Unfolding Method for the Homogenization of Bingham Flow

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Abstract We are interested in the homogenization of a stationary Bingham flow in a porous medium. The model and the formal expansion of this problem are introduced in Lions and Sanchez-Palencia (J. Math. Pures Appl. 60:341–360, 1981) and a rigorous justification of the convergence of the homogenization process is given in Bourgeat and Mikelic (J. Math. Pures Appl. 72:405–414, 1993), by using monotonicity methods coupled with the two-scale convergence method. In order to get the homogenized problem, we apply here the unfolding method in homogenization, method introduced in Cioranescu et al. (SIAM J. Math. Anal. 40:1585–1620, 2008).

1 Introduction

The aim of our chapter is to study the homogenization of the Bingham flow in porous media. The porous media that we consider here are classical periodic porous media containing solid inclusions of the same size as the period, namely ε, where ε is a small real positive parameter.

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In the fluid part of the porous media we consider the stationary flow of the Bingham fluid, under the action of external forces. The Bingham fluid is an incompressible fluid which has a nonlinear constitutive law. So it is a non-Newtonian fluid and it moves like a rigid body when a certain function of the stress tensor is below a given threshold. Beyond this threshold, it obeys a nonlinear constitutive law.

As an example of such fluids we can mention some paints, the mud which can be used for the oil extraction and the volcanic lava.

The mathematical model of the Bingham flow in a bounded domain was introduced in [6] by Duvaut and Lions. The existence of the velocity and of the pressure for such a flow was proved in the case of a bi-dimensional and of a three-dimensional domain.

The homogenization problem was first studied in [8] by Lions and Sanchez-Palencia. The authors did the asymptotic study of the problem by using a multiscale method, involving a "macroscopic" variable x and a "microscopic" variable $y = \frac{x}{\varepsilon}$, and associated to the dimension of the pores. The study is based on a multiscale "ansatz", which allows to get to the limit a nonlinear Darcy law. There is no convergence result proved.

The rigorous justification of the convergence of the homogenization process of the results presented in [8] is given by Bourgeat and Mikelic in [2]. In order to do it, the authors used monotonicity methods coupled with the two-scale convergence method introduced by Nguetseng in [9] and further developed by Allaire in a series of papers, as for example [1]. The limit problem announced in [8] was obtained.

We use in our chapter the unfolding method introduced by Cioranescu et al. in [5] in order to get the homogenized limit problem. The basic idea of the method is to perform a change of scale which blows up the microscopic scale in a periodic fashion. The first advantage of the method is that by using an unfolding operator, functions defined on perforated domains are transformed into functions defined on a fixed domain. The second advantage of the method is that it reduces two-scale convergence to a mere weak convergence in an appropriate space and so general compactness results can be applied. Therefore, no extension operators are required and so the regularity hypotheses on the boundary of the perforated domain, necessary for the existence of such extensions, are not needed. We intend to study some other cases of Bingham flow in porous media, for which we expect that the unfolding method fits better than the two-scale convergence method.

This chapter is organized as follows. In Sect. 2 we describe the problem and we give the preliminary results, namely a priori estimates for the velocity and the pressure on one side and a presentation of the unfolding method introduced in [5], on the other side.

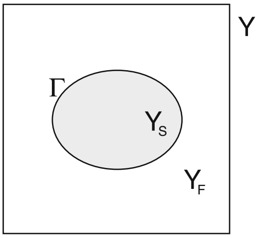
In Sect. 3 we state the main result of the chapter, which is the limit problem obtained after applying the unfolding method for the homogenization of the Bingham flow in the porous media. Mathematically, this corresponds to the passage to the limit as ε tends to zero in the initial problem.

In Sect. 4 we conclude our chapter.

2 Statement of the Problem and Preliminary Results

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and let ε be a small real positive parameter.

We denote by $Y =]0,1[^n$ the unitary cell in $\mathbb{R}^n$, Y_S is an open set strictly included in Y, $Y_F = Y \setminus \overline{Y}_S$ is a connected open set and Γ is the interface between Y_S and Y_F that we assume to be Lipschitz. Let $\varepsilon Y_k = \varepsilon\left(Y+k\right)$, $\varepsilon Y_{S,k} = \varepsilon\left(Y_S+k\right)$, $\varepsilon Y_{F,k} = \varepsilon\left(Y_F+k\right)$, where $k \in \mathbb{Z}^n$.



Elementary cell
Y $=]0,1[^2$.

We consider the set

$$K_\varepsilon = \left\{k \in \mathbb{Z}^n : \varepsilon Y_k \subset \Omega\right\},$$

and we define the fluid part of the porous media, denoted by Ω_ε as follows:

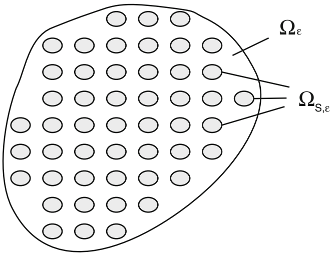
$$\Omega_{S,\varepsilon} = \bigcup_{k \in K_\varepsilon} \varepsilon Y_{S,k}, \;\; \Omega_\varepsilon = \Omega \setminus \overline{\Omega}_{S,\varepsilon}, \;\; \partial\Omega_\varepsilon = \partial\Omega \cup \partial\Omega_{S,\varepsilon}.$$

We assume that Ω_ε is a connected set.

