Chapter 6
Discrete Systems Synchronization

As in Luo [1, 2], a set of concepts on "Ying" and "Yang" in discrete dynamical systems will be presented. Based on the Ying-Yang theory, the complete dynamics of discrete dynamical systems will be presented for an understanding of dynamical behaviors. From the ideas of the Ying-Yang theory of discrete dynamical systems, the companion and synchronization of discrete dynamical systems will be presented herein, and the corresponding conditions will be presented as an integrity part of dynamical system synchronization. The synchronization dynamics of Duffing and Henon maps will be discussed.

6.1 Discrete Systems with a Single Nonlinear Map

Definition 6.1 Consider an implicit vector function $\mathbf{f}: D \to D$ on an open set $D \subset \mathscr{R}^n$ in an n-dimensional discrete dynamical system. For $\mathbf{x}_k, \mathbf{x}_{k+1} \in D$, there is a discrete relation as

$$\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}, \tag{6.1}$$

where the vector function is $\mathbf{f} = (f_1, f_2, \cdots, f_n)^{\mathrm{T}} \in \mathscr{R}^n$ and discrete variable vector is $\mathbf{x}_k = (x_{k1}, x_{k2}, \cdots, x_{kn})^{\mathrm{T}} \in D$ with a parameter vector $\mathbf{p} = (p_1, p_2, \cdots, p_m)^{\mathrm{T}} \in \mathscr{R}^m$.

Definition 6.2 For a discrete dynamical system in Eq. (6.1), the positive and negative discrete sets are defined by

$$\left.\begin{aligned} \Sigma_+ &= \{\mathbf{x}_{k+i} | \mathbf{x}_{k+i} \in \mathscr{R}^n, i \in \mathbb{Z}_+\} \subset D \text{ and} \\ \Sigma_- &= \{\mathbf{x}_{k-i} | \mathbf{x}_{k-i} \in \mathscr{R}^n, i \in \mathbb{Z}_+\} \subset D, \end{aligned}\right\} \tag{6.2}$$

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respectively. The discrete set is

$$\Sigma = \Sigma_+ \cup \Sigma_-. \tag{6.3}$$

A positive mapping is defined as

$$P_+ : \Sigma \to \Sigma_+ \Rightarrow P_+ : \mathbf{x}_k \to \mathbf{x}_{k+1} \tag{6.4}$$

and a negative mapping is defined by

$$P_- : \Sigma \to \Sigma_- \Rightarrow P_- : \mathbf{x}_k \to \mathbf{x}_{k-1}. \tag{6.5}$$

Definition 6.3 For a discrete dynamical system in Eq. (6.1), consider two points $\mathbf{x}_k \in D$ and $\mathbf{x}_{k+1} \in D$, and there is a specific, differentiable, vector function $\mathbf{g} \in \mathscr{R}^n$ to make $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$.

(i) The stable solution based on $\mathbf{x}_{k+1} = P_+\mathbf{x}_k$ for the positive mapping P_+ is called the "Yang" of the discrete dynamical system in Eq. (6.1) in sense of $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ if $\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}$ with $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ have the P_+-1 solutions $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*)$.
(ii) The stable solution based on $\mathbf{x}_k = P_-\mathbf{x}_{k+1}$ for the negative mapping P_- is called the "Ying" of the discrete dynamical system in Eq. (6.1) in sense of $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ if $\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}$ with $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ have the P_--1 solutions $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*)$.
(iii) The solution based on $\mathbf{x}_{k+1} = P_+\mathbf{x}_k$ is called the "Ying-Yang" for the positive mapping P_+ of the discrete dynamical system in Eq. (6.1) in sense of $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ if $\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}$ with $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ have the P_+-1 solutions $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*)$ and the eigenvalues of $DP_+(\mathbf{x}_k^*)$ are distributed inside and outside the unit cycle.
(iv) The solution based on $\mathbf{x}_k = P_-\mathbf{x}_{k+1}$ is called the "Ying-Yang" for the negative mapping P_- of the discrete dynamical system in Eq. (6.1) in sense of $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ if $\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}$ with $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ have the P_--1 solutions $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*)$ and the eigenvalues of $DP_-(\mathbf{x}_{k+1}^*)$ are distributed inside and outside unit cycle.

Consider the positive and negative mappings are

$$\mathbf{x}_{k+1} = P_+\mathbf{x}_k \text{ and } \mathbf{x}_k = P_-\mathbf{x}_{k+1}. \tag{6.6}$$

For the simplest case, consider the constraint condition of $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{x}_{k+1} - \mathbf{x}_k = \mathbf{0}$. Thus, the positive and negative mappings have, respectively, the constraints

$$\mathbf{x}_{k+1} = \mathbf{x}_k \text{ and } \mathbf{x}_k = \mathbf{x}_{k+1}. \tag{6.7}$$

Both positive and negative mappings are governed by the discrete relation in Eq. (6.1). In other words, Eq. (6.6) gives

$$\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0} \text{ and } \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}. \quad (6.8)$$

Setting the period-1 solution $\mathbf{x}_k^*$ and substitution of Eq. (6.7) into Eq. (6.8) gives

$$\mathbf{f}(\mathbf{x}_k^*, \mathbf{x}_k^*, \mathbf{p}) = \mathbf{0} \text{ and } \mathbf{f}(\mathbf{x}_k^*, \mathbf{x}_k^*, \mathbf{p}) = \mathbf{0}. \quad (6.9)$$

From the foregoing equation, the period-1 solutions for the positive and negative mappings are identical. The two relations for positive and negative mappings are illustrated in Fig. 6.1a, b, respectively. To determine the period-1 solution, the fixed points of Eq. (6.7) exist under constraints in Eq. (6.8), as also shown in Fig. 6.1. The two thick lines on the axis are two sets for the mappings from the starting to final states. The relation in Eq. (6.7) is presented by a solid curve. The intersection points of the curves and straight lines for relations in Eqs. (6.7) and (6.8) give the fixed points of Eq. (6.9), which are period-1 solutions, labeled by the circular symbols. However, their stability and bifurcation for the period-1 solutions are different. To determine the stability and bifurcation of the period-1 solution of the positive and negative mappings, the following theorem is stated.

Theorem 6.1 *For a discrete dynamical system in Eq. (6.1), there are two points* $\mathbf{x}_k \in D$ *and* $\mathbf{x}_{k+1} \in D$*, and two positive and negative mappings are*

$$\mathbf{x}_{k+1} = P_+\mathbf{x}_k \text{ and } \mathbf{x}_k = P_-\mathbf{x}_{k+1} \quad (6.10)$$

with

$$\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0} \text{ and } \mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}. \quad (6.11)$$

Suppose a specific, differentiable, vector function $\mathbf{g} \in \mathscr{R}^n$ *makes* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ *hold. If the solutions* $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*)$ *of both* $\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}$ *and* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ *exist, then the following conclusions in the sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}$ *hold.*

(i) *The stable* P_+*-1 solutions are the unstable* P_-*-1 solutions with all eigenvalues of* $DP_-(\mathbf{x}_k^*)$ *outside the unit cycle, vice versa.*
(ii) *The unstable* P_+*-1 solutions with all eigenvalues of* $DP_+(\mathbf{x}_k^*)$ *outside the unit cycle are the stable* P_-*-1 solutions, vice versa.*
(iii) *For the unstable* P_+*-1 solutions with eigenvalue distribution of* $DP_+(\mathbf{x}_k^*)$ *inside and outside the unit cycle, the corresponding* P_-*-1 solution is also unstable with switching the eigenvalue distribution of* $DP_-(\mathbf{x}_k^*)$ *inside and outside the unit cycle, vice versa.*
(iv) *All the bifurcations of the stable and unstable* P_+*-1 solutions are all the bifurcations of the unstable and stable* P_-*-1 solutions, respectively.*

Proof The proof can be referred to Luo [2] (Fig. 6.1). □

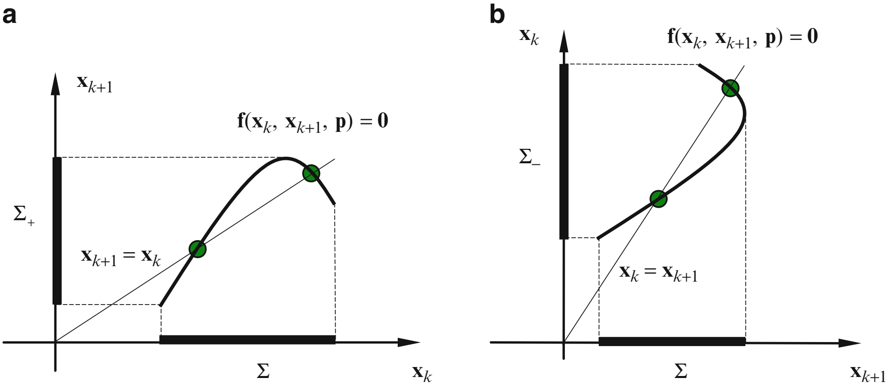

Fig. 6.1 Period-1 solution for (**a**) positive mapping and (**b**) negative mapping. The two *thick lines* on the axis are two sets for the mappings from the starting to final states. The mapping relation is presented by a *solid curve*. The *circular symbols* give period-1 solutions for the positive and negative mappings

From the foregoing theorem, the *Ying*, *Yang* and *Ying*-*Yang* states in discrete dynamical systems exist. To generate the above ideas to $P_+^{(N)}$-1 and $P_-^{(N)}$-1 solutions in discrete dynamical systems in sense of $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$, the mapping structure consisting of N-positive or negative mappings is considered.

Definition 6.4 For a discrete dynamical system in Eq. (6.1), the mapping structures of N-mappings for the positive and negative mappings are defined as

$$\mathbf{x}_{k+N} = \underbrace{P_+ \circ P_+ \circ \cdots \circ P_+}_{N} \mathbf{x}_k = P_+^{(N)} \mathbf{x}_k, \tag{6.12}$$

$$\mathbf{x}_k = \underbrace{P_- \circ P_- \circ \cdots \circ P_-}_{N} \mathbf{x}_{k+N} = P_-^{(N)} \mathbf{x}_{k+N} \tag{6.13}$$

with

$$\mathbf{f}(\mathbf{x}_{k+i-1}, \mathbf{x}_{k+i}, \mathbf{p}) = \mathbf{0} \text{ for } i = 1, 2, \cdots, N \tag{6.14}$$

where $P_+^{(0)} = 1$ and $P_-^{(0)} = 1$ for $N = 0$.

Definition 6.5 For a discrete dynamical system in Eq. (6.1), consider two points $\mathbf{x}_{k+i-1} \in D$ $(i = 1, 2, \cdots, N)$ and $\mathbf{x}_{k+N} \in D$, and there is a specific, differentiable, vector function $\mathbf{g} \in \mathscr{R}^n$ to make $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$.

(i) The stable solution based on $\mathbf{x}_{k+N} = P_+^{(N)} \mathbf{x}_k$ for the positive mapping P_+ is called the "Yang" of the discrete dynamical system in Eq. (6.1) in sense *of*

$\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ if the solutions $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*, \cdots, \mathbf{x}_{k+N}^*)$ of Eq. (6.14) with $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ exist.

(ii) The stable solution based on $\mathbf{x}_k = P_-^{(N)}\mathbf{x}_{k+N}$ for the negative mapping P_- is called the "Ying" of the discrete dynamical system in Eq. (6.1) in sense of $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ if the solutions $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*, \cdots, \mathbf{x}_{k+N}^*)$ of Eq. (6.14) with $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ exist.

(iii) The solution based on $\mathbf{x}_{k+N} = P_+^{(N)}\mathbf{x}_k$ is called the "Ying-Yang" for the positive mapping P_+ of the discrete dynamical system in Eq. (6.1) in sense of $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ if the solutions $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*, \cdots, \mathbf{x}_{k+N}^*)$ of Eq. (6.14) with $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ exist and the eigenvalues of $DP_+^{(N)}(\mathbf{x}_k^*)$ are distributed inside and outside the unit cycle.

(iv) The solution based on $\mathbf{x}_k = P_-^{(N)}\mathbf{x}_{k+N}$ is called the "Ying-Yang" for the negative mapping P_- of the discrete dynamical system in Eq. (6.1) in sense *of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ if the solutions $(\mathbf{x}_k^*, \mathbf{x}_{k+1}^*, \cdots, \mathbf{x}_{k+N}^*)$ of Eq. (6.14) with $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ exist and the eigenvalues of $DP_-^{(N)}(\mathbf{x}_{k+N}^*)$ are distributed inside and outside unit cycle.

To determine the Ying-Yang properties of $P_+^{(N)}$-1 and $P_-^{(N)}$-1 in the discrete mapping system in Eq. (6.1), the corresponding theorem is presented as follows.

Theorem 6.2 *For a discrete dynamical system in Eq. (6.1), there are two points* $\mathbf{x}_k \in D$ *and* $\mathbf{x}_{k+N} \in D$, *and two positive and negative mappings are*

$$\mathbf{x}_{k+N} = P_+^{(N)}\mathbf{x}_k \text{ and } \mathbf{x}_k = P_-^{(N)}\mathbf{x}_{k+N}, \tag{6.15}$$

and $\mathbf{x}_{k+i} = P_+\mathbf{x}_{k+i-1}$ *and* $\mathbf{x}_{k+i-1} = P_-\mathbf{x}_{k+i}$ *can be governed by*

$$\mathbf{f}(\mathbf{x}_{k+i-1}, \mathbf{x}_{k+i}, \mathbf{p}) = \mathbf{0} \text{ for } i = 1, 2, \cdots, N. \tag{6.16}$$

Suppose a specific, differentiable, vector function of $\mathbf{g} \in \mathscr{R}^n$ *makes* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = 0$ *hold. If the solutions* $(\mathbf{x}_k^*, \cdots, \mathbf{x}_{k+i}^*)$ *of Eq. (6.16) with* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ *exist, then the following conclusions in the sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+N}, \boldsymbol{\lambda}) = \mathbf{0}$ *hold.*

(i) *The stable* $P_+^{(N)}$-1 *solution is the unstable* $P_-^{(N)}$-1 *solution with all eigenvalues of* $DP_-^{(N)}(\mathbf{x}_{k+N}^*)$ *outside the unit cycle, vice versa.*

(ii) *The unstable* $P_+^{(N)}$-1 *solution with all eigenvalues of* $DP_+^{(N)}(\mathbf{x}_k^*)$ *outside the unit cycle is the stable* $P_-^{(N)}$-1 *solution, vice versa.*

(iii) *For the unstable* $P_+^{(N)}$-1 *solution with eigenvalue distribution of* $DP_+^{(N)}(\mathbf{x}_k^*)$ *inside and outside the unit cycle, the corresponding* $P_-^{(N)}$-1 *solution is also unstable with switching eigenvalue distribution of* $DP_-^{(N)}(\mathbf{x}_{k+N}^*)$ *inside and outside the unit cycle, vice versa.*

(iv) *All the bifurcations of the stable and unstable* $P_+^{(N)}$-1 *solution are all the bifurcations of the unstable and stable* $P_-^{(N)}$-1 *solution, respectively.*

Proof The proof can be referred to Luo [2]. □

Theorem 6.3 *For a discrete dynamical system in Eq. (6.1), there are two points* $\mathbf{x}_k \in D$ *and* $\mathbf{x}_{k+N} \in D$. *If the period-doubling cascade of the* $P_{+}^{(N)}$-1 *and* $P_{-}^{(N)}$-1 *solution occurs, the corresponding mapping structures are given by*

$$\begin{aligned}
&\mathbf{x}_{k+2N} = P_{+}^{(N)} \circ P_{+}^{(N)}\mathbf{x}_k = P_{+}^{(2N)}\mathbf{x}_k \text{ and } \mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2N}, \boldsymbol{\lambda}) = \mathbf{0};\\
&\mathbf{x}_{k+2^2N} = P_{+}^{(2N)} \circ P_{+}^{(2N)}\mathbf{x}_k = P_{+}^{(2^2N)}\mathbf{x}_k \text{ and } \mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^2N}, \boldsymbol{\lambda}) = \mathbf{0};\\
&\vdots\\
&\mathbf{x}_{k+2^lN} = P_{+}^{(2^{l-1}N)} \circ P_{+}^{(2^{l-1}N)}\mathbf{x}_k = P_{+}^{(2^lN)}\mathbf{x}_k \text{ and } \mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^lN}, \boldsymbol{\lambda}) = \mathbf{0};
\end{aligned} \tag{6.17}$$

for positive mappings and

$$\begin{aligned}
&\mathbf{x}_k = P_{-}^{(N)} \circ P_{-}^{(N)}\mathbf{x}_{k+2N} = P_{-}^{(2N)}\mathbf{x}_{k+2N} \text{ and } \mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2N}, \boldsymbol{\lambda}) = \mathbf{0};\\
&\mathbf{x}_k = P_{-}^{(2N)} \circ P_{-}^{(2N)}\mathbf{x}_{k+2^2N} = P_{-}^{(2^2N)}\mathbf{x}_{k+2^2N} \text{ and } \mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^2N}, \boldsymbol{\lambda}) = \mathbf{0};\\
&\vdots\\
&\mathbf{x}_k = P_{-}^{(2^{l-1}N)} \circ P_{-}^{(2^{l-1}N)}\mathbf{x}_{k+2^lN} = P_{-}^{(2^lN)}\mathbf{x}_{k+2^lN} \text{ and } \mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^lN}, \boldsymbol{\lambda}) = \mathbf{0}
\end{aligned} \tag{6.18}$$

for negative mapping, then the following statements hold, i.e.,

(i) *The stable chaos generated by the limit state of the stable* $P_{+}^{(2^lN)}$-1 *solutions* ($l \to \infty$) *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^lN}, \boldsymbol{\lambda}) = \mathbf{0}$ *is the unstable chaos generated by the limit state of the unstable stable* $P_{-}^{(2^lN)}$-1 *solution* ($l \to \infty$) *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^lN}, \boldsymbol{\lambda}) = \mathbf{0}$ *with all eigenvalue distribution of* $DP_{-}^{(2^lN)}$ *outside unit cycle, vice versa. Such a chaos is the "Yang" chaos in nonlinear discrete dynamical systems.*

(ii) *The unstable chaos generated by the limit state of the unstable* $P_{+}^{(2^lN)}$-1 *solutions* ($l \to \infty$) *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^lN}, \boldsymbol{\lambda}) = \mathbf{0}$ *with all eigenvalue distribution of* $DP_{+}^{(2^lN)}$ *outside the unit cycle is the stable chaos generated by the limit state of the stable* $P_{-}^{(2^lN)}$-1 *solution* ($l \to \infty$) *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^lN}, \boldsymbol{\lambda}) = \mathbf{0}$, *vice versa. Such a chaos is the "Ying" chaos in nonlinear discrete dynamical systems.*

(iii) *The unstable chaos generated by the limit state of the unstable* $P_{+}^{(2^lN)}$-1 *solutions* ($l \to \infty$) *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^lN}, \boldsymbol{\lambda}) = \mathbf{0}$ *with all eigenvalue distribution of* $DP_{+}^{(2^lN)}$ *inside and outside the unit cycle is the unstable chaos generated by the limit state of the unstable* $P_{-}^{(2^lN)}$-1 *solution* ($l \to \infty$) *in sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+2^lN}, \boldsymbol{\lambda}) = \mathbf{0}$ *with switching all eigenvalue distribution of* $DP_{+}^{(2^lN)}$ *inside and outside the unit cycle, vice versa. Such a chaos is the "Ying-Yang" chaos in nonlinear discrete dynamical systems.*

Proof The proof can be referred to Luo [2]. □

6.2 Discrete Systems with Multiple Maps

Definition 6.6 Consider a set of implicit vector functions $\mathbf{f}^{(j)} : D \to D(j = 1, 2, \ldots)$ on an open set $D \subset \mathscr{R}^n$ in an n-dimensional discrete dynamical system. For $\mathbf{x}_k$, $\mathbf{x}_{k+1} \in D$, there is a discrete relation as

$$\mathbf{f}^{(j)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(j)}) = \mathbf{0} \text{ for } j = 1, 2, \cdots \tag{6.19}$$

where the vector function is $\mathbf{f}^{(j)} = (f_1^{(j)}, f_2^{(j)}, \cdots, f_n^{(j)})^{\mathrm{T}} \in \mathscr{R}^n$ and discrete variable vector is $\mathbf{x}_k = (x_{k1}, x_{k2}, \cdots, x_{kn})^{\mathrm{T}} \in \Omega$ with a parameter vector $\mathbf{p}^{(j)} = (p_1^{(j)}, p_2^{(j)}, \cdots, p_{m_j}^{(j)})^{\mathrm{T}} \in \mathscr{R}^{m_j}$.

