Chapter 3
Single Constraint Synchronization

In this chapter, the synchronization of two or more dynamical systems to specific constraints is introduced, which is different from the traditional synchronization of two dynamical systems. For such synchronization, Lyapunov stability method cannot be adopted. The synchronization, desynchronization, and penetration of multiple dynamical systems to a specific constraint are discussed from the theory of discontinuous dynamical systems, and the necessary and sufficient conditions for such synchronicity are presented.

3.1 Introduction to Synchronization

As in Luo [1], consider two dynamic systems as

$$\dot{\mathbf{x}}^{(r)} = \mathbf{F}^{(r)}(\mathbf{x}^{(r)}, t, \mathbf{p}^{(r)}) \in \mathscr{R}^{n_r} \tag{3.1}$$

and

$$\dot{\mathbf{x}}^{(s)} = \mathbf{F}^{(s)}(\mathbf{x}^{(s)}, t, \mathbf{p}^{(s)}) \in \mathscr{R}^{n_s} \tag{3.2}$$

For $\sigma = \{r, s\}$, $\mathbf{F}^{(\sigma)} = (F_1^{(\sigma)}, F_2^{(\sigma)}, \cdots, F_{n_\sigma}^{(\sigma)})^{\mathrm{T}}$, $\mathbf{x}^{(\sigma)} = (x_1^{(\sigma)}, x_2^{(\sigma)}, \cdots, x_{n_\sigma}^{(\sigma)})^T$, and parameter vector $\mathbf{p}^{(\sigma)} = (p_1^{(\sigma)}, p_2^{(\sigma)}, \cdots, p_{k_\sigma}^{(\sigma)})^{\mathrm{T}}$. The vector functions $\mathbf{F}^{(\sigma)}$ can be time-dependent or time-independent. Consider a time interval $I_{12} \equiv (t_1, t_2) \in \mathscr{R}$ and domains $U_{\mathbf{x}^{(\sigma)}} \subseteq \mathscr{R}^{n_\sigma}$ $(\sigma = \{\alpha, \beta\})$. $(t_0, \mathbf{x}_0^{(\sigma)}) \in I_{12} \times U_{\mathbf{x}^{(\sigma)}}$ is initial condition, and the corresponding flows of the two systems are $\mathbf{x}^{(\sigma)}(t) = \mathbf{\Phi}(t, \mathbf{x}_0^{(\sigma)}, t_0, \mathbf{p}^{(\sigma)})$ for $(t, \mathbf{x}^{(\sigma)}) \in I_{12} \times U_{\mathbf{x}^{(\sigma)}}$. The semigroup properties of two flows hold. To discuss the synchronization of the two systems in Eqs. (3.1) and (3.2), the concepts of the slave and master systems are introduced herein.

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Definition 3.1 A system in Eq. (3.2) is called a *master* system if its flow $\mathbf{x}^{(r)}(t)$ is independent. A system in Eq. (3.1) is called a *slave* system of the master system if its flow $\mathbf{x}^{(s)}(t)$ is constrained by a flow $\mathbf{x}^{(r)}(t)$ of the master system.

From the foregoing definition, a *slave* system is constrained by a *master* system via a specific condition, which means that a slave system will be controlled by a master system under a specific constraint. Such a phenomenon is called the synchronization of the slave and master systems under such a specific condition. To make this concept clear, a definition is given as follows.

Definition 3.2 If a flow $\mathbf{x}^{(s)}(t)$ of a slave system in Eq. (3.1) is constrained by a flow $\mathbf{x}^{(r)}(t)$ of a *master* system in Eq. (3.2) through

$$\varphi(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}) = 0, \quad \boldsymbol{\lambda} \in \mathscr{R}^{n_0} \tag{3.3}$$

for time $t \in [t_{m_1}, t_{m_2}]$, then the slave system is said to be *synchronized* with the master system in the sense of Eq. (3.3) for time $t \in [t_{m_1}, t_{m_2}]$, also called an $(n_r : n_s)$-dimensional synchronization of the slave and master systems in the sense of Eq. (3.3). Four special cases are given as follows.

(i) If $t_{m_2} \to \infty$, the slave system is said to be absolutely synchronized with the master system in the sense of Eq. (3.3) for time $t \in [t_{m_1}, \infty)$.
(ii) If $t_{m_1} \to \infty$, the slave system is said to *be asymptotically synchronized* with the master system in the sense of Eq. (3.3).
(iii) For $n_r = n_s$, such a synchronization of the slave and master systems is called an *equidimensional* system synchronization in the sense of Eq. (3.3) for time $t \in [t_{m_1}, t_{m_2}]$.
(iv) For $n_r = n_s$, such a synchronization of the slave and master systems is called an *absolute*, equidimensional system synchronization in the sense of Eq. (3.3) for time $t \in [t_{m_1}, \infty)$.

If $n_r \neq n_s$, the $(n_r : n_s)$-dimensional synchronization is called a non-equidimensional system synchronization. It indicates that the dimension number of a slave system can be less or more than one of the master system. Thus, it is not necessary to require the slave and master systems have the same dimensions for synchronization. Under a certain rule in Eq. (3.3), it is interesting that a slave system can follow another completely different master system to synchronize. From the foregoing definition, it can be seen that a slave system is synchronized with a master system under a constraint condition. In fact, constraints for such a synchronization phenomenon can be more than one. In other words, a slave system is synchronized with a master system under multiple constraints. Thus, the synchronization of a slave system with a master system under multiple constraints is defined.

Definition 3.3 An n_s-dimensional slave system in Eq. (3.1) is called to be synchronized with an n_r-dimensional master system in Eq. (3.2) of the $(n_r : n_s; l)$-*type* (or an $(n_r : n_s; l)$-*synchronization*) if there are l-linearly independent functions

$\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ ($j \in \mathscr{L}$ and $\mathscr{L} = \{1, 2, \cdots, l\}$) to make two flows $\mathbf{x}^{(r)}(t)$ and $\mathbf{x}^{(s)}(t)$of the master and slave systems satisfy

$$\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j) = 0, \quad \boldsymbol{\lambda}_j \in \mathscr{R}^{n_j} \text{ and } j \in \mathscr{L} \tag{3.4}$$

for time $t \in [t_{m_1}, t_{m_2}]$. Eight special cases are given as follows:

(i) If $t_{m_2} \to \infty$, the slave system is said to be *absolutely* synchronized of the $(n_r\!: n_s; l)$*-type* with the master system (or an $(n_r : n_s; l)$-absolute synchronization) in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.

(ii) If $t_{m_1} \to \infty$, the slave system is said to be *asymptotically* synchronized of the $(n_r : n_s; l)$*-type* with the master system (*or* an $(n_r : n_s; l)$*-asymptotic synchronization*) in the sense of Eq. (3.4).

(iii) For $l = n_s$, the slave system is said to be *completely* synchronized of the $(n_r\!: n_s; n_s)$*-type* with the master system (*or an* $(n_r : n_s; n_s)$*-complete synchronization*) in the sense of Eq. (3.4) for time $t \in [t_{m_1}, t_{m_2}]$.

(iv) For $l = n_s$ and $t_{m_2} \to \infty$, the synchronization of the slave and master systems is called *an* $(n_r : n_s; n_s)$*-absolute, complete synchronization* in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.

(v) If $n_r = n_s = n > l$, the synchronization of the slave and master systems is called an equidimensional system synchronization (or an $(n : n; l)$synchronization) in the sense of Eq. (3.4) for time $t \in [t_{m_1}, t_{m_2}]$.

(vi) If $n_r = n_s = n > l$ and $t_{m_1} \to \infty$, the synchronization of the slave and master systems is called an equidimensional, $(n : n; l)$-absolute synchronization in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.

(vii) If $n_r = n_s = n = l$, the synchronization of the slave and master systems is called an equidimensional, complete synchronization (usually called *a synchronization*) in the sense of Eq. (3.4) for time $t \in [t_{m_1}, t_{m_2}]$.

(viii) If $n_r = n_s = n = l$ and $t_{m_2} \to \infty$, the synchronization of the slave and master systems is called an equidimensional, absolute, complete synchronization (or called an *absolute* synchronization) in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.

In the foregoing definition, if the l-nonlinear equations are linearly independent, then there is a set of constants k_j and only $k_j = 0$ for all $j \in \mathscr{L}$ exists to make the following equation hold for all the domains and time,

$$\sum\nolimits_{j=1}^{l} k_j \varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j) = 0. \tag{3.5}$$

In addition, the independence of functions $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ (for all $j \in \mathscr{L}$) is checked through the corresponding normal vectors. The normal vector of $\varphi_j(\mathbf{x}^{(r)}(t),\mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ is computed by

$$\mathbf{n}_{\varphi_j} = \nabla \varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j) = \left(\frac{\partial \varphi_j}{\partial \mathbf{x}^{(r)}}, \frac{\partial \varphi_j}{\partial \mathbf{x}^{(s)}} \right)^{\mathrm{T}}. \tag{3.6}$$

For all domains and time, if all the normal vectors $\mathbf{n}_{\varphi_j}$ $(j \in \mathscr{L})$ are linearly independent, i.e.,

$$\sum\nolimits_{j=1}^{l} k_j \mathbf{n}_{\varphi_j} = 0 \quad \text{only if } k_j = 0 \text{ for all } j \in \mathscr{L}. \tag{3.7}$$

then the functions $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$are linearly independent.

The foregoing definition tells that the slave and master systems are synchronized under l-constraints whatever the state-space dimension of the slave system is higher or lower than the master system. For $l < n_s$, the l-variables of the n-state variables of the slave system can be expressed by the n-state variables of the master system via the l-constraints. Select any l-variables $x_{[j]}$ and the rest $(n_s - l)$ variables $x_{[k]}$ of the n_s-state variables, i.e.,

$$\begin{aligned} &x_{[j]}^{(s)} \in \{x_i, i = 1, 2, \cdots, n_s\} \quad \text{for } j = 1, 2, \cdots, l \\ &x_{(k)} \in \{x_i, i = 1, 2, \cdots, n_s\} \quad \text{for } k = l+1, l+2, \cdots, n_s \end{aligned} \tag{3.8}$$

From Eq. (3.4), due to the linear independence of functions $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ $(j = 1, 2, \cdots, l)$, the constraint conditions give

$$x_{[j]} = f_{[j]}(\mathbf{x}^{(s)}, x_{(l+1)}, x_{(l+2)}, \cdots, x_{(n_s)}, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \cdots, \boldsymbol{\lambda}_l) \quad \text{for } j \in \mathscr{L}. \tag{3.9}$$

In this case, the state variables $x_{[j]}$ for $j \in \mathscr{L}$ can be said to be synchronized with the master system in the conditions of Eq. (3.4). The subscripts $[\cdot]$ and $(\cdot)$ of the state variables of the slave systems stand for the *synchronizable* and *non-synchronizable* variables to the master systems, respectively. If $l = 1$, this definition is reduced to Definition 3.2 and $(n_r : n_s; 1) \equiv (n_r : n_s)$, the $(n_r : n_s; l)$-synchronization reduces to the $(n_r : n_s)$-synchronization. However, for $l = n_s$, the n_s-linearly independent conditions constrain the responses of the master and slave flows in the n_r-dimensional systems. Thus, the n_s-components of the slave flow can be completely determined by the n_r-components of a flow in the master system. Therefore, for the complete synchronization of the slave and master systems, a flow of the slave system is completely controlled by the master system through the constraint conditions in Eq. (3.4). For $l > n_s$, the slave system is overconstrained by the master system. Such a case will be discussed later. For $n_r = n_s = n = l$, an equi-dimensional, complete synchronization of the slave and master systems is obtained. For this case, n-components of a flow in the slave system are controlled by the n-components of a flow in the master system through the n-constraint equations in Eq. (3.4). Because the n-constraint equations in Eq. (3.4) are linearly independent, the determinant of the Jacobian matrix of functions in Eq. (3.4) in neighborhood of the master flow $\mathbf{x}^{(r)}$ is nonzero. Therefore, there is a one-to-one relation between the slave and master flows $\mathbf{x}^{(s)}$ and $\mathbf{x}^{(r)}$. It implies that the slave flow is completely controlled by the master flow. From the above discussion, one obtains

$$\mathbf{x}^{(s)}(t) = \mathbf{h}(\mathbf{x}^{(r)}(t), \boldsymbol{\lambda}) \quad \text{or}$$
$$x_i^{(s)}(t) = h_i(\mathbf{x}^{(r)}(t), \boldsymbol{\lambda}) \quad \text{for } i = 1, 2, \cdots, n. \tag{3.10}$$

Introduce a set of new variables with n-linear, independent relations between the slave and master systems. So one has

$$\mathbf{z}(t) = \mathbf{x}^{(s)}(t) - \mathbf{B}\mathbf{x}^{(r)}(t) = \mathbf{h}(\mathbf{x}^{(r)}) - \mathbf{B}\mathbf{x}^{(r)} \quad \text{or}$$
$$z_i(t) = x_i^{(s)}(t) - b_i x_i^{(r)}(t) = h_i(\mathbf{x}^{(r)}) - b_i x_i^{(r)}(t) \quad \text{for } i = 1, 2, \cdots, n \tag{3.11}$$

where a constant diagonal matrix $\mathbf{B} = diag(b_1, b_2, \cdots, b_n)$. In recent researches on the synchronization of two systems, one likes to make $z_i(t) \to 0$ for $t \to t_{m_1}$ and $z_i(t) = 0$ for $t \in [t_{m_1}, t_{m_2}]$, from which the slave and master system are synchronized. To achieve such synchronization, the fixed points of $b_i x_i^{(r)}(t) = h_i(\mathbf{x}^{(r)})$ for $i = 1, 2, \cdots, n$ can be determined and independent of time. Such a concept can be extended to such linear synchronization, i.e., for $z_i(t) \to c_i$ (constant) for $t \to t_m$ and $z_i(t) = c_i$ for $t \in [t_{m_1}, t_{m_2}]$. The definition is given as follows:

Definition 3.4 For the slave and master in Eqs. (3.1) and (3.2) with $n_r = n_s = n$, if the slave and master flows satisfy

$$\mathbf{x}^{(s)}(t) - \mathbf{B}\mathbf{x}^{(r)}(t) = \mathbf{c} \tag{3.12}$$

with a constant diagonal matrix $\mathbf{B} = diag(b_1, b_2, \cdots, b_n)$ and a constant vector $\mathbf{c} = (c_1, c_2, \cdots, c_n)^{\mathrm{T}}$ for $t \in [t_{m_1}, t_{m_2}]$, then the slave and master systems are equidimensionally synchronized in such a linear sense. If $t_{m+1} \to \infty$, the synchronization of the slave and master systems is absolutely and equidimensionally synchronized in the linear sense for time $t \in [t_{m_1}, \infty)$. Three important synchronizations are also given as follows:

(i) If $\mathbf{c} = \mathbf{0}$ and $b_i = 1$ $(i = 1, 2, \cdots, n)$, the synchronization of the slave and master systems is called an identical synchronization.
(ii) If $\mathbf{c} = \mathbf{0}$ and $b_i = -1$ $(i = 1, 2, \cdots, n)$, the synchronization of the slave and master systems is called an antisymmetric synchronization.
(iii) If $\mathbf{c} = \mathbf{0}$ and $b_i \in \{1, -1\}$ $(i = 1, 2, \cdots, n)$, the synchronization of the slave and master systems is called a mixed, identical and antisymmetric synchronization.

To extend the above idea, new variables are introduced as

$$z_j = \varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j), \quad j \in \mathscr{L}$$
$$\mathbf{z} = \boldsymbol{\varphi}(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}) \tag{3.13}$$

If $z_j = c_j$ (const) or $z_j = 0$, Eq. (3.13) can be used as the constraint condition in Eq. (3.4). If the slave and master systems are not synchronized, the new variables

$(z_j \neq c_j,\ j = 1, 2, \cdots, l)$ will change with time t. The corresponding time-change rate is given by

$$\begin{aligned}
\dot{z}_j &= D\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j) = \frac{\partial \varphi_j}{\partial \mathbf{x}^{(r)}}\dot{\mathbf{x}}^{(\alpha)} + \frac{\partial \varphi_j}{\partial \mathbf{x}^{(s)}}\dot{\mathbf{x}}^{(\beta)} + \frac{\partial \varphi_j}{\partial t} \\
&= \frac{\partial \varphi_j}{\partial \mathbf{x}^{(r)}}\mathbf{F}^{(r)} + \frac{\partial \varphi_j}{\partial \mathbf{x}^{(s)}}\mathbf{F}^{(s)} + \frac{\partial \varphi_j}{\partial t}, \\
\dot{\mathbf{z}} &= D\boldsymbol{\varphi}(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}) = \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}^{(r)}}\dot{\mathbf{x}}^{(r)} + \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}^{(s)}}\dot{\mathbf{x}}^{(s)} + \frac{\partial \boldsymbol{\varphi}}{\partial t} \\
&= \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}^{(r)}}\mathbf{F}^{(r)} + \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}^{(s)}}\mathbf{F}^{(s)} + \frac{\partial \boldsymbol{\varphi}}{\partial t}.
\end{aligned} \tag{3.14}$$

For simplicity, $D\varphi_j = \varphi_j^{(1)}$ and $D^r\varphi_j = \varphi_j^{(r)}$ are adopted from now on. If the slave and master systems are continuous, the time-change rate of the new variables for the constraint conditions in Eq. (3.4) should be zero, i.e., $\dot{z}_j = 0$ $(j \in \mathscr{L})$ or $\dot{\mathbf{z}} = \mathbf{0} \in \mathscr{R}^l$. However, if the slave and master systems are discontinuous to the constraint conditions, the time-change rate of the new variables for the constraint conditions in Eq. (3.4) may not be zero. To investigate the synchronization, the constraints are considered as boundaries in discontinuous dynamical systems.

The slave and master flows $\mathbf{x}^{(s)}(t)$ and $\mathbf{x}^{(r)}(t)$ are determined by differential equations in Eqs. (3.1) and (3.2). Suppose at least there is a point $\mathbf{x}_m$ at time t_m to satisfy the constraint condition in Eq. (3.3), i.e.,

$$z_m = \varphi(\mathbf{x}_m^{(r)}, \mathbf{x}_m^{(s)}, t_m, \boldsymbol{\lambda}) = 0 \tag{3.15}$$

For $t > t_m$, the synchronization between the slave and master systems requires the slave and master flows to satisfy the constraint condition in Eq. (3.3). Because the master flow is independent, only the slave flow can be changed for the condition in Eq. (3.3). If the constraint condition in Eq. (3.3) is treated as a super-surface, the slave system should be switched at the super-surface. If the slave and master systems are C^r-continuous and differentiable $(r \geq 1)$ to the super-surface, the slave and master flows will pass through the super-surface instead of staying on the super-surface because of the continuity and differentiation of the slave and master flows. Otherwise, on the super-surface, one obtains $\dot{z} = \varphi^{(1)} = 0$ for all time $t > t_m$ and $\varphi^{(k)} = 0$ for $k = 1, 2, \cdots$. From a theory of discontinuous dynamical systems in Luo [2, 3], at least the slave system possesses discontinuous vector fields to make the slave and master flows stay on the super-surface, which means that the slave and master systems to the constraint can keep the synchronization on the super-surface. Therefore, the constraints can be used as super-surfaces to investigate the synchronization of slave and master systems.

3.1.1 Generalized Synchronization

As discussed in the previous section, if the number of constraints for slave and master systems is over the dimension of the slave state space (i.e., $l > n_s$), the slave system is overconstrained under the constraint conditions by the master system. In other words, if all the constraint conditions are satisfied, the master system should be partially constrained also for $n_s < l \leq n_r + n_s$. Otherwise, the constraint conditions cannot be satisfied for the synchronization of the slave and master systems. The overconstrained synchronization for slave and master systems can be defined from Definition 3.3, i.e.,

Definition 3.5 If $l > n_s$, an $(n_r : n_s; l)$-synchronization of the slave and master systems in Eqs. (3.1) and (3.2) in sense of Eq. (3.4) for time $t \in [t_{m_1}, t_{m_2}]$ is said to be an $(n_r : n_s; l)$-overconstrained synchronization.

