Conformally Standard Stationary SpaceTimes and Fermat Metrics

Miguel Angel Javaloyes

1 Introduction

Fermat's principle, say, that light rays minimize the arrival time, is linked to general relativity from its very beginning. As early as 1917, Weyl established a version for static spacetimes in [56], and several other authors, as Levi–Civita and Synge [34, 54], gave some attention to the principle. Not much later, in 1927, Levi–Civita stated the stationary version in [35] (see also [50]) that later was included in the book [33]. The general version was formulated by Kovner in 1990 [32] and rigorously established by Perlick in [47] (see also [49] for a version in Finsler spacetimes).

Independently from Fermat's principle, Randers metrics appeared as an attempt of Randers to geometrize electromagnetism in general relativity [51], but it seems that it was Ingarden the first one that thought in Randers metrics as Finsler ones in his PhD thesis [27]. By the way, Miron [45] suggested to name the Randers metrics endowed with a nonlinear Lorentz connection (associated to the Lorentz equation in electrodynamics) as *Ingarden spaces*. Afterwards, they were recovered by M. Matsumoto with the aim of giving examples of the so-called C-reducible Finsler metrics. In order to obtain these examples, he introduced the class of (α,β)-metrics in a manifold M, that is, Finsler metrics that are obtained as a homogeneous combination of the square root of a Riemannian metric h and a one-form β on M (with the notation $\alpha(v)=\sqrt{h(v,v)}$ for $v\in TM$) [40]. In particular, Randers metrics are defined as $\alpha+\beta$. This function is positively homogeneous but not reversible. Moreover, it is positive whenever the h-norm of β is less than 1 in every point. Subsequently, the Japanese school of Finsler geometry spent some time studying Randers metrics, mostly problems related with curvature [41, 53, 57].

M.A. Javaloyes (✉)
Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo,
30100 Espinardo, Murcia, Spain
e-mail: majava@um.es; ma.javaloyes@gmail.com

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Let us point out that the approach of G. Randers himself was somewhat different, since he constructed his metric from a Lorentzian metric and a one-form in the spacetime. It is also remarkable that Lichnerowicz and Thiry obtained a Randers metric when studying Jacobi–Maupertuis principle in general relativity (see [37] and [36, p. 155]).

In this chapter, we will describe some recent results that use techniques of Finsler geometry to study conformally standard stationary spacetimes and vice versa.

2 Finsler and Randers Metrics

There are several definitions of Finsler metrics [30]. But the most general case where you can extend most of the classic Riemannian results is the following. Let $\pi : TM \to M$ be the natural projection from the tangent bundle to the manifold. A *Finsler metric* is a continuous function $F : TM \to [0,+\infty)$ satisfying the following properties:

1. F is C^∞ in $TM \setminus 0$, that is, it is smooth away from the zero section,
2. F is fiberwise positively homogeneous of degree one, that is , $F(\lambda v) = \lambda F(v)$ for every $v \in TM$ and $\lambda > 0$,
3. F^2 is fiberwise strongly convex, that is, the fundamental tensor g_u defined as

$$g_u(v,w) = \frac{\partial^2}{\partial s \partial t} F^2(u+tv+sw)|_{t,s=0}, \tag{1}$$

where $u \in TM \setminus 0$ and $v, w \in T_{\pi(u)}M$, is positively defined for every $u \in TM \setminus 0$.

These conditions imply that F is positive away from the zero section, the triangle inequality holds for F in the fibers (see [3, Sect. 1.2B]) and F^2 is C^1 [55]. Property (3) above is essential to guarantee minimization properties of geodesics. The first geometers that worked with Randers metrics seemed very concerned with computation of curvatures and invariants related with connections, and, apparently, they overlooked the question of strong convexity. Let us recall that a Randers metric on a manifold M is constructed using a Riemannian metric h and a one-form β on M as

$$R(v) = \sqrt{h(v,v)} + \beta(v) \tag{2}$$

for every $v \in TM$. It turns out that it is fiberwise strongly convex if and only if it is positive for every $v \in TM$. This can be easily seen computing the fundamental tensor (see [30, Corollary 4.17]):

$$g_v(w,w) = \frac{R(v)}{\sqrt{h(v,v)}}\left(h(w,w) - \frac{1}{h(v,v)}h(v,w)^2\right) + \left(\frac{h(v,w)}{\sqrt{h(v,v)}} + \beta(w)\right)^2,$$

with $v \in TM \setminus 0$ and $w \in T_{\pi(v)}M$. Up to our knowledge, the first time that a proof of this fact appeared was in [3, Sect. 11.1] published in 2000.

Positive homogeneity of Finsler metrics implies that the length of a piecewise smooth curve $\gamma\colon [a,b] \subseteq \mathbb{R} \to M$ given by

$$\ell_F(\gamma) = \int_a^b F(\dot{\gamma})ds$$

does not depend on the orientation preserving parametrization of the curve. Then you can define the distance between two points $p,q \in M$ as

$$d(p,q) = \inf_{\gamma \in C_{p,q}} \ell_F(\gamma),$$

where $C_{p,q}$ is the space of piecewise smooth curves from p to q. This gives a generalized distance (see [58, p. 5] and also [20, 28]), but not necessarily reversible as the length of a curve depends on the orientation of the parametrization (observe that in general $F(-v) \neq F(v)$). Then, you can define two kind of balls, that is, forward and backward balls, respectively, as

$$B_F^+(p,r) = \{q \in M : d_F(p,q) < r\}, \quad B_F^-(p,r) = \{q \in M : d_F(q,p) < r\},$$

for every $p \in M$ and $r > 0$. Moreover, there exist several definitions for Cauchy sequences.

Definition 1. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called a *forward (resp. backward) Cauchy sequence* if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_F(x_i,x_j) < \varepsilon$ for any $i, j \in \mathbb{N}$ satisfying $N < i < j$ (resp. $N < j < i$).

Moreover, you can also define the energy functional as

$$E_F(\gamma) = \frac{1}{2}\int_a^b F(\dot{\gamma})^2 ds$$

for every piecewise smooth curve $\gamma : [a,b] \subseteq \mathbb{R} \to M$, and geodesics as critical points of this functional. In particular, geodesics must have constant speed (see, e.g., [15, Proposition 2.1]). Let us point out that in some references as [3] geodesics are defined as critical points of the length functional and as a consequence they are not assumed to have constant speed.

3 Fermat's Principle in Conformally Standard Stationary SpaceTimes

Let us recall that a conformally stationary spacetime is a Lorentz manifold (M,g) that admits a timelike conformal vector field K. We refer to the classical books [6, 46] for the basic definitions on Lorentzian geometry and causality. Observe that K determines a time orientation in (M,g) and thus, a spacetime that, with an abuse of notation, we will denote also by (M,g). It can be shown that when K is complete and the spacetime is distinguishing (see [29]), then (M,g) splits as a *conformally standard stationary spacetime*, that is, $M = S\times\mathbb{R}$ and the metric g can be expressed as

$$g((v,\tau),(v,\tau)) = \varphi(g_0(v,v)+2\omega(v)\tau-\tau^2), \tag{3}$$

in $(x,t)\in S\times\mathbb{R}$, where $(v,\tau)\in T_xS\times\mathbb{R}$, φ is a smooth positive function on $S\times\mathbb{R}$ and ω and g_0 are respectively a one-form and a Riemannian metric on the manifold S. In this setting, the vector field K is induced from the natural lifting to M of the canonical vector field d/dt on $\mathbb{R}$, which we will denote in the following by ∂_t. Let us remark that sometimes in literature the one-form ω is expressed using the metrically equivalent vector field δ, that is, $\omega(v) = g_0(v,\delta)$ for every $v\in TS$.