Definition 6.7 Consider a set of implicit vector functions $\mathbf{f}^{(j)} : D \to D(j = 1, 2, \ldots)$ on an open set $D \subset \mathscr{R}^n$ in an n-dimensional discrete dynamical system.

(i) A set for discrete relations is defined as

$$\Phi = \{\mathbf{f}^{(j)} | \mathbf{f}^{(j)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(j)}) = \mathbf{0}, j \in \mathbb{Z}_+; k \in \mathbb{Z}\}. \tag{6.20}$$

(ii) The positive and negative discrete sets are defined as

$$\left.\begin{aligned} \Sigma_+ &= \{\mathbf{x}_{k+i} | \mathbf{x}_{k+i} \in \mathscr{R}^n, i \in \mathbb{Z}_+\} \subset D, \text{ and} \\ \Sigma_- &= \{\mathbf{x}_{k-i} | \mathbf{x}_{k-i} \in \mathscr{R}^n, i \in \mathbb{Z}_+\} \subset D, \end{aligned}\right\} \tag{6.21}$$

respectively, and the total set of the discrete states is

$$\Sigma = \Sigma_+ \cup \Sigma_-. \tag{6.22}$$

(iii) A positive mapping for $\mathbf{f}^{(j)} \in \Phi$ is defined as

$$P_j^+ : \Sigma \to \Sigma_+ \Rightarrow P_j^+ : \mathbf{x}_k \to \mathbf{x}_{k+1}, \tag{6.23}$$

and a negative mapping is defined by

$$P_j^- : \Sigma \to \Sigma_- \Rightarrow P_j^- : \mathbf{x}_k \to \mathbf{x}_{k-1}. \tag{6.24}$$

(iv) Two sets for positive and negative mappings are defined as

$$\begin{aligned} \Theta_+ &= \{P_j^+ | P_j^+ : \mathbf{x}_k \to \mathbf{x}_{k+1} \text{ with } \mathbf{f}^{(j)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(j)}) = \mathbf{0}, \quad j \in \mathbb{Z}_+; k \in \mathbb{Z}\} \\ \Theta_- &= \{P_j^- | P_j^- : \mathbf{x}_{k+1} \to \mathbf{x}_k \text{ with } \mathbf{f}^{(j)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(j)}) = \mathbf{0}, \quad j \in \mathbb{Z}_+; k \in \mathbb{Z}\} \end{aligned} \tag{6.25}$$

with the total mapping sets are

$$\Theta = \Theta_+ \cup \Theta_-. \tag{6.26}$$

Definition 6.8 Consider a discrete dynamical system with a set of implicit vector functions $\mathbf{f}^{(j)}: D \rightarrow D$ $(j = 1, 2, \cdots)$. For a mapping $P_j^+ \in \Theta_+$ with N-actions and $P_j^- \in \Theta_-$ with N-actions. The resultant mapping is defined as

$$P_{j^N}^+ = \underbrace{P_j^+ \circ P_j^+ \circ \cdots \circ P_j^+}_{N} \text{ and } P_{j^N}^- = \underbrace{P_j^- \circ P_j^- \circ \cdots \circ P_j^-}_{N}. \tag{6.27}$$

Definition 6.9 Consider a discrete dynamical system with a set of implicit vector functions $\mathbf{f}^{(j)}: D \rightarrow D$ $(j = 1, 2, \cdots)$. For the m-positive mappings of $P_{j_i}^+ \in \Theta_+ (i = 1, 2, \cdots, m)$ with N_{j_i}-actions ($N_{j_i} \in \{0, \mathbb{Z}_+\}$) and the corresponding m-negative mappings of $P_{j_i}^- \in \Theta_-$ $(i = 1, 2, \cdots, m)$ with N_{j_i}-actions, the resultant nonlinear mapping cluster with pure positive or negative mappings is defined as

$$\left.\begin{aligned} P^+_{(N_{jm}\cdots N_{j_2}N_{j_1})} &= \underbrace{P^{+}_{j_m^{N_{jm}}} \circ \cdots \circ P^{+}_{j_2^{N_{j2}}} \circ P^{+}_{j_1^{N_{j1}}}}_{m-terms}; \\ P^-_{(N_{j_1}N_{j_2}\cdots N_{jm})} &= \underbrace{P^{-}_{j_1^{N_{j1}}} \circ P^{-}_{j_2^{N_{j2}}} \circ \cdots \circ P^{-}_{j_m^{N_{jm}}}}_{m-terms} \end{aligned}\right\} \tag{6.28}$$

in which at least one of mappings ($P_{j_i}^+$ and $P_{j_i}^-$) with $N_{j_i} \in \mathbb{Z}_+$ possesses a nonlinear iterative relation.

Theorem 6.4 *Consider a discrete dynamical system with a set of implicit vector functions* $\mathbf{f}^{(j)}: D \rightarrow D$ $(j = 1, 2, \cdots)$. *For the m-positive mappings of* $P_{j_i}^+ \in \Theta_+$ $(i = 1, 2, \cdots, m)$ *with* N_{j_i}*-actions* ($N_{j_i} \in \{0, \mathbb{Z}_+\}$) *and the corresponding m-negative mappings of* $P_{j_i}^- \in \Theta_-$ $(i = 1, 2, \cdots, m)$ *with* N_{j_i}*-actions, the resultant nonlinear mapping with pure positive and negative mappings*

$$\mathbf{x}_{k+\Sigma_{s=1}^m N_{js}} = P^+_{(N_{jm}\cdots N_{j_2}N_{j_1})}\mathbf{x}_k \text{ and } \mathbf{x}_k = P^-_{(N_{j_1}N_{j_2}\cdots N_{jm})}\mathbf{x}_{k+\Sigma_{s=1}^m N_{js}}, \tag{6.29}$$

and $\mathbf{x}_{k+i} = P_{j_s}^+ \mathbf{x}_{k+i-1}$ *and* $\mathbf{x}_{k+i-1} = P_{j_s}^- \mathbf{x}_{k+i}$ *can be governed by*

$$\mathbf{f}(\mathbf{x}_{k+i-1}, \mathbf{x}_{k+i}, \mathbf{p}) = \mathbf{0} \text{ for } i = 1, 2, \cdots, \Sigma_{s=1}^m N_{j_t}. \tag{6.30}$$

Suppose a differentiable, vector function $\mathbf{g} \in \mathscr{R}^n$ *possesses* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+\Sigma_{s=1}^m N_{js}}, \boldsymbol{\lambda})\boldsymbol{\varphi} = \mathbf{0}$. *If the solutions* $(\mathbf{x}_k^*, \cdots, \mathbf{x}^*_{k+\Sigma_{s=1}^m N_{js}})$ *of Eq. (6.29) with* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+\Sigma_{s=1}^m N_{js}}, \boldsymbol{\lambda}) = \mathbf{0}$ *exist, then the following conclusions in the sense of* $\mathbf{g}(\mathbf{x}_k, \mathbf{x}_{k+\Sigma_{s=1}^m N_{js}}, \boldsymbol{\lambda}) = \mathbf{0}$ *hold.*

(i) *The stable* $P^+_{(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 *solution is the unstable* $P^-_{(N_{j1}N_{j_2}\cdots N_{jm})}$-1 *solutions with all eigenvalues of* $DP^-_{(N_{j_1}N_{j_2}\cdots N_{jm})}(\mathbf{x}^*_{k+\Sigma_{s=1}^m N_{js}})$ *outside the unit cycle, vice versa.*

(ii) *The unstable* $P^+_{(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 *solution with eigenvalues of* $DP^+_{(N_{jm}\cdots N_{j_2}N_{j_1})}(\mathbf{x}_k^*)$ *outside the unit cycle is the stable* $P^-_{(N_{j_1}N_{j_2}\cdots N_{jm})}$-1 *solutions, vice versa.*

(iii) *For the unstable* $P^{+}_{(N_{jm}\cdots N_{j_2}N_{j_1})}$*-1 solution with eigenvalue distribution of* $DP^{+}_{(N_{jm}\cdots N_{j_2}N_{j_1})}(\mathbf{x}^{*}_{k})$ *inside and outside the unit cycle, the corresponding* $P^{-}_{(N_{j_1}N_{j_2}\cdots N_{jm})}$*-1 solution is also unstable with switching eigenvalue distribution of* $DP^{-}_{(N_{jm}\cdots N_{j_2}N_{j_1})}(\mathbf{x}^{*}_{k+\Sigma_{s=1}^{m}N_{js}})$ *inside and outside the unit cycle, vice versa.*

(iv) *All the bifurcations of the stable and unstable* $P^{+}_{(N_{jm}\cdots N_{j_2}N_{j_1})}$*-1 solution are all the bifurcations of the unstable and stable* $P^{-}_{(N_{j_1}N_{j_2}\cdots N_{jm})}$*-1 solution, respectively.*

Proof The proof can be referred to Luo [2]. □

The chaos generated by the period-doubling of the $P^{+}_{(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 and $P^{-}_{(N_{j_1}N_{j_2}\cdots N_{jm})}$-1 solutions can be described through the following theorem.

Theorem 6.5 *Consider a discrete dynamical system with a set of implicit vector functions* $\mathbf{f}^{(j)}: D \to D\ (j = 1, 2, \cdots)$. *For the m-positive mappings of* $P^{+}_{j_i} \in \Theta_{+}$ $(i = 1, 2, \cdots, m)$ *with* N_{j_i}*-actions* $(N_{j_i} \in \{0, \mathbb{Z}_{+}\})$ *and the corresponding m-negative mappings of* $P^{-}_{j_i} \in \Theta_{-}$ $(i = 1, 2, \cdots, m)$ *with* N_{j_i}*-actions, the resultant nonlinear mapping with pure positive and negative mappings*

$$\mathbf{x}_{k+\Sigma_{s=1}^{m}N_{js}} = P^{+}_{(N_{jm}\cdots N_{j_2}N_{j_1})}\mathbf{x}_{k} \text{ and } \mathbf{x}_{k} = P^{-}_{(N_{j_1}N_{j_2}\cdots N_{jm})}\mathbf{x}_{k+\Sigma_{s=1}^{m}N_{js}}; \tag{6.31}$$

and $\mathbf{x}_{k+i} = P^{+}_{j_s}\mathbf{x}_{k+i-1}$ *and* $\mathbf{x}_{k+i-1} = P^{-}_{j_s}\mathbf{x}_{k+i}$ *can be governed by*

$$\mathbf{f}^{(j)}(\mathbf{x}_{k+i-1}, \mathbf{x}_{k+i}, \mathbf{p}^{(j)}) = \mathbf{0} \text{ for } i = 1, 2, \cdots, \Sigma_{s=1}^{m}N_{j_t}. \tag{6.32}$$

Suppose a differentiable, vector function $\mathbf{g} \in \mathscr{R}^{n}$ *possesses* $\mathbf{g}(\mathbf{x}_{k}, \mathbf{x}_{k+\Sigma_{s=1}^{m}N_{js}}, \boldsymbol{\lambda}) = \mathbf{0}$. *If the period-doubling cascade of the* $P^{+}_{(N_{jm}\cdots N_{j_2}N_{j_1})}$*-1 and* $P^{-}_{(N_{jm}\cdots N_{j_2}N_{j_1})}$*-1 solution occurs, the corresponding mapping structures are given by*

$$\left.\begin{aligned}
\mathbf{x}_{k+2\Sigma_{s=1}^{m}N_{js}} &= P^{+}_{(N_{jm}\cdots N_{j_2}N_{j_1})} \circ P^{+}_{(N_{jm}\cdots N_{j_2}N_{j_1})}\mathbf{x}_{k}\\
&= P^{+}_{2(N_{jm}\cdots N_{j_2}N_{j_1})}\mathbf{x}_{k}\\
\mathbf{g}(\mathbf{x}_{k}, \mathbf{x}_{k+2\Sigma_{s=1}^{m}N_{js}}, \boldsymbol{\lambda}) &= \mathbf{0};
\end{aligned}\right\}$$
$$\left.\begin{aligned}
\mathbf{x}_{k+2^{2}\Sigma_{s=1}^{m}N_{js}} &= P^{+}_{2(N_{jm}\cdots N_{j_2}N_{j_1})} \circ P^{+}_{2(N_{jm}\cdots N_{j_2}N_{j_1})}\mathbf{x}_{k}\\
&= P^{+}_{2^{2}(N_{jm}\cdots N_{j_2}N_{j_1})}\mathbf{x}_{k}\\
\mathbf{g}(\mathbf{x}_{k}, \mathbf{x}_{k+2^{2}\Sigma_{s=1}^{m}N_{js}}, \boldsymbol{\lambda}) &= \mathbf{0};
\end{aligned}\right\} \tag{6.33}$$
$$\vdots$$
$$\left.\begin{aligned}
\mathbf{x}_{k+2^{l}\Sigma_{s=1}^{m}N_{js}} &= P^{+}_{2^{l-1}(N_{jm}\cdots N_{j_2}N_{j_1})} \circ P^{+}_{2^{l-1}(N_{jm}\cdots N_{j_2}N_{j_1})}\mathbf{x}_{k}\\
&= P^{+}_{2^{l}(N_{jm}\cdots N_{j_2}N_{j_1})}\mathbf{x}_{k}\\
\mathbf{g}(\mathbf{x}_{k}, \mathbf{x}_{k+2^{l}\Sigma_{s=1}^{m}N_{js}}, \boldsymbol{\lambda}) &= \mathbf{0};
\end{aligned}\right\}$$

for positive mappings and

$$\left.\begin{aligned}\mathbf{x}_k &= P^-_{(N_{j_1}N_{j_2}\cdots N_{jm})} \circ P^-_{(N_{j_1}N_{j_2}\cdots N_{jm})}\mathbf{x}_{k+2\Sigma_{s=1}^m N_{js}}\\ &= P^-_{2(N_{j_1}N_{j_2}\cdots N_{jm})}\mathbf{x}_{k+2\Sigma_{s=1}^m N_{js}}\\ \mathbf{g}&(\mathbf{x}_k,\mathbf{x}_{k+2\Sigma_{s=1}^m N_{js}},\boldsymbol{\lambda}) = \mathbf{0};\end{aligned}\right\}$$
$$\left.\begin{aligned}\mathbf{x}_k &= P^-_{2(N_{j_1}N_{j_2}\cdots N_{jm})} \circ P^-_{2(N_{j_1}N_{j_2}\cdots N_{jm})}\mathbf{x}_{k+2^2\Sigma_{s=1}^m N_{js}}\\ &= P^-_{2^2(N_{j_1}N_{j_2}\cdots N_{jm})}\mathbf{x}_{k+2^2\Sigma_{s=1}^m N_{js}}\\ \mathbf{g}&(\mathbf{x}_k,\mathbf{x}_{k+2^2\Sigma_{s=1}^m N_{js}},\boldsymbol{\lambda}) = \mathbf{0};\end{aligned}\right\}$$
$$\vdots$$
$$\left.\begin{aligned}\mathbf{x}_k &= P^-_{2^{l-1}(N_{j_1}N_{j_2}\cdots N_{jm})} \circ P^-_{2^{l-1}(N_{j_1}N_{j_2}\cdots N_{jm})}\mathbf{x}_{k+2^l\Sigma_{s=1}^m N_{js}}\\ &= P^-_{2^l(N_{j_1}N_{j_2}\cdots N_{jm})}\mathbf{x}_{k+2^l\Sigma_{s=1}^m N_{js}}\\ \mathbf{g}&(\mathbf{x}_k,\mathbf{x}_{k+2^l\Sigma_{s=1}^m N_{js}},\boldsymbol{\lambda}) = \mathbf{0};\end{aligned}\right\} \tag{6.34}$$

for negative mapping, then the following statements hold, i.e.,

(i) *The stable chaos generated by the limit state of the stable* $P^+_{2^l(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 *solutions* $(l \to \infty)$ *in sense of* $\mathbf{g}(\mathbf{x}_k,\mathbf{x}_{k+2^l\Sigma_{s=1}^m N_{js}},\boldsymbol{\lambda}) = \mathbf{0}$ *is the unstable chaos generated by the limit state of the unstable stable* $P^-_{2^l(N_{j_1}N_{j_2}\cdots N_{jm})}$-1 *solution* $(l \to \infty)$ *in sense of* $\mathbf{g}(\mathbf{x}_k,\mathbf{x}_{k+2^l\Sigma_{s=1}^m N_{js}},\boldsymbol{\lambda}) = \mathbf{0}$ *with all eigenvalue distribution of* $DP^-_{2^l(N_{j_1}N_{j_2}\cdots N_{jm})}$ *outside unit cycle, vice versa. Such a chaos is the "Yang" chaos in nonlinear discrete dynamical systems.*

(ii) *The unstable chaos generated by the limit state of the unstable* $P^+_{2^l(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 *solutions* $(l \to \infty)$ *in sense of* $\mathbf{g}(\mathbf{x}_k,\mathbf{x}_{k+2^l\Sigma_{s=1}^m N_{js}},\boldsymbol{\lambda}) = \mathbf{0}$ *with all eigenvalue distribution of* $P^+_{2^l(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 *outside the unit cycle is the stable chaos generated by the limit state of the stable* $P^-_{2^l(N_{j_1}N_{j_2}\cdots N_{jm})}$-1 *solution* $(l \to \infty)$ *in sense of* $\mathbf{g}(\mathbf{x}_k,\mathbf{x}_{k+2^l\Sigma_{s=1}^m N_{js}},\boldsymbol{\lambda}) = \mathbf{0}$*, vice versa. Such a chaos is the "Ying" chaos in nonlinear discrete dynamical systems.*