In Ω_ε we consider a Bingham fluid. If u_ε and p_ε are the velocity and pressure respectively for such a fluid, then the stress tensor is written as

$$\sigma_{ij} = -p_\varepsilon \delta_{ij} + g\frac{D_{ij}(u_\varepsilon)}{(D_{II}(u_\varepsilon))^{\frac{1}{2}}} + 2\mu D_{ij}(u_\varepsilon), \tag{1}$$

where δ_{ij} is the Kronecker symbol and g and μ are real positive constants. The constant g represents the yield stress of the fluid and the constant μ is its viscosity. Relation (1) represents the constitutive law of the Bingham fluid.



Ω_ε: fluid part;
$\Omega_{S,\varepsilon}$: the union of solid inclusions.

Moreover, we define

$$D_{ij}(u_\varepsilon) = \frac{1}{2}\left(\frac{\partial u_{\varepsilon,i}}{\partial x_j} + \frac{\partial u_{\varepsilon,j}}{\partial x_i}\right), 1 \le i,j \le n,$$

$$D_{II}(u_\varepsilon) = \frac{1}{2}\sum_{i,j=1}^{n} D_{ij}(u_\varepsilon) D_{ij}(u_\varepsilon)$$

$$\sigma_{ij}^D = g\frac{D_{ij}}{(D_{II})^{\frac{1}{2}}} + 2\mu D_{ij}$$

$$\sigma_{II} = \frac{1}{2}\sum_{i,j=1}^{n} \sigma_{ij}^D \sigma_{ij}^D.$$

Let us note that the constitutive law (1) is valid only if $D_{II}(u_\varepsilon) \neq 0$. In [6] it is shown that this constitutive law is equivalent with the following one:

$$\begin{cases} (\sigma_{II})^{\frac{1}{2}} < g\varepsilon \Leftrightarrow D_{ij}(u_\varepsilon) = 0 \\ (\sigma_{II})^{\frac{1}{2}} \ge g\varepsilon \Leftrightarrow D_{ij}(u_\varepsilon) = \dfrac{1}{2\mu}\left(1 - \dfrac{g\varepsilon}{(\sigma_{II}^\varepsilon)^{\frac{1}{2}}}\right)\sigma_{ij}^D. \end{cases}$$

We see that this is a threshold law: as long as the shear stress is below $g\varepsilon$, the fluid behaves as a rigid solid. When the value of the shear stress exceeds $g\varepsilon$, the fluid flows and obeys a nonlinear law.

Moreover, the fluid is incompressible, which means that its velocity is divergence free

$$\operatorname{div} u_\varepsilon = 0 \text{ in } \Omega_\varepsilon.$$

In [6] it is shown that the velocity u_ε satisfies the following inequality when we apply to the porous media an external force denoted by f and belonging to $\left(L^2(\Omega)\right)^n$:

$$\begin{cases} a_\varepsilon(u_\varepsilon, v - u_\varepsilon) + j_\varepsilon(v) - j_\varepsilon(u_\varepsilon) \geq (f, v - u_\varepsilon)_{\Omega_\varepsilon}, \ \forall v \in V(\Omega_\varepsilon) \\ u_\varepsilon \in V(\Omega_\varepsilon), \end{cases} \tag{2}$$

where

$$a_\varepsilon(u,v) = 2\mu\varepsilon^2 \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v dx, \ \ j_\varepsilon(v) = g\varepsilon \int_{\Omega_\varepsilon} |\nabla v| \, dx, \ \ (u,v)_{\Omega_\varepsilon} = \int_{\Omega_\varepsilon} u \cdot v dx,$$

$$V(\Omega_\varepsilon) = \left\{ v \in \left(H_0^1(\Omega_\varepsilon)\right)^n : \operatorname{div} v = 0 \text{ in } \Omega_\varepsilon \right\}.$$

If $f \in \left(L^2(\Omega)\right)^n$, we know from [6] that for $n = 2$ or 3 and every fixed ε there exists a unique $u_\varepsilon \in V(\Omega_\varepsilon)$ solution of problem (2) and that if p_ε is the pressure of the fluid in Ω_ε, then the problem (2) is equivalent to the following one:

$$a_\varepsilon(u_\varepsilon, v - u_\varepsilon) + j_\varepsilon(v) - j_\varepsilon(u_\varepsilon) \geq (f, v - u_\varepsilon)_{\Omega_\varepsilon} + (p_\varepsilon, \operatorname{div}(v - u_\varepsilon))_{\Omega_\varepsilon}, \tag{3}$$

for all $v \in \left(H_0^1(\Omega_\varepsilon)\right)^n$, $u_\varepsilon \in V(\Omega_\varepsilon)$ and $p_\varepsilon \in L_0^2(\Omega_\varepsilon)$, which admits a unique solution $(u_\varepsilon, p_\varepsilon)$. Here $L_0^2(\Omega_\varepsilon)$ denotes the space of functions belonging to $L^2(\Omega_\varepsilon)$ and of mean value zero.

The aim of our chapter is to pass to the limit as ε tends to zero in problem (3). In order to do this, we first need to get a priori estimates for the velocity u_ε and the pressure p_ε.