We must observe that, in a Lorentzian manifold, light-like geodesics and their conjugate points are preserved by conformal changes up to parametrization (see, e.g., [44, Theorem 2.36]). This implies that studying light-like geodesics of $(S\times\mathbb{R},g)$ is equivalent to studying light-like geodesics of $(S\times\mathbb{R},\frac{1}{\varphi}g)$. As a consequence, we can assume that the spacetime is a *normalized standard stationary spacetime*, that is, a standard stationary spacetime with a unit Killing vector field and

$$g((v,\tau),(v,\tau)) = g_0(v,v)+2\omega(v)\tau-\tau^2, \tag{4}$$

in $(x,t)\in S\times\mathbb{R}$ for any $(v,\tau)\in T_xS\times\mathbb{R}$. In this case, ∂_t rather than a conformal vector field is a unit Killing vector field.

The advantage of formulating Fermat's principle in (conformally) standard stationary spacetimes is that it is possible to define a global time function given by the second coordinate in $S\times\mathbb{R}$ and it also makes sense to speak about the spatial position, that is, the first coordinate. Now fix two spatial positions x_0 and x_1 in S. Then, Fermat's principle says that the paths of light rays are critical points of the global time function between all the possible trajectories for light rays from (x_0,t_0) to the integral curve of ∂_t through (x_1,t_1), with $t_0,t_1\in\mathbb{R}$. According to general relativity, as photons are massless, the trajectories of light rays must be described by light-like curves. Therefore, the space of curves for the Fermat's principle must be composed of smooth future-pointing light-like curves. Let us observe that as the time orientation is assumed to be given by the Killing vector field ∂_t, a future-pointing causal curve is a curve $\gamma = (x,t) : [a,b]\subseteq\mathbb{R}\to S\times\mathbb{R}$ satisfying $g(\dot\gamma,\dot\gamma)\leq 0$ and $\dot t>0$. If $\gamma=(x,t):[0,1]\to S\times\mathbb{R}$ is a smooth light-like curve from (x_0,t_0) to (x_1,t_1), we need to compute t_1, which is the value of the global time in $\gamma(1)$. As γ is light-like, we have that

$$g_0(\dot x,\dot x)+2\omega(\dot x)\dot t-\dot t^2 = 0,$$

and hence, as γ is assumed to be future-pointing (i.e., $\dot t>0$),

$$\dot t = \sqrt{g_0(\dot x,\dot x)+\omega(\dot x)^2}+\omega(\dot x).$$

Integrating the last equation, we get

$$t(s) = t_0 + \int_0^s \left(\sqrt{g_0(\dot{x},\dot{x}) + \omega(\dot{x})^2} + \omega(\dot{x}) \right) dv. \tag{5}$$

As a consequence, light-like geodesics must be critical points of the functional

$$T(\gamma) = t_1 = t_0 + \int_0^1 \left(\sqrt{g_0(\dot{x},\dot{x}) + \omega(\dot{x})^2} + \omega(\dot{x}) \right) dv.$$

This functional is, up to a constant, the length functional of the Finsler metric in S given by

$$F(v) = \sqrt{g_0(v,v) + \omega(v)^2} + \omega(v), \tag{6}$$

for every $v \in TS$. This metric is of Randers type, that is, the addition of the square of a Riemannian metric and a one-form of norm less than one in every point. We will call this metric the *Fermat metric* associated to the splitting (4) (or in general to the splitting (3)).

Remark 1. With a similar reasoning, we get that past-pointing light-like geodesics are controlled by the reverse metric of (6), that is,

$$\tilde{F}(v) = F(-v)$$

for every $v \in TS$. It is easy to see that:

(i) $d_{\tilde{F}}(p,q) = d_F(q,p)$ for every $p,q \in S$.
(ii) $\gamma\colon [0,1] \subseteq \mathbb{R} \to S$ is a geodesic from p to q of (S,F) if and only if the reverse curve $\tilde{\gamma}\colon [0,1] \to S, t \to \tilde{\gamma}(t) = \gamma(1-t)$ is a geodesic from q to p of $(S,\tilde{F})$.

Then, all the properties of past-pointing light-like geodesics can be also written in terms of the Fermat metric (6).

Remark 2. Let us consider the class of standard static spacetimes $(S \times \mathbb{R}, g_{st})$, with

$$g_{st}((v,\tau),(v,\tau)) = g_0(v,v) - \beta(x)\tau^2,$$

in $(x,t) \in S \times \mathbb{R}$, where $(v,\tau) \in T_xS \times \mathbb{R}$, g_0 is a Riemannian metric on S and β a positive smooth function on S. In particular, they are standard stationary, and the Fermat metric associated to them is Riemannian. Indeed, it is conformal to the metric induced by g_{st} in S, namely, g_0/β. This fact was already pointed out in [35, p. 343]. Up to the name of Fermat metric, other authors have used another name for the same concept, for example, in [18, 25], it is named as *optical metric* and in [2], *optical reference geometry*.

Remark 3. In the stationary case, it must be clarified that our terminology is different from the one introduced by Perlick in [48], where the name of *Fermat metric* is used for the Riemannian metric in S given by

$$h(v,v) = g_0(v,v) + \omega(v)^2, \tag{7}$$

for $v \in TS$. In [48], it is also used the name of *Fermat one-form* for ω. Observe that then our Fermat metric is the addition of the Fermat one-form and the square root of Perlick's Fermat metric. But our Fermat metric contains all the information, and in fact, it allows one to recover the Fermat one-form and Perlick's Fermat metric as

$$h(v,v) = \frac{1}{4}\left(F(v)+F(-v)\right)^2, \qquad \omega(v) = \frac{1}{2}\left(F(v)-F(-v)\right),$$

for any $v \in TS$, where F is given in Eq. (6).

The above computations show that in (conformally) standard stationary spacetimes, Fermat's principle relates future-pointing light-like geodesics of $(S\times\mathbb{R},g)$ as in Eq. (4) with geodesics of the Finsler manifold (S,F) with F given in Eq. (6) up to reparameterizations. Let us state the relation including parameterizations.

Theorem 1 (Fermat's principle). *Let $(S\times\mathbb{R},g)$ be a standard stationary spacetime as in Eq. (4). A curve $\gamma=(x,t):[a,b]\subseteq\mathbb{R}\to S\times\mathbb{R}$ is a light-like geodesic of $(S\times\mathbb{R},g)$ if and only if x is a geodesic for the Fermat metric F in Eq. (6) parameterized to have constant h-Riemannian speed (h as in Eq. (7)) and*

$$t(s) = t(a) + \int_a^s F(\dot{x})\mathrm{d}\nu,$$

for every $s \in [a,b]$.

Proof. The equivalence can be easily obtained computing the critical points of the length functional for F with h-constant Riemannian speed using the Levi–Civita connection ∇ of g_0 and then the light-like critical points of the energy functional of g using again ∇ (see, e.g., [15, Theorem 4.1]). □

Remark 4. Let us point out that Perlick [48] considers a more general case than conformally standard stationary spacetimes. Basically, he considers a conformally stationary spacetime (M,g) where the flow lines of the conformal vector field K of (M,g) have a structure of Hausdorff manifold $\hat{M}$ and the natural projection $\pi: M\to\hat{M}$ is a principal fiber bundle with structure group $\mathbb{R}$, with the action given by the flow of K. Observe that as the fiber is $\mathbb{R}$, there always exists a section of the bundle (see, e.g., [31, page 58]). But the existence of a space-like section is not guaranteed. In fact, assuming that K is complete, this happens if and only if the spacetime is distinguishing (see [29]). Given a section S of the fiber bundle, we can express the metric of (M,g) as in Eq. (4), but with g_0 not necessarily positive definite. In this case, the global function given by the second coordinate is not necessarily a time function, that is, it does not have to be strictly increasing in causal curves. As a consequence, the Fermat metric obtained in Eq. (6) can be non-positive along some directions of the tangent space. In fact, it is not difficult to see that the Fermat metric Eq. (6) is a Finsler metric (with the definition given in Sect. 2) if and only if the section S is space-like.

It can be helpful to restate the Fermat's principle as follows (see [16, Proposition 4.1]).

