The Geometry of Collapsing Isotropic Fluids

Roberto Giambò and Giulio Magli

1 Introduction

Spherical symmetry is a useful test-bed for open problems of astrophysical interest in General Relativity. Among them, a very relevant one is that of the final state of gravitational collapse and, therefore, of the validity of a "Cosmic Censorship" hypothesis. In particular, the study of spherically symmetric spacetimes modeling a collapsing isotropic fluid is a recurrent topic in relativistic literature. What makes it one of the most intriguing problem in gravitational collapse is that perfect fluids are a direct, physically interesting generalization of the so-called *Tolman–Bondi–Lemaitre* (TBL) solution, which is one of the few known-in-detail solutions dynamically collapsing to a singularity. The TBL solution is indeed long known to have naked singularities (the first example discovered in [2], and the complete analysis is due to [10]), while the case of isotropic fluids remains almost open. Some results are actually known from numerical relativity, in particular for barotropic perfect fluids with linear equation of state ($p = \mu\varepsilon$, $\mu \in \mathbb{R}$, see Sect. 4.1 below): for instance, Ori and Piran [13] studied the problem under the assumption of self-similarity, while Harada [7] investigated the same problem, detecting *globally* naked singularities in some cases. Choptuik, whose numerical analysis study on the gravitational collapse of a scalar field [1] remains one of the cornerstones about the cosmic censor problem, worked with Neilsen [11] to the limit case $\mu \to 1^-$, which was also the aim of Snajdr [16]. Unfortunately, outside the realm

R. Giambò (✉)
School of Science and Technology, Mathematics Division, University of Camerino,
Via Madonna delle Carceri, I-62032 Camerino, Italy
e-mail: roberto.giambo@unicam.it

G. Magli
Department of Mathematics, Polytechnic of Milan, Pzza Leonardo da Vinci 32,
I-20133 Milan, Italy
e-mail: giulio.magli@polimi.it

M. Sánchez et al. (eds.), *Recent Trends in Lorentzian Geometry*, Springer Proceedings in Mathematics & Statistics 26, DOI 10.1007/978-1-4614-4897-6_8,

of numerical relativity, little is known about the geometry of these spacetimes: whether a singularity is developed, and if that is the case, what is the causal structure of the solution. There are many studies which do not rely on numerical techniques but either they refer to general situations without specifying the matter properties (see, e.g., [4]) or they deal with anisotropic spacetimes (see, e.g., [5] and references therein).

Some results will be sketched here which shed new light on this problem. In particular, we report here on a quite general analysis of this problem which can be carried out for general equations of state provided that certain regularity assumptions are satisfied. These assumptions essentially require Taylor-expandability of the solution in a special system of coordinates and allow for a quite general picture of barotropic perfect fluids (with pressure proportional to energy density) as well as for some other cases of interest. The problem of how recovering the final state of dust (TBL) collapse from perfect fluids remains, however, open although some hints can be derived. Indeed, it appears that the barotropic solutions found do not converge to TBL solution as the ratio pressure over density goes to zero. Moreover, the qualitative picture emerging from these models is quite different from the TBL case. On the other side, for models where the equation of state is perturbed in a nonlinear way, a qualitative behavior of the singularity similar to the background model is recovered. All these facts likely represent an evidence of the crucial role of pressure in the neighborhood of the singular boundary to determine the causal structure of the spacetime. These results seem to confirm the analysis carried out in [9], where homogeneous dust collapse (Oppenheimer–Snyder model, see Example 31 in Sect. 3) is perturbed adding a small amount of pressure.

The chapter is organized as follows: we review general spherical models in Sect. 2, specializing to the isotropic case in Sect. 3. Qualitative results are presented and discussed in Sect. 4.

2 Relativistic Stars in Spherical Symmetry

Let us consider a generic, non-static, spherically symmetric, 4–dimensional spacetime. Using a comoving coordinate system (t, r, θ, ϕ), the metric is written in the form

$$g = -e^{2\nu}\,\mathrm{d}t^2 + e^{2\lambda}\,\mathrm{d}r^2 + R^2\,\mathrm{d}\Omega^2, \tag{1}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ is the first fundamental form of $\mathbb{S}^2 \subset \mathbb{R}^3$, and ν, λ and R are functions of (t, r) only.

