Chapter 10
Geometric Hitting Set and Disk Cover

Another means of silently lessening the inequality of property is to exempt
all from taxation below a certain point, and to tax the higher portions of
property in geometric progression as they rise.

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10.1 Motivation and Overview

MIN-SENSOR-COVER is a special case of MIN-SET-COVER, which can be seen as MIN-SET-COVER in a geometric case with the base set formed by all targets and all given subsets of targets induced by sensing disks. When all sensing disks have the same size, a classic result indicates that MIN-SENSOR-COVER has PTAS. In this chapter, we introduce some related results in case that sensing disks may have different sizes. Those results may lead us to a sequence of research works on coverage and connected coverage with different sizes of sensing disks.

10.2 Minimum Geometric Hitting Set

Consider a set V of nodes and a set $\mathcal{D}$ of target disks in the plane. A node $v \in V$ is said to *hit* a disk $D \in \mathcal{D}$ if $v \in D$. A subset S of V is said to be a *hitting set* (HS) of $\mathcal{D}$ if each disk in $\mathcal{D}$ is hit by some node in V. The problem of finding a minimum subset of V which is an HS of $\mathcal{D}$ is referred to as MIN-HITTING-SET. In this section, we present a PTAS for MIN-HITTING-SET.

If a target disk is hit by all nodes in V, we simply remove it from $\mathcal{D}$. In addition, if a target disk contains some other target disk, we also remove it from $\mathcal{D}$. Thus, we assume that each target disk in $\mathcal{D}$ is not hit by at least one node in V and does not

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contain any other disk in $\mathcal{D}$. Consequently, the disks in $\mathcal{D}$ have distinct centers. Let T denote the set of the centers of disks in $\mathcal{D}$. For each $t \in T$, we use $D(t)$ to denote the disk in $\mathcal{D}$ centered at t, and $r(t)$ to denote the radius of the disk $D(t)$. Let S be an HS. A set $U \subseteq S$ is said to be a *loose* subset of S if there is a subset U' of V such that $|U'| < |U|$ and $(S \setminus U) \cup U'$ is still an HS, and to be a *tight* subset of S otherwise. S is said to be *k-tight* if every subset $U \subseteq S$ with $|U| \leq k$ is tight. Intuitively, a k-tight HS for sufficiently large k is close to the minimum HS in size. We will formerly prove such relation in the next theorem.

Theorem 10.2.1. *Let c and K be the two universal constants in Theorem 9.3.1. Then, for any k-tight hitting set* $S \subseteq V'$ *with* $k \geq \max\{K, 2\}$, $|C| \leq \left(1 + c/\sqrt{k}\right)$ *opt, where opt is the size of a minimum hitting set.*

Theorem 10.2.1 suggests a local search algorithm for MIN-DISK-COVER, referred to as *k-Local Search* (k-**LS**), where k is a positive integer parameter at least two. It computes a k-tight HS S in two phases:

- *Preprocessing Phase*. Compute an HS S by the well-known greedy algorithm for minimum set cover.
- *Replacement Phase*. While S is not k-tight, find a subset U of S with size at most k and a subset U' of V with size at most $|U| - 1$ satisfying that $(S \setminus U) \cup U'$ is still an HS; replace S by $(S \setminus U) \cup U'$. Finally, we output S.

By Theorem 10.2.1, the algorithm k-**LS** has an approximation ratio at most $1 + O\left(1/\sqrt{k}\right)$ when $k \geq K$. Its running time is dominated by the second phase. Let $m = |V|$. Then, the second phase consists of $O(m)$ iterations. In each iteration, the search for the subset U and its replacement U' takes at most

$$O\left(m^k\right) \cdot O\left(m^{k-1}\right) = O\left(m^{2k-1}\right)$$

time. So, the total running time is

$$O(m) \cdot O\left(m^{2k-1}\right) = O\left(m^{2k}\right).$$

This means that the algorithm k-**LS** is a PTAS.

We move on to the proof of 10.2.1. Let O be a minimum HS. Theorem 10.2.1 holds trivially if $|S| = |O|$. So, we assume that $|S| > |O|$. Let $S' = S \setminus O$ and $O' = O \setminus S$. Then, $|S'| > |O'|$. In addition, $|O'| \geq k$ for otherwise, we can choose a subset of $|O'| + 1$ nodes from S' and replace them by O' to get a smaller HS, which contradicts to the fact that S is k-tight. Let T' be the set of centers of the target disks not hit by $O \cap S$. Then, for each $t \in T'$, $D(t)$ is hit by some node in S' and by some node in O'. In addition, we have the following stronger property.