(iii) *The unstable chaos generated by the limit state of the unstable* $P^+_{2^l(N_{jm}\cdots N_{j_2}N_{j_1})}$-1 *solutions* $(l \to \infty)$ *in sense of* $\mathbf{g}(\mathbf{x}_k,\mathbf{x}_{k+2^l\Sigma_{s=1}^m N_{js}},\boldsymbol{\lambda}) = \mathbf{0}$ *with all eigenvalue distribution of* $DP^+_{2^l(N_{jm}\cdots N_{j_2}N_{j_1})}$ *inside and outside the unit cycle is the unstable chaos generated by the limit state of the unstable* $P^-_{2^l(N_{j_1}N_{j_2}\cdots N_{jm})}$-1 *solution* $(l \to \infty)$ *in sense of* $\mathbf{g}(\mathbf{x}_k,\mathbf{x}_{k+2^l\Sigma_{s=1}^m N_{js}},\boldsymbol{\lambda}) = \mathbf{0}$ *with switching all eigenvalue distribution of* $DP^-_{2^l(N_{j_1}N_{j_2}\cdots N_{jm})}$ *inside and outside the unit cycle, vice versa. Such a chaos is the "Ying-Yang" chaos in nonlinear discrete dynamical systems.*

Proof The proof can be referred to Luo [2]. □

6.3 Complete Dynamics of a Henon Map System

As in Luo and Guo [3], consider the Henon map system as

$$\left.\begin{aligned} f_1(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) &= x_{k+1} - y_k - 1 + a x_k^2 = 0, \\ f_2(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) &= y_{k+1} - b x_k = 0, \end{aligned}\right\} \tag{6.35}$$

where $\mathbf{x}_k = (x_k, y_k)^{\mathrm{T}}$, $\mathbf{f} = (f_1, f_2)^{\mathrm{T}}$ and $\mathbf{p} = (a, b)^{\mathrm{T}}$. Consider two positive and negative mapping structures as

$$\begin{aligned} \mathbf{x}_{k+N} &= P_{+}^{(N)} \mathbf{x}_k = \underbrace{P_{+} \circ \cdots \circ P_{+} \circ P_{+}}_{N-\text{terms}} \mathbf{x}_k, \\ \mathbf{x}_k &= P_{-}^{(N)} \mathbf{x}_{k+N} = \underbrace{P_{-} \circ \cdots \circ P_{-} \circ P_{-}}_{N-\text{terms}} \mathbf{x}_{k+N}. \end{aligned} \tag{6.36}$$

Equations (6.35) and (6.36) give

$$\left.\begin{aligned} &\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}, \\ &\mathbf{f}(\mathbf{x}_{k+1}, \mathbf{x}_{k+2}, \mathbf{p}) = \mathbf{0}, \\ &\vdots \\ &\mathbf{f}(\mathbf{x}_{k+N-1}, \mathbf{x}_{k+N}, \mathbf{p}) = \mathbf{0} \end{aligned}\right\} \tag{6.37}$$

and

$$\left.\begin{aligned} &\mathbf{f}(\mathbf{x}_{k+N-1}, \mathbf{x}_{k+N}, \mathbf{p}) = \mathbf{0}, \\ &\mathbf{f}(\mathbf{x}_{k+N-2}, \mathbf{x}_{k+N-1}, \mathbf{p}) = \mathbf{0}, \\ &\vdots \\ &\mathbf{f}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}) = \mathbf{0}. \end{aligned}\right\} \tag{6.38}$$

The switching of equation order in Eq. (6.38) shows Eqs. (6.37) and (6.38) are identical. For periodic solutions of the positive and negative maps, the periodicity of the positive and negative mapping structures of the Henon map requires

$$\mathbf{x}_{k+N} = \mathbf{x}_k \text{ or } \mathbf{x}_k = \mathbf{x}_{k+N}. \tag{6.39}$$

So the periodic solutions $\mathbf{x}_{k+j}^{*}$ $(j = 0, 1, \cdots, N)$ for the negative and positive mapping structures are the same, which are given by solving Eqs. (6.37) and (6.38) with Eq. (6.39). However, the stability and bifurcation are different because $\mathbf{x}_{k+j}$ varies with $\mathbf{x}_{k+j-1}$ for the jth positive mapping and $\mathbf{x}_{k+j-1}$ varies with $\mathbf{x}_{k+j}$ for the

jth negative mapping. For a small perturbation, Eq. (6.37) for the positive mapping gives

$$[\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j-1}}] + [\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j}}] \cdot [\frac{\partial \mathbf{x}_{k+j}}{\partial \mathbf{x}_{k+j-1}}]|_{(\mathbf{x}^*_{k+j-1},\mathbf{x}^*_{k+j})} = \mathbf{0}, \tag{6.40}$$

where

$$\begin{aligned}[\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j-1}}]_{(\mathbf{x}^*_{k+j-1},\mathbf{x}^*_{k+j})} &= \begin{bmatrix} \frac{\partial f_1}{\partial x_{k+j-1}} & \frac{\partial f_1}{\partial y_{k+j-1}} \\ \frac{\partial f_2}{\partial x_{k+j-1}} & \frac{\partial f_2}{\partial y_{k+j-1}} \end{bmatrix}_{(\mathbf{x}^*_{k+j-1},\mathbf{x}^*_{k+j})} \\ &= \begin{bmatrix} 2ax^*_{k+j-1} & -1 \\ -b & 0 \end{bmatrix},\end{aligned} \tag{6.41}$$

$$\begin{aligned}[\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j}}]_{(\mathbf{x}^*_{k+j-1},\mathbf{x}^*_{k+j})} &= \begin{bmatrix} \frac{\partial f_1}{\partial x_{k+j}} & \frac{\partial f_1}{\partial y_{k+j}} \\ \frac{\partial f_2}{\partial x_{k+j}} & \frac{\partial f_2}{\partial y_{k+j}} \end{bmatrix}_{(\mathbf{x}^*_{k+j-1},\mathbf{x}^*_{k+j})} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.\end{aligned} \tag{6.42}$$

So

$$\begin{aligned}DP_+(\mathbf{x}^*_{k+j-1}) = [\frac{\partial \mathbf{x}_{k+j}}{\partial \mathbf{x}_{k+j-1}}]_{\mathbf{x}^*_{k+j-1}} &= [\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j}}]^{-1}[\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j-1}}]_{\mathbf{x}^*_{k+j-1}} \\ &= \begin{bmatrix} 2ax^*_{k+j-1} & -1 \\ -b & 0 \end{bmatrix}.\end{aligned} \tag{6.43}$$

Similarly, for the negative mapping,

$$[\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j}}] + [\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j-1}}] \cdot [\frac{\partial \mathbf{x}_{k+j-1}}{\partial \mathbf{x}_{k+j}}]|_{(\mathbf{x}^*_{k+j-1},\mathbf{x}^*_{k+j})} = \mathbf{0}. \tag{6.44}$$

With Eqs. (6.41) and (6.42), the foregoing equation gives

$$\begin{aligned}DP_-(\mathbf{x}^*_{k+j}) = [\frac{\partial \mathbf{x}_{k+j-1}}{\partial \mathbf{x}_{k+j}}]_{\mathbf{x}^*_{k+j}} &= [\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j-1}}]^{-1}[\frac{\partial \mathbf{f}}{\partial \mathbf{x}_{k+j}}]_{\mathbf{x}^*_{k+j}} \\ &= -\frac{1}{b}\begin{bmatrix} 0 & 1 \\ b & 2ax^*_{k+j-1} \end{bmatrix}.\end{aligned} \tag{6.45}$$

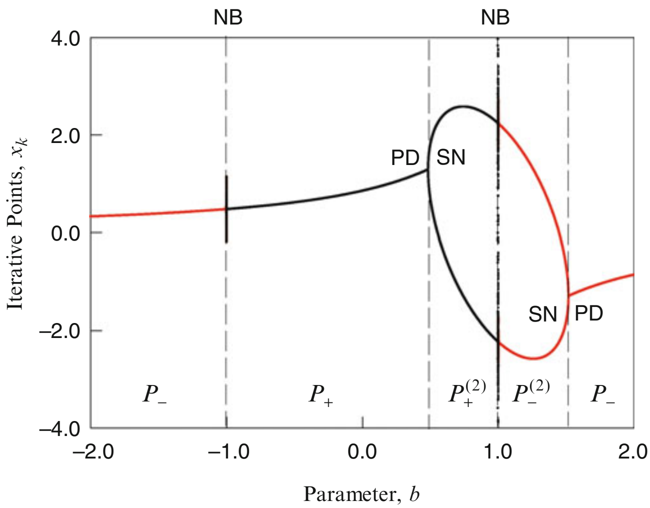

Fig. 6.2 Numerical predictions of periodic solutions of the Henon mapping with negative and positive mappings ($a = 0.2$)

Thus, the resultant perturbation of the mapping structure in Eq. (6.36) gives

$$\begin{aligned} \delta \mathbf{x}_{k+N} &= DP_+^{(N)} \delta \mathbf{x}_k = \underbrace{DP_+ \cdot \ldots \cdot DP_+ \cdot DP_+}_{N-\text{terms}} \delta \mathbf{x}_k, \\ \delta \mathbf{x}_k &= DP_-^{(N)} \delta \mathbf{x}_{k+N} = \underbrace{DP_- \cdot \ldots \cdot DP_- \cdot DP_-}_{N-\text{terms}} \delta \mathbf{x}_{k+N}, \end{aligned} \tag{6.46}$$

where

$$\left. \begin{aligned} DP_+^{(N)} &= \prod\nolimits_{j=1}^{N} DP_+(\mathbf{x}^*_{k+N-j}), \\ DP_-^{(N)} &= \prod\nolimits_{j=1}^{N} DP_-(\mathbf{x}^*_{k+N-j+1}). \end{aligned} \right\} \tag{6.47}$$

From the resultant Jacobian matrix, the eigenvalue analysis can be completed. Before analytical prediction of periodic solution, a numerical prediction of the periodic solutions of the Henon map is presented with varying parameter b for $a = 0.2$, as shown in Fig. 6.2. The dashed vertical lines give the bifurcation points. The acronyms "PD," "SN," and "NB" represented the period-doubling bifurcation, saddle-stable node bifurcation, and Neimark bifurcation, respectively.

From the numerical prediction, the stable periodic solutions of the Henon map are obtained. Herein, through the corresponding mapping structures, the stable and unstable periodic solutions for positive and negative mappings of the Henon maps are presented in Fig. 6.3. The acronyms "PD", "SN", and "NB" represented the period-doubling bifurcation, saddle-stable node bifurcation, and Neimark

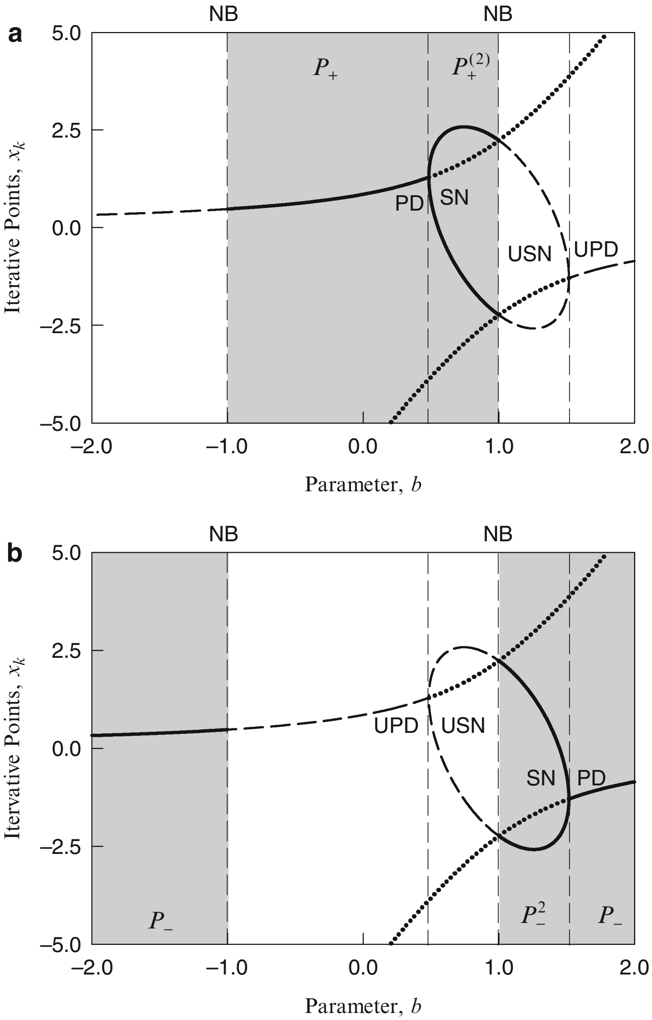

Fig. 6.3 Analytical predictions of *stable* and *unstable* periodic solutions of the Henon map: (**a**) positive mapping (P_+) and (**b**) positive mapping (P_-) ($a = 0.2$ and $b \in (-\infty, +\infty)$)

bifurcation, respectively. The acronyms "UPD" and "USN" represented the period-doubling bifurcation relative to unstable nodes and saddle-unstable node bifurcation, respectively. From the eigenvalue analysis, the stable periodic solutions for positive mapping P_+ lie in $b \in (-1.0, 1.0)$, which is the same as the numerical prediction. In other words, the stable period-1 solution of P_+ is in $b \in (-1, 0.4805)$. For $b \in (0.4805, +\infty)$, the unstable period-1 solution of P_+ is saddle. For $b \in (-\infty, -1.0)$, the unstable period-1 solution of P_+ is unstable focus. The corresponding bifurcations are Neimark bifurcation (NB) and period-doubling bifurcation (PD). However, another unstable period-1 solution of P_+ exists.

For $b \in (1.5215, +\infty)$, the unstable periodic solution of P_+ is unstable node. However, for $b \in (-\infty, 1.5215)$, the unstable periodic solution of P_+ is saddle. Thus, the unstable period-doubling bifurcation (UPD) of the period-1 solution of P_+ occurs at $b \approx 1.5215$. At this point, the unstable periodic solution is from an unstable node to saddle. Because of the unstable period-doubling bifurcation, the unstable periodic solution of $P_+^{(2)}$ is obtained for $b \in (1.0, 1.5215)$. This unstable periodic solution is from unstable focus to unstable node during the parameter of $b \in (1.0, 1.5215)$. At $b \approx 1.5215$, the bifurcation of the unstable periodic solution of $P_+^{(2)}$ occurs between the saddle and unstable node. This bifurcation is called the unstable saddle-node bifurcation. At $b = 1.0$, the Neimark bifurcation (NB) between the periodic solutions of $P_+^{(2)}$ pertaining to the unstable and stable focuses occurs. The stable periodic solution of $P_+^{(2)}$ is from the stable node to the stable focus for $b \in (0.4805, 1.0)$.

Again, from the eigenvalue analysis, the stable periodic solutions for positive mapping P_- lie in $b \in (-\infty, -1.0)$ and $b \in (1.0, +\infty)$, which is the same as in numerical prediction. The stable period-1 solution of P_- is stable focuses in $b \in (-\infty, -1.0)$ and stable nodes in $b \in (1.5215, +\infty)$. For $b \in (-1.0, 0.4805)$, the unstable period-1 solution of P_- is from the unstable focus to unstable node. At $b = -1$, the bifurcation between the stable and unstable period-1 solution of P_- is the Neimark bifurcation (NB). For $b \in (0.4805, +\infty)$, the unstable period-1 solution of P_- is saddle. Thus, the bifurcation between the period-1 solution of P_- between the unstable node and saddle occurs at $b = 0.4805$, which is called the unstable period-doubling bifurcation (UPD). For $b \in (0.4805, +1)$, the unstable period-2 solution of P_- (i.e., $P_-^{(2)}$) is from the unstable node to the unstable focus. For $b \in (1.0, 1.5215)$, the stable period-2 solution of P_- (i.e., $P_-^{(2)}$) is from the stable focus to the stable nodes. Thus, the point at $b \approx 0.4805$ is the bifurcation of the unstable periodic solution of $P_-^{(2)}$ which is the unstable saddle-node bifurcation between the unstable node and saddle (i.e., USN). For the point at $b = 1$, the Neimark bifurcation between the periodic solutions of $P_-^{(2)}$ relative to the unstable and stable focuses occurs. The point at $b \approx 1.5215$ is the bifurcation of the stable periodic solution of $P_-^{(2)}$ which is the saddle bifurcation between the stable node and saddle (SN). For $b \in (-\infty, 1.5215)$, the unstable period-1 solution of P_- is saddle. At $b \approx 1.5215$, the period-doubling bifurcation (PD) of the period-1 solution of P_- takes place.

The strange attractors caused by the period-doubling bifurcation cascade were presented by many researchers. Herein, the strange attractors relative to the Neimark bifurcation between the periodic solutions relative to the unstable and stable focuses are presented. The Poincare mapping relative to the Neimark bifurcation of the period-1 and period-2 solutions of positive mapping (or negative mapping) at $a = 0.2$ and $b = \pm 1$ is presented in Fig. 6.4. In Fig. 6.4a, the most inside point $(x_k^*, y_k^*) \approx (0.4772, -0.4772)$ is the point for the period-1 solution of P_+ or P_- relative to the Neimark bifurcation. With the initial condition $(x_k^*, y_k^*) \approx (1.7188, 0.0)$, the most outside curve is the biggest boundary for the strange attractors around the period-1 solutions with the Neimark bifurcation. The skew symmetry of the strange attractors in the Poincare mapping section is observed. In Fig. 6.4b, the two points $(x_k^*, y_k^*) \approx (2.2361, -2.2361)$

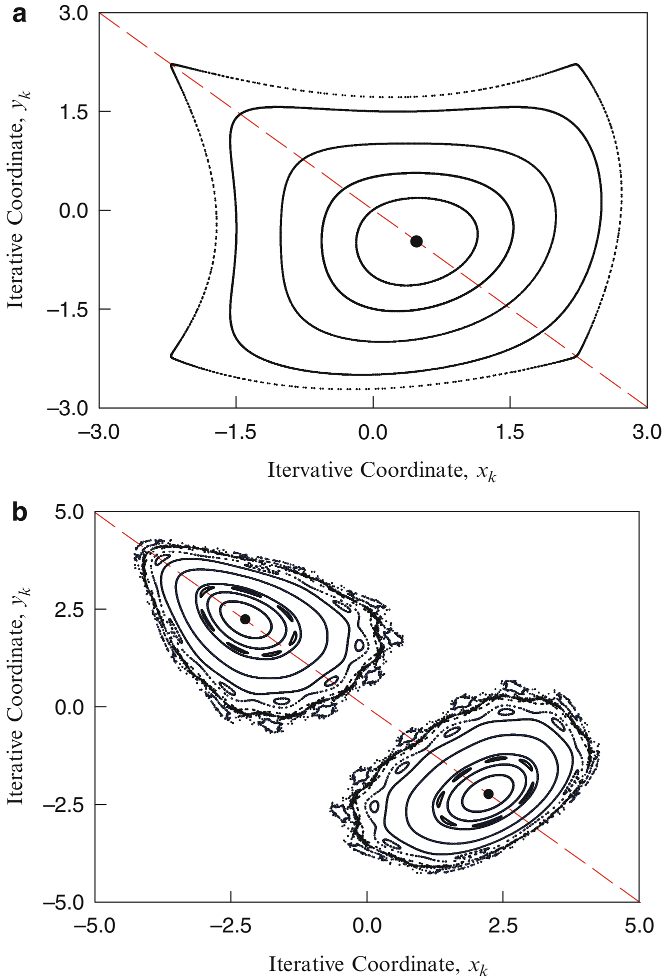

Fig. 6.4 Poincare mappings of the Henon map for the Neimark bifurcation: (**a**) period-1 solution (i.e., P_+-1 or P_--1) ($a = 0.2$ and $b = -1$), and (**b**) period-2 solution (i.e., $P_+^{(2)}$-1 or $P_-^{(2)}$-1) ($a = 0.2$ and $b = 1$)

and $(-2.2361, 2.2361)$ are the points for the period-2 solution of P_+ or P_- relative to the Neimark bifurcation. With the outer chaotic layer, the strange attractor near the periodic solutions of $P_+^{(2)}$-1 (or $P_+^{(2)}$-1) disappears. This chaotic layer possesses eight islands inside the barrier and nine islands outside the barrier. For $(x_k, y_k) \approx (2.9397, -2.2361)$, the seven islands are observed. The skew symmetry of the strange attractors in the Poincare mapping section is observed.