Let us recall that the Poincaré inequality for functions in $\left(H_0^1(\Omega_\varepsilon)\right)^n$ reads

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^n} \leq C\varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)^{n\times n}}.$$

Setting $v = 2u_\varepsilon$ and $v = 0$ successively in (2) and using the Poincaré inequality, we easily find that the velocity satisfies the a priori estimates below:

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)^n} \leq C$$

$$\varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)^{n\times n}} \leq C.$$

Let $v_\varepsilon \in \left(H_0^1(\Omega_\varepsilon)\right)^n$. Setting $v = v_\varepsilon + u_\varepsilon$ in (3) and using estimates on the velocity, we get the estimate for the pressure:

$$\|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^n} \leq C\varepsilon.$$

Then we extend the velocity u_ε by zero to $\Omega \diagdown \Omega_\varepsilon$ and denote the extension by the same symbol and we have the following estimates:

$$\|u_\varepsilon\|_{L^2(\Omega)^n} \leq C$$

$$\varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega)^{n\times n}} \leq C.$$

Moreover $\operatorname{div} u_\varepsilon = 0$ in Ω.

For the pressure p_ε, we know (see [10]) that there exists an extension $\widetilde{p}_\varepsilon \in L^2_0(\Omega)$ such that

$$\|\widetilde{p}_\varepsilon\|_{L^2_0(\Omega)} \leq C$$

$$\|\nabla \widetilde{p}_\varepsilon\|_{H^{-1}(\Omega)^n} \leq C\varepsilon$$

and

$$\langle \nabla p_\varepsilon, v\rangle_{\Omega_\varepsilon} = -\left(\widetilde{p}_\varepsilon, \operatorname{div} v\right)_\Omega ,$$

for every v that is the extension by zero to the whole Ω of a function in $H^1_0(\Omega_\varepsilon)^n$.

For an open set D, the brackets $\langle \cdot,\cdot\rangle_D$ denote the duality product between the spaces $H^{-1}(D)^n$ and $H^1_0(D)^n$, where $H^{-1}(D)^n$ denotes the dual of $H^1_0(D)^n$.

The extension $\widetilde{p}_\varepsilon$ can be defined as in [4] by

$$\widetilde{p}_\varepsilon = p_\varepsilon \text{ in the fluid part } \Omega_\varepsilon,$$

$$\widetilde{p}_\varepsilon(x) = \frac{1}{|Y_F|}\int_{Y_F} p_\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) dy \text{ in the solid part } \Omega_{S,\varepsilon} \text{ of the porous media,}$$

where $\left[\frac{x}{\varepsilon}\right]$ is defined as below.

According to these extensions, problem (3) can be written as

$$\begin{aligned} &2\mu\varepsilon^2 \int_\Omega \nabla u_\varepsilon \cdot \nabla (v - u_\varepsilon)\, dx + g\varepsilon \int_\Omega |\nabla v|\, dx - g\varepsilon \int_\Omega |\nabla u_\varepsilon|\, dx \\ &\geq \int_\Omega f_\varepsilon (v - u_\varepsilon)\, dx + \int_\Omega \widetilde{p}_\varepsilon \operatorname{div}(v - u_\varepsilon)\, dx, \end{aligned} \tag{4}$$

for every v that is the extension by zero to the whole Ω of a function in $H^1_0(\Omega_\varepsilon)^n$.

In order to pass to the limit as ε tends to zero in problem (4), we will use the unfolding method introduced in [5].

The idea of the unfolding method is to transform oscillating functions defined on the domain Ω into functions defined on the domain $\Omega \times Y$, in order to isolate the oscillations in the second variable. This transformation, together with a priori estimates, will allow us to use compactness results and then to get the limits of u_ε and p_ε when ε tends to zero.

We recall the results concerning the unfolding operator that we will use in the sequel.

We know that every real number a can be written as the sum between his integer part $[a]$ and his fractionary part $\{a\}$ which belongs to the interval $[0,1)$.

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we apply this result to every real number $\frac{x_i}{\varepsilon}$ for $i = 1, \dots, n$ and we get

$$x = \varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon \left\{\frac{x}{\varepsilon}\right\}_Y,$$

where $\left[\frac{x}{\varepsilon}\right]_Y \in \mathbb{Z}^n$ and $\left\{\frac{x}{\varepsilon}\right\}_Y \in Y$.

Let $w \in L^2_{loc}(\mathbb{R}^n)$ and let us introduce the operator

$$\widetilde{T}_\varepsilon(w)(x,y) = w\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) \text{ for } x \in \mathbb{R}^n \text{ and } y \in Y.$$

Then, for $w \in L^2(\Omega)$, denoting in the same way its extension by zero outside of Ω, the unfolding operator T_ε is defined by

$$T_\varepsilon(w) = \widetilde{T}_\varepsilon(w)|_{\Omega \times Y}.$$

According to [5], this operator has the following properties:

(p_1) T_ε is linear and continuous from $L^2(\Omega)$ to $L^2(\Omega \times Y)$.
(p_2) $T_\varepsilon(\varphi\phi) = T_\varepsilon(\varphi)T_\varepsilon(\phi)$, $\forall \varphi, \phi \in L^2(\Omega)$.
(p_3) If $\varphi \in L^2(\Omega)$, then $T_\varepsilon(\varphi) \to \varphi$ strongly in $L^2(\Omega \times Y)$.
(p_4) If $\varphi \in L^2(Y)$ is a Y-periodic function and $\varphi^\varepsilon(x) = \varphi\left(\frac{x}{\varepsilon}\right)$, $x \in \mathbb{R}^n$, then

$$T_\varepsilon(\varphi^\varepsilon_{|\Omega}) \to \varphi \text{ strongly in } L^2(\Omega \times Y).$$

(p_5) If $\varphi_\varepsilon \in L^2(\Omega)$ and $\varphi_\varepsilon \to \varphi$ strongly in $L^2(\Omega)$, then

$$T_\varepsilon(\varphi_\varepsilon) \to \varphi \text{ strongly in } L^2(\Omega \times Y).$$