To make an overconstrained slave system be synchronized with a master system, the flow of the master system should be controlled by the constraints. Generally speaking, the slave system can be partially controlled by some constraints in Eq. (3.4), and the master system can be partially controlled by the rest constraints in Eq. (3.4) as well. For some time intervals, the slave system can be controlled by the master system under the constraints. With time varying, for some time intervals, the master system can also be controlled by the slave system. For this case, it is very difficult to know which one of two systems is a slave or master system. In fact, it is not necessary to distinguish slave and master systems from two dynamical systems. For the synchronization of two or more systems, Definition 3.2 can be generalized as follows.

Definition 3.6 If a flow $\mathbf{x}^{(s)}(t)$ of a system in Eq. (3.1) with a flow $\mathbf{x}^{(r)}(t)$ of a system in Eq. (3.2) is constrained by a single constraint in Eq. (3.3) for time $t \in [t_{m_1}, t_{m_2}]$, then the two systems are said to be *synchronized* in the sense of Eq. (3.3) for time $t \in [t_{m_1}, t_{m_2}]$. Five special cases are given as follows.

(i) If $t_{m_2} \to \infty$, the two systems are said to be *absolutely* synchronized in the sense of Eq. (3.3) for time $t \in [t_{m_1}, \infty)$.
(ii) If $t_{m_1} \to \infty$, the two systems are said to be *asymptotically* synchronized in the sense of Eq. (3.3).
(iii) For $n_s = n_r = n$, the two *equidimensional* systems are said to be synchronized in the sense of Eq. (3.3) for time $t \in [t_{m_1}, t_{m_2}]$.
(iv) For $n_s = n_r = n$ and $t_{m_2} \to \infty$, the two *equidimensional* systems are said to be *absolutely* synchronized in the sense of Eq. (3.3) for time $t \in [t_{m_1}, \infty)$.
(v) For $n_s = n_r = n$ and $t_{m_1} \to \infty$, the two *equidimensional* systems are said to be *asymptotically* synchronized in the sense of Eq. (3.3).

In an alike fashion, the synchronization of slave and master systems in Definition 3.3 should be generalized for the synchronization of slave and master systems with or without overconstraints.

Definition 3.7 An n_r-dimensional system in Eq. (3.1) with an n_s-dimensional system in Eq. (3.2) is said to be synchronized with l-constraints (or an l-constraint *synchronization*) for time $t \in [t_{m_1}, t_{m_2}]$ if there are l-linearly independent functions $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ ($j \in \mathscr{L}$ and $\mathscr{L} = \{1, 2, \cdots, l\}$ with $l < n_r + n_s$) to make two flows $\mathbf{x}^{(r)}(t)$ and $\mathbf{x}^{(s)}(t)$ of the two systems satisfy the constraints in Eq. (3.4) for time $t \in [t_{m_1}, t_{m_2}]$. Five special cases are given as follows:

(i) If $t_{m_2} \to \infty$, the two systems are said to be *absolutely* synchronized with l-constraints (or an absolute, l-constraint synchronization) in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.
(ii) If $t_{m_1} \to \infty$, the two systems are said to be *asymptotically* synchronized with l-constraints (*or* an *asymptotic* l-constraint synchronization) in the sense of Eq. (3.4).
(iii) If $n_s = n_r = n$, the two equidimensional systems are said to be synchronized with l-constraints in the sense of Eq. (3.4) for time $t \in [t_{m_1}, t_{m_2}]$.
(iv) If $n_s = n_r = n$ and $t_{m_2} \to \infty$, the two equidimensional systems are said to be *absolutely* synchronized with l-constraints in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.
(v) If $n_s = n_r = n$ and $t_{m_1} \to \infty$, the two equidimensional systems are said to be *asymptotically* synchronized with l-constraints in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.

From the above definition, the number of constraints in Eq. (3.4) can be greater than the dimension number of state space for any one of the two systems in Eqs. (3.1) and (3.2) (i.e., $l > n_s$ or $l > n_r$). For such case, one cannot control only one of the two systems to make them be synchronized through the constraints. In other words, one must control both of two systems to make the corresponding synchronization occur. Of course, if $l \leq n_s$ or $l \leq n_r$, one can control only one of two systems to make them be synchronized through the constraints in Eq. (3.4). If the constraint functions $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ (for all $j \in \mathscr{L}$) is time independent for $l = n_r + n_s$, Eq. (3.4) will give a set of fixed values of $\mathbf{x}^{(r)*}$ and $\mathbf{x}^{(s)*}$, which are independent of time. The constraints yield the values-fixed, static points in the resultant sate space. To make the two systems in Eqs. (3.1) and (3.2) be synchronized at the static points in phase space, such a synchronization can be called *a static synchronization* of two systems in Eqs. (3.1) and (3.2). For $l > n_s + n_r$, the time-independent constraints in Eq. (3.4) will give the statically overconstrained synchronization, which may not be meaningful for practical problems. Such a case will not be discussed any more. If the constraint functions of $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ (for all $j \in \mathscr{L}$) are time dependent for $l = n_r + n_s$, Eq. (3.4) will give a flow of $\mathbf{x}^{(r)*}$ and $\mathbf{x}^{(s)*}$ relative to time. To eliminate time, the constraints in Eq. (3.4) give a one-dimensional flow in the resultant phase space. If the time-dependent constraint functions of $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ (for all $j \in \mathscr{L}$) are *of* l-dimensions with $l = n_s + n_r + 1$, Eq. (3.4) will give a set of fixed values of $\mathbf{x}^{(r)*}$ and $\mathbf{x}^{(s)*}$ at a specific time t^* in the resultant phase space, which is an instantaneous fixed point only at time t^*. For this case, it is very difficult for the two systems to be synchronized for such an instantaneous point. Such a case may not be too meaningful, which will not be discussed. Therefore, the following two definitions are given to describe the above-discussed cases.

Definition 3.8 An n_r-dimensional system in Eq. (3.1) with an n_s-dimensional system in Eq. (3.2) is said to be statically synchronized with l-constraints (or *a static synchronization*) for time $t \in [t_{m_1}, t_{m_2}]$ if there are l-linearly independent and *time-independent* functions $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ $(j \in \mathscr{L}$ and $\mathscr{L} = \{1, 2, \cdots, l\}$ with $l = n_r + n_s)$ to make two flows $\mathbf{x}^{(r)}(t)$ and $\mathbf{x}^{(s)}(t)$ of the two systems satisfy the constraints in Eq. (3.4) for time $t \in [t_{m_1}, t_{m_2}]$. Two special cases are:

(i) If $t_{m_2} \to \infty$, the two systems are said to be *absolutely* and *statically* synchronized with l-constraints (or an *absolute* and *static* synchronization) in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.
(ii) If $t_{m_1} \to \infty$, the two systems are said to be *asymptotically* and *statically* synchronized with l-constraints (*or* an *asymptotic and static* synchronization) in the sense of Eq. (3.4).

Definition 3.9 An n_r-dimensional system in Eq. (3.1) with an n_s-dimensional system in Eq. (3.2) is said to be synchronized with a one-dimensional constraint flow (or a *1D constraint-flow synchronization*) for time $t \in [t_{m_1}, t_{m_2}]$ if there are l linearly independent and *time-dependent* function $\varphi_j(\mathbf{x}^{(r)}(t), \mathbf{x}^{(s)}(t), t, \boldsymbol{\lambda}_j)$ $(j \in \mathscr{L}$ and $\mathscr{L} = \{1, 2, \cdots, l\}$ with $l = n_r + n_s)$ to make two flows $\mathbf{x}^{(r)}(t)$ and $\mathbf{x}^{(s)}(t)$ of the two systems satisfy constraints in Eq. (3.4) for time $t \in [t_{m_1}, t_{m_2}]$. Two special cases are given as follows:

(i) If $t_{m_2} \to \infty$, the two systems are said to be *absolutely* synchronized with a one-dimensional constraint flow (or an *absolute, 1D constraint-flow synchronization*) in the sense of Eq. (3.4) for time $t \in [t_{m_1}, \infty)$.
(ii) If $t_{m_1} \to \infty$, the two systems are said to be *asymptotically* synchronized with a one-dimensional constraint flow (an *asymptotic, 1D constraint-flow synchronization*) in the sense of Eq. (3.4).

3.1.2 Resultant Dynamical Systems

From the theory of discontinuous dynamical systems in Luo [2, 3], the synchronization of two or more dynamical systems with specific constraints can be discussed through a resultant dynamical system. The constraint conditions can be considered as a set of hypersurfaces. If the resultant system to the constraints is discontinuous, the resultant discontinuous dynamical system can be adjusted on both sides of each super-surface for such synchronization. For doing so, a set of new state variables for the resultant discontinuous system will be introduced, and the subdomains and boundaries relative to the constraints will be presented. For synchronization of slave and master systems on the constraint surfaces, only the slave system can be adjusted, and the master system cannot be adjusted. In other words, the slave system can be controlled in order to make it be synchronized with the master system through the constraints. That is, the slave system can be expressed by discontinuous vector fields to all the constraint surfaces for such synchronization, but the master system should keep a continuous vector field to such constraint surfaces. However, for a resultant system formed by two systems with constraints, one can adjust two dynamical systems to make them be synchronized on the constraint conditions in general.

A new vector of state variables of two dynamical systems in Eqs. (3.1) and (3.2) is introduced as

$$\mathbf{y} = (\mathbf{x}^{(r)};\mathbf{x}^{(s)})^{\mathrm{T}} = (x_1^{(r)}, x_2^{(r)}, \cdots, x_{n_r}^{(r)}; x_1^{(s)}, x_2^{(s)}, \cdots, x_{n_s}^{(s)})^{\mathrm{T}} \in \mathscr{R}^{n_r+n_s} \tag{3.16}$$

The notation $(\bullet;\bullet) \equiv (\bullet,\bullet)$ is just for a combined vector of state vectors of two dynamical systems. From the constraint condition in Eq. (3.3), a constraint boundary for the discontinuous description of the synchronization of two dynamical systems in Eqs. (3.1) and (3.2) can be defined, and the corresponding domains separated by such a constraint boundary can be obtained.

Definition 3.10 A constraint boundary in an $(n_r + n_s)$-dimensional phase space for the synchronization of two dynamical systems in Eqs. (3.1) and (3.2) to constraint condition in Eq. (3.3) is defined as

$$\begin{aligned} \partial\Omega_{12} &= \bar{\Omega}_1 \cap \bar{\Omega}_2 \\ &= \left\{ \mathbf{y}^{(0)} \left| \begin{array}{l} \varphi(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) \equiv \varphi(\mathbf{x}^{(r;0)}(t), \mathbf{x}^{(s;0)}(t), t, \boldsymbol{\lambda}) = 0, \\ \varphi \text{ is } C^{r_1}\text{-continuous } (r_1 \geq 1) \end{array} \right. \right\} \subset \mathscr{R}^{n_r+n_s-1}; \end{aligned} \tag{3.17}$$

and two corresponding domains for a resultant system of two dynamical systems in Eqs. (3.1) and (3.2) are defined as

$$\begin{aligned} \Omega_1 &= \left\{ \mathbf{y}^{(0)} \left| \begin{array}{l} \varphi(\mathbf{y}^{(1)}, t, \boldsymbol{\lambda}) \equiv \varphi(\mathbf{x}^{(r;1)}(t), \mathbf{x}^{(s;1)}(t), t, \boldsymbol{\lambda}) > 0, \\ \varphi \text{ is } C^{r_1}\text{-continuous } (r_1 \geq 1) \end{array} \right. \right\} \subset \mathscr{R}^{n_r+n_s} \\ \Omega_2 &= \left\{ \mathbf{y}^{(0)} \left| \begin{array}{l} \varphi(\mathbf{y}^{(2)}, t, \boldsymbol{\lambda}) \equiv \varphi(\mathbf{x}^{(r;2)}(t), \mathbf{x}^{(s;2)}(t), t, \boldsymbol{\lambda}) < 0, \\ \varphi \text{ is } C^{r_1}\text{-continuous } (r_1 \geq 1) \end{array} \right. \right\} \subset \mathscr{R}^{n_r+n_s}; \end{aligned} \tag{3.18}$$

On the two domains, the resultant system of two dynamical systems is discontinuous to the constraint boundary, defined by

$$\dot{\mathbf{y}}^{(\alpha)} = \mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \text{ in } \Omega_\alpha \text{ for } \alpha = 1, 2 \tag{3.19}$$

where $\mathbb{F}^{(\alpha)} = (\mathbf{F}^{(r;\alpha)}, \mathbf{F}^{(s;\alpha)})^{\mathrm{T}}$ and $\boldsymbol{\pi}^{(\alpha)} = (\mathbf{p}^{(r;\alpha)}, \mathbf{p}^{(s;\alpha)})^{\mathrm{T}}$. Suppose there is a vector field $\mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ on the constraint boundary with $\varphi(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) = 0$, and the corresponding dynamical system on such a boundary is expressed by

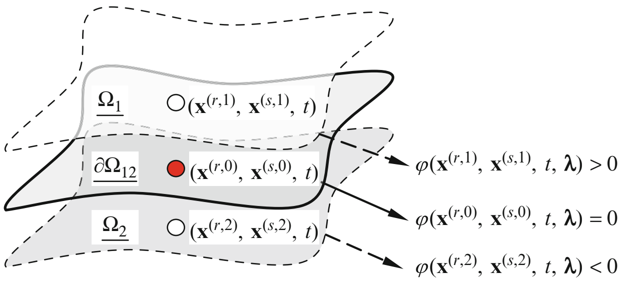

Fig. 3.1 Constraint boundary and domains in $(n_r + n_s)$-dimensional state space

$$\dot{\mathbf{y}}^{(0)} = \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) \text{ on } \partial\Omega_{12}. \tag{3.20}$$

The domains Ω_α $(\alpha = 1, 2)$ are separated by the constraint boundary $\partial\Omega_{12}$, as shown in Fig. 3.1. For a point $(\mathbf{x}^{(r,1)}, \mathbf{x}^{(s,1)}) \in \Omega_1$ at time t, $\varphi(\mathbf{x}^{(r,1)}, \mathbf{x}^{(s,1)}, t, \boldsymbol{\lambda}) > 0$. For a point $(\mathbf{x}^{(r,2)}, \mathbf{x}^{(s,2)}) \in \Omega_2$ at time t, $\varphi(\mathbf{x}^{(r,2)}, \mathbf{x}^{(s,2)}, t, \boldsymbol{\lambda}) < 0$. However, on the boundary $(\mathbf{x}^{(r,0)}, \mathbf{x}^{(s,0)}) \in \partial\Omega_{12}$ at time t, the constraint condition for synchronization should be satisfied (i.e., $\varphi(\mathbf{x}^{(r,0)}, \mathbf{x}^{(s,0)}, t, \boldsymbol{\lambda}) = 0$). If the constraint condition is time independent, the constraint boundary determined by the constraint condition is invariant. The above definition is extended.

Definition 3.11 The jth-constraint boundary in an $(n_r + n_s)$-dimensional phase space for the synchronization of two dynamical systems in Eqs. (3.1) and (3.2), relative to the jth-constraint of the constraint conditions in Eq. (3.4), is defined as

$$\begin{aligned} \partial\Omega_{(12,j)} &= \bar{\Omega}_{(1,j)} \cap \bar{\Omega}_{(2,j)} \\ &= \left\{ \mathbf{y}^{(0;j)} \left| \begin{array}{l} \varphi_j(\mathbf{y}^{(0;j)}, t, \boldsymbol{\lambda}_j) \equiv \varphi_j(\mathbf{x}^{(r,0;j)}(t), \mathbf{x}^{(s,0;j)}(t), t, \boldsymbol{\lambda}_j) = 0, \\ \varphi_j \text{ is } C^{r_j}\text{-continuous } (r_j \geq 1) \end{array} \right. \right\} \\ &\subset \mathscr{R}^{n_r+n_s-1}; \end{aligned} \tag{3.21}$$

and two domains pertaining to the jth-boundary for a resultant system of two dynamical systems in Eqs. (3.1) and (3.2) are defined as

$$\begin{aligned} \Omega_{(1;j)} &= \left\{ \mathbf{y}^{(0)} \left| \begin{array}{l} \varphi_j(\mathbf{y}^{(1;j)}, t, \boldsymbol{\lambda}) \equiv \varphi_j(\mathbf{x}^{(r,1;j)}(t), \mathbf{x}^{(s,1;j)}(t), t, \boldsymbol{\lambda}_j) > 0, \\ \varphi_j \text{ is } C^{r_j}\text{-continuous } (r_j \geq 1) \end{array} \right. \right\} \\ &\subset \mathscr{R}^{n_r+n_s} \\ \Omega_{(2;j)} &= \left\{ \mathbf{y}^{(0)} \left| \begin{array}{l} \varphi_j(\mathbf{y}^{(2;j)}, t, \boldsymbol{\lambda}) \equiv \varphi_j(\mathbf{x}^{(r,2;j)}(t), \mathbf{x}^{(s,2;j)}(t), t, \boldsymbol{\lambda}_j) < 0, \\ \varphi_j \text{ is } C^{r_j}\text{-continuous } (r_j \geq 1) \end{array} \right. \right\} \\ &\subset \mathscr{R}^{n_r+n_s}; \end{aligned} \tag{3.22}$$

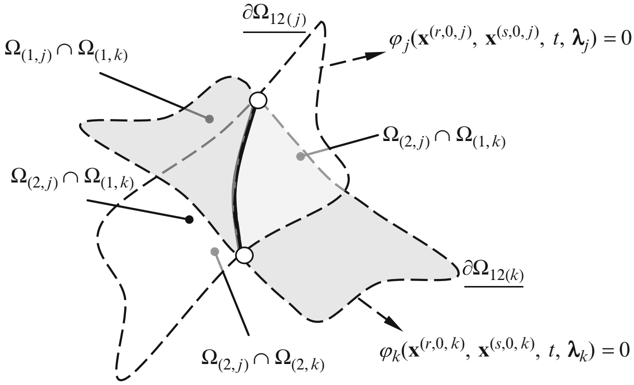

Fig. 3.2 An intersection of two boundaries with $\varphi_j = 0$ and $\varphi_k = 0$ for $j, k \in \mathscr{L}$ and $j \neq k$

On the two domains relative to the jth-constraint boundary, a discontinuous resultant system of two dynamical systems in Eqs. (3.1) and (3.2) with the jth-constraint in Eq. (3.4) is defined by

$$\dot{\mathbf{y}}^{(\alpha_j,j)} = \mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha_j,j)}, t, \boldsymbol{\pi}_j^{(\alpha_j)}) \text{ in } \Omega_{(\alpha_j,j)} \quad \text{for } \alpha_j = 1, 2 \tag{3.23}$$

where $\mathbb{F}^{(\alpha_j,j)} = (\mathbf{F}^{(r,\alpha_j,j)}; \mathbf{F}^{(s,\alpha_j,j)})^{\mathrm{T}}$ and $\boldsymbol{\pi}_j^{(\alpha_j)} = (\mathbf{p}_j^{(r,\alpha_j)}, \mathbf{p}_j^{(s,\alpha_j)})^{\mathrm{T}}$. Suppose there is a vector field of $\mathbb{F}^{(0,j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j)$ on the jth-constraint boundary with $\varphi_j(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j) = 0$, and the corresponding dynamical system on the jth-boundary is

$$\dot{\mathbf{y}}^{(0,j)} = \mathbb{F}^{(0;j)}(\mathbf{y}^{(0,j)}, t, \boldsymbol{\lambda}_j) \text{ in } \Omega_{(12,j)} \quad \text{for } \alpha_j = 1, 2 \tag{3.24}$$