6.4 Companion and Synchronization

This section will extend the concepts presented in the previous section. The companion and synchronization of two discrete dynamical systems will be presented.

Definition 6.10 Consider the αth implicit vector function $\mathbf{f}^{(\alpha)} : D \to D$ ($\alpha = 1, 2, \ldots, N$) on an open set $D \subset \mathscr{R}^n$ in an n-dimensional discrete dynamical system. For $\mathbf{x}_k, \mathbf{x}_{k+1} \in D$, there is a discrete relation as

$$\mathbf{f}^{(\alpha)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(\alpha)}) = \mathbf{0}, \tag{6.48}$$

where the vector function is $\mathbf{f}^{(\alpha)} = (f_1^{(\alpha)}, f_2^{(\alpha)}, \cdots, f_n^{(\alpha)})^{\mathrm{T}} \in \mathscr{R}^n$ and discrete variable vector is $\mathbf{x}_k = (x_{k1}, x_{k2}, \cdots, x_{kn})^{\mathrm{T}} \in D$ with the corresponding parameter vector $\mathbf{p}^{(\alpha)} = (p_1^{(\alpha)}, p_2^{(\alpha)}, \cdots, p_{m_\alpha}^{(\alpha)})^{\mathrm{T}} \in \mathscr{R}^{m_\alpha}$.

Similarly, the discrete sets, positive and negative mappings for discrete dynamical system of $\mathbf{f}^{(\alpha)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(\alpha)}) = \mathbf{0}$ in Eq. (6.48) are defined.

Definition 6.11 For a discrete dynamical system in Eq. (6.48), the positive and negative discrete sets are defined by

$$\left.\begin{aligned} \Sigma_+^{(\alpha)} &= \{\mathbf{x}_{k+i}^{(\alpha)} | \mathbf{x}_{k+i}^{(\alpha)} \in \mathscr{R}^n, i \in \mathbb{Z}_+\} \subset D \text{ and} \\ \Sigma_-^{(\alpha)} &= \{\mathbf{x}_{k-i}^{(\alpha)} | \mathbf{x}_{k-i}^{(\alpha)} \in \mathscr{R}^n, i \in \mathbb{Z}_+\} \subset D, \end{aligned}\right\} \tag{6.49}$$

respectively. The corresponding discrete set is

$$\Sigma^{(\alpha)} = \Sigma_+^{(\alpha)} \cup \Sigma_-^{(\alpha)}. \tag{6.50}$$

A positive mapping for discrete dynamical system is defined as

$$P_{\alpha+} : \Sigma^{(\alpha)} \to \Sigma_+^{(\alpha)} \Rightarrow P_{\alpha+} : \mathbf{x}_k^{(\alpha)} \to \mathbf{x}_{k+1}^{(\alpha)}, \tag{6.51}$$

and a negative mapping is defined by

$$P_{\alpha-} : \Sigma^{(\alpha)} \to \Sigma_-^{(\alpha)} \Rightarrow P_{\alpha-} : \mathbf{x}_k^{(\alpha)} \to \mathbf{x}_{k-1}^{(\alpha)}. \tag{6.52}$$

Definition 6.12 For two discrete dynamical systems in Eq. (6.48), consider two points $\mathbf{x}_k^{(\alpha)}, \mathbf{x}_k^{(\beta)} \in D$ and $\mathbf{x}_{k+1}^{(\alpha)}, \mathbf{x}_{k+1}^{(\beta)} \in D$, and there is a specific, differentiable, vector function $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_l)^{\mathrm{T}} \in \mathscr{R}^l$. For a small number $\varepsilon_k > 0$, there is a small number $\varepsilon_{k+1} > 0$. Suppose there are two sub-domains $\mathrm{U}_k^{(\alpha)} \subset D$ and $\mathrm{U}_k^{(\beta)} \subset D$, then for $\mathbf{x}_k^{(\alpha)} \in \mathrm{U}_k^{(\alpha)}$ and $\mathbf{x}_k^{(\beta)} \in \mathrm{U}_k^{(\beta)}$,

$$||\boldsymbol{\varphi}(\mathbf{x}_k^{(\alpha)}, \mathbf{x}_k^{(\beta)}, \boldsymbol{\lambda})|| \leq \varepsilon_k. \tag{6.53}$$

(i) For $\varepsilon_{k+1}>0$, there are two sub-domains $\mathrm{U}_{k+1}^{(\alpha)} \subset D$ and $\mathrm{U}_{k+1}^{(\beta)} \subset D$. If for $\mathbf{x}_{k+1}^{(\alpha)} \in \mathrm{U}_{k+1}^{(\alpha)}$ and $\mathbf{x}_{k+1}^{(\beta)} \in \mathrm{U}_{k+1}^{(\beta)}$

$$||\boldsymbol{\varphi}(\mathbf{x}_{k+1}^{(\alpha)}, \mathbf{x}_{k+1}^{(\beta)}, \boldsymbol{\lambda})|| \leq \varepsilon_{k+1}, \tag{6.54}$$

then, the discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ are called the companion in sense of $\boldsymbol{\varphi}$ during the kth and $(k+1)$th iteration.

(i.a) The discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ is called the finite companion if for $\mathbf{x}_{k+j}^{(\alpha)} \in \mathrm{U}_{k+j}^{(\alpha)} \subset D$ and $\mathbf{x}_{k+j}^{(\beta)} \in \mathrm{U}_{k+j}^{(\beta)} \subset D$

$$||\boldsymbol{\varphi}(\mathbf{x}_{k+j}^{(\alpha)}, \mathbf{x}_{k+j}^{(\beta)}, \boldsymbol{\lambda})|| \leq \varepsilon_{k+j} \quad \text{for } j = 1, 2, \cdots, N. \tag{6.55}$$

(i.b) The discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ is called the absolute permanent companion if $\mathbf{x}_{k+j}^{(\alpha)} \in \mathrm{U}_{k+j}^{(\alpha)} \subset D$ and $\mathbf{x}_{k+j}^{(\beta)} \in \mathrm{U}_{k+j}^{(\beta)} \subset D$

$$||\boldsymbol{\varphi}(\mathbf{x}_{k+j}^{(\alpha)}, \mathbf{x}_{k+j}^{(\beta)}, \boldsymbol{\lambda})|| \leq \varepsilon_{k+j} \quad \text{for } j = 1, 2, \cdots. \tag{6.56}$$

(i.c) The discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ is called the repeatable finite companion if $\mathbf{x}_{k+jN(-)}^{(\alpha)} \in \mathrm{U}_{k+jN(-)}^{(\alpha)} \subset D$ and $\mathbf{x}_{k+j(-)}^{(\beta)} \in \mathrm{U}_{k+j(-)}^{(\beta)} \subset D$

$$\begin{aligned}
&\Delta\mathbf{I}^{(\alpha)} : \mathbf{x}_{k+jN(-)}^{(\alpha)} \to \mathbf{x}_{k+jN(+)}^{(\alpha)}, \text{ and } \Delta\mathbf{I}^{(\beta)} : \mathbf{x}_{k+jN(-)}^{(\beta)} \to \mathbf{x}_{k+jN(+)}^{(\beta)},\\
&\mathbf{x}_{k+jN(+)}^{(\alpha)} = \mathbf{x}_{k+jN(-)}^{(\alpha)} + \Delta\mathbf{I}_{jN}^{(\alpha)} \text{ and } \mathbf{x}_{k+jN(+)}^{(\beta)} = \mathbf{x}_{k+jN(-)}^{(\beta)} + \Delta\mathbf{I}_{jN}^{(\beta)};\\
&||\boldsymbol{\varphi}(\mathbf{x}_{k+j(+)}^{(\alpha)}, \mathbf{x}_{k+j(+)}^{(\beta)}, \boldsymbol{\lambda})|| \leq \varepsilon_{k+\mathrm{mod}(j,N)} \text{ for } j = 1, 2, \cdots,\\
&\text{with } \mathbf{x}_{k+j(+)}^{(\alpha)} \in \mathrm{U}_{k+\mathrm{mod}(j,N)}^{(\alpha)} \text{ and } \mathbf{x}_{k+j(+)}^{(\beta)} \in \mathrm{U}_{k+\mathrm{mod}(j,N)}^{(\beta)}.
\end{aligned} \tag{6.57}$$

(ii) For $\varepsilon_k>0$, $\varepsilon_{k+(N_\alpha:N_\beta)}>0$ there are there are two sub-domains $\mathrm{U}_{k+N_\alpha}^{(\alpha)} \subset D$ and $\mathrm{U}_{k+N_\beta}^{(\beta)} \subset D$. For $\mathbf{x}_{k+N_\alpha}^{(\alpha)} \in \mathrm{U}_{k+N_\alpha}^{(\alpha)}$ and $\mathbf{x}_{k+N_\beta}^{(\beta)} \in \mathrm{U}_{k+N_\beta}^{(\beta)}$ if

$$||\boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}^{(\alpha)}, \mathbf{x}_{k+N_\beta}^{(\beta)}, \boldsymbol{\lambda})|| \leq \varepsilon_{k+(N_\alpha:N_\beta)}, \tag{6.58}$$

then the discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ from the kth to $(k+N_\alpha)$th iteration and $\mathbf{f}^{(\beta)}$ from the kth to $(k+N_\beta)$th iteration are called the $(N_\alpha : N_\beta)$-companion in sense of $\boldsymbol{\varphi}$.

(ii.a) The discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ is called the finite $(N_\alpha : N_\beta)$ companion if for $\mathbf{x}_{k+jN_\alpha}^{(\alpha)} \in \mathrm{U}_{k+jN_\alpha}^{(\alpha)} \subset D$ and $\mathbf{x}_{k+jN_\beta}^{(\beta)} \in \mathrm{U}_{k+jN_\beta}^{(\beta)} \subset D$

$$||\boldsymbol{\varphi}(\mathbf{x}^{(\alpha)}_{k+jN_\alpha},\mathbf{x}^{(\beta)}_{k+jN_\beta},\boldsymbol{\lambda})|| \leq \varepsilon_{k+j(N_\alpha:N_\beta)} \text{ for } j=1,2,\cdots,N \tag{6.59}$$

(ii.b) The discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ is called the absolute permanent $(N_\alpha : N_\beta)$ companion if $\mathbf{x}^{(\alpha)}_{k+jN_\alpha} \in \mathrm{U}^{(\alpha)}_{k+jN_\alpha} \subset D$ and $\mathbf{x}^{(\beta)}_{k+jN_\beta} \in \mathrm{U}^{(\beta)}_{k+jN_\beta} \subset D$

$$||\boldsymbol{\varphi}(\mathbf{x}^{(\alpha)}_{k+jN_\alpha},\mathbf{x}^{(\beta)}_{k+jN_\beta},\boldsymbol{\lambda})|| \leq \varepsilon_{k+j(N_\alpha:N_\beta)} \text{ for } j=1,2,\cdots, \tag{6.60}$$

(ii.c) The discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ is called the repeatable finite $(N_\alpha : N_\beta)$ companion if $\mathbf{x}^{(\alpha)}_{k+jN_\alpha} \in \mathrm{U}^{(\alpha)}_{k+jN_\alpha} \subset D$ and $\mathbf{x}^{(\beta)}_{k+jN_\beta} \in \mathrm{U}^{(\beta)}_{k+jN_\beta} \subset D$

$$\begin{aligned}
&\Delta\mathbf{I}^{(\alpha)}:\mathbf{x}^{(\alpha)}_{k+jN_\alpha(-)} \to \mathbf{x}^{(\alpha)}_{k+jN_\alpha(+)}, \text{ and } \Delta\mathbf{I}^{(\beta)}:\mathbf{x}^{(\beta)}_{k+jN_\beta(-)} \to \mathbf{x}^{(\beta)}_{k+jN_\beta(+)}\\
&\mathbf{x}^{(\alpha)}_{k+jN_\alpha(+)} = \mathbf{x}^{(\alpha)}_{k+jN_\alpha(-)} + \Delta\mathbf{I}^{(\alpha)}_{jN_\alpha} \text{ and } \mathbf{x}^{(\beta)}_{k+jN_\beta(+)} = \mathbf{x}^{(\beta)}_{k+jN_\beta(-)} + \Delta\mathbf{I}^{(\beta)}_{jN_\beta}\\
&||\boldsymbol{\varphi}(\mathbf{x}^{(\alpha)}_{k+jN_\alpha(+)},\mathbf{x}^{(\beta)}_{k+jN_\beta(+)},\boldsymbol{\lambda})|| \leq \varepsilon_{k+\mathrm{mod}(j,N)(N_\alpha:N_\beta)} \text{ for } j=1,2,\cdots,\\
&\mathbf{x}^{(\alpha)}_{k+jN_\alpha(+)} \in \mathrm{U}^{(\alpha)}_{k+\mathrm{mod}(j,N)N_\alpha} \text{ and } \mathbf{x}^{(\beta)}_{k+jN_\beta(+)} \in \mathrm{U}^{(\beta)}_{k+\mathrm{mod}(j,N)N_\beta}.
\end{aligned} \tag{6.61}$$

Definition 6.13 For two discrete dynamical systems in Eq. (6.48), consider two points $\mathbf{x}^{(\alpha)}_k, \mathbf{x}^{(\beta)}_k \in D$ and $\mathbf{x}^{(\alpha)}_{k+1}, \mathbf{x}^{(\beta)}_{k+1} \in D$, and there is a specific, differentiable, vector function $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_l)^{\mathrm{T}} \in \mathscr{R}^l$. For

$$\boldsymbol{\varphi}(\mathbf{x}^{(\alpha)}_k,\mathbf{x}^{(\beta)}_k,\boldsymbol{\lambda}) = \mathbf{0}, \tag{6.62}$$

(i) If

$$\boldsymbol{\varphi}(\mathbf{x}^{(\alpha)}_{k+1},\mathbf{x}^{(\beta)}_{k+1},\boldsymbol{\lambda}) = \mathbf{0}, \tag{6.63}$$

then, discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ are called the $(1:1)$ synchronization in sense of $\boldsymbol{\varphi}$;

(ii) If

$$\begin{aligned}
&\boldsymbol{\varphi}(\mathbf{x}^{(\alpha)}_{k+1},\mathbf{x}^{(\beta)}_{k+1},\boldsymbol{\lambda}) = \mathbf{0} \text{ with}\\
&\Delta\mathbf{I}^{(\alpha)}:\mathbf{x}^{(\alpha)}_{k+1(-)} \to \mathbf{x}^{(\alpha)}_{k+1(+)} \text{ and } \Delta\mathbf{I}^{(\beta)}:\mathbf{x}^{(\beta)}_{k+1(-)} \to \mathbf{x}^{(\beta)}_{k+1(+)},\\
&\mathbf{x}^{(\alpha)}_{k+1(+)} = \mathbf{x}^{(\alpha)}_{k+1(-)} + \Delta\mathbf{I}^{(\alpha)} \text{ and } \mathbf{x}^{(\beta)}_{k+1(+)} = \mathbf{x}^{(\beta)}_{k+1(-)} + \Delta\mathbf{I}^{(\beta)},\\
&\mathbf{x}^{(\alpha)}_{k+1(+)} = \mathbf{x}^{(\alpha)}_k \text{ and } \mathbf{x}^{(\beta)}_{k+1(+)} = \mathbf{x}^{(\beta)}_k;
\end{aligned} \tag{6.64}$$

then, discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ are called the repeatable $(1:1)$ synchronization in sense of $\boldsymbol{\varphi}$;

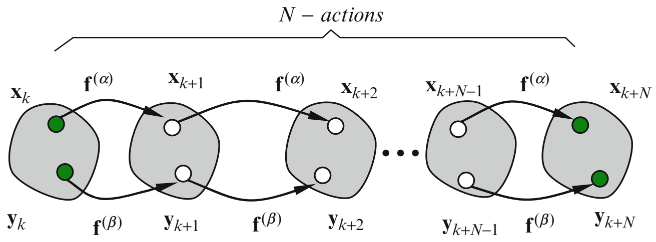

Fig. 6.5 Companion of two discrete dynamical systems

(iii) If

$$\boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}^{(\alpha)}, \mathbf{x}_{k+N_\beta}^{(\beta)}, \boldsymbol{\lambda}) = \mathbf{0} \tag{6.65}$$

then the discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ are called the $(N_\alpha : N_\beta)$-synchronization in sense of $\boldsymbol{\varphi}$;

(iv) If

$$\begin{aligned}
&\boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}^{(\alpha)}, \mathbf{x}_{k+N_\beta}^{(\beta)}, \boldsymbol{\lambda}) = \mathbf{0} \text{ with}\\
&\Delta\mathbf{I}^{(\alpha)} : \mathbf{x}_{k+N_\alpha}^{(\alpha)} \to \mathbf{x}_k^{(\alpha)} \text{ and } \Delta\mathbf{I}^{(\beta)} : \mathbf{x}_{k+N_\beta}^{(\beta)} \to \mathbf{x}_k^{(\beta)},\\
&\mathbf{x}_{k+N_\alpha(+)}^{(\alpha)} = \mathbf{x}_{k+N_\alpha(-)}^{(\alpha)} + \Delta\mathbf{I}^{(\alpha)} \text{ and } \mathbf{x}_{k+N_\beta(+)}^{(\beta)} = \mathbf{x}_{k+N_\beta(-)}^{(\beta)} + \Delta\mathbf{I}^{(\beta)},\\
&\mathbf{x}_{k+N_\beta(+)}^{(\alpha)} = \mathbf{x}_k^{(\alpha)} \text{ and } \mathbf{x}_{k+N_\beta(+)}^{(\beta)} = \mathbf{x}_k^{(\beta)}.
\end{aligned} \tag{6.66}$$

then the discrete dynamical systems of $\mathbf{f}^{(\alpha)}$ and $\mathbf{f}^{(\beta)}$ are called the repeatable $(N_\alpha : N_\beta)$-synchronization in sense of $\boldsymbol{\varphi}$.