Lemma 10.2.2. *There is a planar bipartite graph H on* O' *and* S' *satisfying the following* "locality condition"*: For each* $t \in T$*, there are two adjacent nodes in H, both of which hit* $D(t)$.

Let H be the planar bipartite graph satisfying the property in the above lemma. We claim that for any $U \subseteq S'$, $(S \setminus U) \cup N_H(U)$ is still an HS. Indeed, consider any $t \in T$. If $D(t)$ is hit by $S \setminus U$, then it is also hit by $(S \setminus U) \cup N_H(U)$. If $D(t)$ is not hit by $S \setminus U$, then $D(t)$ is only hit by nodes in U and hence $t \in T'$. By Lemma 10.2.2, there exist two adjacent nodes $u \in S'$ and $v \in O'$, both of which hit $D(t)$. Then, we must have $u \in U$ and hence $v \in N_H(U)$. Thus, $D(t)$ is still hit by $(S \setminus U) \cup N_H(U)$. So, the claim holds.

Now, consider any $U \subseteq S'$ with $|U| \leq k$. Then $|N_H(U)| \geq |U|$, for otherwise $(S \setminus U) \cup N_H(U)$ is an HS smaller than S, which contradicts to the fact that S is k-tight. By Theorem 9.3.1, we have

$$\left|S'\right| \leq (1 + c/\sqrt{k}) \left|O'\right|$$

and hence

$$|S| \leq (1 + c/\sqrt{k}) \, |O| \, .$$

So, Theorem 10.2.1 holds.

In the remaining of this section, we prove Lemma 10.2.2. Let $V' = O' \cup S'$. Consider any $t \in T'$. Define

$$\bar{r}(t) = \min_{v \in V'} \left\{ \frac{\|tv\| + r(t)}{2} : \|tv\| > r(v) \right\}.$$

Clearly, $\bar{r}(t) > r(t)$. Let $D'(t)$ be the disk centered at t of radius $\bar{r}(t)$. Then, for each node $v \in V'$, v hits $D(t)$ if and only if v hits $D'(t)$. Let $\mathcal{D}'$ to denote the collection of disks $D'(t)$ for all $t \in T'$. Consider any $v \in V'$. Define

$$\begin{aligned}
\delta_1(v) &= \min\left\{\bar{r}(t) - \|tv\| : \|tv\| \leq r(t), t \in T'\right\}, \\
\delta_2(v) &= \min\left\{\|vu\| / 3 : u \in V \setminus \{v\}\right\}, \\
\delta(v) &= \min\left\{\delta_1(v), \delta_2(v)\right\}.
\end{aligned}$$

Then, $\delta(v) > 0$ and the disk of radius $\delta(v)$ centered at v is referred to as the perturbation range of v. Then, for any point v' in the perturbation range of v and any $t \in T'$, v hits $D(t)$ if and only if v' hits $D'(t)$. In addition, the perturbation ranges of all nodes in V' are disjoint. A *restricted perturbation* of V' is a mapping σ from V' to the plane such that for each $v \in V'$, $\sigma(v)$ lies within the perturbation range of v.

A set of four points in the plane form a *degenerate quadruple* if they all lie on some circle. The next lemma shows that V' has a restricted perturbation containing no degenerate quadruple.

Lemma 10.2.3. *There exists a restricted perturbation σ of V' such that $\sigma(V')$ contains no degenerate quadruple.*

Proof. We prove the lemma by contradiction. Assume the lemma is not true. Let σ be the "fewest counterexample,"in other words, $\sigma(V')$ contains the least number of degenerate quadruples. Suppose that a node $\sigma(u) \in V'$ is contained in at least one degenerate quadruple in $\sigma(V')$. We show that we can change $\sigma(u)$ to some point in the perturbation range of u which is not involved in any degenerate quadruple. For any triple nodes $\{v_1, v_2, v_3\}$ in $V' \setminus \{u\}$ such that $\sigma(v_1)$, $\sigma(v_2)$, and $\sigma(v_3)$ are not collinear, the circumcircle of $\{\sigma(v_1), \sigma(v_2), \sigma(v_3)\}$ is referred to as a *forbidden circle* of u. As the number of forbidden circles of u is at most $\binom{|V'|-1}{3}$, there is a point u' which lies in the perturbation range of u but not on any forbidden circle of u. Let σ' be the restricted perturbation of $O' \cup S'$ obtained from σ by replacing $\sigma(u)$ with u'. Then, $u' = \sigma'(u)$ is not contained in any degenerate quadruple of $\sigma'(V')$. Thus, $\sigma'(V')$ contains strictly fewer degenerate quadruples than $\sigma(V')$, which contradicts to the choice of σ. Therefore, the lemma holds. □