Proposition 1. *Let* $z_0 = (x_0, t_0)$, $L_{x_1} = \{(x_1, t) : t \in \mathbb{R}\}$ *be, respectively, a point and a vertical line in a standard stationary spacetime. Then,* z_0 *can be joined with* L_{x_1} *by means of a future-pointing light-like pregeodesic* $t \mapsto \gamma(t) = (x_\gamma(t), t)$ *starting at* z_0 *if and only if* x_γ *is a unit speed geodesic of the Fermat metric* F *which joins* x_0 *with* x_1*. In this case,*

$$t_1 - t_0 = \ell_F(x_\gamma|_{[t_0,t_1]}).$$

Let us observe that the Fermat metric depends on the space-like section you choose to obtain the standard splitting (which in some references as [26, 47] is called the gauge choice). The above proposition can be used to obtain the relation between two Fermat metrics associated to different splittings of the same stationary spacetime (with a fixed time-like Killing vector field K). If $(S \times \mathbb{R}, g)$ is one of the splittings (with g as in Eq. (4)), the other one is determined by a section given by a smooth function $f : S \to \mathbb{R}$ as $S_f = \{(x, f(x)) \in S \times \mathbb{R} : x \in S\}$. Then you can define the map $\psi_f : S \times \mathbb{R} \to S \times \mathbb{R}$ given as $\psi(x,t) = (x, t + f(x))$ for every $(x,t) \in S \times \mathbb{R}$. Therefore the other splitting is expressed as $(S \times \mathbb{R}, g_f)$, where $g_f = \psi_f^*(g)$ (here $*$ denotes the pullback operation).

Proposition 2. *With the above notation, the Fermat metric associated to the splitting* $(S \times \mathbb{R}, g_f)$ *is* $F_f = F - df$*, where* F *is the Fermat metric associated to* $(S \times \mathbb{R}, g)$ *and* df *is the differential of the smooth function* f*.*

Proof. Observe that given a curve $\gamma : [-\varepsilon, \varepsilon] \to S$, with $\varepsilon > 0$,

$$\hat{F}(\dot{\gamma}(0)) = \left.\frac{d}{ds}\right|_{s=0} \ell_{\hat{F}}(\gamma|_{[0,s]}),$$

for any Finsler metric $\hat{F}$. Moreover, as a consequence of Proposition 1,

$$\ell_{F_f}(\gamma) = \ell_F(\gamma) + f(\gamma(-\varepsilon)) - f(\gamma(\varepsilon)) = \int_{-\varepsilon}^{\varepsilon} (F(\dot{\gamma}) - df(\dot{\gamma}))ds. \tag{8}$$

Given $v \in TS$, consider $\gamma : [-\varepsilon, \varepsilon] \to S$ such that $\dot{\gamma}(0) = v$. Then,

$$F_f(v) = \left.\frac{d}{ds}\right|_{s=0} \ell_{F_f}(\gamma|_{[0,s]})) = F(v) - df(v)$$

for any $v \in TS$. □

Proposition 3. *Given an arbitrary function* $f : S \to \mathbb{R}$*, the section* S_f *of* $S \times \mathbb{R}$ *is space-like if and only if* $F(v) > df(v)$ *for every* $v \in TS$*.*

Proof. See also [16, Proposition 5.8]. □

Now, we can establish the so-called Stationary to Randers correspondence [16]. Let us call $\mathrm{Stat}(S\times\mathbb{R})$ the space of standard stationary spacetimes with normalized Killing vector field ∂_t and $\mathrm{Rand}(S)$ the space of Randers metrics on S. Then one has the bijective map

$$\mathrm{Stat}(S\times\mathbb{R})\to\mathrm{Rand}(S),\qquad g\mapsto F_g,\tag{9}$$

where F_g is determined as in Eq. (6) by the same stationary data pair (g_0,ω) which determines g in Eq. (4). Moreover, we can define in both sets equivalence relations as

$$\begin{aligned} R\sim R' &\iff R-R'=df \text{ for some smooth function f on S,}\\ g\sim g' &\iff g'=\psi_f^*g \qquad \text{for some change of the initial section } \psi_f, \end{aligned}$$

and consider the corresponding quotient sets $\mathrm{Rand}(S)/\sim$, $\mathrm{Stat}(S\times\mathbb{R})/\sim$. Proposition 2 says that the bijection (9) induces a well-defined bijective map between the quotients

$$(\mathrm{Stat}(S\times\mathbb{R})/\sim)\to(\mathrm{Rand}(S)/\sim).$$

This relation constitutes a very important issue for Randers metrics, because the global invariants in the spacetime must be translated in invariants for the entire class of Randers metrics that differ in the differential of a function.

4 Causality and Fermat Metrics

As, by Proposition 1, geodesics of Fermat metrics contain all the information of light-like geodesics up to reparameterization, it turns out that Fermat metrics can be used to describe the chronological future and past of a given point. As a consequence, we can characterize the causal conditions of a standard stationary spacetime in terms of the Fermat metric. This relation was established in [16] with some previous partial results in [15]. Recall that we say that two events p and q in a spacetime are chronologically related, and write $p\ll q$ (resp. strictly causally related $p<q$) if there exists a future-pointing time-like (resp. causal) curve γ from p to q; p is causally related to q if either $p<q$ or $p=q$, denoted $p\leq q$. Then the *chronological future* (resp. *causal future*) of $p\in M$ is defined as $I^+(p)=\{q\in M: p\ll q\}$ (resp. $J^+(p)=\{q\in M: p\leq q\}$). Analogous notions appear substituting the word "future" by "past" and denoting $I^-(p),J^-(p)$.

Proposition 4. *Let $(S\times\mathbb{R},g)$ be a standard stationary spacetime as in Eq. (4) and $(x_0,t_0)\in S\times\mathbb{R}$. Then*

$$\begin{aligned} I^+(x_0,t_0)&=\cup_{s>0}\{t_0+s\}\times B_F^+(x_0,s),\\ I^-(x_0,t_0)&=\cup_{s<0}\{t_0-s\}\times B_F^-(x_0,s). \end{aligned}$$

Proof. See [16, Proposition 4.2]. □

Using the expression of the chronological future and past in terms of the forward and backward balls of the Fermat metric, we can easily obtain the characterization of the causality conditions in terms of the Fermat metric. For definitions and properties of the different levels of causality, we refer to [44].

Theorem 2. *Let $(S \times \mathbb{R}, g)$ be a standard stationary spacetime as in Eq. (4). Then $(S \times \mathbb{R}, g)$ is causally continuous. Furthermore:*

(a) It is causally simple if and only if one of the following equivalent conditions holds:

  (i) $J^+(p)$ is closed for all p,
  (ii) $J^-(p)$ is closed for all p.
  (iii) The associated Finsler manifold (S, F) is convex.

(b) It is globally hyperbolic if and only if the subsets $B_F^+(x,r) \cap B_F^-(x,r)$ are relatively compact for every $x \in S$ and $r > 0$.

Moreover, a slice $S \times \{t_0\}, t_0 \in \mathbb{R}$, is a Cauchy hypersurface if and only if the Fermat metric F on S is forward and backward complete.

Proof. See [16, Theorems 4.3 and 4.4]. For part (b) see also [16, Proposition 2.2]. □

The static version of the last proposition can be found in [52, Proposition 3.5] (see also [6, Theorem 3.66]). Furthermore, an extension of last theorem characterizing the stationary regions that are causally simple in terms of convex regions for the Fermat metric has been achieved in [11]. Theorem 2 implies some consequences for Randers metrics. In particular we can establish a generalization of the classical Hopf–Rinow theorem.

Theorem 3. *Given a Randers manifold (M, R), the following conditions are equivalent:*

(i) The subsets $B_R^+(x,r) \cap B_R^-(x,r)$ are relatively compact for every $x \in M$ and $r > 0$.
(ii) The subsets that are forward and backward bounded are relatively compact.
(iii) There exists $f : M \to \mathbb{R}$ such that $R + df$ is a forward and backward complete Randers metric.

Moreover, these conditions imply the convexity of (M, R).