From the definition, the companions of two discrete dynamical systems are presented in Figs. 6.5 and 6.6. For each step, if the corresponding relation satisfies Eq. (6.62), the companion is called the $(1:1)$ companion, which is presented in Fig. 6.5. The shaded areas are the companion domain which is controlled by ε_k and $\boldsymbol{\varphi}$. For the repeated companion, for each step, the companion with specific impulses will have the same control domains. Such shaded areas can be overlapped or separated. The $(N_\alpha : N_\beta)$ state for $\mathbf{f}^{(\alpha)}$ with N_α-iterations and $\mathbf{f}^{(\beta)}$ with N_β-iterations satisfy Eq. (6.65) is called the $(N_\alpha : N_\beta)$-companion, which is sketched in Fig. 6.6a. This companion does not require each iteration step to do so. The companion states are shaded. For the repeated companion, the companion state with specific impulses will have the same control domains. The companion for negative maps can be similarly defined, as shown in Fig. 6.6b.

Consider synchronization of two discrete dynamical systems, as shown in Fig. 6.7, with

$$\mathbf{f}^{(\alpha)}(\mathbf{x}_{k+1}, \mathbf{x}_k, \mathbf{p}^{(\alpha)}) = \mathbf{0} \text{ and } \mathbf{f}^{(\beta)}(\mathbf{y}_{k+1}, \mathbf{y}_k, \mathbf{p}^{(\beta)}) = \mathbf{0}. \tag{6.67}$$

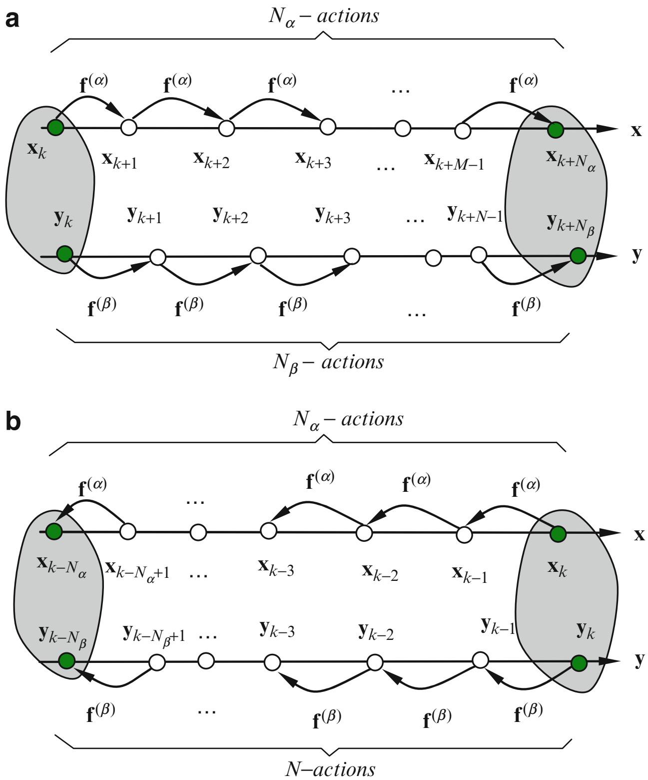

Fig. 6.6 Companion of two discrete nonlinear systems: (**a**) positive companion, (**b**) negative companion

For the initial state, there is a relation as

$$\boldsymbol{\varphi}(\mathbf{x}_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{0}. \tag{6.68}$$

For the positive synchronization, there are N_α-actions with function $\mathbf{f}^{(\alpha)}$ and mapping $P_{\alpha+}$ and N_β-actions with function $\mathbf{f}^{(\beta)}$ and mapping $P_{\beta+}$

$$\begin{aligned} \mathbf{f}^{(\alpha)}(\mathbf{x}_{k+i}, \mathbf{x}_{k+i-1}, \mathbf{p}^{(\alpha)}) &= \mathbf{0} \text{ for } i = 1, 2, \cdots, N_\alpha, \\ \mathbf{f}^{(\beta)}(\mathbf{y}_{k+j}, \mathbf{y}_{k+j-1}, \mathbf{p}^{(\beta)}) &= \mathbf{0} \text{ for } j = 1, 2, \cdots, N_\beta; \end{aligned} \tag{6.69}$$

and the synchronization is based on

$$\boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}, \mathbf{y}_{k+N_\beta}, \boldsymbol{\lambda}) = \mathbf{0}. \tag{6.70}$$

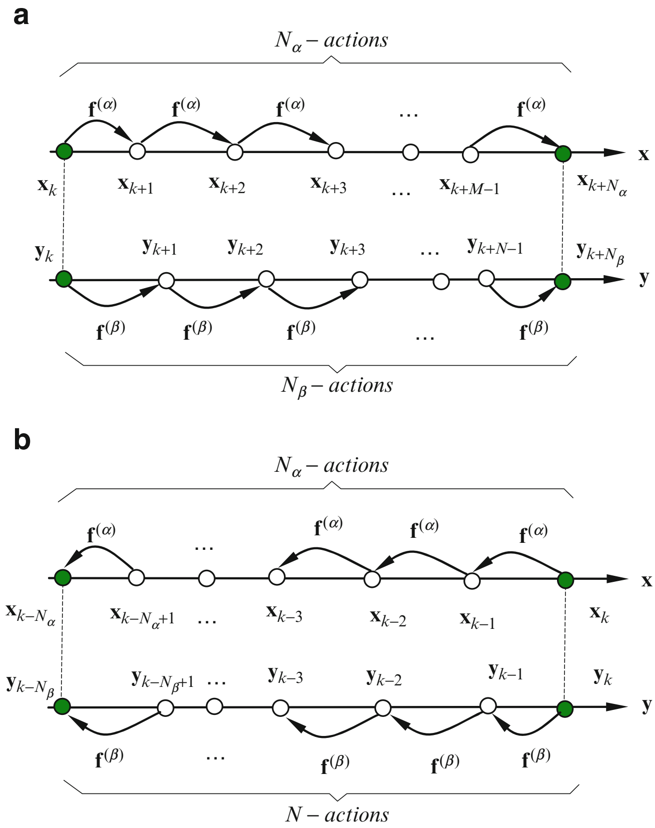

Fig. 6.7 Synchronization of two discrete nonlinear systems: (**a**) positive synchronization, (**b**) negative synchronization

For the negative synchronization, there are N_α-actions with function $\mathbf{f}^{(\alpha)}$ and mapping $P_{\alpha-}$ and N_β-actions with function $\mathbf{f}^{(\beta)}$ and mapping $P_{\beta-}$

$$\begin{aligned}&\mathbf{f}^{(\alpha)}(\mathbf{x}_{k-i+1},\mathbf{x}_{k-i-1},\mathbf{p}^{(\alpha)})=\mathbf{0}\text{ for }i=1,2,\cdots,N_\alpha,\\&\mathbf{f}^{(\beta)}(\mathbf{y}_{k-j},\mathbf{y}_{k-j-1},\mathbf{p}^{(\beta)})=\mathbf{0}\text{ for }j=1,2,\cdots,N_\beta\end{aligned}\tag{6.71}$$

and the synchronization is based on

$$\boldsymbol{\varphi}(\mathbf{x}_{k-N_\alpha},\mathbf{y}_{k-N_\beta},\boldsymbol{\lambda})=\mathbf{0}.\tag{6.72}$$

Thus there is a relation

$$\mathbf{x}_k=P_{\boldsymbol{\varphi}-}\circ P_{\beta-}^{(N_\beta)}\circ P_{\boldsymbol{\varphi}+}\circ P_{\alpha+}^{(N_\alpha)}\mathbf{x}_k,\tag{6.73}$$

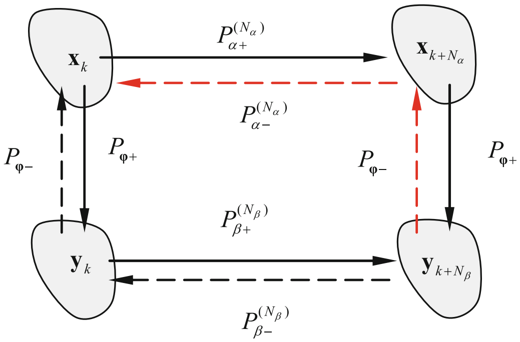

Fig. 6.8 Commutative mapping diagram for synchronization

where

$$
\begin{aligned}
&P_{\boldsymbol{\varphi}-} \circ P_{\beta-}^{(N_\beta)} \circ P_{\boldsymbol{\varphi}+} \circ P_{\alpha+}^{(N_\alpha)} \\
&\quad = P_{\boldsymbol{\varphi}-} \circ \underbrace{P_{\beta-} \circ P_{\beta-} \circ \cdots \circ P_{\beta-}}_{N_\beta-actions} \circ P_{\boldsymbol{\varphi}+} \circ \underbrace{P_{\alpha+} \circ P_{\alpha+} \circ \cdots \circ P_{\alpha+}}_{N_\alpha-actions}.
\end{aligned} \tag{6.74}
$$

From Eq. (6.73), we have

$$
\begin{aligned}
&\mathbf{x}_{k+N_\alpha} = P_{\alpha+}^{(N_\alpha)}\mathbf{x}_k \text{ and } \mathbf{y}_{k+N_\beta} = P_{\boldsymbol{\varphi}+}\mathbf{x}_{k+N_\alpha}, \\
&\mathbf{y}_k = P_{\beta-}^{(N_\beta)}\mathbf{y}_{k+N_\beta} \text{ and } \mathbf{x}_k = P_{\boldsymbol{\varphi}-}\mathbf{y}_k
\end{aligned} \tag{6.75}
$$

and

$$
\begin{aligned}
&\mathbf{x}_{k+N_\alpha} = P_{\alpha+}^{(N_\alpha)}\mathbf{x}_k \text{ and } \mathbf{y}_{k+N_\beta} = P_{\boldsymbol{\varphi}+}\mathbf{x}_{k+N_\alpha}, \\
&P_{\boldsymbol{\varphi}+}\mathbf{x}_k = \mathbf{y}_k \text{ and } P_{\beta+}^{(N_\beta)}\mathbf{y}_k = \mathbf{y}_{k+N_\beta}.
\end{aligned} \tag{6.76}
$$

The corresponding commutative diagram is given in Fig. 6.8. The solid and dashed arrows give the positive and negative mappings, respectively.

From the above discussion on synchronization of $P_{\alpha+}^{(N_\alpha)}$ and $P_{\beta+}^{(N_\beta)}$ under the constraint $\boldsymbol{\varphi}$, the following relations should exist

$$
\begin{aligned}
&\mathbf{x}'_k = P_{\boldsymbol{\varphi}-} \circ P_{\beta-}^{(N_\beta)} \circ P_{\boldsymbol{\varphi}+} \circ P_{\alpha+}^{(N_\alpha)}\mathbf{x}_k, \text{ or} \\
&\mathbf{x}'_k = P_{\alpha-}^{(N_\alpha)} \circ P_{\boldsymbol{\varphi}-} \circ P_{\beta+}^{(N_\beta)} \circ P_{\boldsymbol{\varphi}+}\mathbf{x}_k;
\end{aligned} \tag{6.77}
$$

The above equation forms an iterative mapping. If the fixed point exits, i.e.,

$$
\mathbf{x}'_k = \mathbf{x}_k, \tag{6.78}
$$

then the synchronization of $P_{\alpha+}^{(N_\alpha)}$ and $P_{\beta-}^{(N_\beta)}$ under the constraint $\boldsymbol{\varphi}$ exists

$$\begin{aligned}&\mathbf{x}_{k+N_\alpha} = P_{\alpha+}^{(N_\alpha)}\mathbf{x}_k, \text{ and } \mathbf{y}_{k+N_\beta} = P_{\beta+}^{(N_\beta)}\mathbf{y}_k;\\ &\mathbf{y}_k = P_{\boldsymbol{\varphi}+}\mathbf{x}_k \text{ and } \mathbf{y}_{k+N_\beta} = P_{\boldsymbol{\varphi}+}\mathbf{x}_{k+N_\alpha}.\end{aligned} \tag{6.79}$$

Theorem 6.6 *Consider two discrete dynamical systems $(P_\alpha, \mathbf{f}^{(\alpha)})$ and $(P_\beta, \mathbf{f}^{(\beta)})$ as in Eq. (6.48) with*

$$\begin{aligned}&P_{\alpha+} : \mathbf{x}_k \to \mathbf{x}_{k+1} \text{ and } P_{\alpha-} : \mathbf{x}_{k+1} \to \mathbf{x}_k,\\ &\mathbf{f}^{(\alpha)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(\alpha)}) = \mathbf{0};\end{aligned} \tag{6.80}$$

and

$$\begin{aligned}&P_{\beta+} : \mathbf{y}_k \to \mathbf{y}_{k+1} \text{ and } P_{\alpha-} : \mathbf{y}_{k+1} \to \mathbf{y}_k,\\ &\mathbf{f}^{(\beta)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(\beta)}) = \mathbf{0}.\end{aligned} \tag{6.81}$$

For two points $\mathbf{x}_k \in D_\alpha$ and $\mathbf{y}_k \in D_\beta$, there is a specific, differentiable, vector function $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_l)^{\mathrm{T}} \in \mathscr{R}^l$. The synchronization of two discrete dynamical systems $(P_\alpha, \mathbf{f}^{(\alpha)})$ and $(P_\beta, \mathbf{f}^{(\beta)})$ is under the following constraints

$$\boldsymbol{\varphi}(\mathbf{x}_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{0} \text{ and } \boldsymbol{\varphi}(\mathbf{x}_{k+1}, \mathbf{y}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0}. \tag{6.82}$$

Consider a resultant hybrid mapping relation as

$$\mathbf{x}'_k = P\mathbf{x}_k = P_{\boldsymbol{\varphi}-} \circ P_{\beta-} \circ P_{\boldsymbol{\varphi}+} \circ P_{\alpha+}\mathbf{x}_k \tag{6.83}$$

with

$$\begin{aligned}&P_{\alpha+} : \mathbf{x}_k \to \mathbf{x}_{k+1} \text{ with } \mathbf{f}^{(\alpha)}(\mathbf{x}_{k+1}, \mathbf{x}_k, \mathbf{p}^{(\alpha)}) = \mathbf{0},\\ &P_{\boldsymbol{\varphi}+} : \mathbf{x}_{k+1} \to \mathbf{y}_{k+1} \text{ with } \boldsymbol{\varphi}(\mathbf{x}_{k+1}, \mathbf{y}_{k+1}, \boldsymbol{\lambda}) = \mathbf{0},\\ &P_{\beta-} : \mathbf{y}_{k+1} \to \mathbf{y}_k \text{ with } \mathbf{f}^{(\beta)}(\mathbf{y}_{k+1}, \mathbf{y}_k, \mathbf{p}^{(\alpha)}) = \mathbf{0},\\ &P_{\boldsymbol{\varphi}-} : \mathbf{y}_k \to \mathbf{x}'_k \text{ with } \boldsymbol{\varphi}(\mathbf{x}'_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{0};\\ &\mathbf{x}'_k = \mathbf{x}_k\end{aligned} \tag{6.84}$$

and

$$DP(\mathbf{x}_k^*) = DP_{\boldsymbol{\varphi}-}(\mathbf{y}_k^*) \cdot DP_{\beta-}(\mathbf{y}_{k+1}^*) \cdot DP_{\boldsymbol{\varphi}+}(\mathbf{x}_{k+1}^*) \cdot DP_{\alpha+}(\mathbf{x}_k^*), \tag{6.85}$$

where

$$\begin{aligned}&DP(\mathbf{x}_k^*) = \left[\frac{\partial \mathbf{x}'_k}{\partial \mathbf{x}_k}\right]_{\mathbf{x}_k^*}, DP_{\alpha+}(\mathbf{x}_k^*) = \left[\frac{\partial \mathbf{x}_{k+1}}{\partial \mathbf{x}_k}\right]_{\mathbf{x}_k^*,} , DP_{\boldsymbol{\varphi}+}(\mathbf{x}_{k+1}^*) = \left[\frac{\partial \mathbf{y}_{k+1}}{\partial \mathbf{x}_{k+1}}\right]_{\mathbf{x}_{k+1}^*},\\ &DP_{\beta-}(\mathbf{y}_{k+1}^*) = \left[\frac{\partial \mathbf{y}_k}{\partial \mathbf{y}_{k+1}}\right]_{\mathbf{y}_{k+1}^*}, DP_{\boldsymbol{\varphi}-}(\mathbf{y}_k^*) = \left[\frac{\partial \mathbf{x}'_k}{\partial \mathbf{y}_k}\right]_{\mathbf{y}_k^*}.\end{aligned} \tag{6.86}$$

(i) *The* $(1:1)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is persistent if and only if all the eigenvalues* λ_i $(i = 1, 2, \cdots, n)$ *of* $DP(\mathbf{x}_k^*)$ *lie in the unit circles, i.e.,*

$$|\lambda_i| < 1 \text{ for } i = 1, 2, \cdots, n. \tag{6.87}$$

(ii) *The* $(1:1)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is a saddle-node vanishing if and only if at least one of the real eigenvalues* λ_i *(*$i = 1, 2, \cdots, n_1$ *and* $n_1 \leq n$*) of* $DP(\mathbf{x}_k^*)$ *is positive one (+1) and the other eigenvalues are in the unit circle, i.e.,*

$$\lambda_i = 1 \text{ and } |\lambda_j| < 1 \text{ for } i, j \in \{1, 2, \cdots, n\} \text{ and } j \neq i. \tag{6.88}$$

(iii) *The* $(1:1)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is a period-doubling vanishing if and only if at least one of the real eigenvalues* λ_i *(*$i = 1, 2, \cdots, n_1$ *and* $n_1 \leq n$*) of* $DP(\mathbf{x}_k^*)$ *is negative one (−1) and the other eigenvalues are in the unit circle, i.e.,*

$$\lambda_i = -1 \text{ and } |\lambda_j| < 1 \text{ for } i, j \in \{1, 2, \cdots, n\} \text{ and } j \neq i. \tag{6.89}$$

(iv) *The* $(1:1)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is a Naimark vanishing if and only if one pair of all the complex eigenvalues* $\lambda_i = \alpha_i \pm \beta_i \mathbf{i}$ *(*$i = 1, 2, \cdots, n_1$ *and* $n_1 \leq n/2$*) of* $DP(\mathbf{x}_k^*)$ *are on the unit circle and the other eigenvalues are in the unit circle, i.e.,*

$$|\lambda_i| = \sqrt{\alpha_i^2 + \beta_i^2} = 1 \text{ and } |\lambda_j| < 1 \text{ for } i, j \in \{1, 2, \cdots, n\} \text{ and } j \neq i. \tag{6.90}$$