Moreover, the following results hold (see Proposition 2.9(iii) in [5]):

Proposition 2.1. *Let $\{\varphi_\varepsilon\}_\varepsilon$ be a bounded sequence in $L^2(\Omega)$ such that*

$$T_\varepsilon(\varphi_\varepsilon) \to \widehat{\varphi} \text{ weakly in } L^2(\Omega \times Y).$$

Then

$$\varphi_\varepsilon \to \mathscr{M}_Y(\widehat{\varphi}) \text{ weakly in } L^2(\Omega),$$

where the mean value operator $\mathscr{M}_Y(\widehat{\varphi})$ is defined by

$$\mathscr{M}_Y(\widehat{\varphi}) = \frac{1}{|Y|}\int_Y \widehat{\varphi}(x,y)\,dy \text{ a.e. for } x \in \Omega.$$

Let us observe that for a function $\varphi \in H^1(\Omega)$, one has

$$\nabla_y(T_\varepsilon(\varphi)) = \varepsilon T_\varepsilon(\nabla\varphi) \quad \text{a.e. } (x,y) \in \Omega \times Y.$$

According to Corollary 3.2 in [5], we have

Proposition 2.2. *Let $\{\varphi_\varepsilon\}_\varepsilon$ be a sequence in $H^1(\Omega)$ bounded in $L^2(\Omega)$. Let us assume that*

$$\varepsilon \|\nabla\varphi_\varepsilon\|_{L^2(\Omega)^n} \leq C.$$

Then, there exists $\widehat{\varphi}$ in $L^2(\Omega;H^1(Y))$ such that, up to a subsequence still denoted by ε

$$T_\varepsilon(\varphi_\varepsilon) \to \widehat{\varphi} \text{ weakly in } L^2(\Omega;H^1(Y)),$$

$$\varepsilon T_\varepsilon(\nabla_x\varphi_\varepsilon) \to \nabla_y\widehat{\varphi} \text{ weakly in } (L^2(\Omega \times Y))^n,$$

where $y \mapsto \widehat{\varphi}(\cdot,y) \in L^2(\Omega;H^1_{per}(Y))$, $H^1_{per}(Y)$ being the Banach space of Y-periodic functions in $H^1_{loc}(\mathbb{R}^n)$ with the $H^1(Y)$ norm.

In what follows, in order to replace integrals over the domain Ω by integrals over the domain $\Omega \times Y$, we use the relation below proved in [7]:

$$\int_\Omega \varphi dx \sim \frac{1}{|Y|}\int_{\Omega\times Y} T_\varepsilon(\varphi)\,dxdy,\ \ \forall\varphi \in L^1(\Omega), \tag{5}$$

which is true for ε sufficiently small. Indeed, it is true for every cell $\varepsilon\xi + \varepsilon Y, \xi \in Z$ strictly included in Ω that

$$\int_{\varepsilon\xi+\varepsilon Y} \varphi(x)dx = \varepsilon^n \int_Y \varphi(\varepsilon\xi + \varepsilon y)dy = \frac{1}{|Y|}\int_{(\varepsilon\xi+\varepsilon Y)\times Y} T_\varepsilon(\varphi)(x,y)dxdy.$$

By using this equality for every cell strictly included in Ω and by denoting $\widehat{\Omega}_\varepsilon$ the largest union of such $\varepsilon\xi + \varepsilon Y$ cells strictly included in Ω, the following exact formula is obtained:

$$\int_{\widehat{\Omega}_\varepsilon} \varphi(x)dx = \frac{1}{|Y|}\int_{\widehat{\Omega}_\varepsilon\times Y} T_\varepsilon(\varphi)(x,y)dxdy.$$

This implies

$$\left|\int_\Omega \varphi(x)dx - \frac{1}{|Y|}\int_{\Omega\times Y} T_\varepsilon(\varphi)(x,y)dxdy\right| \leq 2\|\varphi\|_{L^1(\Omega\setminus\widehat{\Omega}_\varepsilon)},$$

and so any integral on Ω of a function from $L^1(\Omega)$ is "almost equivalent" to the integral of its unfolded on $\Omega \times Y$.

3 Main Result

Now we can state the main result of this chapter in the following theorem:

Theorem 3.1. *Let* u_ε *and* $\widetilde{p}_\varepsilon$ *verify relation (4) given in previous section. Then there exist* $\widehat{u} \in L^2\left(\Omega;(H^1_{per}(Y_F))^n\right)$ *and* $\widehat{p} \in L^2_0(\Omega) \cap H^1(\Omega)$ *such that* $u_\varepsilon \to \dfrac{1}{|Y|}\displaystyle\int_{Y_F} \widehat{u}(\cdot,y)\,dy$ *weakly in* $(L^2(\Omega))^n$, $\widetilde{p}_\varepsilon \to \widehat{p}$ *strongly in* $L^2_0(\Omega)$ *and satisfy the limit problem*

$$\begin{aligned}
&2\mu \int_{Y_F} \nabla_y \widehat{u} \cdot \nabla_y (\psi - \widehat{u})\,dy + g\int_{Y_F} \left|\nabla_y(\psi)\right| dy - g\int_{Y_F} \left|\nabla_y \widehat{u}\right| dy \\
&\geq \langle f - \nabla_x \widehat{p}, \psi - \widehat{u}\rangle_{Y_F}
\end{aligned} \tag{6}$$

for every $\psi \in (H^1_{per}(Y))^n$ *such that* $\psi = 0$ *in* $\overline{Y}_S$ *and* $\mathrm{div}_y\,\psi = 0$. *The function* $\widehat{u}$ *satisfies the following conditions:*