Since l-constraint conditions are linearly independent, any two boundaries are intersected each other. Consider two constraint boundaries of $\partial\Omega_{12(j)}$ and $\partial\Omega_{12(k)}$ for synchronization. From Luo [4], the intersection edge of the two constraint boundaries is given by

$$\partial\Omega_{(12,jk)} = \partial\Omega_{(12,j)} \cap \partial\Omega_{(12,k)} \subset \mathscr{R}^{n_r+n_s-2} \tag{3.25}$$

and the corresponding domain in phase space is separated into four subdomains

$$\Omega_{(\alpha_j\alpha_k,jk)} = \Omega_{(\alpha_j,j)} \cap \Omega_{(\alpha_k,k)} \subset \mathscr{R}^{n_r+n_s} \quad \text{for } j, k \in \mathscr{L} \text{ and } \alpha_j, \alpha_k = 1, 2. \tag{3.26}$$

Such a partition of the domain in state space for a resultant system of two dynamical systems is sketched in Fig. 3.2. The intersection of the two constraint boundaries in state space for a resultant system of two dynamical systems is depicted by an $(n_r + n_s - 2)$-manifold, depicted by a dark curve. For the l-linearly independent constraints, the state space partition can be completed via such l-linearly independent constraint boundaries. Based on the l-constraint conditions, the corresponding intersection of boundaries is

$$\partial\Omega_{(12,\mathbf{J})} = \cap_{j=1}^{l}\partial\Omega_{(12,j)} \subset \mathscr{R}^{n_r+n_s-l}. \tag{3.27}$$

which gives an $(n_r + n_s - l)$-dimensional edge manifold. Consider the synchronization of the slave and master systems for discussion. If $n = l$, the intersection manifold of the constraints is an n_s-dimensional state space. Thus, the slave system can be completely controlled through the n_s-constraints to be synchronized with the master system. From the l-constraint conditions in Eq. (3.4), the domain in $(n_r + n_s)$ -dimensional state space is partitioned into many subdomains for the resultant system of two dynamical systems, i.e.,

$$\Omega_{\boldsymbol{\alpha}} = \Omega_{(\alpha_1\alpha_2\cdots\alpha_l)} = \cap_{j=1}^{l}\Omega_{(\alpha_j,j)} \subset \mathscr{R}^{n_r+n_s} \text{ for } \alpha_j = 1,2 \text{ and } j \in \mathscr{L}. \tag{3.28}$$

The total domain $\mho = \cup_{j=1}^{l} \cup_{\alpha_j=1}^{2} \left(\cap_{j=1}^{l}\Omega_{(\alpha_j,j)}\right) \subset \mathscr{R}^{n_r+n_s}$ is a union of all the subdomains. From the foregoing description of a resultant dynamical system, the synchronization of two systems under constraints can be investigated through such a resultant dynamical system with the constraint boundaries as in Luo [2, 3]. The constraint boundaries can be either of one side or of two sides. If the resultant system for the synchronization of two systems can be defined in one of the two subdomains only, such a constraint boundary is called one-side boundary. Otherwise, the constraint boundary is called two-side constraint boundary. If a flow of the resultant system can approach to a constraint flow on the constraint boundaries as $t \to \infty$, for such a case, the synchronization of two systems to the constraint boundaries is asymptotic.

3.2 Synchronization with a Single Constraint

In this section, the synchronicity of two systems to a single constraint will be presented, and the corresponding conditions for such synchronicity will be discussed.

3.2.1 Synchronicity

Before discussing the synchronicity of two dynamical systems to the constraint boundary, the neighborhood of the constraint boundary should be introduced through a typical point on such a constraint boundary for time t_m. For any small $\varepsilon > 0$, the neighborhood of a constraint boundary is defined as follows.

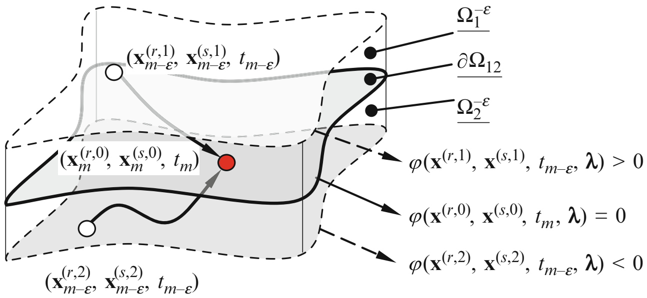

Fig. 3.3 A neighborhood of the constraint boundary and the attractivity of a resultant flow to the constraint boundary in $(n_r + n_s)$-dimensional state space

Definition 3.12 For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ ($\alpha \in \{1,2\}$) and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time $t_m, \mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$ or $(t_m, t_{m+\varepsilon}]$. The ε-neighborhood of the constraint boundary $\partial\Omega_{12}$ is defined as

$$
\begin{aligned}
\Omega_\alpha^{-\varepsilon} &= \left\{ \mathbf{y}^{(\alpha)} \middle| ||\mathbf{y}^{(\alpha)}(t) - \mathbf{y}_m^{(0)}|| \le \delta, \delta > 0, t \in [t_{m-\varepsilon}, t_m) \right\}, \\
\Omega_\alpha^{+\varepsilon} &= \left\{ \mathbf{y}^{(\alpha)} \middle| ||\mathbf{y}^{(\alpha)}(t) - \mathbf{y}_m^{(0)}|| \le \delta, \delta > 0, t \in (t_m, t_{m+\varepsilon}] \right\}.
\end{aligned}
\tag{3.29}
$$

For a point $\mathbf{y}_m^{(0)} = (\mathbf{x}_m^{(r,0)}, \mathbf{x}_m^{(s,0)})^{\mathrm{T}} \in \partial\Omega_{12}$ at time t_m, a surface of the constraint boundary $\partial\Omega_{12}$ at the instantaneous time t_m is governed by $\varphi(\mathbf{x}^{(r,0)}, \mathbf{x}^{(s,0)}, t_m, \boldsymbol{\lambda}) = \varphi(\mathbf{x}_m^{(r,0)}, \mathbf{x}_m^{(s,0)}, t_m, \boldsymbol{\lambda}) = 0$. If the constraint function φ is time independent, such a constraint surface for the synchronization of two dynamical systems is invariant with respect to time. Otherwise, this constraint surface changes with the instantaneous time t_m. In addition to the constraint surface, two boundaries of domain $\Omega_\alpha^{-\varepsilon}$ ($\alpha = 1, 2$) are determined by $\varphi(\mathbf{x}^{(r,\alpha)}, \mathbf{x}^{(s,\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) = \varphi(\mathbf{x}_{m-\varepsilon}^{(r,\alpha)}, \mathbf{x}_{m-\varepsilon}^{(s,\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda})$ $=$ const, as shown in Fig. 3.3. In the ε-neighborhood of a constraint boundary, if the resultant system of two dynamical systems is attractive to such a constraint boundary, any flows in the two ε-domains will approach the constraint boundary. Further, the synchronicity of two dynamical systems to the constraint boundary can be discussed. In other words, the attractivity of the resultant system to the constraint boundary requires that any flow in the two ε-domains of Ω_α ($\alpha = 1, 2$) approach the constraint boundary $\partial\Omega_{12}$ as $t \to t_m$. From Luo [2, 3], the synchronization of two dynamical systems to the constraint needs that any flows of the resultant system in the two ε-domains of Ω_α ($\alpha = 1, 2$) are attractive to the boundary.

Definition 3.13 Consider two dynamical systems in Eqs. (3.1) and (3.2) with a constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ ($\alpha \in \{1,2\}$) and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$. The two systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) are called to be *synchronized* for time $t_m \in [t_{m_1}, t_{m_2}]$ if

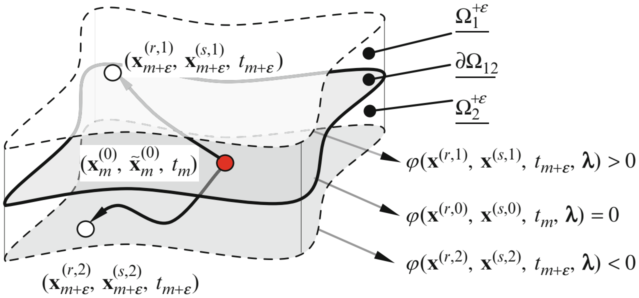

Fig. 3.4 The repulsion of a resultant flow to the constraint boundary in $(n_r + n_s)$-dimensional state space

$$\begin{aligned}&\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\&(-1)^{\alpha}[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0 \quad \text{for } \alpha = 1, 2.\end{aligned} \tag{3.30}$$

In addition to the attractivity of a flow of the resultant system to the constraint boundary, the repulsion of a flow of the resultant system to the constraint boundary can be defined. Because of such a repulsion, any flows of the resultant system in the two ε-domains of Ω_α ($\alpha = 1, 2$) can never approach the constraint boundary. In other words, two dynamical systems in Eqs. (3.1) and (3.2) cannot make the constraint condition in Eq. (3.3) be satisfied. Thus, the repulsion of a flow of the resultant system to the constraint boundary should be introduced. Such a repulsion phenomenon is sketched in Fig. 3.4. The constraint boundary $\partial\Omega_{12}$ is governed by $\varphi(\mathbf{x}^{(r,0)}, \mathbf{x}^{(s,0)}, t_m, \boldsymbol{\lambda}) = 0$. The boundaries of the ε-neighborhood of the constraint boundary are obtained by $\varphi(\mathbf{x}^{(r,\alpha)}, \mathbf{x}^{(s,\alpha)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) = \varphi(\mathbf{x}_{m+\varepsilon}^{(r,\alpha)}, \mathbf{x}_{m+\varepsilon}^{(s,\alpha)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) = \text{const}$. Two flows of the resultant system on both sides of the constraint boundary $\partial\Omega_{12}$ move away in two domains Ω_α ($\alpha = 1, 2$), which means that no any flows of the resultant system can arrive to the constraint boundary. So the synchronization of two dynamical systems in Eqs. (3.1) and (3.2) to the constraint in Eq. (3.3) cannot be achieved. Such a repulsion of a resultant system to the constraint boundary gives the desynchronization of two dynamical systems to the constraint in Eq. (3.3). The desynchronization of two systems to a constraint is defined.

Definition 3.14 Consider two systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ ($\alpha \in \{1, 2\}$) and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $(t_m, t_{m+\varepsilon}]$. The two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) are said to *be repelled* (*or desynchronized*) for $t_m \in [t_{m_1}, t_{m_2}]$ if

$$\begin{aligned}&\varphi(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\&(-1)^{\alpha}[\varphi(\mathbf{y}_{m+\varepsilon}^{(\alpha)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda})] < 0 \quad \text{for } \alpha = 1, 2.\end{aligned} \tag{3.31}$$

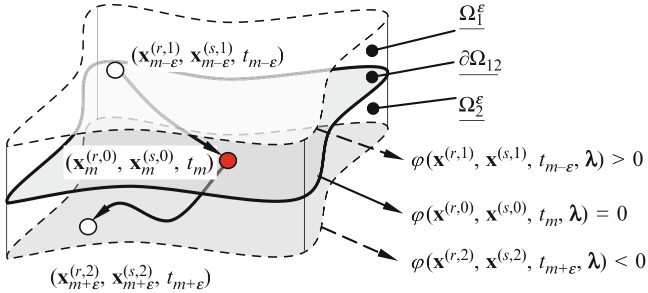

Fig. 3.5 A penetration of a resultant flow to the constraint boundary in $(n_r + n_s)$-dimensional state space

From the theory of discontinuous dynamical systems in Luo [2, 3], a resultant system of two dynamical systems in Eqs. (3.1) and (3.2) may pass through the constraint boundary from a domain to another. For this case, the penetration synchronicity of two dynamical systems can occur, as sketched in Fig. 3.5. Such synchronization can be called *an instantaneous synchronization*. A flow of a resultant system to the constraint boundary for time $t < t_m$ and $t > t_m$ lies in the two domains Ω_1 and Ω_2. In sense of Eq. (3.3), a definition of such penetration synchronicity is given as follows.

Definition 3.15 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. A resultant flow of two dynamical systems in Eqs. (3.1) and (3.2) is said to be *penetrated* to the constraint boundary $\partial\Omega_{\alpha\beta}$ from Ω_α to Ω_β at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$\left.\begin{aligned} &\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0; \\ &(-1)^\alpha[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0; \\ &(-1)^\beta[\varphi(\mathbf{y}_{m+\varepsilon}^{(\beta)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda})] < 0. \end{aligned}\right\} \tag{3.32}$$

In Definition 3.15, the incoming flow with "$-$" and outcome flow with "$+$" to the boundary are prescribed. From the foregoing definition, a penetration flow of the resultant system of two dynamical systems to the constraint boundary can be considered to be formed by the semi-synchronization and semi-desynchronization. Such a penetration flow of the resultant system to the constraint boundary can also be called *an instantaneous synchronization* of two dynamical systems in Eqs (3.1) and (3.2) to constraint in Eq. (3.3). Such an instantaneous synchronization will disappear because the semi-desynchronization exists. From the definition of a penetration flow, a flow of the resultant system in domain Ω_α approaches the constraint boundary. However, in domain Ω_β, such a flow will leave from the constraint boundary. To investigate the

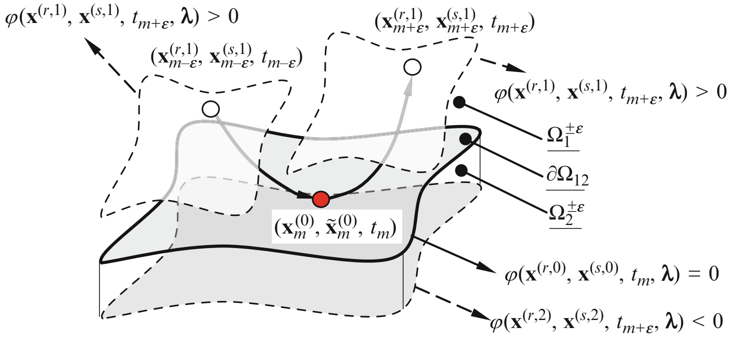

Fig. 3.6 Tangential synchronization to the constraint in an $(n_r + n_s)$-dimensional state space

relations among three types of synchronicity of two dynamical systems to the constraint in Eq. (3.3), the switchability of the synchronization, desynchronization, and penetration is very important, which can be discussed through the singularity of the resultant system to the constraint boundary.

3.2.2 *Singularity to Constraint*

From a theory of discontinuous dynamical systems in Luo [2–4], a flow of a resultant system of two dynamical systems may be tangential to the constraint boundary governed by the constraint condition in Eq. (3.3). For this case, the synchronicity of two dynamical systems to the constraint occurs only at one point and then returns back to the same domain. Such an *instantaneous* synchronization is different from a penetration flow of the resultant system to the constraint boundary. The tangential synchronization of two dynamical systems to the constraint is sketched in Fig. 3.6. In domain Ω_1, the tangential synchronization of the two systems to the constraint boundary $\partial\Omega_{12}$ is presented. The two boundaries at time $t_{m-\varepsilon}$ and $t_{m+\varepsilon}$ are given by the two different surfaces. For such synchronicity, the following definition is given.

Definition 3.16 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}^{(\alpha)}_{m\pm} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}^{(0)}_{m} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}^{(\alpha)}_{m\pm} = \mathbf{y}^{(0)}_{m}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega^{\pm\varepsilon}_\alpha$ for $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. A flow of a resultant system of two dynamical systems in Eqs. (3.1) and (3.2) is said to be *tangential* (or grazing) to the constraint boundary at time t_m if for $\alpha \in \{1,2\}$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(1)}(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = 0;\\
&(-1)^{\alpha}[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\alpha)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.33}$$

In Definition 3.16, the incoming flow with " $-$ " and outcome flow with " $+$ " to the boundary are prescribed. Such a tangency of a resultant flow to the constraint boundary will cause the synchronicity to be changed. The onset and vanishing singularity for synchronizations can be discussed, and the corresponding definition is given as follows.

Definition 3.17 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_m^{(\alpha)} \in \Omega_\alpha$ ($\alpha \in \{1,2\}$) and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_m^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous ($r_\alpha \geq 2$) and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$

(i) The *synchronization* of two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is called to be *vanishing* to form a penetration from domain Ω_α to Ω_β at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(1)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) \neq 0, \varphi^{(1)}(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda}) = 0;\\
&(-1)^{\alpha}[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0;\\
&(-1)^{\beta}[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\beta)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.34}$$

(ii) The synchronization of two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is called to be *onset* from a penetration from domain Ω_α to Ω_β at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(1)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) \neq 0, \varphi^{(1)}(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = 0;\\
&(-1)^{\alpha}[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0;\\
&(-1)^{\beta}[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\beta)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda})] < 0,
\end{aligned} \tag{3.35}$$

In Eq. (3.34), the notation " $\mp$ " represents the synchronization first with " $-$ " and the penetration secondly with "+". This condition is called either the *vanishing* condition of synchronization to form a new penetration or the *onset* condition of *penetration* from the synchronization at the boundary of constraint in Eq. (3.3). However, in Eq. (3.35), the notation " $\pm$ " represents the penetration first with "+"

and the synchronization secondly with " $-$." This condition is called the *onset* condition of *synchronization* from a state of penetration to the boundary, which can also be called the *vanishing* condition of *penetration* to form a synchronization at the constraint boundary at time t_m. The switching conditions between the synchronization and desynchronization are presented as follows.

Definition 3.18 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha\,(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous ($r_\alpha \geq 2$) and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$.

(i) The *synchronization of* two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) is called to be *onset* from a desynchronization at the constraint boundary at time t_m if for $\alpha = 1,2$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(1)}(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = 0;\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\alpha)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.36}$$

(ii) The synchronization of two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) is called to be *vanishing* to form a desynchronization at the constraint boundary at time t_m if for $\alpha = 1,2$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(1)}(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = 0;\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\alpha)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.37}$$

In Eq. (3.36), the notation " $\pm$ " represents the desynchronization first with "+" and the synchronization with " $-$ " second. This condition is called either *the onset condition of synchronization* from the desynchronization on the boundary or *the vanishing condition of desynchronization* to form a new synchronization on the boundary. In Eq. (3.37), the notation " $\mp$ " represents the synchronization first with " $-$ " and the desynchronization second with "+". This condition is called *the vanishing condition of synchronization* to form a new desynchronization, which can also be called *the onset condition of desynchronization* from the synchronization. Similarly, the onset and vanishing conditions of the desynchronization from the penetration can be discussed as for the synchronization. The following definition will give the onset and vanishing conditions of desynchronization.

Definition 3.19 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m,

$\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$.

(i) The desynchronization of two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) is called to be *vanishing* to form a penetration from Ω_α to Ω_β at the constraint boundary at time t_m if for $\alpha, \beta \in \{1, 2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(1)}(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = 0, \varphi^{(1)}(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) \neq 0;\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\alpha)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda})] < 0,\\
&(-1)^\beta[\varphi(\mathbf{y}_{m+\varepsilon}^{(\beta)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.38}$$

(ii) The desynchronization of two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) is called to be *onset* from a penetration from Ω_α to Ω_β at the constraint boundary at time t_m if for $\alpha, \beta \in \{1, 2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(1)}(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = 0,\ \varphi^{(1)}(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) \neq 0;\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\alpha)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda})] < 0;\\
&(-1)^\beta[\varphi(\mathbf{y}_{m+\varepsilon}^{(\beta)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda})] > 0.
\end{aligned} \tag{3.39}$$

Notice that in Eq. (3.38), the notation " $\pm$ " represents the desynchronization first with "+" and the penetration second with " $-$ ". This condition is called the *vanishing* condition of *desynchronization* to form a new penetration on the boundary and can also be called the *onset* condition of penetration from a synchronization state. However, in Eq. (3.39), the notation " $\mp$ " represents the penetration first with " $-$ " and the synchronization second with " $+$ ". This condition is called *the onset condition of desynchronization* from a penetration and also can be called *the vanishing condition of the penetration* to form a desynchronization state. From the previous three definitions, the switching between synchronization and penetration, between desynchronization and penetration, and between desynchronization and synchronization were presented. However, another switching between two penetrations should be discussed.