(v) *The* $(1:1)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is an (*$l_1 : l_2 : l_3$*) vanishing if and only if* l_1 *and* l_2 *real eigenvalues* λ_i *of* $DP(\mathbf{x}_k^*)$ *are (−1) and (+1), respectively, and* l_3*-pairs of complex eigenvalues* $\lambda_i = \alpha_i \pm \beta_i \mathbf{i}$ *(*$i = 1, 2, \cdots, n_1$ *and* $n_1 \leq n/2$*) of* $DP(\mathbf{x}_k^*)$ *are on the unit circle and the other eigenvalues are in the unit circle, i.e.,*

$$\begin{aligned}
&\lambda_i = -1 \text{ for } i = i_1, i_2, \cdots, i_{l_1} \in \{1, 2, \cdots, n\} \\
&\lambda_j = +1 \text{ for } j = j_1, j_2, \cdots, j_{l_2} \in \{1, 2, \cdots, n\} \\
&|\lambda_r| = \sqrt{\alpha_r^2 + \beta_r^2} = 1 \text{ for } r = r_1, r_2, \cdots, r_{l_3} \in \{1, 2, \cdots, n\} \\
&|\lambda_s| < 1 \text{ for } s \in \{1, 2, \cdots, n\} \text{ and } s \notin \{i, j, r\}.
\end{aligned} \tag{6.91}$$

(vi) *The* $(1:1)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is instantaneous if and only if at least one of the eigenvalues* $\lambda_i(i = 1, 2, \ldots, n)$ *of* $DP(\mathbf{x}_k^*)$ *lies out of the unit circle, i.e.,*

$$|\lambda_i| > 1 \text{ for } i \in \{1, 2, \cdots, n\}. \tag{6.92}$$

Proof The proof can be referred to Luo [2]. □

Theorem 6.7 *Consider two discrete dynamical systems* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *as in Eq. (6.48) with*

$$\begin{aligned} &P_{\alpha+} : \mathbf{x}_k \to \mathbf{x}_{k+1} \text{ and } P_{\alpha-} : \mathbf{x}_{k+1} \to \mathbf{x}_k \\ &\mathbf{f}^{(\alpha)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(\alpha)}) = \mathbf{0}, \end{aligned} \tag{6.93}$$

and

$$\begin{aligned} &P_{\beta+} : \mathbf{y}_k \to \mathbf{y}_{k+1} \text{ and } P_{\beta-} : \mathbf{y}_{k+1} \to \mathbf{y}_k, \\ &\mathbf{f}^{(\beta)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(\beta)}) = \mathbf{0}. \end{aligned} \tag{6.94}$$

For two points $\mathbf{x}_k \in D_\alpha$ *and* $\mathbf{y}_k \in D_\beta$*, there is a specific, differentiable, vector function* $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_l)^{\mathrm{T}} \in \mathscr{R}^l$*. The* $(N_\alpha : N_\beta)$*-synchronization of two discrete dynamical systems* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is under the following constraints*

$$\boldsymbol{\varphi}(\mathbf{x}_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{0} \text{ and } \boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}, \mathbf{y}_{k+N_\beta}, \boldsymbol{\lambda}) = \mathbf{0}. \tag{6.95}$$

Consider a resultant hybrid mapping relation as

$$\mathbf{x}'_k = P^{(N_\alpha : N_\beta)}\mathbf{x}_k = P_{\boldsymbol{\varphi}-} \circ P_{\beta-}^{(N_\beta)} \circ P_{\boldsymbol{\varphi}+} \circ P_{\alpha+}^{(N_\alpha)}\mathbf{x}_k \tag{6.96}$$

with

$$\begin{aligned} &P_{\alpha+}^{(N_\alpha)} : \mathbf{x}_k \to \mathbf{x}_{k+N_\alpha} \text{with} \\ &\left.\begin{aligned} &\mathbf{f}^{(\alpha)}(\mathbf{x}_{k+1}, \mathbf{x}_k, \mathbf{p}^{(\alpha)}) = \mathbf{0}, \\ &\mathbf{f}^{(\alpha)}(\mathbf{x}_{k+2}, \mathbf{x}_{k+1}, \mathbf{p}^{(\alpha)}) = \mathbf{0}, \\ &\vdots \\ &\mathbf{f}^{(\alpha)}(\mathbf{x}_{k+N_\alpha}, \mathbf{x}_{k+N_\alpha-1}, \mathbf{p}^{(\alpha)}) = \mathbf{0}, \end{aligned}\right\} \\ &P_{\boldsymbol{\varphi}+} : \mathbf{x}_{k+1} \to \mathbf{y}_{k+1} \text{ with } \boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}, \mathbf{y}_{k+N_\beta}, \boldsymbol{\lambda}) = \mathbf{0}; \\ &P_{\beta-}^{(N_\beta)} : \mathbf{y}_{k+N_\beta} \to \mathbf{y}_k \text{with} \\ &\left.\begin{aligned} &\mathbf{f}^{(\beta)}(\mathbf{y}_{k+N_\beta}, \mathbf{y}_{k+N_\beta-1}, \mathbf{p}^{(\beta)}) = \mathbf{0}, \\ &\vdots \\ &\mathbf{f}^{(\beta)}(\mathbf{y}_{k+2}, \mathbf{y}_{k+1}, \mathbf{p}^{(\beta)}) = \mathbf{0}, \\ &\mathbf{f}^{(\beta)}(\mathbf{y}_{k+1}, \mathbf{y}_k, \mathbf{p}^{(\beta)}) = \mathbf{0}, \end{aligned}\right\} \\ &P_{\boldsymbol{\varphi}-} : \mathbf{y}_k \to \mathbf{x}'_k \text{ with } \boldsymbol{\varphi}(\mathbf{x}'_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{0}; \\ &\mathbf{x}'_k = \mathbf{x}_k \end{aligned} \tag{6.97}$$

and

$$DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*) = DP_{\boldsymbol{\varphi}-}(\mathbf{y}_k^*) \cdot DP_{\beta-}^{(N_\beta)}(\mathbf{y}_{k+N_\alpha}^*) \cdot DP_{\boldsymbol{\varphi}+}(\mathbf{x}_{k+N_\alpha}^*) \cdot DP_{\alpha+}^{(N_\alpha)}(\mathbf{x}_k^*), \quad (6.98)$$

where

$$\begin{aligned} DP_{\alpha+}^{(N_\alpha)}(\mathbf{x}_k^*) &= DP_{\alpha+}(\mathbf{x}_{k+N_\alpha-1}^*) \cdot \ldots \cdot DP_{\alpha+}(\mathbf{x}_{k+1}^*) \cdot DP_{\alpha+}(\mathbf{x}_k^*), \\ DP_{\beta-}^{(N_\beta)}(\mathbf{y}_{k+N_\beta}^*) &= DP_{\beta-}(\mathbf{x}_{k+1}^*) \cdot \ldots \cdot DP_{\beta-}(\mathbf{x}_{k+N_\beta-1}^*) \cdot DP_{\beta-}(\mathbf{x}_{k+N_\beta}^*), \end{aligned} \quad (6.99)$$

$$\begin{aligned} &DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*) = \left[\frac{\partial \mathbf{x}'_k}{\partial \mathbf{x}_k}\right]_{\mathbf{x}_k^*}, \\ &DP_{\alpha+}^{(N_\alpha)}(\mathbf{x}_k^*) = \prod_{j=N_\alpha}^{1} \left[\frac{\partial \mathbf{x}_{k+j}}{\partial \mathbf{x}_{k+j-1}}\right]_{\mathbf{x}_{k+j-1}^*}, DP_{\boldsymbol{\varphi}+}(\mathbf{x}_{k+N_\alpha}^*) = \left[\frac{\partial \mathbf{y}_{k+N_\beta}}{\partial \mathbf{x}_{k+N_\alpha}}\right]_{\mathbf{x}_{k+N_\alpha}^*}, \\ &DP_{\beta-}^{(N_\beta)}(\mathbf{y}_{k+N_\beta}^*) = \prod_{j=1}^{N_\beta} \left[\frac{\partial \mathbf{y}_{k+N_\beta-j}}{\partial \mathbf{y}_{k+N_\beta-j+1}}\right]_{\mathbf{y}_{k+N_\beta-j+1}^*}, DP_{\boldsymbol{\varphi}-}(\mathbf{y}_k^*) = \left[\frac{\partial \mathbf{x}'_k}{\partial \mathbf{y}_k}\right]_{\mathbf{y}_k^*}. \end{aligned} \quad (6.100)$$

(i) *The* $(N_\alpha : N_\beta)$*-synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is persistent if and only if all the eigenvalues* λ_i *(*$i = 1, 2, \cdots, n$*) of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *lie in the unit circles, i.e.,*

$$|\lambda_i| < 1 \text{ for } i = 1, 2, \cdots, n. \quad (6.101)$$

(ii) *The* $(N_\alpha : N_\beta)$*-synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is a saddle-node vanishing if and only if at least one of the real eigenvalues* λ_i *(*$i = 1, 2, \cdots, n_1$ *and* $n_1 \le n$*) of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *is positive one (+1) and the other eigenvalues are in the unit circle, i.e.,*

$$\lambda_i = 1 \text{ and } |\lambda_j| < 1 \text{ for } i,j \in \{1, 2, \cdots, n\} \text{ and } j \ne i. \quad (6.102)$$

(iii) *The* $(N_\alpha : N_\beta)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is a period-doubling vanishing if and only if at least one of the real eigenvalues* λ_i *(*$i = 1, 2, \cdots, n_1$ *and* $n_1 \le n$*) of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *is negative one (−1) and the other eigenvalues are in the unit circle, i.e.,*

$$\lambda_i = -1 \text{ and } |\lambda_j| < 1 \text{ for } i,j \in \{1, 2, \cdots, n\} \text{ and } j \ne i. \quad (6.103)$$

(iv) *The* $(N_\alpha : N_\beta)$*-synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is a Naimark vanishing if and only if at least one pair of all* *the complex eigenvalues* $\lambda_i = \alpha_i \pm \beta_i \mathbf{i}$ *(* $i = 1, 2, \cdots, n_1$ *and* $n_1 \leq n/2$ *) of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *are on the unit circle and the other eigenvalues are in the unit circle, i.e.,*

$$|\lambda_i| = \sqrt{\alpha_i^2 + \beta_i^2} = 1 \text{ and } |\lambda_j| < 1 \text{ for } i, j \in \{1, 2, \cdots, n\} \text{ and } j \neq i. \tag{6.104}$$

(v) *The* $(N_\alpha : N_\beta)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is an* ($l_1 : l_2 : l_3$) *vanishing if and only if* l_1 *and* l_2 *real eigenvalues* λ_i *of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *are (−1) and (+1), respectively, and* l_3*-pairs of complex eigenvalues* $\lambda_i = \alpha_i \pm \beta_i \mathbf{i}$ *(*$i = 1, 2, \cdots, n_1$ *and* $n_1 \leq n/2$*) of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *are on the unit circle and the other eigenvalues are in the unit circle, i.e.,*

$$\begin{aligned} \lambda_i &= -1 \text{ for } i = i_1, i_2, \cdots, i_{l_1} \in \{1, 2, \cdots, n\}, \\ \lambda_j &= +1 \text{ for } j = j_1, j_2, \cdots, j_{l_2} \in \{1, 2, \cdots, n\}, \\ |\lambda_r| &= \sqrt{\alpha_r^2 + \beta_r^2} = 1 \text{ for } r = r_1, r_2, \cdots, r_{l_3} \in \{1, 2, \cdots, n\}, \\ |\lambda_s| &< 1 \text{ for } s \in \{1, 2, \cdots, n\} \text{ and } s \notin \{i, j, r\}. \end{aligned} \tag{6.105}$$

(vi) *The* $(N_\alpha : N_\beta)$ *synchronization of two discrete dynamical systems of* $(P_\alpha, \mathbf{f}^{(\alpha)})$ *and* $(P_\beta, \mathbf{f}^{(\beta)})$ *is instantaneous if and only if at least one of the eigenvalues* λ_i *(*$i = 1, 2, \cdots, n$*) of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *lies out of the unit circle, i.e.,*

$$|\lambda_i| > 1 \text{ for } i \in \{1, 2, \cdots, n\}. \tag{6.106}$$

Proof The proof can be referred to Luo [2]. □

Fixed points in nonlinear discrete dynamical systems possess many types of unstable states from eigenvalue analysis. From the similar ideas, the instantaneous $(N_\alpha : N_\beta)$ synchronization of two discrete dynamical systems can be classified. Therefore, such instantaneous synchronization classification will not be presented herein. If $N_\alpha \to \infty$ and $N_\beta \to \infty$, the $(N_\alpha : N_\beta)$ synchronization of two discrete dynamical systems should be chaotic. Consider two hybrid maps

$$\begin{aligned} P_+^{(\Sigma_{i=1}^n N_\beta^i \oplus N_\alpha^i)} &= \underbrace{P_{\beta+}^{(N_\beta^n)} \circ P_{\alpha+}^{(N_\alpha^n)} \circ \cdots \circ P_{\beta+}^{(N_\beta^1)} \circ P_{\alpha+}^{(N_\alpha^1)}}_{n-terms}, \\ P_+^{(\Sigma_{j=1}^m M_\beta^j \oplus M_\alpha^j)} &= \underbrace{P_{\beta+}^{(M_\beta^m)} \circ P_{\alpha+}^{(M_\alpha^m)} \circ \cdots \circ P_{\beta+}^{(M_\beta^1)} \circ P_{\alpha+}^{(M_\alpha^1)}}_{m-terms}. \end{aligned} \tag{6.107}$$

$$\begin{aligned} P_-^{(\Sigma_{i=n}^1 N_\alpha^i \oplus N_\beta^i)} &= \underbrace{P_{\alpha-}^{(N_\alpha^1)} \circ P_{\beta-}^{(N_\beta^1)} \circ \cdots \circ P_{\alpha-}^{(N_\alpha^m)} \circ P_{\beta-}^{(N_\beta^m)}}_{n-terms}, \\ P_-^{(\Sigma_{j=m}^1 M_\alpha^j \oplus M_\beta^j)} &= \underbrace{P_{\alpha-}^{(M_\alpha^1)} \circ P_{\beta-}^{(M_\beta^1)} \circ \cdots \circ P_{\alpha-}^{(M_\alpha^m)} \circ P_{\beta-}^{(M_\beta^m)}}_{m-terms}. \end{aligned} \tag{6.108}$$

The $(N_\beta \oplus N_\alpha : M_\beta \oplus M_\alpha)$-hybrid synchronization of two discrete systems with two maps $P_+^{(\Sigma_{i=1}^n N_\beta^i \oplus N_\alpha^i)}$ and $P_+^{(\Sigma_{j=1}^m M_\beta^j \oplus M_\alpha^j)}$ can be investigated via the following map

$$P^{(N_\beta \oplus N_\alpha : M_\beta \oplus M_\alpha)}\mathbf{x}_k = P_{\boldsymbol{\varphi}-} \circ P_-^{(\Sigma_{j=m}^1 M_\alpha^j \oplus M_\beta^j)} \circ P_{\boldsymbol{\varphi}+} \circ P_+^{(\Sigma_{i=1}^n N_\beta^i \oplus N_\alpha^i)}\mathbf{x}_k, \text{ or}$$
$$P^{(N_\beta \oplus N_\alpha : M_\beta \oplus M_\alpha)}\mathbf{x}_k = P_{\boldsymbol{\varphi}-} \circ P_-^{(\Sigma_{i=n}^1 N_\alpha^i \oplus N_\beta^i)} \circ P_{\boldsymbol{\varphi}+} \circ P_+^{(\Sigma_{j=1}^m M_\beta^j \oplus M_\alpha^j)}\mathbf{x}_k. \tag{6.109}$$

Thus,

$$\mathbf{x}'_k = P^{(N_\beta \oplus N_\alpha : M_\beta \oplus M_\alpha)}\mathbf{x}_k. \tag{6.110}$$

Similar to the $(N_\alpha : N_\beta)$-synchronization in Theorem 6.7, the corresponding fixed point and the stability conditions of Eq. (6.110) gives the $(N_\beta \oplus N_\alpha : M_\beta \oplus M_\alpha)$-hybrid synchronization of two discrete systems. This concept can be extended to the discrete dynamical systems with multiple maps.

As in discrete dynamical systems with multiple maps in Section 6.2, the synchronization for the resultant mappings in multiple different maps can be developed.

Definition 6.14 Consider two sets of discrete dynamical systems $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ and $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ as in Eq. (6.48) for each discrete system with

$$P_{\alpha_i+} : \mathbf{x}_k \to \mathbf{x}_{k+1} \text{ and } P_{\alpha_i-} : \mathbf{x}_{k+1} \to \mathbf{x}_k,$$
$$\mathbf{f}^{(\alpha_i)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(\alpha_i)}) = \mathbf{0}, \tag{6.111}$$

and

$$P_{\beta_j+} : \mathbf{y}_k \to \mathbf{y}_{k+1} \text{ and } P_{\beta_j-} : \mathbf{y}_{k+1} \to \mathbf{y}_k,$$
$$\mathbf{f}^{(\beta_j)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(\beta_j)}) = \mathbf{0}. \tag{6.112}$$

For the two sets of discrete dynamical systems, the resultant mappings are

$$P^+_{(N_{\alpha_m} \cdots N_{\alpha_2} N_{\alpha_1})} = \underbrace{P^+_{\alpha_m^{N_{\alpha m}}} \circ \cdots \circ P^+_{\alpha_2^{N_{\alpha 2}}} \circ P^+_{\alpha_1^{N_{\alpha 1}}}}_{m-terms},$$
$$P^-_{(N_{\alpha_1} N_{\alpha_2} \cdots N_{\alpha_m})} = \underbrace{P^-_{\alpha_1^{N_{\alpha 1}}} \circ P^-_{\alpha_2^{N_{\alpha 2}}} \circ \cdots \circ P^-_{\alpha_m^{N_{\alpha m}}}}_{m-terms}; \tag{6.113}$$

and

$$P^+_{(N_{\beta_n} \cdots N_{\beta_2} N_{\beta_1})} = \underbrace{P^+_{\beta_n^{N_{\beta n}}} \circ \cdots \circ P^+_{\beta_2^{N_{\beta 2}}} \circ P^+_{\beta_1^{N_{\beta 1}}}}_{n-terms},$$
$$P^-_{(N_{\beta_1} N_{\beta_2} \cdots N_{\beta_m})} = \underbrace{P^-_{\beta_1^{N_{\beta 1}}} \circ P^-_{\beta_2^{N_{\beta 2}}} \circ \cdots \circ P^-_{\beta_n^{N_{\beta n}}}}_{n-terms}, \tag{6.114}$$

where

$$N_\alpha = \sum\nolimits_{i=1}^{m} N_{\alpha_i} \text{ and } N_\beta = \sum\nolimits_{j=1}^{n} N_{\beta_j}. \tag{6.115}$$

For two points $\mathbf{x}_k \in D_{\alpha_1}$ and $\mathbf{y}_k \in D_{\beta_1}$, there is a specific, differentiable, vector function $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_l)^{\mathrm{T}} \in \mathscr{R}^l$.

(i) If

$$\boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}^{(\alpha_m)}, \mathbf{x}_{k+N_\beta}^{(\beta_n)}, \boldsymbol{\lambda}) = \mathbf{0}, \tag{6.116}$$

then the two discrete dynamical systems $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ and $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ are called the $(N_\alpha : N_\beta)$-synchronization in sense of $\boldsymbol{\varphi}$.