Now, we fix a restricted perturbation σ of V' satisfying that $\sigma(V')$ contains no degenerate quadruple. Let G' be the graph obtained from Voronoi dual of $\sigma(V')$ by removing all edges between two nodes in O' and all edges between two nodes in S'. Then, G' is planar. In the next, we show that G' satisfies the locality condition: For each target $t \in T'$, there are two adjacent nodes in G', both of which hit $D'(t)$. We consider two cases:

Case 1: t lies in the Voronoi cell of $\sigma(u)$ for some $u \in O'$. Then, $\sigma(u)$ must hit $D'(t)$ as $D'(t)$ is hit by $\sigma(O')$. Let v be a node in S' such that $\sigma(v)$ has the shortest distance from t. Then, $\sigma(v)$ must also hit $D'(t)$ as $D'(t)$ is hit by $\sigma(O')$. If $\sigma(u)$ and $\sigma(v)$ are adjacent, then lemma holds trivially. So, we assume that $\sigma(u)$ and $\sigma(v)$ are nonadjacent. Then t lies outside the Voronoi cell of $\sigma(v)$. We walk from t to $\sigma(v)$ along the straight line segment $t\sigma(v)$. During this walk, we may cross some Voronoi cells, and at some point before reaching $\sigma(v)$, we will enter the Voronoi cell of $\sigma(v)$ the first time. Let x be the point at which we first enter the Voronoi cell of $\sigma(v)$. We must enter this cell from another cell, and we assume the cell is the Voronoi cell of $\sigma(w)$. Then, $\sigma(w)$ does not lie in the ray $x\sigma(v)$, and hence

$$\|t\sigma(w)\| < \|tx\| + \|x\sigma(w)\| = \|tx\| + \|x\sigma(v)\| = \|t\sigma(v)\|.$$

Since $\sigma(v)$ hits $D'(t)$, $\sigma(w)$ hits $D'(t)$ as well; and by the choice of v, $w \in O'$. As $\sigma(w)$ is adjacent to $\sigma(v)$, the locality condition is satisfied.

Case 2: t lies in the Voronoi cell of $\sigma(u)$ for some $u \in S'$. The proof is the same as in Case 1 and is thus omitted.

Finally, we define a graph H on V' such that two nodes u and v are adjacent if and only if $\sigma(u)$ and $\sigma(v)$ are adjacent in G'. Then, H is also a planar bipartite graph. In addition, for any target $t \in T'$, let $\sigma(u)$ and $\sigma(v)$ be two adjacent nodes in G', both of which hit $D'(t)$. Then, u and v are two adjacent nodes in H, both of which hit $D(t)$. This completes the proof of Lemma 10.2.2.

10.3 Minimum Disk Cover

Consider a set $\mathcal{D}$ of disks and a set T of target points in the plane. A disk $D \in \mathcal{D}$ is said to *cover* a target $t \in T$ if $t \in D$. A subset $\mathcal{D}'$ of $\mathcal{D}$ is said to be a *cover* of T if each target in T is covered by some node in V. The problem of finding a minimum subset of $\mathcal{D}$ which is a cover of T is referred to as MIN-DISK-COVER. In this section, we present a PTAS for MIN-DISK-COVER.

Suppose that each disk in $\mathcal{D}$ has a unique ID for tie-breaking. A disk $D \in \mathcal{D}$ is said to be *redundant* if there exists another disk $D' \in \mathcal{D}$ satisfying that either D only covers a proper subset of targets covered by D', or D covers exactly the same set of targets as D' but has a larger ID than D. If a disk in $\mathcal{D}$ is redundant, we simply remove it from $\mathcal{D}$. Thus, we assume that no disk in $\mathcal{D}$ is redundant. Consequently, we can identify the disks in $\mathcal{D}$ by their centers. Let V denote the set of the centers of disks in $\mathcal{D}$. For each $v \in V$, we use $D(v)$ to denote the disk in $\mathcal{D}$ centered at v, and $r(v)$ to denote the radius of the disk $D(v)$. For simplicity, a node $v \in V$ is said to *cover* a target $t \in T$ if $D(v)$ covers t, a subset C of V is said to be a *cover* of T if the set of disks $\{D(v) : v \in C\}$ is a cover of T.