Proof. The equivalence between the two first conditions is standard, and it holds for any Finsler metric. For $(i) \Rightarrow (iii)$, first observe that any Randers metric can be obtained as the Fermat metric of a standard stationary spacetime (see [8, Proposition 3.1]). Now let $(S \times \mathbb{R}, g)$ be the standard stationary spacetime having as a Fermat metric R. By Proposition 2, this spacetime is globally hyperbolic, but then using [7], we obtain that there exists a smooth space-like Cauchy hypersurface S_f. Consider the splitting associated to the Cauchy hypersurface. By Proposition 2, the Fermat metric associated to the new splitting is of the form $R - df$ for a certain smooth function $f : S \to \mathbb{R}$. Moreover, by Proposition 2, $R - df$ must be forward and backward complete. For $(iii) \Rightarrow (i)$, observe that part (b) of Theorem 2 implies that the stationary spacetime associated to $R + df$ is globally hyperbolic. By Proposition 2, the stationary spacetime associated to R is the same as the one associated to $R + df$, but considering another splitting. Therefore it is globally hyperbolic and (i) follows from part (b) of Theorem 2. The convexity can be obtained from the Avez–Seifert Theorem applied to $(S \times \mathbb{R}, g)$ (see e.g., [6, Theorem 6.1]). □

It turns out that the condition of forward or backward completeness can be substituted by one of the two first equivalent conditions in Theorem 3 in some classical results of Finsler geometry as, for example, the theorems of Bonnet–Myers and Synge or the sphere theorem in its nonreversible version by Rademacher (see [16, Remark 5.3]).

Theorem 2 has been used in [17] to obtain some conditions that ensure global hyperbolicity. Recall that h is defined in Eq. (7). Given any Riemannian metric g in S, we will denote by d_g the distance in S associated to g. We say that a positive function f in S grows at most linearly with respect to d_g if given a point x_0, there exist positive constants A, B such that $f(x) \leq A \cdot d_g(x_0, x) + B$ for every $x \in S$. This condition does not depend on x_0. We also will denote

$$\|\omega\|_g = \sup_{v \in T_x S} \frac{|\omega(v)|}{\sqrt{g(v,v)}}$$

the g-norm of a one-form ω in $x \in S$ for any Riemannian metric g on S.

Theorem 4. *Let $(S \times \mathbb{R}, g)$ be a conformally standard stationary spacetime with g as in Eq. (3). Then the slices $S \times \{t\}$, $t \in \mathbb{R}$, are Cauchy hypersurfaces if one of the following conditions holds:*

(i) The metric $\frac{1}{(1+\|\omega\|^2_{g_0})^2} h$ is complete.

(ii) The metric g_0 is complete and $\|\omega\|_{g_0}$ grows at most linearly in d_{g_0}.

(iii) There exists a proper function $f : S \to \mathbb{R}$ such that the product $\|df\|_{g_0} \cdot \|\omega\|_{g_0}$ grows at most linearly in $d_{g_0 + df \otimes df}$

Moreover, if $(S \times \mathbb{R}, g)$ is globally hyperbolic:

(iv) The slices $S \times \{t\}$, $t \in \mathbb{R}$, are Cauchy hypersurfaces if $\|\omega\|^2_{g_0}$ grows at most linearly in d_h.

(v) For any proper function $f : S \to \mathbb{R}$, $\|\omega\|_{g_0}$ grows at most linearly in $d_{g_0 + df \otimes df}$.

Proof. For (i) and (iv), see [17, Theorem 2]. For (ii), see part (1) of Proposition 2 in [17] and for (iii) and (v), [17, Theorem 4]. □

Indeed, in [17], the authors obtain several interesting pinching inequalities as

$$\frac{\sqrt{h(v,v)}}{2(1 + \|\omega\|^2_{g_0})} \leq F(v) \leq 2\sqrt{h(v,v)}$$

and

$$\frac{\sqrt{g_0(v,v)}}{\sqrt{1+\|\omega\|_{g_0}^2}+\|\omega\|_{g_0}} \le F(v) \le \left(\sqrt{1+\|\omega\|_{g_0}^2}+\|\omega\|_{g_0}\right)\sqrt{g_0(v,v)}$$

for every $v \in TS$ (see [17, Propositions 1 and 2]). We point out that in [52] (especially in Corollary 3.5) there are some results in the same direction as the last theorem.

As a further relation between causality of a standard stationary spacetime and Randers metrics, Cauchy developments will be constructed in terms of the Fermat metric. A subset A of a spacetime M is *achronal* if no $x, y \in A$ satisfies $x \ll y$; in this case, the *future (resp. past) Cauchy development* of A, denoted by $D^+(A)$ (resp. $D^-(A)$), is the subset of points $p \in M$ such that every past- (resp. future)-inextendible causal curve through p meets A. The union $D(A) = D^+(A) \cup D^-(A)$ is the *Cauchy development* of A. The *future (resp. past) Cauchy horizon* $H^+(A)$ (*resp.* $H^-(A)$) is defined as

$$H^{\pm}(A) = \{p \in \bar{D}^{\pm}(A) : I^{\pm}(p) \text{ does not meet } D^{\pm}(A)\}.$$

Intuitively, $D(A)$ is the region of M a priori predictable from data in A, and its *horizon* $H(A) = H^+(A) \cup H^-(A)$, the boundary of this region.

Proposition 5. *Let $(S \times \mathbb{R}, g)$ be a standard stationary spacetime as in Eq. (4) such that $S \times \{t_0\}$ is a Cauchy hypersurface, $A \subset S$, and $A_{t_0} = A \times \{t_0\}$ the corresponding (necessarily achronal) subset of $S \times \{t_0\}$. Then*

$$D^+(A_{t_0}) = \{(y,t) : d_F(x,y) > t - t_0 \text{ for every } x \notin A \text{ and } t \ge t_0\}, \tag{10}$$

$$D^-(A_{t_0}) = \{(y,t) : d_F(y,x) > t_0 - t \text{ for every } x \notin A \text{ and } t \le t_0\}. \tag{11}$$

Moreover, the Cauchy horizons can be described as

$$H^+(A_{t_0}) = \{(y,t) : \inf_{x \notin A} d_F(x,y) = t - t_0\} \tag{12}$$

$$H^-(A_{t_0}) = \{(y,t) : \inf_{x \notin A} d_F(y,x) = t_0 - t\}. \tag{13}$$

Proof. See [16, Proposition 4.7]. □

Last proposition can be used to study the differentiability of the Cauchy horizon in terms of the distance computed with the Fermat metric from a closed subset [16, Theorem 4.10] and vice versa (see [16, Sect. 5.4].

5 Causal Boundaries and Fermat Metrics

In general relativity, it is important to complete the spacetime with some kind of boundary. One way to obtain an intrinsic completion is using the causal structure. This completion has been largely discussed because of some topological pathologies, but it seems that after [21], the definition and the topology is now satisfactory. As the causal completion (or c-completion for short) depends only on the causal structure of the spacetime, it is expectable that, in conformally standard stationary spacetimes, can be computed in terms of the Fermat metric. Let us recall that the c-completion is constructed in strongly causal spacetimes by adding some ideal points to the spacetime in such a way that time-like curves always have an endpoint in the new space. This is done by identifying the points of the space with PIP's (resp. PIF's), namely, proper indecomposable past (resp. future) sets; in other words, every point $p \in M$ is identified with $I^-(p)$ and $I^+(p)$. Then, we add to the spacetime the TIP's (resp. TIF's), namely, terminal indecomposable past (resp. future) sets. Then, the future (resp. past) c-completion $\hat{\partial} M$ (resp. $\check{\partial} M$) is given by the TIP's (resp. TIF's). Moreover, $\hat{M} := M \cup \hat{\partial} M$ and $\check{M} := M \cup \check{\partial} M$. In order to obtain the causal completion of M, we must identify some TIP's and TIF's. This is done by means of the S-relation. Denote $\hat{M}_\emptyset = \hat{M} \cup \{\emptyset\}$ (resp. $\check{M}_\emptyset = \check{M} \cup \{\emptyset\}$). The S-relation is defined in $\hat{M}_\emptyset \times \check{M}_\emptyset$ as follows. If $(P,F) \in \hat{M} \times \check{M}$, then $P \sim_S F$ if and only if:

(i) F is included and a maximal indecomposable future set in $\uparrow P$ (the common future of P)
(ii) P is included and a maximal indecomposable past set in $\downarrow F$ (the common past of F).