$$\widehat{u}(x,y) = 0 \text{ in } Y_S, \text{ a.e. in } \Omega, \tag{7}$$

$$\widehat{u}(x,y) = 0 \text{ on } \Gamma, \text{ a.e. in } \Omega, \tag{8}$$

$$\mathrm{div}_y\,\widehat{u}(x,y) = 0 \text{ in } Y_F, \text{ a.e. in } \Omega, \tag{9}$$

$$\mathrm{div}_x \int_{Y_F} \widehat{u}(x,y)dy = 0 \text{ in } \Omega, \tag{10}$$

$$\nu \cdot \int_{Y_F} \widehat{u}(x,y)dy = 0 \text{ on } \partial\Omega. \tag{11}$$

Proof. Taking into account the a priori estimates and using Propositions 2.1 and 2.2 we have the following convergences for the velocity and for the pressure:

$$\|u_\varepsilon\|_{L^2(\Omega)^n} \leq C \Rightarrow T_\varepsilon(u_\varepsilon) \to \widehat{u} \text{ weakly in } (L^2(\Omega \times Y))^n,$$

$$\varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega)^{n\times n}} \leq C \Rightarrow \varepsilon T_\varepsilon(\nabla u_\varepsilon) \to \nabla_y \widehat{u} \text{ weakly in } (L^2(\Omega \times Y))^{n\times n},$$

$$\widehat{u} \in L^2\left(\Omega;\left(H^1_{per}(Y)\right)^n\right),$$

$$u_\varepsilon \to \frac{1}{|Y|}\int_Y \widehat{u}(\cdot,y)\,dy \text{ weakly in } (L^2(\Omega))^n,$$

and according to [10], we have

$$\widetilde{p}_\varepsilon \to \widehat{p} \text{ strongly in } L_0^2(\Omega).$$

Using property p_5 of the unfolding method we get

$$T_\varepsilon(\widetilde{p}_\varepsilon) \to \widehat{p} \text{ strongly in } L_0^2(\Omega \times Y).$$

In order to prove relation (7) let us recall that

$$\lim_{\varepsilon \to 0} \int\limits_{\Omega \times Y} T_\varepsilon(u_\varepsilon)(x,y) T_\varepsilon(\Psi)(x,y) T_\varepsilon(\psi)(x,y)\,dxdy = \int\limits_{\Omega \times Y} \widehat{u}(x,y)\Psi(x)\psi(y)\,dxdy,$$

for all $\Psi \in \mathscr{D}(\Omega)$, the space of infinitely differentiable functions with compact support in Ω and for all $\psi \in (H^1_{per}(Y))^n$. By choosing a function $\psi(y)$ such that $\psi = 0$ in Y_F we deduce

$$\int\limits_{\Omega \times Y_S} \widehat{u}(x,y)\Psi(x)\psi(y)\,dxdy = 0,$$

which proves (7).

Relation (8) is a consequence of the fact that $u_\varepsilon = 0$ at the interface between the fluid and the solid part and of the definition and properties of the unfolding boundary operator. This operator was first defined in [3] and we refer to it for the proof.

In order to prove relation (9), let us observe that $\operatorname{div} u_\varepsilon = 0$ implies $\varepsilon T_\varepsilon (\operatorname{div} u_\varepsilon) = 0$. But

$$\varepsilon T_\varepsilon(\operatorname{div} u_\varepsilon) = \varepsilon T_\varepsilon \left(\sum_{i=1}^n \frac{\partial u_{\varepsilon,i}}{\partial x_i} \right) = \varepsilon T_\varepsilon \left(\sum_{i=1}^n \frac{1}{\varepsilon} \frac{\partial u_{\varepsilon,i}}{\partial y_i} \right) = \operatorname{div}_y T_\varepsilon(u_\varepsilon)$$

and so $\operatorname{div}_y T_\varepsilon(u_\varepsilon) = 0$.

We pass to the limit as ε tends to zero in this last equality and by using (7) we get $\operatorname{div}_y \widehat{u} = 0$ in Y_F, a.e. in Ω.

In order to prove relation (10), let us take $\Psi \in \mathscr{D}(\Omega)$.

We have

$$0 = \int_\Omega \operatorname{div} u_\varepsilon \Psi\,dx = \int_\Omega u_\varepsilon \nabla \Psi\,dx.$$

By applying the unfolding we get

$$0 = \int_\Omega \int_Y T_\varepsilon(u_\varepsilon)\, T_\varepsilon(\nabla \Psi)\,dxdy.$$

We pass to the limit as ε tends to zero and taking into account relation (7) we get

$$0 = \int_\Omega \int_{Y_F} \widehat{u} \nabla_x \Psi dxdy,$$

$$0 = \int_\Omega \mathrm{div}_x \left(\int_{Y_F} \widehat{u}(x,y)\, dy \right) \Psi dx, \quad \forall\, \Psi \in \mathscr{D}(\Omega),$$

which implies (10).

In order to prove relation (11), we use the following assertions:

$$\widehat{u}(x,y) = 0 \text{ in } Y_S, \text{ a.e. in } \Omega,$$

$$u_\varepsilon \to \frac{1}{|Y|} \int_{Y_F} \widehat{u}(x,y)\, dy \text{ weakly in } (L^2(\Omega))^n,$$

the linearity and continuity of the normal trace from the space

$$H(\mathrm{div}, \Omega) = \left\{ \varphi \in \left(L^2(\Omega) \right)^n : \mathrm{div}\, \varphi \in L^2(\Omega) \right\}$$

into $H^{-1/2}(\partial\Omega)$.