Definition 3.20 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha (\alpha \in \{1, 2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. The penetration of the two dynamical systems in

Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) is called to be *switched* at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m}^{(0)}, t_{m}, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(1)}(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi^{(1)}(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = 0;\\
&(-1)^{\alpha}[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\alpha)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda})] < 0,\\
&(-1)^{\beta}[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\beta)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.40}$$

Based on the definitions of the tangential (or grazing) and switching singularity, there is a critical parameter $\boldsymbol{\lambda}_{cr}$ from which $\partial\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda})/\partial\boldsymbol{\lambda}|_{\boldsymbol{\lambda}_{cr}} \neq 0$, such a singularity is called the corresponding bifurcation at $\boldsymbol{\lambda}_{cr}$for parameter $\boldsymbol{\lambda}$.

3.3 Synchronicity with Singularity

As similar to discontinuous dynamical systems in Luo [2–4], the above synchronicity of two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) can be extended to the case of higher order singularity. The corresponding definitions can be presented. The definition for the $(2k_\alpha : 2k_\beta)$ synchronization of two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) at the corresponding constraint boundary for time $t_m \in [t_{m_1}, t_{m_2}]$ is presented first.

Definition 3.21 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \boldsymbol{\Omega}_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_{m}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_{m}^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}_\alpha^{-\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_m)$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2k_\alpha + 1)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. The two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is called to be synchronized with the $(2k_1 : 2k_2)$-type to the constraint in Eq. (3.3) for time $t_m \in [t_{m_1}, t_{m_2}]$ if for $\alpha = 1, 2$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m}^{(0)}, t_{m}, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha;\\
&(-1)^{\alpha}[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.41}$$

As in the definition for the $(2k_1 : 2k_2)$-synchronization, the definition for the $(2k_1 : 2k_2)$-desynchronization of two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) on the corresponding constraint boundary for time $t_m \in [t_{m_1}, t_{m_2}]$ is also presented.

Definition 3.22 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \boldsymbol{\Omega}_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_{m}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_{m}^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}_\alpha^{+\varepsilon}$ for time

$t \in (t_{m+}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous ($r_\alpha \geq 2k_\alpha + 1$) and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. The two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is said to be desynchronized (or repelled) with the $(2k_1 : 2k_2)$-type to the constraint in Eq. (3.3) for $t_m \in [t_{m_1}, t_{m_2}]$ if for $\alpha = 1, 2$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0,\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) = 0, s_\alpha = 1, 2, \cdots, 2k_\alpha,\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m+\varepsilon}^{(\alpha)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.42}$$

As discussed before, the penetration on the boundary of constraint is composed of the semi-synchronization and semi-desynchronization. From the foregoing two definitions, the $(2k_\alpha : 2k_\beta)$-penetration of two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) at time t_m is described.

Definition 3.23 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ ($\alpha \in \{1, 2\}$) and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous ($r_\alpha \geq 2k_\alpha$) and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. A flow of two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is said to be *penetrated with the* $(2k_\alpha : 2k_\beta)$*-type* from domain Ω_α to domain Ω_β at the constraint boundary at time t_m if

$$\begin{aligned}
&\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha;\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = 0 \text{ for } s_\beta = 1, 2, \cdots, 2k_\beta;\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0 \text{ for } \alpha \in \{1, 2\} \text{ and}\\
&(-1)^\beta[\varphi(\mathbf{y}_{m+\varepsilon}^{(\beta)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda})] < 0 \text{ for } \alpha \neq \beta \in \{1, 2\}.
\end{aligned} \tag{3.43}$$

From the three definitions, the higher singularity is used for description of the synchronization, desynchronization, and penetration at the constraint boundary, and the switching among the three synchronous states can be discussed through the higher order singularity as well.

3.4 Higher Order Singularity

From the previous descriptions of the synchronization, desynchronization, and penetration with the higher order singularity for two dynamical systems to the constraint, the higher order singularity of the two dynamical systems to the constraint boundary is discussed as follows.

Definition 3.24 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2k_\alpha)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. A resultant flow of the two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is said to be tangential to the constraint boundary with the $(2k_\alpha - 1)$th-order at time t_m if for $\alpha \in \{1,2\}$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = 0 \;\; s_\alpha = 1, 2, \cdots, 2k_\alpha - 1;\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0,\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m+\varepsilon}^{(\alpha)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.44}$$

The foregoing definition gives the definition of the $(2k_\alpha - 1)$th tangential condition to the constraint boundary. Based on the similar ideas, the switchability of the synchronization, desynchronization, and penetration of two dynamical systems to the constraint boundary can be described.

Definition 3.25 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2k_\alpha + 1)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$.

(i) The $(2k_\alpha : 2k_\beta)$-synchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is said to be *vanishing* to form a $(2k_\alpha : 2k_\beta)$-penetration from domain Ω_α to domain Ω_β at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda}) = 0 \text{ for } s_\beta = 1, 2, \cdots, 2k_\beta + 1;\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0,\\
&(-1)^\beta[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\beta)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.45}$$

(ii) The $(2k_\alpha : 2k_\beta)$-synchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is said to be *onset* from the $(2k_\alpha : 2k_\beta)$-penetration from Ω_α to Ω_β at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m}^{(0)}, t_{m}, \boldsymbol{\lambda}) = 0;\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\beta = 1, 2, \cdots, 2k_\beta + 1;\\
&(-1)^{\alpha}[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0,\\
&(-1)^{\beta}[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\beta)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.46}$$

From this definition, the condition in Eq. (3.45) for *the onset of the* $(2k_\alpha : 2k_\beta)$ *-synchronization* from the $(2k_\alpha : 2k_\beta)$-penetration on the constraint boundary can also be called *the vanishing condition of the* $(2k_\alpha : 2k_\beta)$*-penetration* to form a new $(2k_\alpha : 2k_\beta)$-synchronization on the constraint boundary. In Eq. (3.46), the *vanishing condition of the* $(2k_\alpha : 2k_\beta)$-synchronization to form a new $(2k_\alpha : 2k_\beta)$-penetration can also be called *the onset condition of the* $(2k_\alpha : 2k_\beta)$-penetration from the synchronization. The onset and vanishing conditions of the $(2k_\alpha : 2k_\beta)$-desynchronization from the $(2k_\alpha : 2k_\beta)$-penetration can be discussed. The following definition will give the onset and vanishing conditions of the $(2k_\alpha : 2k_\beta)$-desynchronization.

Definition 3.26 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1, 2\})$ and $\mathbf{y}_{m}^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_{m}^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2k_\alpha + 1)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$.

(i) The $(2k_\alpha : 2k_\beta)$-synchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is called to be *vanishing* to form a $(2k_\alpha : 2k_\beta)$-desynchronization at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m}^{(0)}, t_{m}, \boldsymbol{\lambda}) = 0,\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha + 1,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\beta = 1, 2, \cdots, 2k_\beta + 1,\\
&(-1)^{\alpha}[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\alpha)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda})] < 0,\\
&(-1)^{\beta}[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\beta)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\beta)}, t_{m\mp}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.47}$$

(ii) The $(2k_\alpha : 2k_\beta)$-synchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is said to be *onset* from the $(2k_\alpha : 2k_\beta)$-desynchronization at the constraint boundary at time t_m if for $\alpha, \beta \in \{1, 2\}$ and $\alpha \neq \beta$

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m}^{(0)}, t_{m}, \boldsymbol{\lambda}) = 0,\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha + 1,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\beta = 1, 2, \cdots, 2k_\beta + 1,\\
&(-1)^{\alpha}[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\alpha)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda})] < 0,\\
&(-1)^{\beta}[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\beta)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.48}$$

The conditions in Eqs. (3.47) and (3.48) are inversely switched. The condition in Eq. (3.47) for *the onset condition of the* $(2k_\alpha : 2k_\beta)$*-synchronization* from the $(2k_\alpha : 2k_\beta)$-desynchronization on the constraint boundary can be called *the vanishing condition of the* $(2k_\alpha : 2k_\beta)$-desynchronization to form a new $(2k_\alpha : 2k_\beta)$-synchronization on such a constraint boundary. However, the condition in Eq. (3.48) for *the vanishing condition of the* $(2k_\alpha : 2k_\beta)$-synchronization to form a new $(2k_\alpha : 2k_\beta)$-penetration can be called *the onset condition of the* $(2k_\alpha : 2k_\beta)$-desynchronization from the synchronization. The switching of desynchronization and penetration on the boundary will be discussed as follows.

Definition 3.27 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2k_\alpha + 1)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$.

(i) The $(2k_\alpha : 2k_\beta)$-desynchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is called to be *vanishing* to form a $(2k_\alpha : 2k_\beta)$-penetration from domain Ω_α to domain Ω_β at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$
\begin{aligned}
&\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0,\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha + 1,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\beta = 1, 2, \cdots, 2k_\beta,\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\alpha)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda})] < 0,\\
&(-1)^\beta[\varphi(\mathbf{y}_{m+\varepsilon}^{(\beta)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.49}
$$

(ii) The $(2k_\alpha : 2k_\beta)$-desynchronization of the two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is said to be *onset* from the $(2k_\alpha : 2k_\beta)$-penetration from domain Ω_α to domain Ω_β at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$
\begin{aligned}
&\varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0,\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha + 1,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\beta = 1, 2, \cdots, 2k_\beta,\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\alpha)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda})] < 0,\\
&(-1)^\beta[\varphi(\mathbf{y}_{m+\varepsilon}^{(\beta)}, t_{m+\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda})] < 0.
\end{aligned} \tag{3.50}
$$

In Eq. (3.49), *the onset condition of the* $(2k_\alpha : 2k_\beta)$*-desynchronization* from the $(2k_\alpha : 2k_\beta)$-penetration on the constraint boundary can be called *the vanishing condition of the* $(2k_\alpha : 2k_\beta)$-penetration to form a new $(2k_\alpha : 2k_\beta)-$desynchronization on the constraint boundary. However, in Eq. (3.50), *the vanishing condition of the* $(2k_\alpha : 2k_\beta)$-synchronization to form a new $(2k_\alpha : 2k_\beta)$-penetration can be called *the onset condition of the* $(2k_\alpha : 2k_\beta)$-penetration from the $(2k_\alpha : 2k_\beta)$-desynchronization.

Definition 3.28 Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha\,(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous $(r_\alpha \geq 2k_\alpha + 1)$ and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. The $(2k_\alpha : 2k_\beta)$-penetration of the two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3) is called to be *switched* to a new $(2k_\beta : 2k_\alpha)$-penetration at the constraint boundary at time t_m if for $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$

$$
\begin{aligned}
&\varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0,\\
&\varphi^{(s_\alpha)}(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha + 1,\\
&\varphi^{(s_\beta)}((\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda}) = 0 \text{ for } s_\beta = 1, 2, \cdots, 2k_\beta + 1,\\
&(-1)^\alpha[\varphi(\mathbf{y}_{m\mp\varepsilon}^{(\alpha)}, t_{m\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\mp}^{(\alpha)}, t_{m\mp}, \boldsymbol{\lambda})] < 0,\\
&(-1)^\beta[\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\beta)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m\pm}^{(\beta)}, t_{m\pm}, \boldsymbol{\lambda})] < 0.
\end{aligned}
\tag{3.51}
$$

In the foregoing definition, the condition for the $(2k_\alpha : 2k_\beta)$-penetration switching to the $(2k_\beta : 2k_\alpha)$-penetration at the boundary is presented.

3.5 Synchronization to Constraint

In the previous section, the definitions for the synchronicity and the corresponding singularity of two dynamical systems to the constraint were discussed. What conditions can guarantee such synchronicity of the two dynamical systems to the constraint exists? In this section, necessary and sufficient conditions for the synchronization of two dynamical systems to the specific constraint will be presented. The synchronicity switching is discussed through the singularity of a flow of the resultant system to the constraint boundary.

Theorem 3.1 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha\,(\alpha \in \{1,2\})$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *at time* t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. *At* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, *the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}-*continuous* $(r_\alpha \geq 3)$ *and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. *For* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ *and* $\mathbf{y}^{(0)} \in \partial\Omega_{12}$, *suppose* $D^{s_\alpha}\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = D^{s_\alpha}\mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ $(s_\alpha = 0, 1, 2, \cdots)$ *for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$. *The two dynamical systems in Eqs. (3.1) and (3.2) to the constraint in Eq. (3.3) are synchronized for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with any time* t_m

$$\begin{aligned}&\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}, \varphi^{(r_\alpha)}(\mathbf{y}_{m\pm}^{(\alpha)}, t_m, \boldsymbol{\lambda}) = 0\\ &\text{for } \alpha = 1,2 \text{ and } r_\alpha = 0,1,2,\cdots;\end{aligned} \tag{3.52}$$

(ii) *for* $\mathbf{y}_\kappa^{(\alpha)} \in \Omega_\alpha^{-\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_m)$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with* $t_m \in (t_{m_1}, t_{m_2})$

$$\begin{aligned}&\mathbf{y}_\kappa^{(\alpha)} \neq \mathbf{y}_m^{(0)}, (-1)^\alpha \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^-, \boldsymbol{\lambda}) > 0,\\ &\lim_{t_\kappa^- \to t_m} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^-, \boldsymbol{\lambda}) = 0 \text{ for } \alpha = 1,2;\end{aligned} \tag{3.53}$$

(iii) *for* $\mathbf{y}_\kappa^{(\alpha)} \in \Omega_\alpha^{+\varepsilon}$ *at time* $t_\kappa^+ \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with* $t_m \notin [t_{m_1}, t_{m_2}]$

$$\begin{aligned}&\mathbf{y}_\kappa^{(\alpha)} \neq \mathbf{y}_m^{(0)}, (-1)^\alpha \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^+, \boldsymbol{\lambda}) < 0,\\ &\lim_{t_\kappa^+ \to t_m} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^+, \boldsymbol{\lambda}) = 0 \text{ for } \alpha = 1,2;\end{aligned} \tag{3.54}$$

(iv) *for* $\mathbf{y}_\kappa^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_{m-})$ *or* $t_\kappa^+ \in (t_{m+}, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with* $t_m = t_{m_1}$ and t_{m_2}

$$\begin{aligned}&\mathbf{y}_\kappa^{(\alpha)} \neq \mathbf{y}_m^{(0)}, \lim_{t_\kappa^\pm \to t_{m\pm}} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) = 0,\\ &\lim_{t_\kappa^\pm \to t_{m\pm}} (-1)^\alpha \varphi^{(2)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) < 0 \text{ for } \alpha = 1,2;\end{aligned} \tag{3.55}$$

Proof (i) Consider two dynamical systems in Eqs. (3.1) and (3.2) with a constraint condition in Eq. (3.3). From Definition 3.10, one has for $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)} \in \partial\Omega_{12}$,

$$\varphi(\mathbf{y}^{(\alpha)}(t), t, \boldsymbol{\lambda}) = \varphi(\mathbf{y}^{(0)}(t), t, \boldsymbol{\lambda}) = 0.$$

Because $D^{s_\alpha}\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = D^{s_\alpha}\mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ $(s_\alpha = 0,1,2,\cdots)$ on the constraint boundary $\partial\Omega_{12}$, one obtains $d^{r_\alpha}\mathbf{y}^{(\alpha)}/dt^{r_\alpha} = d^{r_\alpha}\mathbf{y}^{(0)}(t)/dt^{r_\alpha}$ $(r_\alpha = 1,2,3,\cdots)$. The foregoing equation gives

$$\varphi^{(r_\alpha)}(\mathbf{y}^{(\alpha)}(t), t, \boldsymbol{\lambda}) = \varphi^{(r_\alpha)}(\mathbf{y}^{(0)}(t), t, \boldsymbol{\lambda}) = 0.$$

(ii) and (iii) For $\mathbf{y}_\kappa^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ at time $t_\kappa^- \in [t_{m-\varepsilon}, t_m)$ or $t_\kappa^+ \in (t_m, t_{m+\varepsilon}]$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ with $t_m \in (t_{m_1}, t_{m_2})$,

$$\varphi(\mathbf{y}_\kappa^{(1)}, t_\kappa^\pm, \boldsymbol{\lambda}) > 0 \text{ and } \varphi(\mathbf{y}_\kappa^{(2)}, t_\kappa^\pm, \boldsymbol{\lambda}) < 0.$$

Introduce $0 < \varepsilon_1 = |t_{m\pm\varepsilon} - t_\kappa^\pm| < |t_{m\pm\varepsilon} - t_m| = \varepsilon$ for $t_m > t_\kappa^-$ and $t_m < t_\kappa^+$. Because of

$$\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\alpha)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) = \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1) + o(\varepsilon_1)$$

and once higher order terms drop, the foregoing equation leads to

$$\varphi(\mathbf{y}_{m\pm\varepsilon}^{(\alpha)}, t_{m\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) = \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1)$$

From Definition 2.13 for $t_m \in (t_{m_1}, t_{m_2})$ with t_κ^-, we have

$$\begin{aligned}
&\lim_{t_\kappa^- \to t_{m-}} (-1)^\alpha \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) > 0,\\
&\lim_{t_\kappa^- \to t_m} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^-, \boldsymbol{\lambda}) = \varphi^{(1)}(\mathbf{y}_m^{(\alpha)}, t_m, \boldsymbol{\lambda}) = 0.
\end{aligned}$$

However, using Eq. (3.53), the condition in Definition 3.13 is obtained.
From Definition 3.14 for $t_m \notin [t_{m_1}, t_{m_2}]$ with t_κ^+, we have

$$\begin{aligned}
&\lim_{t_\kappa^+ \to t_{m+}} (-1)^\alpha \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) < 0,\\
&\lim_{t_\kappa^+ \to t_m} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^+, \boldsymbol{\lambda}) = \varphi^{(1)}(\mathbf{y}_m^{(\alpha)}, t_m, \boldsymbol{\lambda}) = 0.
\end{aligned}$$

However, using Eq. (3.54), the condition in Definition 3.14 is obtained.
(iv) For $\mathbf{y}_\kappa^{(\alpha)} \in \boldsymbol{\Omega}_\alpha^{\pm\varepsilon}$ at time $t_\kappa^- \in [t_{m-\varepsilon}, t_{m-})$ or $t_\kappa^+ \in (t_{m+}, t_{m+\varepsilon}]$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ with $t_m = t_{m_1}$ and t_{m_2},

$$\begin{aligned}
&\lim_{t_\kappa^\pm \to t_{m\pm}} [\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})]\\
&\quad = \lim_{t_\kappa^\pm \to t_{m\pm}} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1)\\
&\quad + \lim_{t_\kappa^\pm \to t_{m\pm}} \frac{1}{2!}\varphi^{(2)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1)^2 + o({\varepsilon_1}^2)
\end{aligned}$$

Ignoring the third-order term and the higher order terms of ε_1, we have

$$\begin{aligned}
&\lim_{t_\kappa^\pm \to t_{m\pm}} [\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})]\\
&\quad = \lim_{t_\kappa^\pm \to t_{m\pm}} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1)\\
&\quad + \lim_{t_\kappa^\pm \to t_{m\pm}} \frac{1}{2!}\varphi^{(2)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1)^2
\end{aligned}$$

Using $\lim_{\kappa_i \to m_i\pm} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) = 0$, the foregoing equation gives

$$\begin{aligned}
&\lim_{t_\kappa^\pm \to t_{m\pm}} [\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})]\\
&\quad = \lim_{t_\kappa^\pm \to t_{m\pm}} \frac{1}{2!}\varphi^{(2)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1)^2
\end{aligned}$$

If $\lim_{t_\kappa^\pm \to t_{m\pm}} (-1)^\alpha \varphi^{(2)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) < 0$, we have

$$\lim_{t_\kappa^\pm \to t_{m\pm}} [\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})] < 0$$

From Definition 3.18, the point $(\mathbf{x}_{m_i\pm}^{(\alpha)}, t_{m_i\pm})$ $(i = 1, 2)$ is tangential point to the constraint. The synchronization at such a point appears or disappears. However, from the conditions in Definition 3.18, Eq. (3.55) can be obtained. This theorem is proved. □

For the point $(\mathbf{y}_{m_1}^{(\alpha)}, t_{m_1})$, the synchronization will be onset. However, for the point $(\mathbf{y}_{m_2}^{(\alpha)}, t_{m_2})$, the synchronization will vanish. For $t_m \in (t_{m_1}, t_{m_2})$, the synchronization at point $(\mathbf{y}_m^{(\alpha)}, t_m)$ on the constraint boundary can be formed. For $t_m \notin [t_{m_1}, t_{m_2}]$, the desynchronization at point $(\mathbf{y}_m^{(\alpha)}, t_m)$ on the constraint boundary can be formed. If $t_{m_1} \to -\infty$ and $t_{m_2} \to \infty$, the synchronization is absolute. The synchronization of two dynamical systems to the constraint can occur at any time t_m. Once the synchronization is formed on the constraint boundary, such synchronization on the constraint boundary will not disappear. If the higher order singularity on the boundary exists, the corresponding theorem is presented in a similar fashion.