Moreover, we also put

$$P \sim_S \emptyset, \qquad \emptyset \sim_S F.$$

In particular, the only S-relations between PIP's and PIF's are $I^+(p) \sim_S I^-(p)$. Then, the c-completion $\bar{M}$ is the quotient set $\hat{M}_\emptyset \times \check{M}_\emptyset / \sim_S$ endowed with the *chronological topology* (see Definition 2.2 and the paragraph below in [20]). We can identify $M \equiv \{(I^-(p), I^+(p)) : p \in M\}$ and define the *c-boundary* as $\partial M := \bar{M} \setminus M$. We say that the c-completion is simple as a point set when every TIP (resp. TIF) determines a unique pair in ∂M (for the definition of topologically simple see [20, Definition 2.4]).

Up to the completion of the Finsler manifold (S,F), there are several nonequivalent ways to do it. You can compute the forward (resp. backward) Cauchy boundary $\partial_C^+ S$ (resp. $\partial_C^- S$) by adding ideal points in such a way that you can always obtain the convergence of forward (resp. backward) Cauchy sequences. Then the forward (resp. backward) *Cauchy completion* is $S_C^+ := S \cup \partial_C^+ S$ (resp. $S_C^- := S \cup \partial_C^- S$). Moreover, denote $\partial_C^s S := \partial_C^+ S \cap \partial_C^- S$ and $S_C^s = S \cup \partial_C^s S$. The map

$$d_Q : S_C^+ \times (S_C^+ \cup S_C^-) \to [0, \infty]$$

defined by

$$d_Q([\{x_n\}],[\{y_m\}]) := \lim_n(\lim_m d_F(x_n,y_m))$$

is a quasi-distance (see [20, Propositions 3.25 and 3.32]).

The *Gromov completion* is obtained by considering the subset $\mathscr{L}_1(S,d_F)$ of d_F-Lipschitz functions on S [20, Definition 5.2] and $\mathscr{L}_1(S,d_F)_* = \mathscr{L}_1(S,d_F)/\mathbb{R}$ (two functions are related when they differ in a constant). Then define the maps

$$j^+ : S_C^+ \to \mathscr{L}_1(S,d_F), \quad x \to -d_x^+, \quad \text{where } d_x^+ = d_Q(\cdot,x),$$
$$j^- : S_C^- \to \mathscr{L}_1(S,d_F), \quad x \to +d_x^-, \quad \text{where } d_x^- = d_Q(x,\cdot),$$

which are injective (see [20, Proposition 5.7]). We can identify the points of S with the class of (minus) the distance function to (resp. from) the point, which is denoted by $j^+(S)$ (resp. $j^-(S)$). The forward (resp. backward) Gromov completion S_G^+ (resp. S_G^-) is the closure of S in $\mathscr{L}_1(S,d_F)_*$ considering the compact-open topology. Observe that this topology is equivalent to that of uniform convergence on compact subsets and to that of pointwise convergence.

Let $C^+(S)$ be the set of piecewise smooth curves $c : [\alpha,+\Omega) \to S$, $\Omega \leq \infty$, such that $F(\dot{c}) < 1$. For $c \in C^+(S)$, the associated (forward) Busemann function $b_c^+ : S \to (-\infty,\infty]$ is $b_c^+(\cdot) = \lim_{s\to\Omega}(s - d_F(\cdot,c(s)))$, which always exists because $s \mapsto s - d_F(\cdot,c(s))$ is increasing [20, Lemma 4.14]. Observe that b_c^+ is finite or infinite everywhere. Denote

$$B^+(S) := \{b_c^+ < \infty : c \in C^+(S)\}.$$

Moreover, if $\Omega < \infty$, then there exists some $\bar{x} \in S_C^+$ such that $b_c^+(x) = \Omega - d_F(x,\bar{x})$ for all $x \in S$ (here d_F is extended to S_C^+), and we denote

$$b_c^+ = d_p^+ := \Omega - d_F(\cdot,\bar{x}),$$

with $p = (\bar{x},\Omega) \in S_C^+ \times \mathbb{R}$. If $\Omega = \infty$, we say that b_c^+ is a properly Busemann function and we write

$$\mathscr{B}^+(S) := \{b_c^+ < \infty : c \in C^+(S), \Omega = \infty\}.$$

The *Busemann completion* as a point set is the quotient $S_B^+ := B^+(S)/\mathbb{R} \subset S_G^+$ and the (forward) *Busemann boundary* $\partial_B^+ S := S_B^+ \setminus S (\subset \partial_G^+ S)$. Furthermore, the (forward) *properly Busemann boundary* is defined as $\partial_{\mathscr{B}}^+ S := \mathscr{B}^+(S)/\mathbb{R}$. Then $\partial_B^+ S = j^+(\partial_C^+ S) \cup \partial_{\mathscr{B}}^+ S$. S_B^+ will be endowed with the chronological topology (see [20, Sect. 5.2.2]).

Recall that given a topological space T, the *forward cone* with base T is constructed as the quotient topological space $(T \times (-\infty,\infty])/\sim$, where the unique non-trivial identifications are $(x,\infty) \sim (x',\infty)$ for all $x,x' \in T$. Moreover, the class of (x,∞) is called the *apex* of the cone.

Finally, given a future-pointing time-like curve $\gamma : [\alpha,\Omega) \to M$, parameterized as $\gamma(t) = (c(t),t)$, observe that $I^-[\gamma] = \{(x',t') \in M : t' < b_c^+(x')\}$. Therefore the indecomposable past sets ($\neq M$) can be identified with the elements of $B^+(S)$. When $b_c^+ \equiv \infty$, then $I^-[\gamma] = M$, and it will be denoted with i^+. Given a d_F-Lipschitz function $f : S \to \mathbb{R}$, we define its past $P(f)$ (resp. future $F(f)$) as

$$P(f) := \{(x,t) \in M : t < f(x)\} \subset M$$

(resp. $F(f) := \{(x,t) \in M : t > -f(x)\} \subset M$). If $(P,F) \in \partial M$ with $\emptyset \neq P = P(b_c^+)$, we define the line over (P,F), denoted as $L(P,F)$, as

- If $F = \emptyset$ then $L(P,\emptyset) := \{(P',\emptyset) : P' = P(b_c^+ + k), k \in \mathbb{R}\}$,
- If $F \neq \emptyset$, it follows that $P = P(d_p^+)$ and $F = F(d_{p'}^-)$, where $p \in \partial_c^+ S \times \mathbb{R}$, $p' \in \partial_c^- S \times \mathbb{R}$ (see [20, Theorem 6.15]), and then

$$L(P,F) := \{(P',F') : P' = P(d_p^+ + k), F' = F(d_{p'}^- + k), k \in \mathbb{R}\}.$$

A dual definition is assumed for $(P,F) \in \partial M$ with $\emptyset \neq F = F(b_c^-)$.

Let us first describe the structure of the c-completion as a point set.

Theorem 5. *Let $(S \times \mathbb{R}, g)$ be a (conformally) standard stationary spacetime as in Eq. (3) and denote $M = S \times \mathbb{R}$. Then, the c-boundary ∂M has the following structure:*

(i) The future (resp. past) c-boundary $\hat{\partial} M$ (resp. $\check{\partial} M$) is naturally a point set cone with base $\partial_B^+ S$ (resp. $\partial_B^- S$) and apex i^+ (resp. i^-).