(ii) If

$$\begin{aligned}
&\boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}^{(\alpha_m)}, \mathbf{x}_{k+N_\beta}^{(\beta_n)}, \boldsymbol{\lambda}) = \mathbf{0} \text{ with}\\
&\Delta\mathbf{I}^{(\alpha_m\alpha_1)} : \mathbf{x}_{k+N_\alpha(-)}^{(\alpha_m)} \to \mathbf{x}_{k+N_\alpha(+)}^{(\alpha_m)} \text{ and } \Delta\mathbf{I}^{(\beta_n\beta_1)} : \mathbf{x}_{k+N_\beta(-)}^{(\beta_n)} \to \mathbf{x}_{k+N_\beta(+)}^{(\beta_n)},\\
&\mathbf{x}_{k+N_\alpha(+)}^{(\alpha_m)} = \mathbf{x}_{k+N_\alpha(-)}^{(\alpha_m)} + \Delta\mathbf{I}^{(\alpha_m\alpha_1)} \text{ and } \mathbf{x}_{k+N_\beta(+)}^{(\beta_n)} = \mathbf{x}_{k+N_\beta(-)}^{(\beta_n)} + \Delta\mathbf{I}^{(\beta_n\beta_1)},\\
&\mathbf{x}_{k+N_\beta(+)}^{(\alpha_m)} = \mathbf{x}_k^{(\alpha_1)} \text{ and } \mathbf{x}_{k+N_\beta(+)}^{(\beta_n)} = \mathbf{x}_k^{(\beta_1)}.
\end{aligned} \tag{6.117}$$

then the two discrete dynamical systems $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ and $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ are called the repeatable $(N_\alpha : N_\beta)$-synchronization in sense of $\boldsymbol{\varphi}$.

The corresponding theorem can be presented as in Theorem 6.7. For convenience, the statement is given as follows.

Theorem 6.8 *Consider two sets of discrete dynamical systems* $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ *and* $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ *as in Eq. (6.48) for each discrete system with*

$$\begin{aligned}
&P_{\alpha_i +} : \mathbf{x}_k \to \mathbf{x}_{k+1} \text{ and } P_{\alpha_i -} : \mathbf{x}_{k+1} \to \mathbf{x}_k,\\
&\mathbf{f}^{(\alpha_i)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(\alpha_i)}) = \mathbf{0}
\end{aligned} \tag{6.118}$$

and

$$\begin{aligned}
&P_{\beta_j +} : \mathbf{y}_k \to \mathbf{y}_{k+1} \text{ and } P_{\beta_j -} : \mathbf{y}_{k+1} \to \mathbf{y}_k,\\
&\mathbf{f}^{(\beta_j)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(\beta_j)}) = \mathbf{0}.
\end{aligned} \tag{6.119}$$

For two points $\mathbf{x}_k \in D_{\alpha_1}$ *and* $\mathbf{y}_k \in D_{\beta_1}$*, there is a specific, differentiable, vector function* $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_l)^{\mathrm{T}} \in \mathscr{R}^l$*. The* $(N_\alpha : N_\beta)$*-synchronization of two sets of discrete dynamical systems* $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ *and* $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ *is under the following constraints*

$$\boldsymbol{\varphi}(\mathbf{x}_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{0} \text{ and } \boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}, \mathbf{y}_{k+N_\beta}, \boldsymbol{\lambda}) = \mathbf{0}. \tag{6.120}$$

Consider a resultant hybrid mapping relation as

$$\mathbf{x'}_k = P^{(N_\alpha:N_\beta)}\mathbf{x}_k = P_{\boldsymbol{\varphi}-} \circ P_{N_\beta-} \circ P_{\boldsymbol{\varphi}+} \circ P_{N_\alpha+}\mathbf{x}_k \tag{6.121}$$

with

$$P_{N_\alpha+} = P^{+}_{(N_{\alpha_m}\cdots N_{\alpha_2}N_{\alpha_1})} \text{ and } P_{N_\beta-} = P^{-}_{(N_{\beta_1}N_{\beta_2}\cdots N_{\beta_n})}, \tag{6.122}$$

$$\begin{array}{l}
P_{N_{\alpha_i}+} : \mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha_r}} \to \mathbf{x}_{k+\Sigma_{r=1}^{i}N_{\alpha_r}} \text{ with} \\
\left.\begin{array}{l}
\mathbf{f}^{(\alpha_i)}(\mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha_r}+1}, \mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha_r}}, \mathbf{p}^{(\alpha)}) = \mathbf{0}, \\
\mathbf{f}^{(\alpha_i)}(\mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha_r}+2}, \mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha_r}+1}, \mathbf{p}^{(\alpha)}) = \mathbf{0}, \\
\vdots \\
\mathbf{f}^{(\alpha_i)}(\mathbf{x}_{k+\Sigma_{r=1}^{i}N_{\alpha_r}}, \mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha_r}-1}, \mathbf{p}^{(\alpha_i)}) = \mathbf{0}
\end{array}\right\} \\
\text{for } i = 1, 2, \cdots, m \\
P_{\boldsymbol{\varphi}+} : \mathbf{x}_{k+N_\alpha} \to \mathbf{y}_{k+N_\beta} \text{ with } \boldsymbol{\varphi}(\mathbf{x}_{k+N_\alpha}, \mathbf{y}_{k+N_\beta}, \boldsymbol{\lambda}) = \mathbf{0}; \\
P_{N_{\beta_j}} : \mathbf{y}_{k+N_\beta-\Sigma_{r=j}^{n}N_{\beta_r}} \to \mathbf{y}_{k+N_\beta-\Sigma_{r=j-1}^{j}N_{\beta_r}} \text{ with} \\
\left.\begin{array}{l}
\mathbf{f}^{(\beta_j)}(\mathbf{y}_{k+N_\beta-\Sigma_{r=j}^{n}N_{\beta_r}}, \mathbf{y}_{k+N_\beta-\Sigma_{r=j}^{n}N_{\beta_r}-1}, \mathbf{p}^{(\beta_j)}) = \mathbf{0}, \\
\vdots \\
\mathbf{f}^{(\beta_j)}(\mathbf{y}_{k+N_\beta-\Sigma_{r=j-1}^{j}N_{\beta_r}+2}, \mathbf{y}_{k+N_\beta-\Sigma_{r=j-1}^{j}N_{\beta_r}+1}, \mathbf{p}^{(\beta_j)}) = \mathbf{0}, \\
\mathbf{f}^{(\beta_j)}(\mathbf{y}_{k+N_\beta-\Sigma_{r=j-1}^{j}N_{\beta_r}+1}, \mathbf{y}_{k+N_\beta-\Sigma_{r=j-1}^{j}N_{\beta_r}}, \mathbf{p}^{(\beta_j)}) = \mathbf{0}
\end{array}\right\} \\
\text{for } j = n, n-1, \cdots, 1 \\
P_{\boldsymbol{\varphi}-} : \mathbf{y}_k \to \mathbf{x'}_k \text{ with } \boldsymbol{\varphi}(\mathbf{x'}_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{0}; \\
\mathbf{x'}_k = \mathbf{x}_k
\end{array} \tag{6.123}$$

and

$$DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*) = DP_{\boldsymbol{\varphi}-}(\mathbf{y}_k^*) \cdot DP_{\beta-}^{(N_\beta)}(\mathbf{y}_{k+N_\alpha}^*) \cdot DP_{\boldsymbol{\varphi}+}(\mathbf{x}_{k+N_\alpha}^*) \cdot DP_{\alpha+}^{(N_\alpha)}(\mathbf{x}_k^*) \tag{6.124}$$

where

$$\begin{array}{l}
DP_{\beta-}^{(N_\beta)}(\mathbf{y}_{k+N_\alpha}^*) = \prod_{i=m}^{1} DP_{\alpha_i+}^{(N_{\alpha_i})}(\mathbf{x}_k^*), \\
DP_{\alpha_i+}^{(N_{\alpha_i})}(\mathbf{x}_k^*) = DP_{\alpha_i+}(\mathbf{x}^*_{k+\Sigma_{r=1}^{i}N_{\alpha_i}-1}) \cdot \ldots \cdot DP_{\alpha_i+}(\mathbf{x}^*_{k+\Sigma_{r=1}^{i-1}N_{\alpha_i}+1}) \cdot DP_{\alpha_i+}(\mathbf{x}^*_{k+\Sigma_{r=1}^{i-1}N_{\alpha_i}}),
\end{array} \tag{6.125}$$

$$\begin{aligned}
DP_{\beta-}^{(N_\beta)}(\mathbf{y}_{k+N_\alpha}^*) &= \prod\nolimits_{j=1}^{m} DP_{\beta_j+}^{(N_{\beta_j})}(\mathbf{y}_{k+N_\beta-\Sigma_{r=1}^{j-1}N_{\alpha_i}}^*),\\
DP_{\beta_j+}^{(N_{\beta_j})}(\mathbf{y}_{k+N_\beta-\Sigma_{r=1}^{j-1}N_{\alpha_i}}^*) &= DP_{\beta-}(\mathbf{x}_{k+N_\beta-\Sigma_{r=1}^{j-1}N_{\alpha_i}}^*)\cdot \ldots \cdot DP_{\beta-}(\mathbf{x}_{k+N_\beta-\Sigma_{r=1}^{j-1}N_{\alpha_i}-1}^*)\\
&\quad \cdot DP_{\beta-}(\mathbf{x}_{k+N_\beta-\Sigma_{r=1}^{j}N_{\alpha_i}}^*),
\end{aligned} \tag{6.126}$$

$$DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*) = \left[\frac{\partial \mathbf{x}'_k}{\partial \mathbf{x}_k}\right]_{\mathbf{x}_k^*}, \tag{6.127}$$

$$\begin{aligned}
DP_{\alpha_i+}^{(N_{\alpha_i})}(\mathbf{x}_{k+\Sigma_{r=1}^{i}N_{\alpha_i}}^*) &= \prod\nolimits_{s=N_{\alpha_i}}^{1}\left[\frac{\partial \mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha_i}+s}}{\partial \mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha_i}+s-1}}\right]_{\mathbf{x}_{k+\Sigma_{r=1}^{i-1}N_{\alpha_i}+s-1}^*},\\
DP_{\varphi+}(\mathbf{x}_{k+N_\alpha}^*) &= \left[\frac{\partial \mathbf{y}_{k+N_\beta}}{\partial \mathbf{x}_{k+N_\alpha}}\right]_{\mathbf{x}_{k+N_\alpha}^*},\\
DP_{\beta_j-}^{(N_{\beta_j})}(\mathbf{y}_{k+N_\beta-\Sigma_{r=1}^{j-1}N_{\beta_j}}^*) &= \prod\nolimits_{s=1}^{N_{\beta_j}}\left[\frac{\partial \mathbf{y}_{k+N_\beta-\Sigma_{r=1}^{j}N_{\beta_j}-s}}{\partial \mathbf{y}_{k+N_\beta-\Sigma_{r=1}^{j}N_{\beta_j}-s+1}}\right]_{\mathbf{y}_{k+N_\beta-\Sigma_{r=1}^{j}N_{\beta_j}-s+1}^*},\\
DP_{\varphi-}(\mathbf{y}_k^*) &= \left[\frac{\partial \mathbf{x}'_k}{\partial \mathbf{y}_k}\right]_{\mathbf{y}_k^*}.
\end{aligned} \tag{6.128}$$

(i) *The* $(N_\alpha : N_\beta)$ *-synchronization of two sets of discrete dynamical systems* $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ *and* $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ *is persistent if and only if all the eigenvalues* λ_i *(*$i = 1, 2, \cdots, n$*) of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *lie in the unit circles, i.e.,*

$$|\lambda_i|<1 \text{ for } i = 1, 2, \cdots, n. \tag{6.129}$$

(ii) *The* $(N_\alpha : N_\beta)$ *-synchronization of two sets of discrete dynamical systems* $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ *and* $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ *is a saddle-node vanishing if and only if at least one of the real eigenvalues* λ_i *(*$i = 1, 2, \cdots, n_1$ *and* $1 \le n_1 \le n$*) of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *is positive one (+1) and the other eigenvalues are in the unit circle, i.e.,*

$$\lambda_i = 1 \text{ and } |\lambda_j|<1 \text{ for } i,j \in \{1, 2, \cdots, n\} \text{ and } j \neq i. \tag{6.130}$$

(iii) *The* $(N_\alpha : N_\beta)$ *synchronization of two sets of discrete dynamical systems* $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ *and* $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ *is a period-doubling vanishing if and only if at least one of the real eigenvalues* λ_i *(*$i = 1, 2, \cdots, n_1$ *and* $1 \le n_1 \le n$*) of*

$DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *is negative one (−1) and the other eigenvalues are in the unit circle, i.e.,*

$$\lambda_i = -1 \text{ and } |\lambda_j|<1 \text{ for } i,j \in \{1,2,\cdots,n\} \text{ and } j \neq i. \tag{6.131}$$

(iv) *The* $(N_\alpha : N_\beta)$ *-synchronization of two sets of discrete dynamical systems* $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ *and* $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ *is a Naimark vanishing if and only if at least one pair of all the complex eigenvalues* $\lambda_i = \alpha_i \pm \beta_i\mathbf{i}$ *(*$i = 1,2,\cdots,n_1$ *and* $1 \leq n_1 \leq n/2$*) of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *are on the unit circle and the other eigenvalues are in the unit circle, i.e.,*

$$|\lambda_i| = \sqrt{\alpha_i^2+\beta_i^2} = 1 \text{ and } |\lambda_j|<1 \text{ for } i,j \in \{1,2,\cdots,n\} \text{ and } j \neq i. \tag{6.132}$$

(v) *The* $(N_\alpha : N_\beta)$ *synchronization of two sets of discrete dynamical systems* $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ *and* $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ *is an (*$l_1 : l_2 : l_3$*) vanishing if and only if* l_1 *and* l_2 *real eigenvalues* λ_i *of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *are (−1) and (+1), respectively, and* l_3*-pairs of complex eigenvalues* $\lambda_i = \alpha_i \pm \beta_i\mathbf{i}$ *(*$i = 1,2,\cdots,n_1$ *and* $1 \leq n_1 \leq n/2$*) of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *are on the unit circle and the other eigenvalues are in the unit circle, i.e.,*

$$\begin{aligned}
\lambda_i &= -1 \text{ for } i = i_1, i_2, \cdots, i_{l_1} \in \{1,2,\cdots,n\}, \\
\lambda_j &= +1 \text{ for } j = j_1, j_2, \cdots, j_{l_2} \in \{1,2,\cdots,n\}, \\
|\lambda_r| &= \sqrt{\alpha_r^2+\beta_r^2} = 1 \text{ for } r = r_1, r_2, \cdots, r_{l_3} \in \{1,2,\cdots,n\}, \\
|\lambda_s| &<1 \text{ for } s \in \{1,2,\cdots,n\} \text{ and } s \notin \{i,j,r\}.
\end{aligned} \tag{6.133}$$

(vi) *The* $(N_\alpha : N_\beta)$ *synchronization of two sets of discrete dynamical systems* $\cup_{i=1}(P_{\alpha_i}, \mathbf{f}^{(\alpha_i)})$ *and* $\cup_{j=1}(P_{\beta_j}, \mathbf{f}^{(\beta_j)})$ *is instantaneous if and only if at least one of the eigenvalues* λ_i *(*$i = 1,2,\cdots,n$*) of* $DP^{(N_\alpha:N_\beta)}(\mathbf{x}_k^*)$ *lies out of the unit circle, i.e.,*

$$|\lambda_i|>1 \text{ for } i \in \{1,2,\cdots,n\}. \tag{6.134}$$

Proof The proof can be referred to Luo [2]. □

6.5 An Application of Discrete Systems Synchronization

As in Luo and Guo [4], consider an identical synchronization of the Duffing and Henon maps as an example. The Duffing map is

$$x_{1(k+1)} = x_{2(k)} \text{ and } x_{2(k+1)} = -dx_{1(k)} + cx_{2(k)} - x_{2(k)}^3. \tag{6.135}$$

and the Henon map is

$$y_{1(k+1)} = y_{2(k)} + 1 - ay_{1(k)}^2 \text{ and } y_{2(k+1)} = by_{1(k)}. \tag{6.136}$$

Introduce the vectors as

$$\begin{aligned} &\mathbf{x}_k = (x_{1(k)}, x_{2(k)})^{\mathrm{T}} \text{ and } \mathbf{y}_k = (y_{1(k)}, y_{2(k)})^{\mathrm{T}} \\ &\mathbf{f}^{(\alpha)} = (f_1^{(\alpha)}, f_2^{(\alpha)})^{\mathrm{T}} \text{ for } \alpha = 1, 2. \end{aligned} \tag{6.137}$$

Herein $\alpha = 1$ for the Duffing map and $\alpha = 2$ for the Henon map. Thus, the Duffing map is described by

$$P_1 : \mathbf{x}_k \to \mathbf{x}_{k+1} \text{ and } \mathbf{f}^{(1)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(1)}) = \mathbf{0}, \tag{6.138}$$

where

$$\begin{aligned} &f_1^{(1)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(1)}) = x_{1(k+1)} - x_{2(k)}, \\ &f_2^{(1)}(\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{p}^{(1)}) = x_{2(k+1)} + dx_{1(k)} - cx_{2(k)} + x_{2(k)}^3; \\ &\mathbf{p}^{(1)} = (c, d)^{\mathrm{T}}. \end{aligned} \tag{6.139}$$

The Henon map is described by

$$P_2 : \mathbf{y}_k \to \mathbf{y}_{k+1} \text{ and } \mathbf{f}^{(2)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(2)}) = \mathbf{0}, \tag{6.140}$$

where

$$\begin{aligned} &f_1^{(2)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(2)}) = y_{1(k+1)} - y_{2(k)} - 1 + ay_{1(k)}^2, \\ &f_2^{(2)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(2)}) = y_{2(k+1)} - by_{1(k)}; \\ &\mathbf{p}^{(2)} = (a, b)^{\mathrm{T}}. \end{aligned} \tag{6.141}$$

Consider the $(N_1 : N_2)$ synchronization of the Duffing and Henon maps with

$$\begin{aligned} &\boldsymbol{\varphi}(\mathbf{x}_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{x}_k - \mathbf{y}_k = \mathbf{0}, \\ &\boldsymbol{\varphi}(\mathbf{x}_{k+N_1}, \mathbf{y}_{k+N_2}, \boldsymbol{\lambda}) = \mathbf{x}_{k+N_1} - \mathbf{y}_{k+N_2} = \mathbf{0}, \end{aligned} \tag{6.142}$$

where

$$\begin{aligned} &\mathbf{x}_{k+N_1} = P_1^{(N_1)}\mathbf{x}_k = \underbrace{P_1 \circ P_1 \circ \cdots \circ P_1}_{N_1} \mathbf{x}_k \text{ with} \\ &f_1^{(1)}(\mathbf{x}_{k+i-1}, \mathbf{x}_{k+i}, \mathbf{p}^{(1)}) = x_{1(k+i)} - x_{2(k+i-1)} = 0, \\ &f_2^{(1)}(\mathbf{x}_{k+i-1}, \mathbf{x}_{k+i}, \mathbf{p}^{(1)}) = x_{2(k+i)} + dx_{1(k+i-1)} - cx_{2(k+i-1)} + x_{2(k+i-1)}^3 = 0 \\ &\text{for } i = 1, 2, \cdots, N_1; \end{aligned} \tag{6.143}$$

$$\begin{aligned}
&\mathbf{y}_{k+N_2} = P_2^{(N_2)}\mathbf{y}_k = \underbrace{P_2 \circ P_2 \circ \cdots \circ P_2}_{N_2}\mathbf{y}_k \text{ with} \\
&f_1^{(2)}(\mathbf{y}_{k+j}, \mathbf{y}_{k+j-1}, \mathbf{p}^{(2)}) = y_{1(k+j)} - y_{2(k+j-1)} - 1 + ay_{1(k+j-1)}^2 = 0, \\
&f_2^{(2)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(2)}) = y_{2(k+j)} - by_{1(k+j-1)} = 0 \\
&\text{for } j = 1, 2, \cdots, N_2.
\end{aligned} \tag{6.144}$$