Theorem 3.2 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}_{m\pm}^{(\alpha)} \in \boldsymbol{\Omega}_\alpha$ $(\alpha \in \{1, 2\})$ *and* $\mathbf{y}_m^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *at time* $t_m, \mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. *At* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}_\alpha^{\pm\varepsilon}$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, *the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}*-continuous* $(r_\alpha \geq 2k_\alpha + 1)$ *and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. *For* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}_\alpha$ *and* $\mathbf{y}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$, *suppose* $D^{s_\alpha}\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = D^{s_\alpha}\mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ $(s_\alpha = 0, 1, 2, \cdots)$ *for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$. *The two dynamical systems in Eqs. (3.1) and (3.2) to the constraint in Eq. (3.3) are synchronized for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}_m^{(\alpha)} \in \boldsymbol{\Omega}_\alpha$ *and* $\mathbf{y}_m^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *with any time* t_m

$$\begin{aligned} &\mathbf{y}_m^{(\alpha)} = \mathbf{y}_m^{(0)}, \varphi^{(r_\alpha)}(\mathbf{y}_m^{(\alpha)}, t_m, \boldsymbol{\lambda}) = 0 \\ &\text{for } \alpha = 1, 2 \text{ and } r_\alpha = 0, 1, 2, \cdots; \end{aligned} \tag{3.56}$$

(ii) *for* $\mathbf{y}_\kappa^{(\alpha)} \in \boldsymbol{\Omega}_\alpha^{-\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_m)$ *and* $\mathbf{y}_m^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *with* $t_m \in (t_{m_1}, t_{m_2})$

$$\left.\begin{aligned} &\mathbf{y}_\kappa^{(\alpha)} \neq \mathbf{y}_m^{(0)}, \lim_{t_\kappa^- \to t_{m-}} \varphi^{(s_\alpha)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^-, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha, \\ &\lim_{t_\kappa^- \to t_{m-}} (-1)^\alpha \varphi^{(2k_\alpha+1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^-, \boldsymbol{\lambda}) > 0, \\ &\lim_{t_\kappa^- \to t_m} \varphi^{(2k_\alpha+1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^-, \boldsymbol{\lambda}) = 0 \text{ for } \alpha = 1, 2; \end{aligned}\right\} \tag{3.57}$$

(iii) *for* $\mathbf{y}_{\kappa}^{(\alpha)} \in \Omega_{\alpha}^{+\varepsilon}$ *at time* $t_{\kappa}^{+} \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with* $t_m \notin [t_{m_1}, t_{m_2}]$

$$\left.\begin{aligned}
&\mathbf{y}_{\kappa}^{(\alpha)} \neq \mathbf{y}_m^{(0)}, \lim_{t_{\kappa}^{+}\to t_{m+}} \varphi^{(s_\alpha)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{+}, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha,\\
&\lim_{t_{\kappa}^{+}\to t_{m+}} (-1)^{\alpha}\varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{+}, \boldsymbol{\lambda}) < 0,\\
&\lim_{t_{\kappa}^{+}\to t_{m}} \varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{+}, \boldsymbol{\lambda}) = 0 \text{ for } \alpha = 1, 2;
\end{aligned}\right\} \tag{3.58}$$

(iv) *for* $\mathbf{y}_{\kappa}^{(\alpha)} \in \Omega_{\alpha}^{+\varepsilon}$ *at time* $t_{\kappa}^{-} \in [t_{m-\varepsilon}, t_{m-})$ *or* $t_{\kappa}^{+} \in (t_{m+}, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with* $t_m = t_{m_1}$ and t_{m_2}

$$\left.\begin{aligned}
&\mathbf{y}_{\kappa}^{(\alpha)} \neq \mathbf{y}_m^{(0)}, \lim_{t_{\kappa}^{\pm}\to t_{m\pm}} \varphi^{(s_\alpha)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha+1,\\
&\lim_{t_{\kappa}^{\pm}\to t_{m\pm}} (-1)^{\alpha}\varphi^{(2k_\alpha+2)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda}) < 0 \text{ for } \alpha = 1, 2.
\end{aligned}\right\} \tag{3.59}$$

Proof (i) Consider two dynamical systems in Eqs. (3.1) and (3.2) with a constraint condition in Eq. (3.3). From Definition 3.10, one has for $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)} \in \partial\Omega_{12}$,

$$\varphi(\mathbf{y}^{(\alpha)}(t), t, \boldsymbol{\lambda}) = \varphi(\mathbf{y}^{(0)}(t), t, \boldsymbol{\lambda}) = 0.$$

Because $D^{s_\alpha}\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = D^{s_\alpha}\mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})(s_\alpha = 0, 1, 2, \cdots)$ on the constraint boundary $\partial\Omega_{12}$, one obtains $d^{r_\alpha}\mathbf{y}^{(\alpha)}/dt^{r_\alpha} = d^{r_\alpha}\mathbf{y}^{(0)}(t)/dt^{r_\alpha}$ $(r_\alpha = 1, 2, 3, \cdots)$. The foregoing equation gives

$$\varphi^{(r_\alpha)}(\mathbf{y}^{(\alpha)}(t), t, \boldsymbol{\lambda}) = \varphi^{(r_\alpha)}(\mathbf{y}^{(0)}(t), t, \boldsymbol{\lambda}) = 0.$$

(ii) and (iii) For $\mathbf{y}_{\kappa}^{(\alpha)} \in \Omega_{\alpha}^{\pm\varepsilon}$ at $t_{\kappa}^{-} \in [t_{m-\varepsilon}, t_m)$ or $t_{\kappa}^{+} \in (t_m, t_{m+\varepsilon}]$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ with $t_m \in (t_{m_1}, t_{m_2})$,

$$(-1)^{\alpha}\varphi(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda}) < 0.$$

Introduce $0 < \varepsilon_1 = |t_{m\pm\varepsilon} - t_{\kappa}^{\pm}| < |t_{m\pm\varepsilon} - t_m| = \varepsilon$ for $t_m > t_{\kappa}^{-}$ and $t_m < t_{\kappa}^{+}$. Because of

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda})\\
&= \sum\nolimits_{s_\alpha=1}^{2k_\alpha} \frac{1}{s_\alpha!}\varphi^{(s_\alpha)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{s_\alpha}\\
&+ \frac{1}{(2k_\alpha+1)!}\varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{2k_\alpha+1} + o((\varepsilon_1)^{2k_\alpha+1}),
\end{aligned}$$

and once the $(2k_\alpha + 2)$ and higher order terms drop, one obtains

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda}) \\
&= \sum\nolimits_{s_\alpha=1}^{2k_\alpha} \frac{1}{s_\alpha!} \varphi^{(s_\alpha)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{s_\alpha} \\
&+ \frac{1}{(2k_\alpha+1)!} \varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{2k_\alpha+1},
\end{aligned}$$

$$\begin{aligned}
&\lim_{t_\kappa^{\pm}\to t_{m\pm}} [\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda})] \\
&= \sum\nolimits_{s_\alpha=1}^{2k_\alpha} \lim_{t_\kappa^{\pm}\to t_{m\pm}} \frac{1}{s_\alpha!} \varphi^{(s_\alpha)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{s_\alpha} \\
&+ \lim_{t_\kappa^{\pm}\to t_{m\pm}} \frac{1}{(2k_\alpha+1)!} \varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{2k_\alpha+1}.
\end{aligned}$$

Definition 2.21 for $t_m \in (t_{m_1}, t_{m_2})$ with t_κ^- gives

$$\begin{aligned}
&\lim_{t_\kappa^{-}\to t_{m-}} (-1)^\alpha \varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{-}, \boldsymbol{\lambda}) > 0, \\
&\lim_{t_\kappa^{-}\to t_{m}} \varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{-}, \boldsymbol{\lambda}) = \varphi^{(2k_\alpha+1)}(\mathbf{y}_{m}^{(\alpha)}, t_m, \boldsymbol{\lambda}) = 0.
\end{aligned}$$

However, using Eq. (3.57), the condition in Definition 3.13 is obtained. Definition 3.22 for $t_m \notin [t_{m_1}, t_{m_2}]$ with t_κ^+ leads to

$$\begin{aligned}
&\lim_{t_\kappa^{+}\to t_{m+}} (-1)^\alpha \varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{+}, \boldsymbol{\lambda}) > 0, \\
&\lim_{t_\kappa^{+}\to t_{m}} \varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{+}, \boldsymbol{\lambda}) = \varphi^{(2k_\alpha+1)}(\mathbf{y}_{m}^{(\alpha)}, t_m, \boldsymbol{\lambda}) = 0.
\end{aligned}$$

However, using Eq. (3.58), the condition in Definition 3.14 is obtained.

(iv) Similarly, for $\mathbf{y}_\kappa^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ at time $t_\kappa^- \in [t_{m-\varepsilon}, t_{m-})$ or $t_\kappa^+ \in (t_{m+}, t_{m+\varepsilon}]$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ with $t_m = t_{m_1}$ and t_{m_2},

$$\begin{aligned}
&\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda}) \\
&= \sum\nolimits_{s_\alpha=1}^{2k_\alpha+1} \frac{1}{s_\alpha!} \varphi^{(s_\alpha)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{s_\alpha} \\
&+ \frac{1}{(2k_\alpha+2)!} \varphi^{(2k_\alpha+2)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{2k_\alpha+2} + o((\varepsilon_1)^{2k_\alpha+2})
\end{aligned}$$

Ignoring the $(2k_\alpha + 3)$term or higher order terms, one obtains

$$\begin{aligned}
&\lim_{t_\kappa^{\pm}\to t_{m\pm}} [\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda})] \\
&= \sum\nolimits_{s_\alpha}^{2k_\alpha+1} \lim_{t_\kappa^{\pm}\to t_{m\pm}} \frac{1}{(s_\alpha)!} \varphi^{(s_\alpha)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{s_\alpha} \\
&+ \lim_{t_\kappa^{\pm}\to t_{m\pm}} \frac{1}{(2k_\alpha+2)!} \varphi^{(2k_\alpha+2)} \varphi^{(2k_\alpha+2)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda})(\pm\varepsilon_1)^{2k_\alpha+2}
\end{aligned}$$

Using $\lim_{\kappa_i \to m_i \pm} \varphi^{(s_\alpha)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) = 0$ $(s_\alpha = 1, 2, \cdots 2k_\alpha + 1)$, the foregoing equation gives

$$\begin{aligned} &\lim_{t_\kappa^\pm \to t_{m\pm}} [\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})] \\ &= \lim_{t_\kappa^\pm \to t_{m\pm}} \frac{1}{(2k_\alpha + 2)!} \varphi^{(2k_\alpha+2)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})(\pm\varepsilon_1)^{2k_\alpha+2} \end{aligned}$$

If $\lim_{t_\kappa^\pm \to t_{m\pm}} \varphi^{(2k_\alpha+2)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda}) < 0$, one obtains

$$\lim_{t_\kappa^\pm \to t_{m\pm}} [\varphi(\mathbf{y}_{m\pm\varepsilon_1}^{(\alpha)}, t_{m\pm\varepsilon_1}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^\pm, \boldsymbol{\lambda})] < 0$$

From Definition 2.24, the point $(\mathbf{x}_{m_i\pm}^{(\alpha)}, t_{m_i\pm})$ $(i = 1, 2)$ is tangential point to the constraint. The synchronization at such a point appears or disappears. However, from the conditions in Definition 2.24, Eq. (3.59) can be obtained. This theorem is proved. □

Consider the foregoing two theorems with $t_{m_1} \to -\infty$ and $t_{m_2} \to \infty$. For this case, once the two dynamical systems to the constraint are synchronized, such synchronization can keep forever. To explain the two theorems, the synchronization of the flows of two dynamical systems on the boundary $\partial\Omega_{12}$ is in Fig. 3.7. Any point of a constraint flow on the constraint boundary is expressed by $(\mathbf{y}_m^{(0)}, t_m)$ for synchronization. In the two domains, the resultant flows in the vicinity of the constraint boundary are expressed by $(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon})$ $(\alpha = 1, 2)$. The onset point is denoted by $(\mathbf{y}_{m_1}^{(0)}, t_{m_1})$. For $t_m > t_{m_1}$ and $t_{m_2} \to \infty$, all the flows of the resultant system of two dynamical systems will be on the constraint boundary. Thus, the synchronization of the two dynamical systems to the constraint is an absolute synchronization. The starting point of a resultant flow for the synchronization can occur at any time $t_m > t_{m_1}$. However, if t_{m_2} is finite, the two dynamical systems to the constraint can be synchronized only in a finite time interval of $t \in (t_{m_1}, t_{m_2})$. To the point on the boundary at time $t = t_{m_2}$, such synchronization will disappear. Further, the resultant flow on the constraint boundary for synchronization vanishing will enter into the domain, which cannot be synchronized any more in sense of Eq. (3.3). Such synchronization is very easily realized through the discontinuous vector fields to the two dynamical systems to the constraint boundary. For the synchronization of slave and master systems to the constraint, a slave system is controlled by discontinuous, external vector fields in order to make it synchronize with the master system.

For $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ at $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$ $(\alpha \in \{1, 2\})$, the synchronization of two dynamical systems to a specific constraint requires $D^k\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda}) = D^k\varphi(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) = 0$. If a resultant system of two different dynamical systems is continuous to the constraint boundary, it is very difficult to make the two different dynamical systems be synchronized with a specific constraint. Most of such synchronization is *asymptotic* as $t \to \infty$. To make the synchronization of two dynamical systems to a specific constraint possible, one often considers control

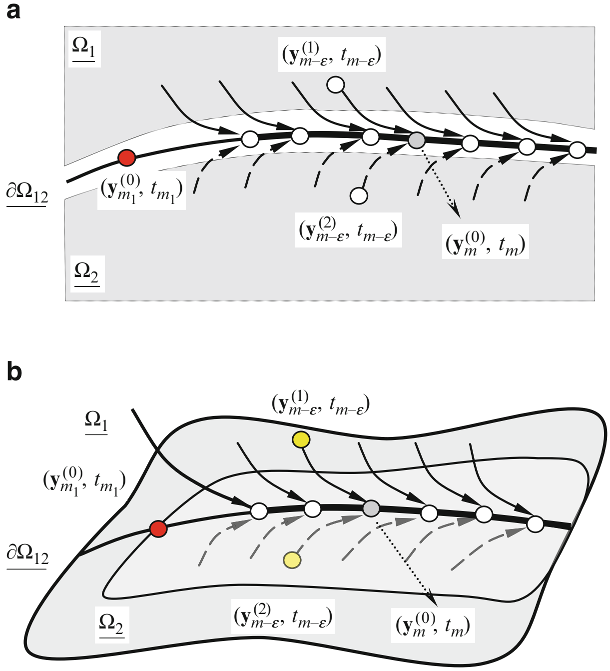

Fig. 3.7 (**a**) A cross-section view and (**b**) a three-dimensional view for an absolute synchronization of two dynamical systems to the constraint in vicinity of the constraint boundary $\partial\Omega_{12}$ in $(n_r + n_s)$-dimensional state space. Any point for synchronization on the constraint boundary is expressed by $(\mathbf{y}^{(0)}_m, t_m)$. In two domains, the resultant flows in the vicinity of the constraint boundary are expressed by $(\mathbf{y}^{(\alpha)}_{m-\varepsilon}, t_{m-\varepsilon})$ $(\alpha = 1, 2)$. The onset point on the constraint boundary is $(\mathbf{y}^{(0)}_{m_1}, t_{m_1})$, depicted by a *red circular symbol*

schemes to realize the synchronization via adjusting vector fields. Next, consider the resultant system of two different dynamical systems to be discontinuous to the constraint boundary.

For $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ at $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$ $(\alpha \in \{1, 2\})$, the synchronization of two dynamical systems with a specific constraint satisfies

$$\frac{d^k}{dt^k}\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda}) \neq \frac{d^k}{dt^k}\varphi(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) = 0 \quad \text{for } k = 1, 2, \cdots. \tag{3.60}$$

To distinguish $\mathbf{y}^{(\alpha)}_{s-}$ from $\mathbf{y}^{(0)}_s$ at time $t_s \in [t_m, t_{m+1}]$, a point $\mathbf{y}^{(\alpha)}_{s-} \in \Omega^{-\varepsilon}_\alpha$ in the domain infinitesimally approaches a point $\mathbf{y}^{(0)}_s \in \partial\Omega_{12}$ on the constraint boundary at time t. For $\mathbf{y}^{(\alpha)}_{s-} \in \Omega^{-\varepsilon}_\alpha$ (or $\mathbf{y}^{(\alpha)}_{s-} \notin \partial\Omega_{12}$), the corresponding differentiation of vector fields with respect to state variables can be carried out. For $\mathbf{y}^{(0)}_s \in \partial\Omega_{12}$ on the constraint boundary, such differentiation cannot be done for $t' \in (t_s - \varepsilon, t_s)$ (any small $\varepsilon > 0$) because the vector fields $(\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}),\ \alpha \in \{1, 2\})$ to the constraint boundary $\partial\Omega_{12}$ are discontinuous (i.e., $\mathbb{F}^{(0)}(\mathbf{y}^{(0)}_s, t_s, \boldsymbol{\lambda}) \neq \mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}_{s-}, t_{s-}, \boldsymbol{\pi}^{(\alpha)})$

for $\mathbf{y}_{s-}^{(\alpha)} = \mathbf{y}_s^{(0)}$ at time $t_s = t_{s-}$). Therefore, the time t_s will be replaced by $t_{s-} = t_s - 0$ for a point $\mathbf{y}_{s-}^{(\alpha)} \in \Omega_\alpha$. Under the constraint condition in Eq. (3.3), the corresponding theorem is presented for the synchronization of two dynamical systems in Eqs. (3.1) and (3.2) as follows.