(ii) A pair $(P,F) \in \partial M$ with $P \neq \emptyset$ satisfies that $P = P(b_c^+)$ for some $c \in C^+(S)$ and:

  (a) If $b_c^+ \equiv \infty$ then $P = M, F = \emptyset$.

  (b) If $b_c^+ \in \mathscr{B}^+(S) (\equiv \partial_{\mathscr{B}}^+ S \times \mathbb{R})$, then $F = \emptyset$.

  (c) If $b_c^+ \in B^+(S) \setminus \mathscr{B}^+(S)$, then $b_c^+ = d_p^+$ with $p = (x^+, \Omega^+) \in \partial_C^+ S \times \mathbb{R}$, $P = P(d_p^+)$ and $F \subset F(d_p^-)$. In this case, there are two exclusive possibilities:

    (c1) Either $F = \emptyset$.

    (c2) Or $F = F(d_{p'}^-)$ with $p' = (x^-, \Omega^-) \in \partial_C^- S \times \mathbb{R}$ and satisfying

$$\Omega^- - \Omega^+ = d_Q(x^+, x^-)$$

    (in this case, p' is not necessarily unique).

  Moreover, if $x^+ \in \partial_C^s S$, then $p' = p$, $\uparrow P = F(d_p^-)$ and P is univocally S-related with $F = F(d_p^-)$.

A dual result holds for pairs (P,F) with $F \neq \emptyset$. So, the total c-boundary is the disjoint union of lines $L(P,F)$.

When ∂M is simple as a point set, it is the quotient set $\hat{\partial} M \cup_d \check{\partial} M / \sim_S$ of the partial boundaries $\hat{\partial} M, \check{\partial} M$ under the S-relation.

Proof. See [20, Theorem 1.2]. □

Let us finally describe the causal and topological structures. As to the completions of (S,F), let us remark that in the description of the c-completion, we only need Busemann and Cauchy completions, while Gromov completion is useful to define the Busemann one. Observe that Gromov completion is a compact metrizable topological space and the Busemann one is T_1, sequentially compact but not necessarily Hausdorff. This is because, in S_B^+, the topology inherited from the Gromov completion is finer than the chronological topology (otherwise, the Busemann completion would not be sequentially compact). As a matter of fact, Busemann and Gromov completions coincide both as a point set and as a topological space when S_B^+ is Hausdorff.

Theorem 6. *Let $(S\times\mathbb{R},g)$ be a (conformally) standard stationary spacetime as in Eq. (3) and denote $M=S\times\mathbb{R}$. Then, for each $(P,F)\in\partial M$, the line $L(P,F)$ is:*

(i) Time like if $P=P(d_p^+)$ and $F=F(d_p^-)$ for some $p\in\partial_C^s M\times\mathbb{R}$
(ii) Horismotic if either P or F are empty
(iii) Locally horismotic otherwise

(see [20, Definition 6.22]).

As to the topology of the c-completion:

(iv) If S_B^+ (resp. S_B^-) is Hausdorff, the future (resp. past) causal boundary has the structure of a (topological) cone with base $\partial_B^+ S$ (resp. $\partial_B^- S$) and apex i^+ (resp. i^-).
(v) If S_C^s is locally compact and d_Q^+ is a generalized distance, then $\overline{M}$ is simple, and so, it coincides with the quotient topological space $\hat{M}\cup_d\check{M}/\sim_S$ of the partial completions $\hat{M}$ and $\check{M}$ under the S-relation.

Summarizing, if S_C^s is locally compact, d_Q is a generalized distance, and $S_B^\pm$ is Hausdorff; ∂M coincides with the quotient topological space $(\hat{\partial}M\cup_d\check{\partial}M)/\sim_S$, where $\hat{\partial}M$ and $\check{\partial}M$ have the structure of cones with bases $\partial_B^+S,\partial_B^-S$ and apexes i^+,i^-, respectively.

Proof. See [20, Theorem 1.2]. □

See the contribution by Flores and Herrera [19] to these proceedings for a more detailed study.

6 Existence of Light-Like Geodesics

The study of multiplicity of light-like geodesics between an event and a vertical line was the original scope of the use of Fermat metrics. For example, in [39], the authors use the shortening method applied to the Fermat metric to give some existence results. It is remarkable that in [39], the authors refer to the Fermat metric as a pseudo-Finsler metric and they are concerned about the local existence, uniqueness and regularity of minimizers of the length functional (see [39, Appendix A.1]).

Of course, this is because, in that moment, they were not aware of the fact that Randers metrics are fiberwise strongly convex. By the way, it seems that this fact is not collected in the classical books of Finsler geometry available at the time that [39] was published. This was done just two years later in [3, Sect. 11.1].

Once you know that Randers metrics are fiberwise strongly convex, the local existence and uniqueness of geodesics are guaranteed. Moreover, studying light-like geodesics between an event and a vertical line or ligth-like geodesics spatially closed in a conformally standard stationary spacetime is equivalent to studying the existence and multiplicity of geodesics between two points or closed geodesics of the Fermat metric, respectively (up to the case with boundary see [11] and [9, Proposition 4.9] in these proceedings). This can be done by applying the theories of Lyusternik–Schnirelmann and Morse to the energy functional of a Finsler manifold (M,F). In fact, you can consider the space of curves of Sobolev class H^1 on M. Recall that this space does not depend on the Riemannian metric that you fix on M. Thus, we fix an auxiliary Riemannian metric h on M. Moreover, fix a smooth submanifold N of $M\times M$ and consider the collection $\Lambda_N(M)$ of the curves $x:[0,1]\to M$, having H^1 regularity, that is, x is absolutely continuous and $\int_0^1 h(\dot x,\dot x)\mathrm{d}s$ is finite, and with $(x(0),x(1))\in N\subseteq M\times M$. Then, it is well known that $\Lambda_N(M)$ is a Hilbert manifold modeled on any of the equivalent Hilbert spaces of all the H^1-sections with endpoints in TN of the pulled back bundle x^*TM, with x a regular curve in $\Lambda_N(M)$. Let us observe that even when the strong convexity condition is available, we must pay some attention to the fact that F^2 is not even C^2 on the zero section unless F^2 is quadratic, that is, a Riemannian metric (see [55]).

Proposition 6. *A nonconstant curve $\gamma\in\Lambda_N(M)$ is a geodesic for the Finsler manifold (M,F) satisfying*

$$g_{\dot\gamma(0)}(V,\dot\gamma(0))=g_{\dot\gamma(1)}(W,\dot\gamma(1)) \tag{14}$$

for any $(V,W)\in T_{(\gamma(0),\gamma(1))}N$ if and only if it is a (non-constant) critical point of the energy functional E_F on $\Lambda_N(M)$.

Proof. See, for example, [15, Proposition 2.1]. □

Moreover, recall that a functional J defined on a Banach manifold $(X,\|\cdot\|)$ satisfies the Palais–Smale condition if every sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\{J(x_n)\}_{n\in\mathbb{N}}$ is bounded and $\|dJ(x_n)\|\to 0$ contains a convergent subsequence. This condition is fundamental to apply the theories of Lyusternik–Schnirelmann and Morse, which study the relation between the number of critical points and the topology of the manifold. Palais–Smale is satisfied by the energy functional precisely when one of the equivalence conditions of the generalized Hopf–Rinow Theorem in 3 holds.

Theorem 7. *Let (M,F) be a Finsler manifold with $B_F^+(x,r)\cap B_F^-(x,r)$ relatively compact for every $x\in M$ and $r>0$, and N, a closed submanifold on $M\times M$ such that the first or the second projection of N to M is compact, then E_F is a $C^{1,1}$ functional on $\Lambda_N(M)$ and it satisfies the Palais–Smale condition on $\Lambda_N(M)$.*

Proof. See [15, Theorem 3.1] or [43] and the comments before [16, Theorem 5.2]. □

Again, the most difficult part to prove Palais–Smale for the energy functional of a Finsler metric is the lack of differentiability of F^2 in the zero section. As F^2 is not C^2 on the zero section, we can only apply the mean value theorem to the derivatives of F^2 away from the zero section. Once Palais–Smale condition is available, we can apply Lyusternik–Schnirelmann theory to obtain the existence of infinitely many geodesics between two arbitrary points when the manifold is non-contractible (see [15, Proposition 3.1]). With a different approach, it is possible to prove the existence of only a finite number of geodesics between two nonconjugate points in the presence of a convex function for the Finsler metric [13, Theorem 2.4].