In the following, we will be interested in those matter models admitting a well-defined description in terms of the standard relativistic mechanics of continua. This implies that g must satisfy Einstein field equations (hereafter, Greek indices like μ, ν run from 0 to 3)

$$G^{\mu}_{\nu} = 8\pi T^{\mu}_{\nu} \tag{2}$$

where G is the Einstein tensor of g ($G = Ric - \frac{1}{2}Sg$, S being the scalar curvature) and T is the stress–energy tensor embodying the matter properties, that in the comoving system takes the form

$$8\pi T = -\varepsilon\, \mathrm{d}t \otimes \frac{\partial}{\partial t} + p_r\, \mathrm{d}r \otimes \frac{\partial}{\partial r} + p_t \left(\mathrm{d}\theta \otimes \frac{\partial}{\partial \theta} + \mathrm{d}\phi \otimes \frac{\partial}{\partial \phi} \right) \tag{3}$$

where ε, p_r and p_t are the *energy density*, the *radial* and the *tangential pressure*, respectively, and again are all functions of (t,r) only.

Using Eqs. (1) and (3), and introducing *Misner–Sharp mass* function

$$m = \frac{R}{2}(1 - g(\nabla R, \nabla R)) = \frac{R}{2}\left(1 - (R'e^{-\lambda})^2 + (\dot{R}e^{-\nu})^2\right) \tag{4}$$

(where a prime and a dot denote differentiation with respect to r and t, respectively, and ∇ is the gradient operator indicted by g) a complete set for Einstein field equation (2) is given by

$$m' = 4\pi\varepsilon R^2 R', \tag{5}$$

$$\dot{m} = -4\pi p_r R^2 \dot{R}, \tag{6}$$

$$\dot{R}' = \dot{\lambda} R' + \nu' \dot{R}, \tag{7}$$

$$p_r' = -(\varepsilon + p_r)\nu' - \tfrac{2R'}{R}(p_r - p_t). \tag{8}$$

In particular, Eq. (7) is equivalent to Eq. (2) for $(\mu,\nu) = (0,1)$, and using it in Eq. (2) for $(\mu,\nu) = (0,0)$ and $(\mu,\nu) = (1,1)$ we get Eqs. (5) and (6) respectively. Finally, Bianchi identity div $T = 0$ implies relation (8).

We are interested in spacetimes modeling collapsing (spherical) objects, with the aim to investigate the behavior especially near the center of symmetry of the system ($r = 0$). We stress that the "central shell" $r = 0$ is regular at initial time of observation and may possibly develop a singularity after some amount of comoving time. In view of that we will consider r as defined in a right neighborhood of $r = 0$, say $[0, r_b]$, and we will need

1. To assume some geometrical and physical reasonableness hypotheses on the metric.
2. To smoothly match the solution with an external spacetime.

Let us briefly review these assumptions.

2.1 Geometrical and Physical Assumptions

- We first demand that the internal solution satisfies the *dominant energy condition (dec)*, which means that

$$\varepsilon \geq 0, \quad |p_r| \leq \varepsilon, \quad |p_t| \leq \varepsilon, \tag{9}$$

  throughout the evolution.

- We require that there exists an initial time (say, $t = 0$) such that the solution is regular: in particular, the energy density measured on the internal solution must be finite and outward decreasing:

$$\lim_{r\to 0^+} \varepsilon(0,r) \in \mathbb{R}, \quad \frac{\mathrm{d}}{\mathrm{d}r}\varepsilon(0,r) \le 0 \,\text{on}\, [0,r_b]. \tag{10}$$

Without loss of generality, on the initial time, the following initial condition can be imposed

$$R(0,r) = r. \tag{11}$$

- We will impose some local cartesianity conditions on the metric, to prevent bad behavior on the center of symmetry due to polar coordinate choice and to impose isotropy at the center of symmetry. This is equivalent to state

$$R(t,0) = 0, \quad R'(t,0) = e^{\lambda}(t,0), \quad p_r(t,0) = p_t(t,0) \tag{12}$$