Let $C \subseteq V$ be a cover of T. A set $U \subseteq C$ is said to be a *loose* subset of C if there is a subset U' of V such that $|U'| < |U|$ and $(C \setminus U) \cup U'$ is still a cover, and to be a *tight* subset of C otherwise. C is said to be *k-tight* if every subset $U \subseteq C$ with $|U| \leq k$ is tight. Intuitively, a k-tight cover for sufficiently large k is close to the minimum cover in size. We will formerly prove such relation in the next theorem.

Theorem 10.3.1. *Let c and K be the two universal constants in Theorem 9.3.1. Then, for any k-tight cover $C \subseteq V'$ with $k \geq \max\{K, 2\}$, $|C| \leq \left(1 + c/\sqrt{k}\right) opt$, where opt is the size of a minimum cover.*

Theorem 10.3.1 suggests a local search algorithm for MIN-DISK-COVER, referred to as *k-Local Search* (k**-LS**), where k is a positive integer parameter at least two. It computes a k-tight cover C in two phases:

- *Preprocessing Phase*. Compute a cover $C \subseteq V$ by the well-known greedy algorithm for minimum set cover.
- *Replacement Phase*. While C is not k-tight, find a subset U of C with size at most k and a subset U of V with size at most $|U| - 1$ satisfying that $(C \setminus U) \cup U'$ is still a cover; replace C by $(C \setminus U) \cup U'$. Finally, we output C.

By Theorem 10.3.1, the algorithm k**-LS** has an approximation ratio at most $1 + O\left(1/\sqrt{k}\right)$ when $k \geq K$. Its running time is dominated by the second phase. Let $m = |V|$. Then, the second phase consists of $O(m)$ iterations. In each iteration, the search for the subset U and its replacement U' takes at most

$$O\left(m^k\right) \cdot O\left(m^{k-1}\right) = O\left(m^{2k-1}\right)$$

time. So, the total running time is

$$O(m) \cdot O\left(m^{2k-1}\right) = O\left(m^{2k}\right).$$

This means that the algorithm k-**LS** is a PTAS.

We move on to the proof of 10.3.1. Let $O \subseteq V$ be a minimum cover. Theorem 10.3.1 holds trivially if $|C| = |O|$. So, we assume that $|C| > |O|$. Let $C' = C \setminus O$ and $O' = O \setminus C$. Then, $|C'| > |O'|$. In addition, $|O'| \geq k$ for otherwise, we can choose a subset of $|O'| + 1$ nodes from C' and replace them by O' to get a smaller cover, which contradicts to the fact that C is k-tight. Let T' be the set of targets not covered by $O \cap C$. Then, each $t \in T'$ is covered by some node in O' and by some node in C'. In addition, we have the following stronger property.

Lemma 10.3.2. *There is a planar bipartite graph H on O' and C' satisfying the following* "locality condition"*: For each $t \in T$, there are two adjacent nodes in H, both of which cover t.*

Let H be the planar bipartite graph satisfying the property in the above lemma. We claim that for any $U \subseteq C'$, $(C \setminus U) \cup N_H(U)$ is still a cover. Indeed, consider any $t \in T$. If t is covered by $C \setminus U$, then it is also covered by $(C \setminus U) \cup N_H(U)$. If t is not covered by $C \setminus U$, then it is only covered by nodes in U and hence $t \in T'$. By Lemma 10.3.2, there exist two adjacent nodes $u \in C'$ and $v \in O'$, both of which cover t. Then, we must have $u \in U$ and hence $v \in N_H(U)$. Thus, t is still covered by $(C \setminus U) \cup N_H(U)$. So, the claim holds.

Now, consider any $U \subseteq C'$ with $|U| \leq k$. Then $|N_H(U)| \geq |U|$, for otherwise $(C \setminus U) \cup N_H(U)$ is a cover smaller than C, which contradicts to the fact that C is k-tight. By Theorem 9.3.1, we have

$$\left|C'\right| \leq (1 + c/\sqrt{k}) \left|O'\right|$$

and hence

$$|C| \leq (1 + c/\sqrt{k}) |O|.$$

So, Theorem 10.3.1 holds.