Theorem 3.3 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ and $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous ($r_\alpha \geq 3$) and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. For $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ and $\mathbf{y}^{(0)} \in \partial\Omega_{12}$, suppose $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ for $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$. The two dynamical systems in Eqs. (3.1) and (3.2) to the constraint in Eq. (3.3) are synchronized for time $t \in [t_{m_1}, t_{m_2}]$ if and only if*

(i) *for $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$ and $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ ($\alpha \in \{1,2\}$) at time $t = t_m \in [t_{m_1}, t_{m_2}]$*

$$\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0 \tag{3.61}$$

(ii) *for time $t_m \in (t_{m_1}, t_{m_2})$,*

$$\mathbf{y}_{m-}^{(\alpha)} = \mathbf{y}_m^{(0)} \text{ and } (-1)^\alpha \varphi^{(1)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) > 0 \text{ for } \alpha = 1,2 \tag{3.62}$$

(iii) *with penetration at time $t = t_{m_i}$, $\mathbf{y}_{m_i\pm}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)}$ ($i = 1,2$)*

$$\begin{aligned} &\varphi^{(1)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \text{ and } (-1)^\alpha \varphi^{(2)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0, \\ &(-1)^\alpha \varphi^{(1)}(\mathbf{y}_{m_i-}^{(\beta)}, t_{m_i-}, \boldsymbol{\lambda}) > 0 \quad \text{for } \alpha,\beta \in \{1,2\} \text{ and } \beta \neq \alpha \end{aligned} \tag{3.63}$$

or with desynchronization at time $t = t_{m_i}$, $\mathbf{y}_{m_i\pm}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)}$ ($i = 1,2$)

$$\begin{aligned} &\varphi^{(1)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \text{ and } (-1)^\alpha \varphi^{(2)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0, \\ &\varphi^{(1)}(\mathbf{y}_{m_i\pm}^{(\beta)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \text{ and } (-1)^\beta \varphi^{(2)}(\mathbf{y}_{m_i\pm}^{(\beta)}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0 \\ &\text{for } \alpha,\beta \in \{1,2\} \text{ and } \beta \neq \alpha. \end{aligned} \tag{3.64}$$

Proof (i) Consider two dynamical systems in Eqs. (3.1) and (3.2) with a constraint condition in Eq. (3.3). From Definition 3.10, the constraint functions for the constraint boundary $\partial\Omega_{12}$ and domains Ω_α ($\alpha = 1,2$) are given by

$$\begin{aligned} &\varphi(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) = 0 \quad \text{for } \mathbf{y}^{(0)} \in \partial\Omega_{12}, \\ &(-1)^\alpha \varphi(\mathbf{y}^{(\alpha)}, t, \lambda) < 0 \quad \text{for } \mathbf{y}^{(\alpha)} \in \Omega_\alpha, \quad \alpha = 1,2. \end{aligned}$$

For $t = t_{m-}$ and $\mathbf{y}^{(\alpha)} = \mathbf{y}_{m-}^{(\alpha)} \in \Omega_\alpha$ ($\alpha \in \{1,2\}$), we have $\mathbf{y}_{m-}^{(\alpha)} = \mathbf{y}_m^{(0)} \in \partial\Omega_{12}$. Further,

$$\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0.$$

Equation (3.61) is obtained, vice versa. Because $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ on the constraint boundary $\partial\Omega_{12}$, one obtains $d^{r_\alpha}\mathbf{y}^{(\alpha)}/dt^{r_\alpha} \neq d^{r_\alpha}\mathbf{y}^{(0)}/dt^{r_\alpha}$ for all time t. Thus, the following equation cannot always hold for all $r_\alpha = 1, 2, \cdots$

$$\varphi^{(r_\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda}) \neq \varphi^{(r_\alpha)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) = 0.$$

(ii) For time $t_m \in (t_{m_1}, t_{m_2})$, $\mathbf{y}^{(\alpha)}_{m-} = \mathbf{y}^{(0)}_m \in \partial\Omega_{12}$. Consider a point $\mathbf{y}^{(\alpha)}_{m-\varepsilon} \in \Omega^{\varepsilon}_{\alpha}$ for $t_{m-\varepsilon} = t_m - \varepsilon$ in the neighborhood of $\mathbf{y}^{(0)}_m \in \partial\Omega_{12}$ and $\varepsilon > 0$. We have

$$\varphi(\mathbf{y}^{(\alpha)}_{m-\varepsilon}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m-}, t_{m-}, \boldsymbol{\lambda}) = -\varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m-}, t_{m-}, \boldsymbol{\lambda})\varepsilon + o(\varepsilon).$$

Because of any selection of $\varepsilon > 0$, if

$$(-1)^{\alpha}\varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m-}, t_{m-}, \boldsymbol{\lambda}) > 0 \text{ for } \alpha = 1, 2$$

then

$$(-1)^{\alpha}[\varphi(\mathbf{y}^{(\alpha)}_{m-\varepsilon}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m-}, t_{m-}, \boldsymbol{\lambda})] < 0.$$

From Definition 3.15, the two dynamical systems to a specific constraint are synchronized for time interval of $t_m \in (t_{m_1}, t_{m_2})$. However, if the foregoing equation is satisfied, Eq. (3.62) is achieved.

(iii) At time $t = t_{m_i}$, $\mathbf{y}^{(\alpha)}_{m_i\pm} = \mathbf{y}^{(0)}_{m_i} \in \partial\Omega_{12}$. Consider a point $\mathbf{y}^{(\alpha)}_{m_i\pm\varepsilon} \in \Omega_{\alpha}$ ($\alpha = 1, 2$) for $t_{m_i\pm\varepsilon} = t_{m_i} \pm \varepsilon$ in the neighborhood of $\mathbf{y}^{(0)}_m \in \partial\Omega_{12}$ and $\varepsilon > 0$. The Taylor series expansion gives

$$\begin{aligned}&\varphi(\mathbf{y}^{(\alpha)}_{m_2\pm\varepsilon}, t_{m_2\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})\\ &= \pm\varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})\varepsilon + \frac{1}{2!}\varphi^{(2)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})\varepsilon^2 + o(\varepsilon^2)\end{aligned}$$

If the third and higher order terms are dropped in the foregoing equation in Ω_{α} ($\alpha = 1, 2$), with the condition

$$\varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0$$

the following equation is achieved.

$$\varphi(\mathbf{y}^{(\alpha)}_{m_2\pm\varepsilon}, t_{m_2\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = \frac{1}{2!}\varphi^{(2)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})\varepsilon^2.$$

If $\varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) \neq 0$ and only the first-order term in the Taylor series expansion is considered, one gets

$$\varphi(\mathbf{y}^{(\alpha)}_{m_2\pm\varepsilon}, t_{m_2\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = \pm\varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})\varepsilon$$

For $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$, from Definition 3.19, the disappearance and appearance of synchronization with the penetration require

$$(-1)^{\alpha}[\varphi(\mathbf{y}^{(\alpha)}_{m_2\pm\varepsilon}, t_{m_2\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})] < 0,$$
$$(-1)^{\beta}[\varphi(\mathbf{y}^{(\beta)}_{m_2-\varepsilon}, t_{m_2-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\beta)}_{m_i-}, t_{m_i-}, \boldsymbol{\lambda})] < 0,$$

from which Eq. (3.63) is obtained, vice versa.

(iv) For $\alpha, \beta \in \{1,2\}$ and $\alpha \neq \beta$, from Definition 18, the disappearance and onset of synchronization with the desynchronization require

$$(-1)^{\alpha}[\varphi(\mathbf{y}^{(\alpha)}_{m_2\pm\varepsilon}, t_{m_2\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})] < 0,$$
$$(-1)^{\beta}[\varphi(\mathbf{y}^{(\beta)}_{m_2\pm\varepsilon}, t_{m_2\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\beta)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})] < 0,$$

from which Eq. (3.64) is obtained, vice versa. Therefore, this theorem is proved. □

From the foregoing theorem, the synchronization of two dynamical systems to a special constraint requires that the first-order derivative of the constraint function be less than zero. The *onset and vanishing* conditions of the synchronization in Eqs. (3.61) and (3.62) are the *vanishing and onset* conditions relative to the penetration and desynchronization, respectively. If the first-order derivative is zero, under what conditions can two dynamical systems to a special constraint be synchronized together in sense of Eq. (3.3)? The following theorem will consider the synchronization of two dynamical systems to a special constraint with higher order singularity.

Theorem 3.4 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}^{(\alpha)}_{m\pm} \in \boldsymbol{\Omega}_{\alpha}$ $(\alpha \in \{1,2\})$ *and* $\mathbf{y}^{(0)}_{m} \in \partial\boldsymbol{\Omega}_{12}$ *at time* t_m, $\mathbf{y}^{(\alpha)}_{m\pm} = \mathbf{y}^{(0)}_{m}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. *At* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}^{\pm\varepsilon}_{\alpha}$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, *the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}*-continuous* $(r_\alpha \geq 2k_\alpha + 1)$ *and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. *For* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}$ *and* $\mathbf{y}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$, *suppose* $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ *for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$. *The two dynamical systems in Eqs. (3.1) and (3.2) to the constraint in Eq. (3.3) are synchronized of the* $(2k_\alpha : 2k_\beta)$*-type for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}^{(\alpha)}_{m\pm} = \mathbf{y}^{(0)}_{m}$ *and* $\mathbf{y}^{(\alpha)}(t) \in \boldsymbol{\Omega}_{\alpha}$ $(\alpha \in \{1,2\})$ *at time* $t = t_m \in [t_{m_1}, t_{m_2}]$

$$\varphi(\mathbf{y}^{(\alpha)}_{m\pm}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}^{(0)}_{m}, t_m, \boldsymbol{\lambda}) = 0 \tag{3.65}$$

(ii) *for time* $t_m \in (t_{m_1}, t_{m_2})$,

$$\begin{aligned} &\mathbf{y}^{(\alpha)}_{m-} = \mathbf{y}^{(0)}_{m} \text{ and } \varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{m-}, t_{m-}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha, \\ &(-1)^{\alpha}\varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{m-}, t_{m-}, \boldsymbol{\lambda}) > 0 \quad \text{for } \alpha = 1, 2. \end{aligned} \tag{3.66}$$

(iii) *with the* $(2k_\alpha : 2k_\beta)$*-penetration for time* $t = t_{m_i}$, $\mathbf{y}_{m_i}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)}$ $(i = 1, 2)$,

$$\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2, \cdots, 2k_\alpha + 1),\\
&(-1)^\alpha \varphi^{(2k_\alpha+2)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) < 0,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m_i-}^{(\beta)}, t_{m_i-}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots 2k_\beta),\\
&(-1)^\beta \varphi^{(2k_\beta+1)}(\mathbf{y}_{m_i-}^{(\beta)}, t_{m_i-}, \boldsymbol{\lambda}) > 0 \quad \text{for } \alpha, \beta \in \{1,2\} \text{ and } \beta \neq \alpha.
\end{aligned} \tag{3.67}$$

or with the $(2k_\alpha : 2k_\beta)$*-desynchronization for time* $t = t_{m_i}$, $\mathbf{y}_{m_i}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)}$ $(i = 1, 2)$,

$$\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2, \cdots, 2k_\alpha + 1),\\
&(-1)^\alpha \varphi^{(2k_\alpha+2)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) < 0,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m_i-}^{(\beta)}, t_{m_i-}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots 2k_\beta + 1),\\
&(-1)^\beta \varphi^{(2k_\beta+2)}(\mathbf{y}_{m_i-}^{(\beta)}, t_{m_i-}, \boldsymbol{\lambda}) < 0 \quad \text{for } \alpha, \beta \in \{1,2\} \text{ and } \beta \neq \alpha.
\end{aligned} \tag{3.68}$$

Proof Consider two dynamical systems in Eqs. (3.1) and (3.2) with a constraint condition in Eq. (3.3).

(i) From Definition 3.10, the constraint functions for the constraint boundary $\partial\Omega_{12}$ and domains Ω_α $(\alpha = 1, 2)$ are given by

$$\begin{aligned}
&\varphi(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) = 0 \text{ for } \mathbf{y}^{(0)} \in \partial\Omega_{12},\\
&(-1)^\alpha \varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda}) < 0 \text{ for } \mathbf{y}^{(\alpha)} \in \Omega_\alpha,\ \alpha = 1, 2.
\end{aligned}$$

For $t = t_m \in [t_{m_1}, t_{m_2}]$ and $\mathbf{y}^{(\alpha)} = \mathbf{y}_m^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$, we have $\mathbf{y}_{m-}^{(\alpha)} = \mathbf{y}_m^{(0)} \in \partial\Omega_{12}$. Further,

$$\varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0,$$

Equation (3.65) is obtained, vice versa. Because $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ on the constraint boundary $\partial\Omega_{12}$, one obtains $d^{r_\alpha}\mathbf{y}^{(\alpha)}/dt^{r_\alpha} \neq d^{r_\alpha}\mathbf{y}^{(0)}/dt^{r_\alpha}$ for all time t. Thus, the following equation cannot always hold for all $r_\alpha = 1, 2, \cdots$

$$\varphi^{(r_\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda}) \neq \varphi^{(r_\alpha)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda}) = 0.$$

(ii) For time $t_m \in (t_{m_1}, t_{m_2})$, $\mathbf{y}_{m-}^{(\alpha)} = \mathbf{y}_m^{(0)} \in \partial\Omega_{12}$. Consider a point $\mathbf{y}_{m-\varepsilon}^{(\alpha)} \in \Omega_\alpha^\varepsilon$ for $t_{m-\varepsilon} = t_m - \varepsilon$ in the neighborhood of $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ and $\varepsilon > 0$. The following Taylor series expansion is achieved.

$$\begin{aligned}
\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) &= \sum\nolimits_{s_\alpha=1}^{2k_\alpha} \frac{1}{s_\alpha!} \varphi^{(s_\alpha)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})(-\varepsilon)^{s_\alpha}\\
&- \frac{1}{(2k_\alpha + 1)!} \varphi^{(2k_\alpha+1)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})\varepsilon^{2k_\alpha+1} + o(\varepsilon^{2k_\alpha+1}).
\end{aligned}$$

Due to the higher order singularity, i.e.,

$$\varphi^{(s_\alpha)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha$$

and by ignoring of the $(2k_\alpha + 2)$-order and higher order terms, the Taylor series expansion gives

$$\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) = -\frac{1}{(2k_\alpha + 1)!}\varphi^{(2k_\alpha+1)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})\varepsilon^{2k_\alpha+1}.$$

From Definition 3.22, the synchronization of two dynamical systems to a specific constraint for time $t_m \in (t_{m_1}, t_{m_2})$ requires

$$(-1)^\alpha[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0.$$

Thus,

$$(-1)^\alpha\varphi^{(2k_\alpha+1)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) > 0.$$

However, if $(-1)^\alpha\varphi^{(2k_\alpha+1)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) > 0$,

$$(-1)^\alpha[\varphi(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda})] < 0.$$

is achieved, which implies the two dynamical systems to the specific constraint are synchronized for time $t_m \in (t_{m_1}, t_{m_2})$.

(iii) At time $t = t_{m_i}$, $\mathbf{y}_{m_i\pm}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)} \in \partial\Omega_{12}$. Consider a point $\mathbf{y}_{m_i\pm\varepsilon}^{(\alpha)} \in \Omega_\alpha$ for $t_{m_i\pm\varepsilon} = t_{m_i} \pm \varepsilon$ in the neighborhood of $\mathbf{y}_{m_i}^{(0)} \in \partial\Omega_{12}$ and $\varepsilon > 0$. The Taylor series expansion gives

$$\begin{aligned}\varphi(\mathbf{y}_{m_i\pm\varepsilon}^{(\alpha)}, t_{m_i\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) &= \sum\nolimits_{s_\alpha=1}^{2k_\alpha+1}\frac{1}{s_\alpha!}\varphi^{(s_\alpha)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda})(\pm\varepsilon)^{s_\alpha}\\ &+ \frac{1}{(2k_\alpha + 2)!}\varphi^{(2k_\alpha+2)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda})\varepsilon^{2k_\alpha+2} + o(\varepsilon^{2k_\alpha+2})\end{aligned}$$

Because of the higher order singularity of the constraint function in domain Ω_α, i.e.,

$$\varphi^{(s_\alpha)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha$$

and once the higher order terms of $\varepsilon^{2k_\alpha+1}$ are dropped, one obtains

$$\varphi(\mathbf{y}_{m_i\pm\varepsilon}^{(\alpha)}, t_{m_i\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = \pm\frac{1}{(2k_\alpha + 1)!}\varphi^{(2k_\alpha+1)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda})\varepsilon^{2k_\alpha+1}.$$

If the following equation exists

$$\varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \quad \text{for } s_\alpha = 1, 2, \cdots, 2k_\alpha + 1$$

and the higher order term of $\varepsilon^{2k_\alpha+2}$ will not be considered, the Taylor series expansion gives

$$\varphi(\mathbf{y}^{(\alpha)}_{m_i\pm\varepsilon}, t_{m_i\pm\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = \frac{1}{(2k_\alpha + 2)!}\varphi^{(2k_\alpha+2)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda})\varepsilon^{2k_\alpha+2}.$$

From Definition 2.25, the onset and vanishing conditions of the $(2k_\alpha : 2k_\beta)$-synchronization of the two dynamical systems with a corresponding penetration on the constraint boundary $\partial\Omega_{\alpha\beta}$ are

$$\begin{aligned}
&(-1)^\alpha[\varphi(\mathbf{y}^{(\alpha)}_{m_i\mp\varepsilon}, t_{m_i\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda})] < 0,\\
&(-1)^\beta[\varphi(\mathbf{y}^{(\beta)}_{m_i-\varepsilon}, t_{m_i-\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\beta)}_{m_i-}, t_{m_i-}, \boldsymbol{\lambda})] < 0,
\end{aligned}$$

with

$$\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2, \cdots 2k_\alpha + 1),\\
&\varphi^{(s_\beta)}(\mathbf{y}^{(\beta)}_{m_i-}, t_{m_i-}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots 2k_\beta).
\end{aligned}$$

Thus, one gets

$$(-1)^\alpha\varphi^{(2k_\alpha+2)}(\mathbf{y}^{(\alpha)}_{m_i\mp\varepsilon}, t_{m_i\mp\varepsilon}, \boldsymbol{\lambda}) < 0 \text{ and } (-1)^\beta\varphi^{(2k_\beta+1)}(\mathbf{y}^{(\beta)}_{m_i-}, t_{m_i-}, \boldsymbol{\lambda}) > 0.$$

In other words, Eq. (3.67) is obtained. If Eq. (3.67) holds, the conditions in Definition 3.25 can be obtained for the onset and vanishing condition for synchronization from the penetration.