By applying now the unfolding operator to the inequality (4), we get

$$\begin{aligned} & 2\mu\varepsilon^2 \int_{\Omega \times Y_F} T_\varepsilon(\nabla u_\varepsilon) \cdot T_\varepsilon(\nabla(v - u_\varepsilon))\, dxdy \\ & + g\varepsilon \int_{\Omega \times Y_F} T_\varepsilon(|\nabla v|)\, dxdy - g\varepsilon \int_{\Omega \times Y_F} T_\varepsilon(|\nabla u_\varepsilon|)\, dxdy \\ & \geq \int_{\Omega \times Y_F} T_\varepsilon(f_\varepsilon) T_\varepsilon(v - u_\varepsilon)\, dxdy + \int_{\Omega \times Y_F} T_\varepsilon(\widetilde{p}_\varepsilon) T_\varepsilon(\mathrm{div}(v - u_\varepsilon))\, dxdy. \end{aligned} \tag{12}$$

In order to pass to the limit in relation (12), we will consider a test function $v = v^\varepsilon$ of the form

$$v^\varepsilon(x) = \Psi(x)\, \psi\left(\frac{x}{\varepsilon}\right), \text{ with } \Psi \in D(\Omega) \text{ and } \psi \in V(Y_F), \tag{13}$$

where $V(Y_F) = \left\{ \varphi \in (H^1_{per}(Y))^n : \varphi = 0 \text{ on } \overline{Y}_S \text{ and } \mathrm{div}_y\, \varphi = 0 \right\}$.

We have

$$\nabla_x v^\varepsilon = \nabla_x \left(\Psi(x)\, \psi\left(\frac{x}{\varepsilon}\right) \right) = \nabla_x \Psi(x)\, \psi\left(\frac{x}{\varepsilon}\right) + \Psi(x)\, \nabla_x \psi\left(\frac{x}{\varepsilon}\right). \tag{14}$$

Let us remark that due to condition (7) and to the choice of the test function v^ε, we can write the integrals either on $\Omega \times Y$ or on $\Omega \times Y_F$.

By using this test function we get for the first term in relation (12):

$$\begin{aligned}
&2\mu\varepsilon^2\int_{\Omega\times Y_F} T_\varepsilon(\nabla u_\varepsilon)\cdot T_\varepsilon(\nabla(v-u_\varepsilon))\,dxdy\\
&=2\mu\varepsilon^2\int_{\Omega\times Y_F} T_\varepsilon(\nabla u_\varepsilon)\cdot T_\varepsilon(\nabla v)\,dxdy-2\mu\varepsilon^2\int_{\Omega\times Y_F} T_\varepsilon(\nabla u_\varepsilon)\cdot T_\varepsilon(\nabla u_\varepsilon)\,dxdy\\
&=2\mu\varepsilon^2\int_{\Omega\times Y_F} T_\varepsilon(\nabla u_\varepsilon)\cdot\left[T_\varepsilon(\nabla_x\Psi)\,T_\varepsilon(\psi)+\frac{1}{\varepsilon}T_\varepsilon(\Psi)\,T_\varepsilon(\nabla_y\psi)\right]dxdy\\
&\quad-2\mu\varepsilon^2\int_{\Omega\times Y_F} T_\varepsilon(\nabla u_\varepsilon)\cdot T_\varepsilon(\nabla u_\varepsilon)\,dxdy\\
&=2\mu\int_{\Omega\times Y_F}\varepsilon T_\varepsilon(\nabla u_\varepsilon)\cdot\varepsilon T_\varepsilon(\nabla_x\Psi)\,\psi dxdy+2\mu\int_{\Omega\times Y_F}\varepsilon T_\varepsilon(\nabla u_\varepsilon)\cdot T_\varepsilon(\Psi)\,\nabla_y\psi dxdy\\
&\quad-2\mu\int_{\Omega\times Y_F}|\varepsilon T_\varepsilon(\nabla u_\varepsilon)|^2\,dxdy.
\end{aligned}$$

According to the general convergence results for the unfolding, we have that the first term tends to zero and the second one to the following limit:

$$2\mu\int_{\Omega\times Y_F}\nabla_y\widehat{u}\cdot\Psi\nabla_y\psi(y)\,dxdy.$$

By using now the fact that the function $B(\varphi)=|\varphi|^2$ is proper convex continuous, we have for the third term

$$\liminf_{\varepsilon\to 0} 2\mu\int\limits_{\Omega\times Y_F}|\varepsilon T_\varepsilon(\nabla u_\varepsilon)|^2dxdy\geq 2\mu\int\limits_{\Omega\times Y_F}|\nabla_y\widehat{u}|^2dxdy.$$