6.1 Morse Theory for Light-Like Geodesics

As to the Morse theory for the energy functional in the space of H^1 curves, the main difficulty is that E_F is not twice differentiable at a curve γ, even if γ is a geodesic, unless the restriction of F^2 to the geodesic is a quadratic function with respect to the velocities (see [1] and also [10]). As a consequence, Morse Lemma cannot be proved in the curves of class H^1 with the standard techniques. Even if Morse Theory works for $C^{1,1}$-functionals in Hilbert manifolds (see, e.g., [42, Chap. 8]), the Morse Lemma is essential to compute the critical groups in terms of the index of the critical point. In [14], this problem is circumvented using that the space of curves with C^1 regularity is a Banach manifold densely immersed in the Hilbert space of H^1 curves and E_F, restricted to the C^1 class, admits second differential in regular curves of C^1. To be more precise, consider the second differential of E_F in the space of C^1-curves, assume for simplicity that the kernel is trivial, and extend it by density to H^1. This gives a functional that it is represented by the identity plus a compact operator in a certain scalar product [14, Lemma 2]. Moreover, the restriction of this operator to the space of C^1-curves gives an invertible operator [14, Lemma 5], and then one can obtain a Morse Lemma for this restriction and the scalar product of H^1 [14, Theorem 7]. Finally, we show that the critical groups of the C^1-class coincide with those of H^1 using a classical result by Palais. As the geometrical index of a light-like geodesic coincides with the one of its projection as a Fermat geodesic [14, Theorem 13], the Morse relations for light-like geodesics in conformally standard stationary spacetimes follow.

Theorem 8. *Let $(S\times\mathbb{R},g)$ be a globally hyperbolic conformally standard stationary spacetime with g as in Eq. (3), $p=(p_0,t_0)\in S\times\mathbb{R}$ and $L_{q_0}=\{(q_0,s)\in S\times\mathbb{R}: s\in\mathbb{R}\}$. Assume that for each $s\in\mathbb{R}$ the points p and (q_0,s) are non-conjugate along every future-pointing light-like geodesic connecting them. Then there exists a formal series $Q(r)$ with coefficients in $\mathbb{N}\cup\{+\infty\}$ such that*

$$\sum_{z\in G_{p,L_{q_0}}} r^{\mu(z)} = P(r,\Lambda_{(p_0,q_0)}(S)) + (1+r)Q(r),$$

where $G_{p,L_{q_0}}$ *is the set of all the future-pointing light-like geodesics connecting* p *to* L_{q_0}*,* $\mu(z)$ *is the number of conjugate points of* z *counted with multiplicity, and* $P(r,\Lambda_{(p_0,q_0)}(S))$ *is the Poincaré polynomial of* $\Lambda_{(p_0,q_0)}(S)$*.*

Proof. See [14, Theorem 15]. □

Recall that the Gromoll–Meyer theorem ensures the existence of infinitely many geometrically distinct closed geodesics whenever $\limsup_{k\to\infty}\beta_k(\Lambda(M)) = +\infty$, where $\beta_k(\Lambda(M))$ are the Betti numbers of the loop space of M. Using the same hypothesis, in [12], it is obtained the existence of an infinite number of geometrically distinct geodesics joining two nonconjugate points p and q. By geometrically distinct, we mean that they do not come from the iterations of a finite number of closed geodesics that go through p and q (as in the round sphere).

Let us observe that even if the problem of existence of normal geodesics between two arbitrary submanifolds in a standard stationary spacetime cannot be reduced to a problem for the Fermat metric in general, in [5], the authors use completeness of the Fermat metric to prove a result of this type with some hypotheses in the submanifolds [5, Theorem 1.1].

6.2 t-Periodic Light-Like Geodesics and the Closed Geodesic Problem

Let us recall that a light-like geodesic $\gamma = (x,t) : \mathbb{R} \to S\times\mathbb{R}$ in a standard stationary spacetime is said t-periodic if there exists $T \geq 0$ and $s_0 > 0$ such that x is periodic, that is, x and its derivatives coincide in 0 and s_0, $t(s_0) = t(0)+T$ and $\dot t(s_0) = \dot t(0)$. In this case, T is called the *universal period*. They are related with closed geodesics for the Fermat metric.

Proposition 7. *Let* (M,g) *be a conformally standard stationary spacetime as in Eq. (3). Then,* $\gamma = (x,t) : \mathbb{R}\to S\times\mathbb{R}$ *is a t-periodic light-like geodesic if and only if* $x : \mathbb{R}\to S$ *is a closed geodesic of the Fermat metric.*

Proof. The implication to the right follows from Proposition 1. For the other one, first observe that $g(\dot\gamma,\partial_t)$ is constant. To see this, recall that as ∂_t is a conformal field, it satisfies

$$g(\nabla_V\partial_t,W)+g(\nabla_W\partial_t,V) = \lambda g(V,W) \tag{15}$$

for every $V,W\in\mathfrak{X}(M)$, where ∇ is the Levi–Civita connection of (M,g) and λ a smooth function from M to $\mathbb{R}$. Then, using that γ is a light-like geodesic and Eq. (15),

$$\frac{d}{ds}g(\dot\gamma,\partial_t) = g(\dot\gamma,\nabla_{\dot\gamma}\partial_t) = \frac{1}{2}\lambda g(\dot\gamma,\dot\gamma) = 0.$$

Using again Proposition 1, we deduce that there exists $s_0 > 0$ such that $\gamma(s_0) = \gamma(0)$ and $\dot\gamma(s_0) = \mu\dot\gamma(0)$ for some $\mu > 0$, but the fact that $g(\dot\gamma, \partial_t)$ is a non null constant (it cannot be zero because $\dot\gamma$ is light-like and ∂_t time-like) implies that $\mu = 1$ as required. □

As to the closed geodesic problem for compact manifolds, most of the classical results for Riemannian metrics, such as Gromoll–Meyer and Bangert–Hingston theorems, which assume, respectively, that $\limsup_{k\to\infty} \beta_k(\Lambda(M)) = +\infty$ and that the fundamental group of the manifold is infinite abelian, are available in the Finslerian setting under the same topological hypothesis, obtaining the corresponding results of multiplicity for t-periodic light-like geodesics in conformally standard stationary spacetimes (see [8] and references therein). As an exception, there are Finsler metrics with a finite number of geometrically distinct closed geodesics, the so-called Katok metrics. Remarkably, these Finsler metrics are of Randers type, and they have constant flag curvature. Let us observe that the classification of Randers metrics of constant flag curvature has been obtained in [4] using the expression of a Randers metric as a Zermelo one, that is, a metric defined from a Riemannian metric g and a vector field W in a manifold M as

$$Z(v) = \sqrt{\frac{1}{\lambda}g(v,v) + \frac{1}{\lambda^2}g(v,W)^2} - \frac{1}{\lambda}g(v,W),$$

where $\lambda = 1 - g(W,W)$ must be positive. Indeed, (M,Z) has constant flag curvature if and only if W is a homothety and g has constant curvature [4]. We can then construct standard stationary spacetimes with compact orbit manifold S and a finite number of geometrically distinct t-periodic light rays (see [8, Propositions 3.1 and 3.4]).

6.3 *Alternative Functional to Energy*

Existence and multiplicity of Fermat geodesics can be studied by means of other functionals rather than the energy one. In [22], the authors use the functional defined as

$$J(x) = \sqrt{\int_0^1 h(\dot x, \dot x)\mathrm{d}s} + \int_0^1 \omega(\dot x)\mathrm{d}s$$

for every curve $x : [0,1] \to S$ of class H^1 with h as in Eq. (7). The advantage of this functional is that it is C^2 on geodesics. Its critical points are Fermat geodesics parameterized with h-constant speed (see also [38]). This functional has also been used in [23, 24] to obtain a result of genericity of stationary spacetimes without conjugate light-like geodesics between a fix event p and a fixed vertical line.