If the $(2k_\alpha : 2k_\beta)$-synchronization of two dynamical systems to a specific constraint vanishes and appears with a $(2k_\alpha : 2k_\beta)$-desynchronization, the following conditions are required

$$\begin{aligned}
&(-1)^\alpha[\varphi(\mathbf{y}^{(\alpha)}_{m_i\mp\varepsilon}, t_{m_i\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\alpha)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda})] < 0,\\
&(-1)^\beta[\varphi(\mathbf{y}^{(\beta)}_{m_i\mp\varepsilon}, t_{m_i\mp\varepsilon}, \boldsymbol{\lambda}) - \varphi(\mathbf{y}^{(\beta)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda})] < 0,
\end{aligned}$$

with the singularity conditions

$$\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2, \cdots 2k_\alpha + 1),\\
&\varphi^{(s_\beta)}(\mathbf{y}^{(\beta)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots 2k_\beta + 1).
\end{aligned}$$

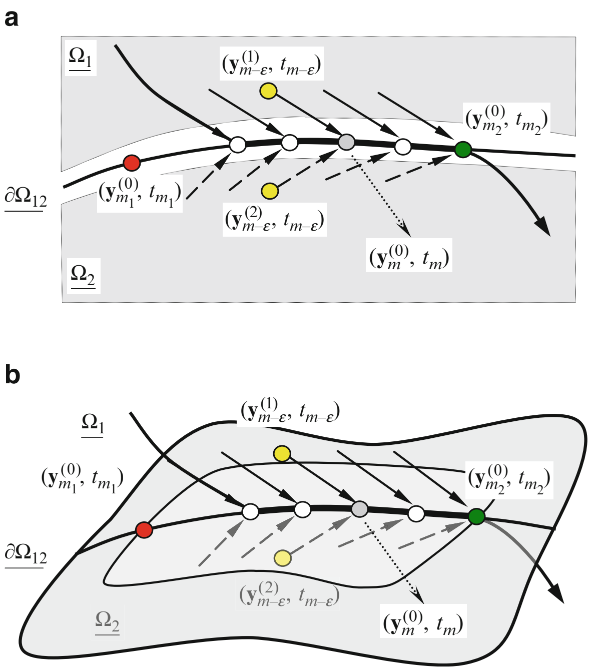

Fig. 3.8 (**a**) A cross-section view and (**b**) a three-dimensional view of the synchronization of resultant flows in vicinity of the constraint boundary $\partial\Omega_{12}$ in $(n_s + n_r)$-dimensional state space. On the constraint boundary, any point for synchronization is expressed by $(\mathbf{y}_m^{(0)}, t_m)$. In two domains, the resultant flows in the vicinity of the constraint boundary are expressed by $(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon})$ $(\alpha = 1, 2)$. The onset and vanishing points are $(\mathbf{y}_{m_1}^{(0)}, t_{m_1})$ and $(\mathbf{y}_{m_2}^{(0)}, t_{m_2})$ with *red* and *blue circular symbols*

So one obtains

$$(-1)^{\alpha}\varphi^{(2k_\alpha+2)}(\mathbf{y}_{m_i\mp\varepsilon}^{(\alpha)}, t_{m_i\mp\varepsilon}, \boldsymbol{\lambda}) < 0 \text{ and } (-1)^{\beta}\varphi^{(2k_\beta+2)}(\mathbf{y}_{m_i-}^{(\beta)}, t_{m_i-}, \boldsymbol{\lambda}) < 0.$$

i.e., Eq. (3.68) is obtained, vice versa. Therefore, this theorem is proved. □

In the foregoing theorem, the *onset and vanishing* conditions of the $(2k_\alpha : 2k_\beta)$-synchronization in Eqs. (3.67) and (3.68) for time $t = t_{m_i}$ $(i = 1, 2)$ are also the *vanishing and onset* conditions of the $(2k_\alpha : 2k_\beta)$-penetration and the $(2k_\alpha : 2k_\beta)$-desynchronization, respectively. To explain the synchronization of the two dynamical systems under the condition in Eq. (3.3) in the previous two theorems, such synchronization is sketched in Fig. 3.8. On the constraint boundary, any point for synchronization is expressed by $(\mathbf{y}_m^{(0)}, t_m)$. In the two domains, any flows in the vicinity of the boundary are expressed by $(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon})$ $(\alpha = 1, 2)$. The onset and vanishing points are $(\mathbf{y}_{m_1}^{(0)}, t_{m_1})$ and $(\mathbf{y}_{m_2}^{(0)}, t_{m_2})$ with red and blue circular symbols. Both of the two points belong to a submanifold on the boundary in the $(n_r + n_s)$-dimensional phase space. Once a flow of the resultant system of two dynamical systems from domain Ω_1 comes to any point of the subregion on the constraint boundary, the synchronization of the two dynamical systems to the constraint occurs until the point $(\mathbf{y}_{m_2}^{(0)}, t_{m_2})$is reached. If $t_{m_2} \to \infty$, such synchronization will not disappear forever. For $t_m > t_{m_1}$, once the resultant flows are on the constraint boundary, the synchronization of the two dynamical systems to the constraint will keep forever.

3.6 Desynchronization to Constraint

The synchronization for two dynamical systems to the constraint in Eq. (3.3) is discussed. The desynchronization of two dynamical systems is opposite to the synchronization. Similarly, for a case of $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ on the constraint boundary, the desynchronization will be discussed, and the desynchronization for $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ on the constraint boundary will be addressed. The desynchronization with $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ is stated.

Theorem 3.5 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ $(\alpha \in \{1,2\})$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *at time* t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. *At* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, *the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}*-continuous* $(r_\alpha \geq 3)$ *and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. *For* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ *and* $\mathbf{y}^{(0)} \in \partial\Omega_{12}$, *suppose* $D^{s_\alpha}\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = D^{s_\alpha}\mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ $(s_\alpha = 0, 1, 2, \cdots)$ *for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$. *The two dynamical systems in Eqs. (3.1) and (3.2) to the constraint in Eq. (3.3) are desynchronized for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}_m^{(\alpha)} \in \Omega_\alpha$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with any time* t_m

$$\begin{aligned} &\mathbf{y}_m^{(\alpha)} = \mathbf{y}_m^{(0)}, \varphi^{(r_\alpha)}(\mathbf{y}_m^{(\alpha)}, t_m, \boldsymbol{\lambda}) = 0 \\ &\text{for } \alpha = 1, 2 \text{ and } r_\alpha = 0, 1, 2, \cdots \end{aligned} \tag{3.69}$$

(ii) *for* $\mathbf{y}_\kappa^{(\alpha)} \in \Omega_\alpha^{+\varepsilon}$ *at time* $t_\kappa^+ \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with* $t_m \in (t_{m_1}, t_{m_2})$

$$\begin{aligned} &\mathbf{y}_\kappa^{(\alpha)} \neq \mathbf{y}_m^{(0)}, (-1)^\alpha \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^+, \boldsymbol{\lambda}) < 0, \\ &\lim_{t_\kappa^+ \to t_m} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^+, \boldsymbol{\lambda}) = 0 \text{ for } \alpha = 1, 2 \end{aligned} \tag{3.70}$$

(iii) *for* $\mathbf{y}_\kappa^{(\alpha)} \in \Omega_\alpha^{-\varepsilon}$ *at time* $t_\kappa^- \in [t_{m-\varepsilon}, t_m)$ *and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *with* $t_m \notin [t_{m_1}, t_{m_2}]$

$$\begin{aligned} &\mathbf{y}_\kappa^{(\alpha)} \neq \mathbf{y}_m^{(0)}, (-1)^\alpha \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^-, \boldsymbol{\lambda}) > 0, \\ &\lim_{t_\kappa^- \to t_m} \varphi^{(1)}(\mathbf{y}_\kappa^{(\alpha)}, t_\kappa^-, \boldsymbol{\lambda}) = 0 \text{ for } \alpha = 1, 2 \end{aligned} \tag{3.71}$$

(iv) *for* $\mathbf{y}_{\kappa}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}^{+\varepsilon}$ *at time* $t_{\kappa}^{-} \in [t_{m-\varepsilon}, t_{m-})$ *or* $t_{\kappa}^{+} \in (t_{m+}, t_{m+\varepsilon}]$ *and* $\mathbf{y}_{m}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *with* $t_m = t_{m_1}$ and t_{m_2}

$$\mathbf{y}_{\kappa}^{(\alpha)} \neq \mathbf{y}_{m}^{(0)}, \lim_{t_{\kappa}^{\pm} \to t_{m\pm}} \varphi^{(1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda}) > 0,$$
$$\lim_{t_{\kappa}^{\pm} \to t_{m\pm}} (-1)^{\alpha} \varphi^{(2)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{\pm}, \boldsymbol{\lambda}) < 0 \text{ for } \alpha = 1,2 \tag{3.72}$$

Proof Once Definitions 3.13, 3.14, 3.17, and 3.18 are used, the proof of this theorem is similar to the proof of Theorem 3.1. □

Theorem 3.6 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}_{m\pm}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}$ $(\alpha \in \{1,2\})$ *and* $\mathbf{y}_{m}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *at time* t_m, $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_{m}^{(0)}$. *For any small* $\varepsilon > 0$, *there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. *At* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}^{\pm\varepsilon}$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, *the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}*-continuous* $(r_\alpha \geq 2k_\alpha + 1)$ *and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. *For* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}$ *and* $\mathbf{y}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$, *suppose* $D^{s_\alpha}\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = D^{s_\alpha}\mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ $(s_\alpha = 0,1,2,\cdots)$ *for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$. *The two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) are desynchronized for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}_{m\pm}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}$ *and* $\mathbf{y}_{m}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *with any time* t_m

$$\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_{m}^{(0)}, \varphi^{(r_\alpha)}(\mathbf{y}_{m\pm}^{(\alpha)}, t_m, \boldsymbol{\lambda}) = 0$$
$$\text{for } \alpha = 1,2 \text{ and } r_\alpha = 0,1,2,\cdots \tag{3.73}$$

(ii) *for* $\mathbf{y}_{\kappa}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}^{+\varepsilon}$ *at time* $t_{\kappa}^{+} \in (t_m, t_{m+\varepsilon}]$ *and* $\mathbf{y}_{m}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *with* $t_m \in (t_{m_1}, t_{m_2})$

$$\mathbf{y}_{\kappa}^{(\alpha)} \neq \mathbf{y}_{m}^{(0)}, \lim_{t_{\kappa}^{+} \to t_{m+}} \varphi^{(s_\alpha)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{+}, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1,2,\cdots,2k_\alpha;$$
$$(-1)^{\alpha} \varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{+}, \boldsymbol{\lambda}) < 0 \text{ and}$$
$$\lim_{t_{\kappa}^{+} \to t_m} \varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{+}, \boldsymbol{\lambda}) = 0 \text{ for } \alpha = 1,2 \tag{3.74}$$

(iii) *for* $\mathbf{y}_{\kappa}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}^{-\varepsilon}$ *at time* $t_{\kappa}^{-} \in [t_{m-\varepsilon}, t_m)$ *and* $\mathbf{y}_{m}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *with* $t_m \notin [t_{m_1}, t_{m_2}]$

$$\mathbf{y}_{\kappa}^{(\alpha)} \neq \mathbf{y}_{m}^{(0)}, \lim_{t_{\kappa}^{-} \to t_{m-}} \varphi^{(s_\alpha)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{-}, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1,2,\cdots,2k_\alpha;$$
$$(-1)^{\alpha} \varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{-}, \boldsymbol{\lambda}) > 0 \text{ and}$$
$$\lim_{t_{\kappa}^{-} \to t_m} \varphi^{(2k_\alpha+1)}(\mathbf{y}_{\kappa}^{(\alpha)}, t_{\kappa}^{-}, \boldsymbol{\lambda}) = 0 \text{ for } \alpha = 1,2 \tag{3.75}$$

(iv) *for* $\mathbf{y}_{\kappa}^{(\alpha)} \in \boldsymbol{\Omega}_{\alpha}^{+\varepsilon}$ *at time* $t_{\kappa}^{-} \in [t_{m-\varepsilon}, t_{m-})$ *or* $t_{\kappa}^{+} \in (t_{m+}, t_{m+\varepsilon}]$ *and* $\mathbf{y}_{m}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$ *with* $t_m = t_{m_1}$ and t_{m_2}

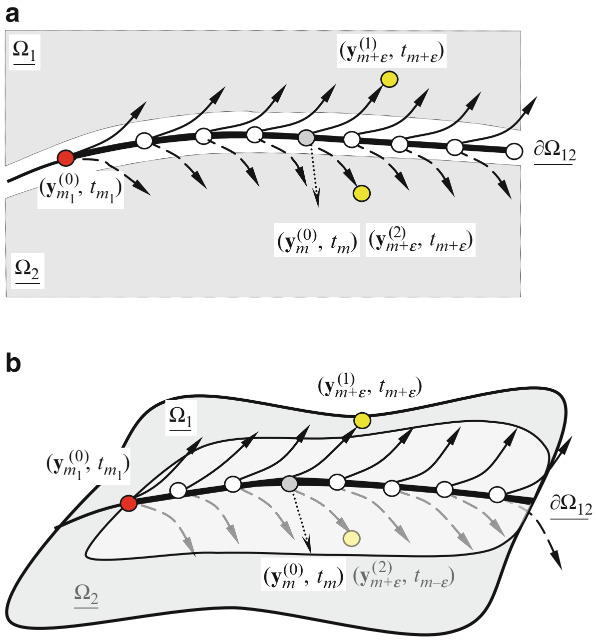

Fig. 3.9 (**a**) Cross-section view and (**b**) three-dimensional view for the desynchronization of slave and master flows in vicinity of the boundary $\partial\Omega_{12}$ in $(n_r + n_s)$-dimensional state space. On the boundary, any point for desynchronization is expressed by $(\mathbf{y}^{(0)}_m, t_m)$. In the two domains, the flows in the vicinity of the boundary are expressed by $(\mathbf{y}^{(\alpha)}_{m+\varepsilon}, t_{m+\varepsilon})$ $(\alpha = 1, 2)$. The onset point is $(\mathbf{y}^{(0)}_{m_1}, t_{m_1})$, depicted by a *red circular symbol*

$$\begin{aligned}
&\mathbf{y}^{(\alpha)}_{\kappa} \neq \mathbf{y}^{(0)}_{m}, \lim_{t^{\pm}_{\kappa} \to t_{m\pm}} \varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{\kappa}, t^{\pm}_{\kappa}, \boldsymbol{\lambda}) = 0 \text{ for } s_\alpha = 1, 2, \cdots, 2k_\alpha + 1; \\
&\lim_{t^{\pm}_{\kappa} \to t_{m\pm}} (-1)^{\alpha} \varphi^{(2k_\alpha+2)}(\mathbf{y}^{(\alpha)}_{\kappa}, t^{\pm}_{\kappa}, \boldsymbol{\lambda}) < 0 \text{ for } \alpha = 1, 2
\end{aligned} \tag{3.76}$$

Proof Once Definitions 3.21, 3.22, 3.25, and 3.26 are used, the proof of this theorem is similar to the proof of Theorem 3.2. □

If $t_{m_1} \to -\infty$ and $t_{m_2} \to \infty$, such a desynchronization of two dynamical systems to constraint in Eq. (3.3) is absolute. Once the resultant flows on the constraint boundary are repelled, such a desynchronization can keep forever. To explain the two foregoing theorems, the desynchronization of two dynamical systems to a specific constraint is sketched in Fig. 3.9 through the resultant flows in the vicinity of the constraint boundary $\partial\Omega_{12}$. Any point for desynchronization on the constraint boundary is expressed by $(\mathbf{y}^{(0)}_m, t_m)$. In the two domains, the resultant flows in the vicinity of the boundary are expressed by $(\mathbf{y}^{(\alpha)}_{m+\varepsilon}, t_{m+\varepsilon})$ $(\alpha = 1, 2)$. The onset point for the desynchronization is denoted by $(\mathbf{y}^{(0)}_{m_1}, t_{m_1})$. For $t_m > t_{m_1}$ and $t_m \to \infty$, all the resultant flows leave from the constraint boundary. However, if $t_{m_2} > t_{m_1}$ is finite, such desynchronization to the constraint will disappear at a point $(\mathbf{y}^{(0)}_{m_2}, t_{m_2})$.

For $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$, the desynchronization of two dynamical systems to a specific constraint is different from those for $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$. Thus, the conditions for the desynchronization of two dynamical systems with discontinuous vector fields are discussed as follows.

Theorem 3.7 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For $\mathbf{y}^{(\alpha)}_{m\pm} \in \Omega_\alpha$ ($\alpha \in \{1,2\}$) and $\mathbf{y}^{(0)}_{m} \in \partial\Omega_{12}$ at time t_m, $\mathbf{y}^{(\alpha)}_{m\pm} = \mathbf{y}^{(0)}_{m}$. For any small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_{m+\varepsilon}]$. At $\mathbf{y}^{(\alpha)} \in \Omega^{\pm\varepsilon}_\alpha$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, the constraint function $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ is C^{r_α}-continuous ($r_\alpha \geq 3$) and $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$. For $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ and $\mathbf{y}^{(0)} \in \partial\Omega_{12}$, suppose $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ for $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$. The two dynamical systems in Eqs. (3.1) and (3.2) to the constraint in Eq. (3.3) are desynchronized for time $t \in [t_{m_1}, t_{m_2}]$ if and only if*

(i) *for $\mathbf{y}^{(\alpha)}_{m+} = \mathbf{y}^{(0)}_{m}$ and $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ ($\alpha \in \{1,2\}$) at time $t = t_m \in [t_{m_1}, t_{m_2}]$*

$$\varphi(\mathbf{y}^{(\alpha)}_{m+}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}^{(0)}_{m}, t_m, \boldsymbol{\lambda}) = 0 \tag{3.77}$$

(ii) *for time $t_m \in [t_{m_1}, t_{m_2})$,*

$$\mathbf{y}^{(\alpha)}_{m+} = \mathbf{y}^{(0)}_{m} \text{ and } (-1)^\alpha \varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m+}, t_{m+}, \boldsymbol{\lambda}) < 0 \text{ for } \alpha = 1,2 \tag{3.78}$$

(iii) *with an penetration for time $t = t_{m_i}$, $\mathbf{y}^{(\alpha)}_{m_i\pm} = \mathbf{y}^{(0)}_{m_i} = \mathbf{y}^{(\beta)}_{m_i+}$ ($i = 1,2$),*

$$\begin{aligned}&\varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \text{ and } (-1)^\alpha \varphi^{(2)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0,\\&(-1)^\beta \varphi^{(1)}(\mathbf{y}^{(\beta)}_{m_i+}, t_{m_i+}, \boldsymbol{\lambda}) < 0 \text{ for } \alpha,\beta \in \{1,2\} \text{ and } \beta \neq \alpha,\end{aligned} \tag{3.79}$$

or with a synchronization for time $t = t_{m_i}$, $\mathbf{y}^{(\alpha)}_{m_i\pm} = \mathbf{y}^{(0)}_{m_i} = \mathbf{y}^{(\beta)}_{m_i\pm}$,

$$\begin{aligned}&\varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \text{ and } (-1)^\alpha \varphi^{(2)}(\mathbf{y}^{(\alpha)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0,\\&\varphi^{(1)}(\mathbf{y}^{(\beta)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \text{ and } (-1)^\beta \varphi^{(2)}(\mathbf{y}^{(\beta)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0\\&\text{for } \alpha,\beta \in \{1,2\} \text{ and } \beta \neq \alpha.\end{aligned} \tag{3.80}$$

Proof By using Definitions 3.13, 3.17–3.19, the proof of this theorem is similar to the proof of Theorem 3.3. □

From the foregoing theorem, the desynchronization of two dynamical systems to a specific constraint requires that the first-order derivative of the constraint function be greater than zero. In addition, the *onset and vanishing* conditions of desynchronization in Eqs. (3.79) and (3.80) are the *vanishing and onset* conditions for onset of the penetration and synchronization with the desynchronization, respectively. The following theorem will give the corresponding conditions for the desynchronization of two dynamical systems to a specific constraint with the higher order singularity.