In order to pass to the limit in the nonlinear terms, let us first remark that for a function v in $\left(H^1(\Omega)\right)^n$, we have

$$\begin{aligned}
\left[T_\varepsilon(|\nabla v|)\right]^2&=T_\varepsilon(|\nabla v|)T_\varepsilon(|\nabla v|)=T_\varepsilon(|\nabla v|^2)=T_\varepsilon\Big(\sum_{i,j=1}^n\Big(\frac{\partial v_i}{\partial x_j}\Big)^2\Big)=\\
&=\sum_{i,j=1}^n\Big(T_\varepsilon\Big(\frac{\partial v_i}{\partial x_j}\Big)\Big)^2=\sum_{i,j=1}^n\Big(\frac{1}{\varepsilon}\frac{\partial}{\partial y_j}T_\varepsilon(v_i)\Big)^2=\frac{1}{\varepsilon^2}|\nabla_y T_\varepsilon(v)|^2,
\end{aligned}$$

and we deduce

$$\varepsilon T_\varepsilon(|\nabla v|)=|\nabla_y T_\varepsilon(v)|. \tag{15}$$

In order to pass to the limit in the first nonlinear term, by using the previous identity for the function v^ε given by (13), we have

$$
\begin{aligned}
&\left| g\varepsilon \int_{\Omega\times Y_F} T_\varepsilon\left(|\nabla v_\varepsilon|\right) dxdy - g\int_{\Omega\times Y_F} \left|\nabla_y\left(\Psi\psi\right)\right| dxdy \right| \\
&\quad = \left| g\int_{\Omega\times Y_F} \left|\nabla_y T_\varepsilon\left(v_\varepsilon\right)\right| dxdy - g\int_{\Omega\times Y_F} \left|\nabla_y\left(\Psi\psi\right)\right| dxdy \right| \\
&\quad \leq g\int_{\Omega\times Y_F} \left|\nabla_y T_\varepsilon\left(v_\varepsilon\right) - \nabla_y\left(\Psi\psi\right)\right| dxdy = \\
&\quad = g\int_{\Omega\times Y_F} \left|\varepsilon T_\varepsilon\left(\nabla_x\Psi\right)(x,y)\cdot\psi(y) + T_\varepsilon\left(\Psi\right)(x,y)\nabla_y\psi(y) - \Psi(x)\nabla_y\psi(y)\right| dxdy \\
&\quad \leq g\int_{\Omega\times Y_F} \left|T_\varepsilon\left(\varepsilon\nabla_x\Psi\right)(x,y)\cdot\psi(y)\right| dxdy \\
&\qquad + \int_{\Omega\times Y_F} \left|\left(T_\varepsilon\left(\Psi\right)(x,y) - \Psi(x)\right)\nabla_y\psi(y)\right| dxdy \\
&\quad \leq g\left\|T_\varepsilon\left(\varepsilon\nabla_x\Psi\right)\right\|_{\left(L^2(\Omega\times Y_F)\right)^n}\left\|\psi\right\|_{\left(L^2(\Omega\times Y_F)\right)^n} + \\
&\qquad\qquad + \left\|T_\varepsilon\left(\Psi\right) - \Psi\right\|_{L^2(\Omega\times Y_F)}\left\|\nabla_y\left(\psi\right)\right\|_{\left(L^2(\Omega\times Y_F)\right)^{n\times n}}.
\end{aligned}
$$

Passing to the limit as $\varepsilon \to 0$, by property p_5, we have that $T_\varepsilon\left(\varepsilon\frac{\partial\Psi}{\partial x_i}\right) \to 0$ strongly in $L^2\left(\Omega\times Y_F\right)$ and so

$$\left\|T_\varepsilon\left(\varepsilon\nabla_x\Psi\right)\right\|_{\left(L^2(\Omega\times Y_F)\right)^n} \to 0.$$

Moreover, by property p_3, $T_\varepsilon(\Psi) \to \Psi$ strongly in $L^2\left(\Omega\times Y_F\right)$ and so

$$\left\|T_\varepsilon\left(\Psi\right) - \Psi\right\|_{L^2(\Omega\times Y_F)} \to 0.$$

Then

$$\lim_{\varepsilon\to 0} g\varepsilon\int_{\Omega\times Y_F} T_\varepsilon\left(|\nabla v_\varepsilon|\right) dxdy = g\int_{\Omega\times Y_F}\left|\nabla_y\left(\Psi\psi\right)\right| dxdy.$$

In order to pass to the limit in the second nonlinear term, we use identity (15) for the function u_ε and the fact that the function $E(\varphi) = |\varphi|$ is proper convex continuous. We then deduce

$$\liminf_{\varepsilon\to 0} g\varepsilon\int_{\Omega\times Y_F} T_\varepsilon\left(|\nabla u_\varepsilon|\right) dxdy \geq g\int_{\Omega\times Y_F}\left|\nabla_y\widehat{u}\right| dxdy.$$

Moreover,

$$\int_{\Omega\times Y_F} T_\varepsilon\left(f_\varepsilon\right) T_\varepsilon\left(v\right) dxdy - \int_{\Omega\times Y_F} T_\varepsilon\left(f_\varepsilon\right) T_\varepsilon\left(u_\varepsilon\right) dxdy \to \int_{\Omega\times Y_F} f\Psi\psi dxdy - \int_{\Omega\times Y_F} f\widehat{u} dxdy.$$

We consider now the term $\int_{\Omega\times Y_F} T_\varepsilon(\widetilde{p}_\varepsilon)\,T_\varepsilon(\operatorname{div}(v-u_\varepsilon))\,dxdy$. Using $\operatorname{div}_x u_\varepsilon = 0$, we obtain

$$\int_{\Omega\times Y_F} T_\varepsilon(\widetilde{p}_\varepsilon)\,T_\varepsilon(\operatorname{div}_x(v-u_\varepsilon))\,dxdy = \int_{\Omega\times Y_F} T_\varepsilon(\widetilde{p}_\varepsilon)\,T_\varepsilon(\operatorname{div}_x v)\,dxdy$$

$$= \int_{\Omega\times Y_F} T_\varepsilon(\widetilde{p}_\varepsilon)\,T_\varepsilon\left(\operatorname{div}_x\left(\Psi(x)\,\psi\left(\frac{x}{\varepsilon}\right)\right)\right)dxdy$$

$$= \int_{\Omega\times Y_F} T_\varepsilon(\widetilde{p}_\varepsilon)\,T_\varepsilon\left(\nabla_x\Psi(x)\,\psi\left(\frac{x}{\varepsilon}\right) + \Psi(x)\operatorname{div}_x\psi\left(\frac{x}{\varepsilon}\right)\right)dxdy$$

$$= \int_{\Omega\times Y_F} T_\varepsilon(\widetilde{p}_\varepsilon)\,T_\varepsilon(\nabla_x\Psi)\,\psi dxdy.$$

