \frac{2}{\sqrt{3}}A(\Delta) + \frac{1}{2}P(\Delta) + \chi(\Delta),$$

where $|\Delta|$ is the number of vertices in Δ.

To prove this inequality, the following two lemmas is proved at the first.

Lemma 3.6.2. *Let a, b, c be lengths of three edges of a triangle Δ. Suppose $a \geq b \geq c \geq 1$. Then*

$$\frac{4}{\sqrt{3}}A(\Delta) + a \geq b + c.$$

Proof. By Hero's formula,

$$\begin{aligned} A(\Delta) &= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)} \\ &\geq \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)} \\ &\geq \frac{1}{4}\sqrt{3c(-a+b+c)} \\ &\geq \frac{1}{4}\sqrt{3(-a+b+c)^2} \\ &= \frac{\sqrt{3}}{4}(-a+b+c). \end{aligned}$$

Hence,

$$\frac{4}{\sqrt{3}}A(\Delta)+a \geq b+c. \qquad \square$$

Lemma 3.6.3. *Let BCDE be a quadrilateral in the Euclidean plane with area A and perimeter P. Suppose the length of every diagonal of BCDE is not less than the length of every edge of BCDE and the length of every edge is at least one. Then*

$$\frac{4}{\sqrt{3}}A-P+2 \geq 0.$$

Proof. Without loss of generality, assume $\angle B+\angle D \leq \pi$. Note that in this case, one must have $A = A(\triangle BCE)+A(\triangle CDE)$. Since diagonal CE is the longest edge in $\triangle BCE$ and in $\triangle DEC$, one has $\angle B \geq \pi/3$ and $\angle D \geq \pi/3$. Hence, $\pi/3 \leq \angle B \leq 2\pi/3$ and $\pi/3 \leq \angle D \leq 2\pi/3$. Therefore,

$$\begin{aligned} A &= \frac{1}{2}(|BC|\cdot|BE|\cdot\sin\angle B+|DC|\cdot|DE|\cdot\sin\angle D) \\ &\geq \frac{\sqrt{3}}{4}(|BC|\cdot|BE|+|DC|\cdot|DE|). \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{4}{\sqrt{3}}A-P+2 \\ &\geq |BC|\cdot|BE|+|DC|\cdot|DE|-(|BC|+|BE|+|DC|+|DE|)+2 \\ &= (|BC|-1)(|BE|-1)+(|DC|-1)(|DE|-1) \\ &\geq 0. \qquad \square \end{aligned}$$

Now, it is ready to prove Theorem 3.6.1.

Proof of Theorem 3.6.1. The proof is an induction on the number of one-dimensional simplexes contained in Δ. First, suppose Δ contains no one-dimensional simplex. Then $A(\Delta) = P(\Delta) = 0$ and $\chi(\Delta)$ is equal to the number of vertices. Therefore, the Folkman–Graham Inequality is true.

Next, suppose Δ contains k one-dimensional simplexes and $k \geq 1$. Assume that for every simplicial complex with less than k one-dimensional complexes, the Folkman–Graham inequality holds. Let $union(\Delta)$ denote the union of simplexes in Δ. Then it is easy to see that for two simplicial complexes Δ and Γ, if $union(\Delta) = union(\Gamma)$, then $A(\Delta) = A(\Gamma)$, $P(\Delta) = P(\Gamma)$ and $\chi(\Delta) = \chi(\Gamma)$. Therefore, it suffices to show the Folkman–Graham Inequality holds for one of simplicial complexes with the same union. Without loss of generality, suppose that Δ has the minimum total length of one-dimensional complexes among those simplicial simplexes with the same union and the same number of one-dimensional simplexes as Δ has.

Consider a one-dimensional complex σ in Δ with the longest length. There are three cases in the following.

Case 1. $\varepsilon(\sigma,\Delta)=0$. In this case, $\Delta-\{\sigma\}$ is a simplicial complex and $P(\Delta-\{\sigma\})=P(\Delta)-2m(\sigma)$. Therefore, by induction hypothesis,

$$\begin{aligned}|\Delta| &= |\Delta-\{\sigma\}|\\ &\leq \frac{2}{\sqrt{3}}A(\Delta-\{\sigma\})+\frac{1}{2}P(\Delta-\{\sigma\})+\chi(\Delta-\{\sigma\})\\ &= \frac{2}{\sqrt{3}}A(\Delta)+\frac{1}{2}P(\Delta)-m(\sigma)+\chi(\Delta)+1\\ &\leq \frac{2}{\sqrt{3}}A(\Delta)+\frac{1}{2}P(\Delta)+\chi(\Delta).\end{aligned}$$