In the remaining of this section, we prove Lemma 10.3.2. Let $V' = O' \cup C'$. For each $v \in V'$, define

$$\bar{r}(v) = \min_{t \in T} \{\|tv\| : \|tv\| > r(v)\},$$

Clearly, $\bar{r}(v) > r(v)$, and if we increase the radius of v to any value below $\bar{r}(v)$, the set of targets covered by v remains the same. A function ρ on V' is said to be *coverage-preserving* if $r(v) \leq \rho(v) < \bar{r}(v)$ for each $v \in V'$. For each coverage-preserving function ρ, we use $\mathcal{D}_\rho$ to denote the collection of disks centered at v of radius $\rho(v)$ for all $v \in V'$.

Lemma 10.3.3. *There exists a coverage-preserving function ρ on V' such that $\mathcal{D}_\rho$ contains no degenerate quadruple.*

Now, fix a coverage-preserving function ρ on V' such that $\mathcal{D}_\rho$ contains no degenerate quadruple. For each node $v \in V'$, let $D'(v)$ denote the disk centered at v of radius $\rho(v)$. We claim that any pair of disks in $\mathcal{D}_\rho$ are geometrically nonredundant. Indeed, assume to the contrary that there exist two nodes in u and v such that $D'(u) \subseteq D'(v)$. Since ρ is coverage-preserving, all targets covered by u are also covered by v, which is a contradiction. Thus, our claim holds. Let H be the graph obtained from the Voronoi dual of $\mathcal{D}_\rho$ by removing all edges between two nodes in O' and all edges between two nodes in C'. Then, H is a planar bipartite graph on O' and C'.

Next, we show that H satisfies the locality condition: For each $t \in T$, there are two adjacent nodes in H, both of which cover t. Clearly, t is covered by a node $v \in V'$ if and only if $\ell(t,v) \leq 0$ where $\ell(t,v) = \|tv\| - \rho(v)$ is the shifted distance from t to v. Thus, if $\ell(t,u) \leq \ell(t,v)$ for some two nodes u and v in V' and t is covered by v, then t is also covered by u. We consider two cases:

Case 1: t lies in the Voronoi cell of $D'(u)$ for some $u \in O'$. Then, u must cover t as t is covered by O'. Let v be a node in C' to which t has the smallest shifted distance. Then, v must also cover t, as t is covered by C'. If u and v are adjacent, then the locality condition holds trivially. So, we assume that u and v are nonadjacent. Then, t lies outside the Voronoi cell of $D'(v)$. We walk from t to v along the straight line segment tv. During this walk, we may cross some Voronoi cells of the disks in $\mathcal{D}_\rho$, and at some point before reaching v we will enter the Voronoi cell of $D'(v)$ the first time. Let x be the point at which we first enter the Voronoi cell of $D'(v)$. We must enter this cell from another cell, and we assume this cell the Voronoi cell of $D'(w)$. Then, $\ell(t,w) \leq \ell(t,v)$ as

$$
\begin{aligned}
\ell(t,w) &= \|tw\| - \rho(w) \\
&\leq \|tx\| + \|xw\| - \rho(w) \\
&= \|tx\| + \ell(x,w) \\
&= \|tx\| + \ell(x,v) \\
&= \|tx\| + \|xv\| - \rho(v) \\
&= \|tv\| - \rho(v) \\
&= \ell(t,v).
\end{aligned}
$$

We further claim that $\ell(t,w) < \ell(t,v)$. Indeed, assume to the contrary that $\ell(t,w) = \ell(t,v)$. Then, we must have $\|tw\| = \|tx\| + \|xw\|$, in other words, w lies in the ray tv. As $\ell(t,w) = \ell(t,v)$, either $D'(v) \subseteq D'(w)$ or $D'(w) \subseteq D'(v)$, which is a contradiction. Therefore, our claim is true. By the choice of v, $w \in O'$ and w is adjacent to v. In addition, w covers t since $\ell(t,w) < \ell(t,v)$ and v dominates t. Thus, the locality condition is satisfied.

Case 2: t lies in the Voronoi cell of $D'(u)$ for some $u \in C'$. The proof is the same as in Case 1 and is thus omitted.

Since ρ is coverage-preserving, Lemma 10.3.2 holds.