7 Further Applications

7.1 *Randers Metrics of Constant Flag Curvature and Stationary SpaceTimes*

Flag curvature plays a similar role in Finsler geometry as sectional curvature in the Riemannian setting, that is, it is an important invariant related to the behaviour of geodesics. Let us recall that Randers metrics with constant flag curvature have been classified in [4]. These metrics have already appeared in the context of Fermat metrics to provide examples of spacetimes with a finite number of geometrically distinct t-periodic light-like geodesics [8, Propositions 3.1 and 3.4]. Subsequently, these spacetimes were studied in [26].

Proposition 8. *Let* $(S\times\mathbb{R},g)$ *be a conformally standard stationary spacetime as in Eq. (3) whose Fermat metric is of constant flag curvature. Then* $(S\times\mathbb{R},g)$ *is locally conformally flat.*

Proof. See [26, Sect. II.E.2]. ☐

The converse of the last proposition is not true in general, because, for example, you can find a Randers metric of the form $\sqrt{g_0}+df$ with g_0 the Euclidean metric in $\mathbb{R}^n$ and $f:\mathbb{R}^n\to\mathbb{R}$ a smooth function, such that $\sqrt{g_0}+df$ does not have constant flag curvature (see [3, Sect. 3.9B]), and $\sqrt{g_0}+df$ is the Fermat metric associated to a certain splitting of Minkowski spacetime (see Proposition 2). Anyway, it is expectable, as commented in [26, Sect. II.E.2], that given a conformally flat stationary spacetime, you can find a space-like section having as a Fermat metric a Randers metric with constant flag curvature. Let us point out that in [26], the authors give several examples of Randers metrics coming from well-known stationary spacetimes. They also recall the relation between magnetic Lagrangians and Randers metrics.

7.2 *Time-Like Geodesics with Fixed Arrival Proper Time*

First of all, let us recall that the proper time of a time-like curve $\alpha:[a,b]\to M$ in a Lorentzian manifold (M,g) is defined as $\int_a^b\sqrt{-g(\dot\gamma,\dot\gamma)}\mathrm{d}s$. Let us also remark that existence of time-like geodesics with fixed arrival proper time between an event and a vertical line in a standard stationary spacetime can be reduced to existence of light-like geodesics in a one-dimensional higher standard stationary spacetime. Observe that, in this case, as time-like geodesics are not preserved by conformal changes, we cannot consider conformally standard stationary spacetimes as in Eq. (3), but standard stationary spacetimes $(S\times\mathbb{R},g)$ such that

$$g((v,\tau),(v,\tau))=g_0(v,v)+2\omega(v)\tau-\beta\tau^2,$$

in $(x,t) \in S \times \mathbb{R}$ for any $(v,\tau) \in T_xS \times \mathbb{R}$, where ω and g_0 are respectively a one-form and a Riemannian metric on S and β is a positive function on S. Then we can define the one-dimensional higher spacetime $(S \times \mathbb{R}^2, \eta)$, with η defined as

$$\eta((v,y,\tau),(v,y,\tau)) = g_0(v,v) + y^2 + 2\omega(v)\tau - \beta\tau^2,$$

in $(x,v,t) \in S \times \mathbb{R}^2$, where $(v,y,\tau) \in T_xS \times \mathbb{R}^2$. A curve from the event $(x_0,t_0) \in S \times \mathbb{R}$ to the line $L_{x_1} = \{(x_1,s) \in S \times \mathbb{R} : s \in \mathbb{R}\}$ is a time-like geodesic $\gamma = (x,t) : [0,1] \to S \times \mathbb{R}$ of $(S \times \mathbb{R}, g)$ with arrival proper time T if and only if $[0,1] \ni s \to (x(s), s, t(s)) \in S \times \mathbb{R}^2$ is a light-like geodesic of $(S \times \mathbb{R}^2, \eta)$ from the event $(x_0, 0, t_0)$ to the line $\{(x_1, T, s) \in S \times \mathbb{R}^2 : s \in \mathbb{R}\}$. Moreover, the Fermat metric of this standard stationary spacetime is given as

$$\tilde{F}(v,y) = \sqrt{\frac{1}{\beta} g_0(v,v) + \frac{v^2}{\beta} + \frac{1}{\beta^2}\omega(v)^2} + \frac{1}{\beta}\omega(v),$$

in $(x,v) \in S \times \mathbb{R}$, where $(v,y) \in T_xS \times \mathbb{R}$. As completeness conditions for the original Fermat metric in Eq. (6), which in this case is expressed as

$$F(v) = \sqrt{\frac{1}{\beta} g_0(v,v) + \frac{1}{\beta^2}\omega(v)^2} + \frac{1}{\beta}\omega(v),$$

imply completeness conditions for $\tilde{F}$ (see the proof of [15, Proposition 4.2]), some multiplicity results [15, Proposition 4.2] and Morse relations [14, Theorem 18] are available when the spacetime is globally hyperbolic. Observe that in [11], the existence of such time-like geodesics under sharp conditions (weaker than global hyperbolicity) is obtained.

7.3 Conformal Maps and Almost Isometries

Another interesting relation between Fermat metrics and conformally standard stationary spacetimes occurs at the level of transformations (see [28]). As Fermat metrics remain invariant by conformal changes in the conformally stationary spacetime, we need to consider conformal maps in the spacetime. Moreover, as we want to project these maps into maps of the orbit manifold S, they have to preserve the conformal vector field K. Summing up, they have to be K-conformal maps, denoted by $\mathrm{Conf}_K(S \times \mathbb{R}, g)$, that is, they must preserve the metric up to a positive constant in every point and the conformal vector field K. As to general relativity, these maps are precisely those that preserve the causal structure and the observers along K. Their counterpart in Fermat metrics are the so-called *almost isometries*, which are maps $\varphi : S \to S$ such that $\varphi^*(F) = F + df$ for a certain smooth function

$f : S \to \mathbb{R}$ (here $*$ denotes the pullback operator). Let us denote by $\widetilde{\mathrm{Iso}}(S,F)$ the group of almost isometries of (S,F), which is a Lie group [28].

Theorem 9. *Let* $\psi : S \times \mathbb{R} \to S \times \mathbb{R}$ *be a* K*-conformal map of a conformally standard stationary spacetime as in Eq. (3). Then, there exist functions* $\varphi : S \to S$ *and* $f : S \to \mathbb{R}$ *such that* $\psi(x,t) = \big(\varphi(x), t + f(x)\big)$ *and* φ *is an almost isometry for the Fermat metric of* $(S \times \mathbb{R}, g)$*. Moreover,* φ *is a Riemannian isometry for the metric* h *in Eq. (7), and the map* $\pi : \mathrm{Conf}_K(S \times \mathbb{R}, g) \to \widetilde{\mathrm{Iso}}(S,F)$*, defined as* $\pi(\psi) = \varphi$*, is a Lie group homomorphism. The map can be projected to the quotient*

$$\bar{\pi} : \mathrm{Conf}_K(S \times \mathbb{R}, g)/\mathscr{K} \to \widetilde{\mathrm{Iso}}(S,F),$$

(where $\mathscr{K}$ *is the subgroup generated by the flow of* K*) and gives an isomorphism of Lie groups.*

Proof. See [28]. □

As a consequence of this relation, it follows a result of genericity of stationary spacetimes with discrete K-conformal group.

Corollary 1. *Given a manifold* S*, for a* generic *set of data* (g_0, ω)*, the stationary metric* $g = g(g_0, \omega)$ *given in Eq. (4) on* $S \times \mathbb{R}$ *has discrete* K*-conformal group* $\mathrm{Conf}_K(S \times \mathbb{R}, g)$.

Proof. See [28]. □

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