Case 2. $\varepsilon(\sigma,\Delta)=1$. Let τ be the two-dimensional simplex in Δ having σ as a face. Let σ' and σ'' be other two one-dimensional faces of τ. Without loss of generality, assume $m(\sigma')\geq m(\sigma'')$. From the choice of σ, it can be seen that $m(\sigma)\geq m(\sigma')\geq m(\sigma'')\geq 1$. By Lemma 3.6.2,

$$\frac{4}{\sqrt{3}}m(\tau)+m(\sigma)\geq m(\sigma')+\sigma(\sigma'').$$

Note that $\Gamma=\Delta-\{\tau,\sigma\}$ is a simplicial complex. By induction hypothesis,

$$\begin{aligned}&\frac{2}{\sqrt{3}}A(\Delta)+\frac{1}{2}P(\Delta)+\chi(\Delta)\\ &\quad= \frac{2}{\sqrt{3}}A(\Gamma)+\frac{1}{2}P(\Gamma)+\chi(\Gamma)+\frac{2}{\sqrt{3}}A(\tau)+\frac{1}{2}(m(\sigma)-m(\sigma')-m(\sigma''))\\ &\quad\geq \frac{2}{\sqrt{3}}A(\Gamma)+\frac{1}{2}P(\Gamma)+\chi(\Gamma)\\ &\quad\geq |\Gamma|=|\Delta|.\end{aligned}$$

Case 3. $\varepsilon(\sigma,\Delta)=2$. Let τ and τ' be the two-dimensional simplexes in Δ having σ as a face. Let B and D be two vertices of σ. Suppose C is the third vertex of τ and E is the third vertex of τ'. Because σ is the longest edge in τ and in τ', it can be seen that $\angle BCE\leq\pi/2$, $\angle CEB\leq\pi/2$, $\angle ECD\leq\pi/2$ and $\angle DEC\leq\pi/2$. Thus, $\angle BCD\leq\pi$ and $\angle DEB\leq\pi$. This implies that $BCDE$ is a convex quadrilateral. It follows that a simplicial complex can be obtained from Δ by replacing τ and τ' by $\triangle BCD$ and $\triangle DEB$ with the same union and the same number of one-dimensional simplexes as Δ has. By the choice of Δ, one has $|BD|\geq|CE|=m(\sigma)$. By Lemma 3.6.3,

$$\frac{4}{\sqrt{3}}(A(\tau)+A(\tau'))-P(\tau\cup\tau')+2\geq 0.$$

Note that $\Gamma = \Delta - \{\sigma, \tau, \tau'\}$ is a simplicial complex. By induction hypothesis,

$$\begin{aligned}
&\frac{2}{\sqrt{3}}A(\Delta) + \frac{1}{2}P(\Delta) + \chi(\Delta) \\
&= \frac{2}{\sqrt{3}}A(\Gamma) + \frac{1}{2}P(\Gamma) + \chi(\Gamma) + \frac{2}{\sqrt{3}}(A(\tau) + A(\tau')) - P(\tau \cup \tau') + 1 \\
&\geq \frac{2}{\sqrt{3}}A(\Gamma) + \frac{1}{2}P(\Gamma) + \chi(\Gamma) \\
&\geq |\Gamma| = |\Delta|.
\end{aligned}$$

□

Corollary 3.6.4 (Zassenhaus–Groemer–Oler). *Let X be a compact convex region in the Euclidean plane. Suppose X contains a set V of centers of n non-overlapping unit circles. Then*

$$n \leq \frac{2}{\sqrt{3}}A(X) + \frac{1}{2}P(X) + 1,$$

where $A(X)$ and $P(X)$ denote the area and perimeter of X, respectively.

Proof. Let H be the convex hull of V. Then $A(X) \geq A(H)$ and $P(X) \geq P(H)$. Let Δ be a simplicial complex with vertex set V, whose two-dimensional faces form a triangulation of H. Then the union of Δ equals H and $\chi(\Delta) = 1$. By Folkman–Graham Inequality,

$$\frac{2}{\sqrt{3}}A(X) + \frac{1}{2}P(X) + 1 \geq \frac{2}{\sqrt{3}}A(\Delta) + \frac{1}{2}P(\Delta) + 1 \geq |\Delta| = |V|.$$

□

Corollary 3.6.5. *A disk* $\text{disk}_r(o)$ *can contain at most*

$$\frac{2}{\sqrt{3}}\pi r^2 + \pi r + 1$$

independent points.

Proof. It follows immediately from Zassenhaus–Graemer–Oler inequality. □