For the $(N_1 : N_2)$ synchronization, the equivalent mapping structure is

$$\mathbf{x}'_k = P_{\boldsymbol{\varphi}-} \circ P_{2-}^{(N_2)} \circ P_{\boldsymbol{\varphi}+} \circ P_{1+}^{(N_1)}\mathbf{x}_k. \tag{6.145}$$

If $\mathbf{x}'_k = \mathbf{x}_k,$ we have

$$\begin{aligned}
&\left.\begin{aligned}
&f_1^{(1)}(\mathbf{x}_{k+i-1}, \mathbf{x}_{k+i}, \mathbf{p}^{(1)}) = x_{1(k+i)} - x_{2(k+i-1)} = 0 \\
&f_2^{(1)}(\mathbf{x}_{k+i-1}, \mathbf{x}_{k+i}, \mathbf{p}^{(1)}) = x_{2(k+i)} + dx_{1(k+i-1)} - cx_{2(k+i-1)} + x_{2(k+i-1)}^3 = 0
\end{aligned}\right\} \\
&\text{for } i = 1, 2, \cdots, N_1; \\
&\boldsymbol{\varphi}(\mathbf{x}_{k+N_1}, \mathbf{y}_{k+N_2}, \boldsymbol{\lambda}) = \mathbf{x}_{k+N_1} - \mathbf{y}_{k+N_2} = \mathbf{0}; \\
&\left.\begin{aligned}
&f_1^{(2)}(\mathbf{y}_{k+j}, \mathbf{y}_{k+j-1}, \mathbf{p}^{(2)}) = y_{1(k+j)} - y_{2(k+j-1)} - 1 + ay_{1(k+j-1)}^2 = 0 \\
&f_2^{(2)}(\mathbf{y}_k, \mathbf{y}_{k+1}, \mathbf{p}^{(2)}) = y_{2(k+j)} - by_{1(k+j-1)} = 0
\end{aligned}\right\} \\
&\text{for } j = N_2, \cdots, 2, 1; \\
&\boldsymbol{\varphi}(\mathbf{x}_k, \mathbf{y}_k, \boldsymbol{\lambda}) = \mathbf{x}_k - \mathbf{y}_k = \mathbf{0}.
\end{aligned} \tag{6.146}$$

From which the fixed points of Eq. (6.145) [i.e., $\mathbf{x}_{k+i}^*$ $(i = 1, 2, \cdots, N_1)$ and $\mathbf{y}_{k+j}^*$ $(j = 1, 2, \cdots N_2)$] can be obtained. The corresponding stability boundary of such fixed points is given the eigenvalue analysis, i.e.,

$$\Delta\mathbf{x}'_k = DP_{\boldsymbol{\varphi}+} \cdot DP_{2-}^{(N_2)} \cdot DP_{\boldsymbol{\varphi}+} \cdot DP_{1+}^{(N_1)}\Delta\mathbf{x}_k; \tag{6.147}$$

where

$$\begin{aligned}
DP_{\boldsymbol{\varphi}-}(\mathbf{y}_k^*) &= \left[\frac{\partial \mathbf{x}'_k}{\partial \mathbf{y}_k}\right]_{\mathbf{y}_k^*} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \\
DP_{2-}^{(N_2)} &= \prod\nolimits_{j=1}^{N_2} DP_{2-}(\mathbf{y}_{k+j}^*), \\
DP_{2-}(\mathbf{y}_{k+j}^*) &= \left[\frac{\partial \mathbf{y}_{k+j-1}}{\partial \mathbf{y}_{k+j}}\right]_{\mathbf{y}_{k+j}^*} = -\frac{1}{b}\begin{bmatrix} 0 & 1 \\ b & 2ay_{1(k+j-1)}^* \end{bmatrix}; \\
DP_{\boldsymbol{\varphi}+}(\mathbf{x}_{k+N_1}^*) &= \left[\frac{\partial \mathbf{y}_{k+N_2}}{\partial \mathbf{x}_{k+N_1}}\right]_{\mathbf{x}_{k+N_1}^*} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \\
DP_{1+}^{(N_1)} &= \prod\nolimits_{i=N_1}^{1} DP_{2-}(\mathbf{x}_{k+j}^*), \\
DP_{1+}(\mathbf{x}_{k+j-1}^*) &= \left[\frac{\partial \mathbf{x}_{k+j}}{\partial \mathbf{x}_{k+j-1}}\right]_{\mathbf{x}_{k+j-1}^*} = \begin{bmatrix} 0 & 1 \\ -d & -c + 3(x_{2(k+j-1)}^*)^2 \end{bmatrix}.
\end{aligned} \tag{6.148}$$

Through the above analysis procedure, the $(N_1 : N_2)$ synchronization domains and boundaries can be determined from Theorem 6.7. In Eq. (6.145), we can form a new map iteration

$$\begin{aligned} &\mathbf{x}_{J+1} = P\mathbf{x}_J \text{ with} \\ &\mathbf{x}_J \equiv \mathbf{x}_k \text{ and } P \equiv P_{\boldsymbol{\varphi}-} \circ P_{2-}^{(N_2)} \circ P_{\boldsymbol{\varphi}+} \circ P_{1+}^{(N_1)}. \end{aligned} \tag{6.149}$$

Using Eq. (6.149), numerical iteration can be done to observe the $(N_1 : N_2)$ identical synchronization of the Duffing and Henon maps.

As in Luo and Guo [4], consider parameters of $a = 0.8$, $c = 2.75$ and $d = 0.2$. From the mapping in Eq. (6.149), the $(1 : 1)$ -identical synchronization of the Duffing and Henon maps is simulated, as shown in Fig. 6.9. The bifurcation scenario alike plots for $x_{1(k)}$ and $x_{2(k)}$ with $y_{1(k)}$ and $y_{2(k)}$. The shaded regions are for the $(1 : 1)$ synchronization. PD and SN represent period-doubling and saddle-node vanishing of the $(1 : 1)$ synchronization, respectively. The synchronization range is $b \in (-\infty, -30.84)$ and $b \in (33.88, \infty)$ in Fig. 6.9a, b. In Fig. 6.9c–f, the zoomed view for small parameter ranges are presented. The parameter ranges are given by $b \in (1.2431,\ 1.3687)$ and $b \in (-1.7667, -1.4216)$, respectively. The analytical predictions of the $(1 : 1)$-synchronization is presented in Fig. 6.9. The solid curves are the $(1 : 1)$ synchronizations. PD and SN represent period-doubling and saddle-node vanishing of the $(1 : 1)$ synchronization, respectively. The instantaneous $(1 : 1)$ synchronizations are represented by dashed curves. For numerical simulations, the instantaneous synchronization state cannot be achieved. The $(1 : 1)$ synchronization given by the analytical prediction matches with the numerical prediction. The large parameter ranges for the $(1 : 1)$ synchronization are presented in Fig. 6.10a, b. The small parameter ranges for the $(1 : 1)$-synchronization are arranged in Fig. 6.10c–f. The corresponding parameter maps for $(1 : 1)$-synchronization are presented in Fig. 4.13. The shaded regions are for the $(1 : 1)$ synchronization. PD and SN represent period-doubling and saddle-node vanishing of the $(1 : 1)$ synchronization, respectively. The intersected points of the PD and SN vanishing are $(1, 1, 0)$-critical synchronization vanishing with $\lambda_1 = -1$ and $\lambda_2 = 1$. Figure 6.11a, c, e is for overall parameter maps, and Fig. 6.11b, d, f is for the zoomed views of parameter maps. Figure 6.11a, b shows parameter map (a, b) for $c = 2.75$ and $d = 0.2$. Figure 6.11c, d presents the parameter maps (d, b) for $a = 0.8$ and $c = 2.75$. Figure 6.11e, f gives the parameter (c, b) for $a = 0.8$ and $d = 0.2$. For the parameter maps, the $(1 : 1)$ synchronizations exist in different regions with many cusp points, and such cusp points will be very difficult to be analyzed by the catastrophe analysis. Other discrete dynamical system synchronization can be carried out from the theory of discrete dynamical system synchronization, which is presented in this chapter.

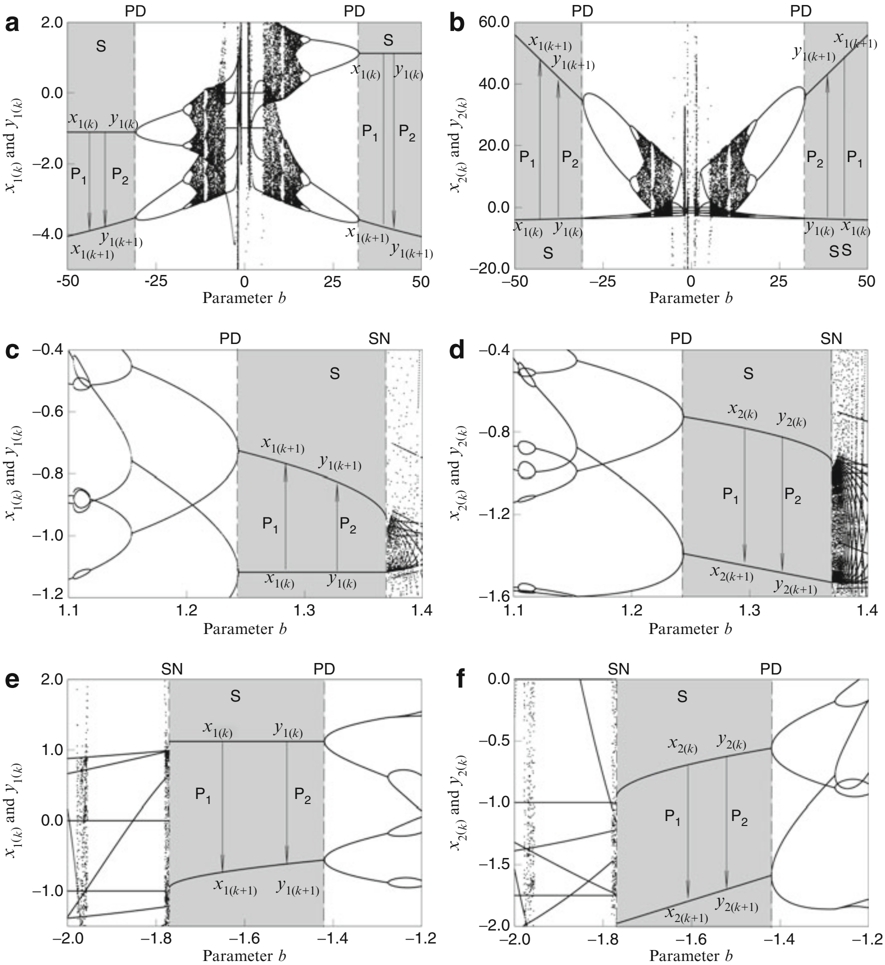

Fig. 6.9 The numerical iteration for the (1:1) synchronization of two discrete dynamical systems with the Duffing and Henon maps. Bifurcation scenario alike plots for $x_{1(k)}$ and $x_{2(k)}$ with $y_{1(k)}$ and $y_{2(k)}$: (**a**) and (**b**) for $b \in (-\infty, -30.84)$ and $b \in (33.88, \infty)$; (**c**) and (**d**) for $b \in (1.2431,\ 1.3687)$; (**e**) and (**f**) for $b \in (-1.7667, -1.4216)$. The *shaded regions* are for the (1:1) synchronization. PD and SN represent period-doubling and saddle-node vanishing of the (1:1) synchronization, respectively ($a = 0.8$, $c = 2.75$ and $d = 0.2$)

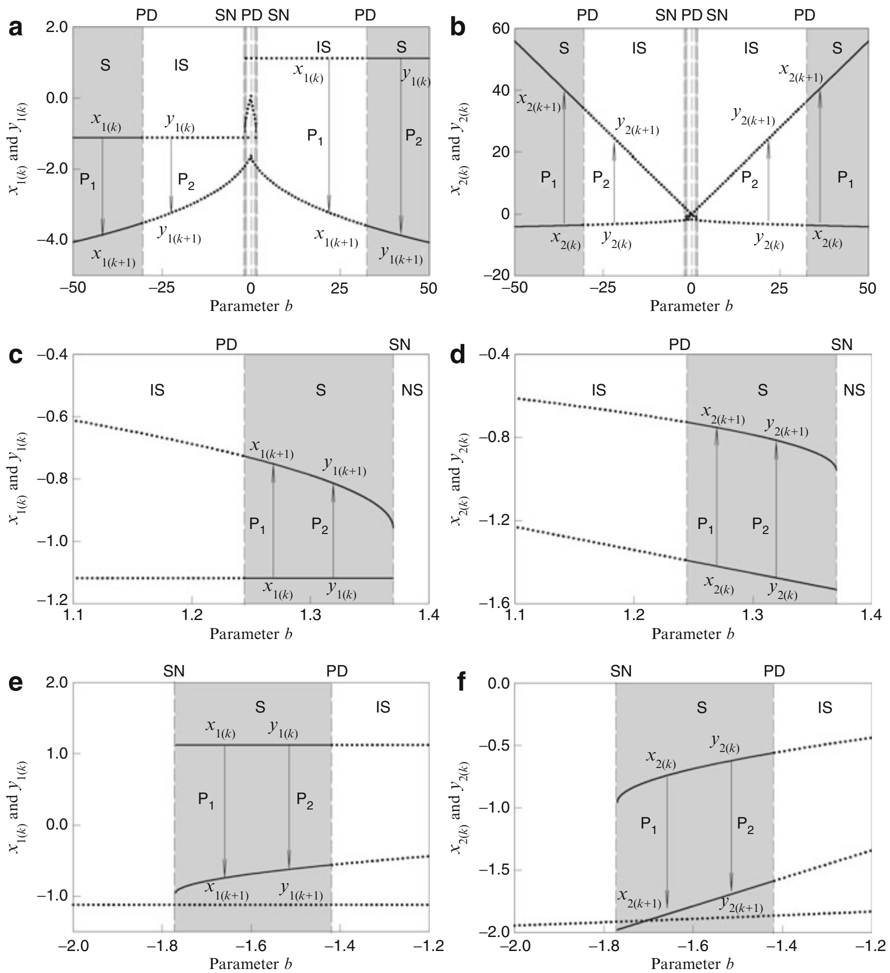

Fig. 6.10 The analytical prediction of the (1:1) synchronization of two discrete dynamical systems with the Duffing and Henon maps. The iterative states $x_{1(k)}$ and $x_{2(k)}$ with $y_{1(k)}$ and $y_{2(k)}$ are presented: (**a**) and (**b**) for $b \in (-\infty, -30.84)$ and $b \in (33.88, \infty)$; (**c**) and (**d**) for $b \in (1.2431, 1.3687)$; (**e**) and (**f**) for $b \in (-1.7667, -1.4216)$. The *shaded regions* are for the (1:1) synchronization. PD and SN represent period-doubling and saddle-node vanishing of the (1:1) synchronization, respectively. The instantaneous (1:1) synchronizations are represented by the *dotted curves* ($a = 0.8$, $c = 2.75$, and $d = 0.2$)

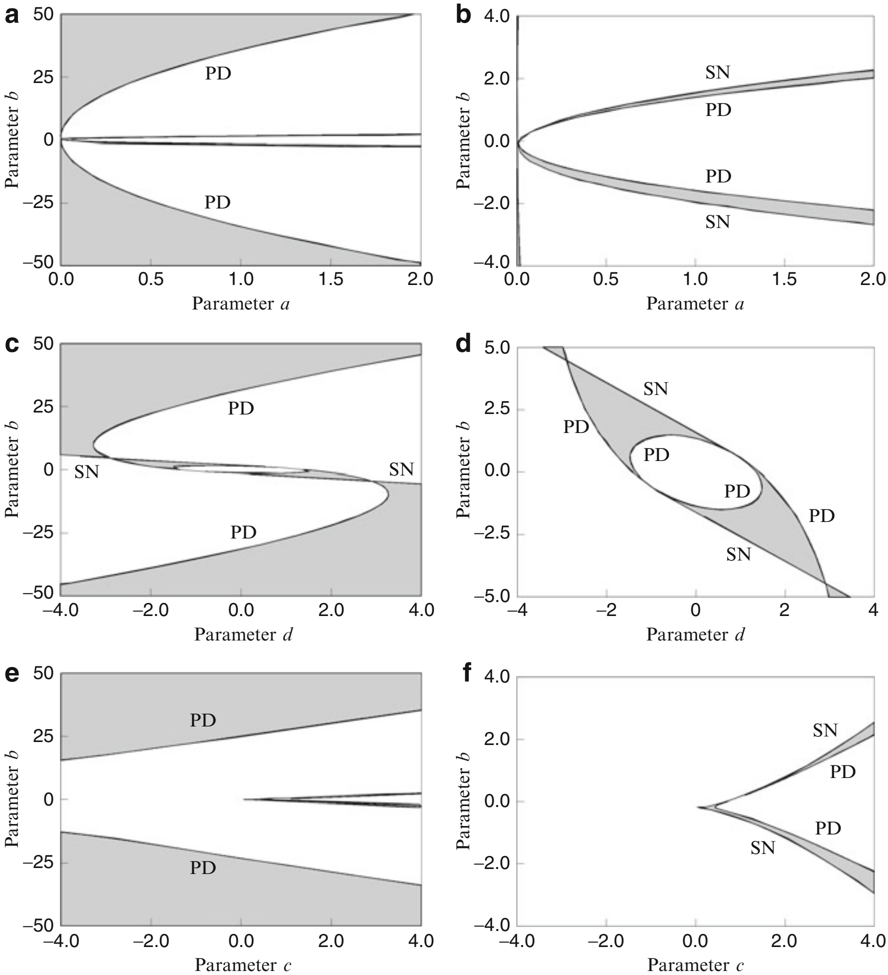

Fig. 6.11 Parameter maps of the $(1:1)$ synchronization of two discrete dynamical systems with the Duffing and Henon maps: (**a**) and (**b**) parameter map (a, b) for $c = 2.75$ and $d = 0.2$; (**c**) and (**d**) parameter maps (d, b) for $a = 0.8$ and $c = 2.75$; (**e**) and (**f**) parameter (c, b) for $a = 0.8$ and $d = 0.2$. The overall views are given on the *left-hand side*, and the zoomed views are given on the *right-hand side*. The *shaded regions* are for the $(1:1)$ synchronization. PD and SN represent period-doubling and saddle-node vanishing of the $(1:1)$ synchronization, respectively

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