Theorem 3.8 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ *(*$\alpha \in \{1,2\}$*) and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *at time* t_m*,* $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$*. For any small* $\varepsilon > 0$*, there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$*. At* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$*, the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}*-continuous (*$r_\alpha \geq 2k_\alpha + 1$*) and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$*. For* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ *and* $\mathbf{y}^{(0)} \in \partial\Omega_{12}$*, suppose* $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ *for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$*. The two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) are desynchronized of the* $(2k_1 : 2k_2)$*-type for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}_{m+}^{(\alpha)} = \mathbf{y}_m^{(0)}$ *and* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ *(*$\alpha \in \{1,2\}$*) at time* $t = t_m \in [t_{m_1}, t_{m_2}]$

$$\varphi(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0 \tag{3.81}$$

(ii) *for time* $t_m \in (t_{m_1}, t_{m_2})$*,* $\mathbf{y}_{m+}^{(\alpha)} = \mathbf{y}_m^{(0)} = \mathbf{y}_{m+}^{(\beta)}$

$$\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) < 0 \quad (s_\alpha = 1, 2, \cdots, 2k_\alpha),\\
&(-1)^\alpha \varphi^{(2k_\alpha+1)}(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) < 0,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots, 2k_\beta),\\
&(-1)^\beta \varphi^{(2k_\beta+1)}(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) < 0 \quad \text{for } \beta \in \{1,2\} \text{ and } \alpha \neq \beta
\end{aligned} \tag{3.82}$$

(iii) *with a* $(2k_\alpha : 2k_\beta)$*-penetration flow for time* $t = t_{m_i}$*,* $\mathbf{y}_{m_i\pm}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)} = \mathbf{y}_{m_i+}^{(\beta)}$

$$\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \ (s_\alpha = 1, 2, \cdots, 2k_\alpha + 1),\\
&(-1)^\alpha \varphi^{(2k_\alpha+2)}(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) < 0,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m_i+}^{(\beta)}, t_{m_i+}, \boldsymbol{\lambda}) = 0 \ (s_\beta = 1, 2, \cdots, 2k_\beta),\\
&(-1)^\beta \varphi^{(2k_\beta+1)}(\mathbf{y}_{m_i+}^{(\beta)}, t_{m_i+}, \boldsymbol{\lambda}) < 0 \quad \text{for } \alpha,\beta \in \{1,2\} \text{ and } \alpha \neq \beta
\end{aligned} \tag{3.83}$$

or with a $(2k_1 : 2k_2)$*-synchronization for time* $t = t_{m_i}$*,* $\mathbf{y}_{m_i\pm}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)} = \mathbf{y}_{m_i\pm}^{(\beta)}$

$$\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}_{m_i\pm}^{(\alpha)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2, \cdots, 2k_\alpha + 1),\\
&(-1)^\alpha \varphi^{(2k_\alpha+2)}(\mathbf{y}_{m+}^{(\alpha)}, t_{m+}, \boldsymbol{\lambda}) < 0,\\
&\varphi^{(s_\beta)}(\mathbf{y}_{m_i\pm}^{(\beta)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots, 2k_\beta + 1),\\
&(-1)^\beta \varphi^{(2k_\beta+2)}(\mathbf{y}_{m_i\pm}^{(\beta)}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0 \quad \text{for } \alpha,\beta \in \{1,2\} \text{ and } \alpha \neq \beta.
\end{aligned} \tag{3.84}$$

Proof Using Definitions 3.23, 3.25–3.27, the proof of this theorem is similar to Theorem 3.4. □

The *onset and vanishing* conditions of the $(2k_1 : 2k_2)$-desynchronization in Eqs. (3.83) and (3.84) are the *vanishing and onset* conditions of the $(2k_\alpha : 2k_\beta)$

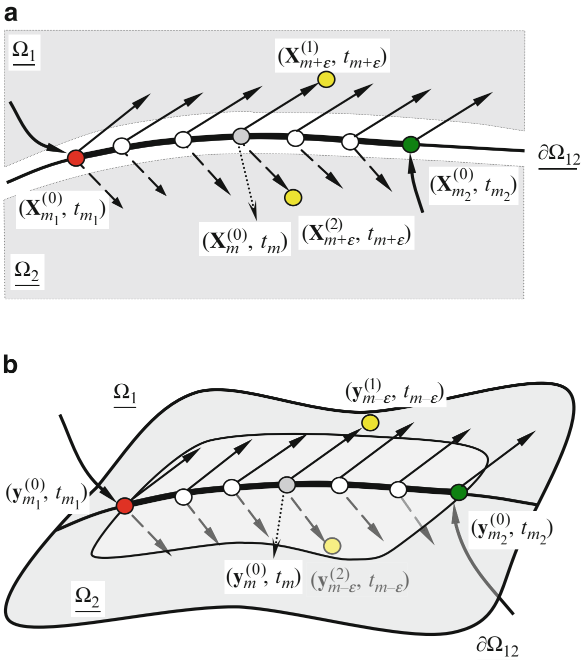

Fig. 3.10 (**a**) A cross-section view and (**b**) a three-dimensional view of the desynchronization of resultant flows in vicinity of the constraint boundary $\partial\Omega_{12}$ in $(n_r + n_s)$-dimensional state space. On the constant boundary, any point for desynchronization is expressed by $(\mathbf{y}^{(0)}_m, t_m)$. In two domains, the resultant flows in the vicinity of the constant boundary are expressed by $(\mathbf{y}^{(\alpha)}_{m-\varepsilon}, t_{m-\varepsilon})$ $(\alpha = 1, 2)$. The onset and vanishing points are $(\mathbf{y}^{(0)}_{m_1}, t_{m_1})$ and $(\mathbf{y}^{(0)}_{m_2}, t_{m_2})$ with *red* and *green circular symbols*

penetration and the $(2k_1 : 2k_2)$-synchronization, respectively. The $(2k_1 : 2k_2)$-desynchronization requires that all the $(2k_1 + 1 : 2k_2 + 1)$-order derivative of the constraint function should be greater than zero. The desynchronization of two dynamical systems to a specific constraint is presented in the previous two theorems, as sketched in Fig. 3.10 through the resultant flows in the vicinity of the constraint boundary. On the constraint boundary, any point relative to desynchronization is expressed by $(\mathbf{y}^{(0)}_m, t_m)$. In the two domains, the flows in the vicinity of the constraint boundary are expressed by $(\mathbf{y}^{(\alpha)}_{m+\varepsilon}, t_{m+\varepsilon})$ $(\alpha = 1, 2)$. The *onset and vanishing* points are $(\mathbf{y}^{(0)}_{m_1}, t_{m_1})$ and $(\mathbf{y}^{(0)}_{m_2}, t_{m_2})$ with red and green circular symbols, which are generated by the two penetrations. The points $(\mathbf{y}^{(0)}_{m_1}, t_{m_1})$ and $(\mathbf{y}^{(0)}_{m_2}, t_{m_2})$are starting and vanishing points of the resultant flow relative to desynchronization.

If $t_{m_2} \to \infty$, once the desynchronization exists, no any synchronization of two systems to a specific constraint can be achieved. For a case of $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ and $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(\beta)}(\mathbf{y}^{(\beta)}, t, \boldsymbol{\pi}^{(\beta)})$, the desynchronization can be determined through the two foregoing theorems.

3.7 Penetration to Constraint

The synchronization and desynchronization of two dynamical systems to a specific constraint have been discussed. The penetration of two dynamical systems to a specific constraint is also very important for the onset and vanishing of synchronization and desynchronization. For $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) = \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ with $\alpha = 1, 2$, the penetration of two dynamical systems to a specific constraint cannot exist. However, if two dynamical systems to a specific constraint possess discontinuous vector fields, the penetration can occur at the constraint boundary. The corresponding theorems are presented as follows.

Theorem 3.9 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}_{m\pm}^{(\alpha)} \in \Omega_\alpha$ *(*$\alpha \in \{1, 2\}$*) and* $\mathbf{y}_m^{(0)} \in \partial\Omega_{12}$ *at time* t_m*,* $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$*. For any small* $\varepsilon > 0$*, there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$*. At* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha^{\pm\varepsilon}$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$*, the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}*-continuous (*$r_\alpha \geq 3$*) and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$*. For* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ *and* $\mathbf{y}^{(0)} \in \partial\Omega_{12}$*, suppose* $D^{s_\alpha}\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq D^{s_\alpha}\mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ s_α *(* $= 0, 1, 2, \cdots$ *) for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$*. The two dynamical systems in Eqs. (3.1) and (3.2) to the constraint in Eq. (3.3) is penetrated at time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}_{m\pm}^{(\alpha)} = \mathbf{y}_m^{(0)}$ *and* $\mathbf{y}^{(\alpha)} \in \Omega_\alpha$ *(*$\alpha \in \{1, 2\}$*) at time* $t = t_m \in [t_{m_1}, t_{m_2}]$

$$\varphi(\mathbf{y}_{m\pm}^{(\alpha)}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}_m^{(0)}, t_m, \boldsymbol{\lambda}) = 0 \tag{3.85}$$

(ii) *at time* $t = t_m \in (t_{m_1}, t_{m_2})$*,* $\mathbf{y}_{m-}^{(\alpha)} = \mathbf{y}_m^{(0)} = \mathbf{y}_{m+}^{(\beta)}$

$$\begin{aligned}&(-1)^\alpha \varphi^{(1)}(\mathbf{y}_{m-}^{(\alpha)}, t_{m-}, \boldsymbol{\lambda}) > 0 \text{ and } (-1)^\beta \varphi^{(1)}(\mathbf{y}_{m+}^{(\beta)}, t_{m+}, \boldsymbol{\lambda}) < 0\\ &\text{for } \alpha, \beta \in \{1, 2\} \text{ and } \alpha \neq \beta.\end{aligned} \tag{3.86}$$

(iii) *with a synchronization at time* $t = t_{m_i}$*,* $\mathbf{y}_{m_i-}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)} = \mathbf{y}_{m_i\pm}^{(\beta)}$ *(*$i \in \{1, 2\}$*),*

$$\begin{aligned}&(-1)^\alpha \varphi^{(1)}(\mathbf{y}_{m_i-}^{(\alpha)}, t_{m_i-}, \boldsymbol{\lambda}) > 0,\\ &\varphi^{(1)}(\mathbf{y}_{m_i\pm}^{(\beta)}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \text{ and } (-1)^\beta \varphi^{(2)}(\mathbf{y}_{m_i\pm}^{(\beta)}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0\\ &\text{for } \alpha, \beta \in \{1, 2\} \text{ and } \alpha \neq \beta,\end{aligned} \tag{3.87}$$

or with a desynchronization at time $t = t_{m_i}$, $\mathbf{y}_{m_i\mp}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)} = \mathbf{y}_{m_i+}^{(\beta)}$ ($i \in \{1, 2\}$),

$$\begin{aligned}&\varphi^{(1)}(\mathbf{y}_{m_i\mp}^{(\alpha)}, t_{m_i\mp}, \boldsymbol{\lambda}) = 0, \text{ and } (-1)^\alpha \varphi^{(2)}(\mathbf{y}_{m_i\mp}^{(\alpha)}, t_{m_i\mp}, \boldsymbol{\lambda}) < 0\\ &(-1)^\beta \varphi^{(1)}(\mathbf{y}_{m_i+}^{(\beta)}, t_{m_i+}, \boldsymbol{\lambda}) < 0 \text{ for } \alpha, \beta \in \{1, 2\} \text{ and } \alpha \neq \beta,\end{aligned} \tag{3.88}$$

or with a switching penetration at time $t = t_{m_i}$, $\mathbf{y}_{m_i\mp}^{(\alpha)} = \mathbf{y}_{m_i}^{(0)} = \mathbf{y}_{m_i\pm}^{(\beta)}$ ($i \in \{1, 2\}$)

$$\varphi^{(1)}(\mathbf{y}^{(\alpha)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda}) = 0, \text{ and } (-1)^{\alpha}\varphi^{(2)}(\mathbf{y}^{(\alpha)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda}) < 0,$$
$$\varphi^{(1)}(\mathbf{y}^{(\beta)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \text{ and } (-1)^{\beta}\varphi^{(2)}(\mathbf{y}^{(\beta)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0$$
$$\text{for } \alpha, \beta \in \{1,2\} \text{ and } \alpha \neq \beta. \tag{3.89}$$

Proof By using Definitions 3.15, 3.17, 3.18, and 3.20, the proof of this theorem is similar to the proof of Theorem 3.3. □

Theorem 3.10 *Consider two dynamical systems in Eqs. (3.1) and (3.2) with constraint in Eq. (3.3). For* $\mathbf{y}^{(\alpha)}_{m\pm} \in \boldsymbol{\Omega}_\alpha$ *(*$\alpha \in \{1,2\}$*) and* $\mathbf{y}^{(0)}_m \in \partial\boldsymbol{\Omega}_{12}$ *at time* t_m*,* $\mathbf{y}^{(\alpha)}_{m\pm} = \mathbf{y}^{(0)}_m$*. For any small* $\varepsilon > 0$*, there is a time interval* $[t_{m-\varepsilon}, t_{m+\varepsilon}]$*. At* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}^{\pm\varepsilon}_\alpha$ *for time* $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$*, the constraint function* $\varphi(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})$ *is* C^{r_α}*-continuous (*$r_\alpha \geq 2k_\alpha + 1$*) and* $|\varphi^{(r_\alpha+1)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\lambda})| < \infty$*. For* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}_\alpha$ *and* $\mathbf{y}^{(0)} \in \partial\boldsymbol{\Omega}_{12}$*, suppose* $\mathbb{F}^{(\alpha)}(\mathbf{y}^{(\alpha)}, t, \boldsymbol{\pi}^{(\alpha)}) \neq \mathbb{F}^{(0)}(\mathbf{y}^{(0)}, t, \boldsymbol{\lambda})$ *for* $\mathbf{y}^{(\alpha)} = \mathbf{y}^{(0)}$*. The two dynamical systems in Eqs. (3.1) and (3.2) to constraint in Eq. (3.3) are penetrated of the* $(2k_1 : 2k_2)$*-type for time* $t \in [t_{m_1}, t_{m_2}]$ *if and only if*

(i) *for* $\mathbf{y}^{(\alpha)}_m = \mathbf{y}^{(0)}_m$ *and* $\mathbf{y}^{(\alpha)} \in \boldsymbol{\Omega}_\alpha$ *(*$\alpha \in \{1,2\}$*) at time* $t = t_m \in [t_{m_1}, t_{m_2}]$

$$\varphi(\mathbf{y}^{(\alpha)}_{m\pm}, t_{m\pm}, \boldsymbol{\lambda}) = \varphi(\mathbf{y}^{(0)}_m, t_m, \boldsymbol{\lambda}) = 0 \tag{3.90}$$

(ii) *at time* $t = t_m \in (t_{m_1}, t_{m_2})$*,* $\mathbf{y}^{(\alpha)}_{m-} = \mathbf{y}^{(0)}_m = \mathbf{y}^{(\beta)}_{m+}$

$$\varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{m-}, t_{m-}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2, \cdots, 2k_\alpha),$$
$$(-1)^{\alpha}\varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{m-}, t_{m-}, \boldsymbol{\lambda}) > 0;$$
$$\varphi^{(s_\beta)}(\mathbf{y}^{(\beta)}_{m+}, t_{m+}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots, 2k_\beta),$$
$$(-1)^{\beta}\varphi^{(2k_\beta+1)}(\mathbf{y}^{(\beta)}_{m+}, t_{m+}, \boldsymbol{\lambda}) < 0 \text{ for } \beta \in \{1{,}2\} \text{ and } \alpha \neq \beta; \tag{3.91}$$

(iii) *with a* $(2k_1 : 2k_2)$*-synchronization at time* $t = t_{m_i}$*,* $\mathbf{y}^{(\alpha)}_{m_i-} = \mathbf{y}^{(0)}_{m_i} = \mathbf{y}^{(\beta)}_{m_i\pm}$ *(*$i \in \{1,2\}$*)*

$$\varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{m_i-}, t_{m_i-}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2, \cdots, 2k_\alpha),$$
$$(-1)^{\alpha}\varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{m_i-}, t_{m_i-}, \boldsymbol{\lambda}) > 0,$$
$$\varphi^{(s_\beta)}(\mathbf{y}^{(\beta)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots, 2k_\beta + 1),$$
$$(-1)^{\beta}\varphi^{(2k_\beta+2)}(\mathbf{y}^{(\beta)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0 \text{ for } \beta \in \{1{,}2\} \text{ and } \alpha \neq \beta, \tag{3.92}$$

or with a $(2k_1 : 2k_2)$ *desynchronization at time* $t = t_{m_i}$*,* $\mathbf{y}^{(\alpha)}_{m_i\mp} = \mathbf{y}^{(0)}_{m_i} = \mathbf{y}^{(\beta)}_{m_i+}$ *(*$i \in \{1,2\}$*),*

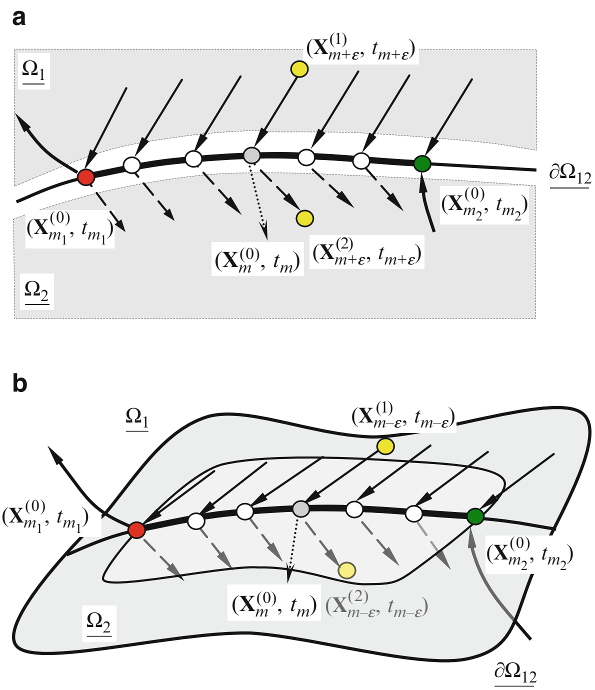

Fig. 3.11 (**a**) A cross-section view and (**b**) a three-dimensional view of the penetration of resultant flows in vicinity of the constraint boundary $\partial\Omega_{12}$ in $(n_r + n_s)$-dimensional state space. On the constraint boundary, any point for penetration is expressed by $(\mathbf{y}^{(0)}_m, t_m)$. In two domains, the resultant flows in the vicinity of the constraint boundary are expressed by $(\mathbf{y}^{(\alpha)}_{m-\varepsilon}, t_{m-\varepsilon})$ $(\alpha = 1, 2)$. The onset and vanishing points are $(\mathbf{y}^{(0)}_{m_1}, t_{m_1})$ and $(\mathbf{y}^{(0)}_{m_2}, t_{m_2})$ with *red* and *blue circular symbols*

$$\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2, \cdots, 2k_\alpha + 1),\\
&(-1)^\alpha \varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda}) < 0,\\
&\varphi^{(s_\beta)}(\mathbf{y}^{(\beta)}_{m+}, t_{m+}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots, 2k_\beta),\\
&(-1)^\beta \varphi^{(2k_\beta+1)}(\mathbf{y}^{(\beta)}_{m+}, t_{m+}, \boldsymbol{\lambda}) < 0 \text{ for } \beta \in \{1,2\} \text{ and } \alpha \neq \beta,
\end{aligned} \tag{3.93}$$

or with a $(2k_\beta : 2k_\alpha)$*-penetration at time* $t = t_{m_i}$, $\mathbf{y}^{(\alpha)}_{m_i\mp} = \mathbf{y}^{(0)}_{m_i} = \mathbf{y}^{(\beta)}_{m_i\pm} (i \in \{1, 2\})$,

$$\begin{aligned}
&\varphi^{(s_\alpha)}(\mathbf{y}^{(\alpha)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda}) = 0 \quad (s_\alpha = 1, 2, \cdots, 2k_\alpha + 1),\\
&(-1)^\alpha \varphi^{(2k_\alpha+1)}(\mathbf{y}^{(\alpha)}_{m_i\mp}, t_{m_i\mp}, \boldsymbol{\lambda}) < 0,\\
&\varphi^{(s_\beta)}(\mathbf{y}^{(\beta)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) = 0 \quad (s_\beta = 1, 2, \cdots, 2k_\beta + 1),\\
&(-1)^\beta \varphi^{(2k_\beta+2)}(\mathbf{y}^{(\beta)}_{m_i\pm}, t_{m_i\pm}, \boldsymbol{\lambda}) < 0 \text{ for } \beta \in \{1,2\} \text{ and } \alpha \neq \beta.
\end{aligned} \tag{3.94}$$

Proof Using Definitions 3.21, 3.23, 3.24, and 3.26, the proof of this theorem is similar to the proof of Theorem 3.4. □

The penetration of the two dynamical systems to a specific constraint is sketched in Fig. 3.11. The *onset and vanishing* conditions of the $(2k_\alpha : 2k_\beta)$-penetration of the $t \in [t_{m_1}, t_{m_2}]$ to a specific constraint are the *vanishing and onset* conditions of the $(2k_\alpha : 2k_\beta)$-synchronization, the $(2k_\alpha : 2k_\beta)$-desynchronization, and the $(2k_\beta : 2k_\alpha)$-penetration, respectively. On the constraint boundary, any point for penetration is expressed by $(\mathbf{y}_m^{(0)}, t_m)$. In two domains, the incoming and output resultant flows in the vicinity of the constraint boundary are expressed by $(\mathbf{y}_{m-\varepsilon}^{(\alpha)}, t_{m-\varepsilon})$ and $(\mathbf{y}_{m+\varepsilon}^{(\beta)}, t_{m+\varepsilon})$ $(\alpha, \beta \in \{1, 2\}$ and $\alpha \neq \beta)$. The *onset and vanishing* points are $(\mathbf{y}_{m_1}^{(0)}, t_{m_1})$ and $(\mathbf{y}_{m_2}^{(0)}, t_{m_2})$ with red and blue circular symbols.

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