Passing to the limit as ε tends to zero and using (10) the last term tends to

$$\int_{\Omega\times Y_F} \widehat{p}\nabla_x\Psi(x)\,\psi(y)\,dxdy = \int_{\Omega\times Y_F} \widehat{p}\nabla_x\Psi(x)\,\psi(y)\,dxdy$$

$$- \int_\Omega \widehat{p}\left(\operatorname{div}_x\int_{Y_F}\widehat{u}dy\right)dx$$

$$= -\left\langle \nabla_x\widehat{p}, \int_{Y_F}(\Psi(x)\,\psi(y)-\widehat{u})\,dy\right\rangle_\Omega .$$

Finally we obtain

$$2\mu\int_{\Omega\times Y_F}\nabla_y\widehat{u}\cdot\nabla_y(\Psi\psi-\widehat{u})\,dxdy + g\int_{\Omega\times Y_F}\left|\nabla_y(\Psi\psi)\right|dxdy - g\int_{\Omega\times Y_F}\left|\nabla_y\widehat{u}\right|dxdy$$

$$\geq \left\langle f-\nabla_x\widehat{p}, \int_{Y_F}(\Psi(x)\,\psi(y)-\widehat{u})\,dy\right\rangle_\Omega,\ \forall\Psi\in\mathscr{D}(\Omega),\ \psi\in V(Y_F),$$

relation which by density is always true for a test function $\widehat{v}\in L^2(\Omega, V(Y_F))$.

Then we easily find that the function $\widehat{u}$ is the unique solution of the problem

$$2\mu\int_{\Omega\times Y_F}\nabla_y\widehat{u}\cdot\nabla_y(\widehat{v}-\widehat{u})\,dxdy + g\int_{\Omega\times Y_F}\left|\nabla_y(\widehat{v})\right|dxdy - g\int_{\Omega\times Y_F}\left|\nabla_y\widehat{u}\right|dxdy$$

$$\geq \int_{\Omega\times Y_F} f(\widehat{v}-\widehat{u})\,dxdy,$$

for every $\widehat{v} \in L^2(\Omega, V(Y_F))$ such that $div_x \int_{Y_F} \widehat{v}(x,y)dy = 0$ and $\nu \cdot \int_{Y_F} \widehat{v}(x,y)dy = 0$ on $\partial\Omega$.

The pressure $\widehat{p} \in H^1(\Omega)$, nonunique, and relation (6) are recovered as in [8]. □

4 Conclusion

We gave in this chapter the proof of the homogenization of the Bingham flow in porous media, by using the unfolding method, an alternative method to the two-scale convergence method, which was already used in [2] in order to solve the same problem. Our aim is to continue to work on the homogenization of the Bingham flow with different boundary conditions than the one treated in this chapter and for which we expect that the unfolding method will fit better than the two-scale convergence method.

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References

1. Allaire, G.: Homogenization of the unsteady Stokes equations in porous media. In: Bandle, C., Bemelmans, J., Chipot, M., Grüter, M., Saint Jean Paulin, J. (eds.) Progress in Partial Differential Equations: Calculus of Variations, Applications, pp. 109–123. Longman Scientific & Technical, New York (1992)
2. Bourgeat, A., Mikelic, A.: A note on homogenization of Bingham flow through a porous medium. J. Math. Pures Appl. **72**, 405–414 (1993)
3. Cioranescu, D., Damlamian, A., Donato, P., Griso, G., Zaki, R.: The periodic unfolding method in domains with holes. hal.archives-ouvertes.fr/docs/00/59/16/32/PDF/TrousDV2010-12-11.pdf
4. Cioranescu, D., Damlamian, A., Griso, G.: The Stokes problem in perforated domains by the periodic unfolding method. In: Mihailescu-Suliciu, M. (ed.) New Trends in Continuum Mechanics, pp. 67–80. Theta Foundation, Romania (2003)
5. Cioranescu, D., Damlamian, A., Griso, G.: The periodic unfolding method in homogenization. SIAM J. Math. Anal. **40**, 1585–1620 (2008)
6. Duvaut, G., Lions, J.L.: Les Inéquations en Mécanique et en Physique. Dunod, Paris (1972)
7. Griso, G.: Habilitation à Diriger les Recherches. Université de Lille I, Lille (2005)
8. Lions, J.L., Sanchez-Palencia, E.: Ecoulement d'un fluide viscoplastique de Bingham dans un milieu poreux. J. Math. Pures Appl. **60**, 341–360 (1981)
9. Nguetseng, G.: A general convergence result for a functional related to the theory of homogenization. SIAM J. Math. Anal. **20**, 608–623 (1989)
10. Tartar, L.: Convergence of the homogenization process. In: Sanchez-Palencia, E. (ed.) Non-homogeneous Media and Vibration Theory. Lecture Notes in Physics, vol. 127. Springer, Heidelberg (1980)