for all $t \ge 0$ up to (possibly) singularity formation.
- We also ask for the solution to be free from *shell–crossing singularities*, that are caused by the vanishing of $R'(t,r)$. Shell-crossing singularities usually correspond to Tipler-weak divergences of the curvature, and for this reason are considered as "less important" although extendibility proofs of a spacetime beyond a shell-crossing singularity is available in literature only in some particular cases (see [12] and reference therein as [14]). We will require that no shell-crossing singularity appears prior to (possible) singularity formation due to the vanishing of $R(t,r)$. The latter are usually called *shell–focusing* singularities and are those we will be interested in. Then we assume that

$$R(t,r) > 0 \Rightarrow R'(t,r) > 0. \tag{13}$$

- Finally, in order to obtain a global model, a matching with an external space will be performed at $\Sigma = \{r = r_b\}$, requiring that the first and the second fundamental forms of the two metrics at Σ coincide (Israel–Darmois junction conditions). From Eq. (6) we observe that the radial pressure p_r in general does not vanish along Σ, which as well known is a necessary and sufficient condition to match the solution with a Schwarzschild exterior. In this more general case, a natural choice for the exterior metric is the generalized Vaidya spacetime written in radiative coordinates (V is a null coordinate)

$$g_{\text{ext}} = -\left(1 - \frac{2\mu(V,S)}{S}\right) \mathrm{d}V^2 + 2\,\mathrm{d}V\,\mathrm{d}S + S^2 \mathrm{d}\Omega^2, \tag{14}$$

where $\mu(V,S)$ is an arbitrary (non negative) C^2 function. The immersion of Σ in the two spacetimes can be parameterized respectively by $(\tau,\theta,\phi) \hookrightarrow (\tau,r_b,\theta,\phi)$ and $(\tau,\theta,\phi) \hookrightarrow (V(\tau),S(\tau),\theta,\phi)$, and junction conditions are found to be

$$S(\tau) = R(\tau, r_b), \quad \dot{V}(\tau) = \frac{e^{\nu}}{R' e^{-\lambda} + \dot{R} e^{-\nu}}|_{(\tau, r_b)} \tag{15}$$

$$\mu(V(\tau), S(\tau)) = m(\tau, r_b), \qquad \frac{\partial \mu}{\partial V}(V(\tau), S(\tau)) = 0. \tag{16}$$

Equation (15) can be solved for $S(\tau)$ and $V(\tau)$ to give the parameterization of Σ into the outer spacetime, once that the inner spacetime is known on Σ. Instead, Eq. (16) impose a constraint on the mass function μ on Σ – for example, if one requires the outer solution to be a Schwarzschild exterior and then μ constant, then $\dot{m}(\tau, r_b) = 0$ which is equivalent, in view of Eq. (6), that $p_r = 0$ on Σ as stated before.

2.2 Singularity Formation and Cosmic Censor

In addition to the assumptions above reviewed, we require the interior spacetime to model a collapsing spherical object, which implies that we will be interested in those solutions such that $\dot{R}(t, r) \leq 0$ during the evolution. The collapse may either produce an asymptotically regular solution existing for all times $t \geq 0$ or a (shell-focusing) singularity, due to the vanishing of $R(t, r)$. Actually, recalling Eq. (12), singularities will be detected by the relation

$$a(t, r) := \frac{R(t, r)}{r} = 0, \tag{17}$$

thus defining a singularity curve $t_s(r)$ such that $a(t_s(r), r) = 0$. In view of Eq. (10), $t_s(t)$ (if it exists) is strictly positive $\forall r \in [0, r_b]$. Let us consider the case when $t_s(r)$ is well defined $\forall r \in [0, r_b]$. To get information about the causal structure of the spacetime, we will perform a study of radial null geodesics which can be extended (in the comoving past) up to the singular curve $t_s(r)$. For this aim a crucial role is played by the *apparent horizon*, implicitly defined by the relation $R = 2m$, which is the boundary of the *trapped region* $R < 2m$.

We will review in the final section some particular cases occurring during the collapse of isotropic models. For the moment, as an example, we sketch a situation happening, for instance, in the collapse of a spherical dust cloud (i.e., $p_r = p_t = 0$) as well as for a more general class of spacetimes [5]. It can be seen that the relation $R = 2m$ implicitly defines a curve $t_h(r)$ such that $t_h(0) = t_s(0)$ and $t_h(r) < t_s(r)$, $\forall r > 0$. Then all geodesics that can be extended in the past up to the points $t_s(r)$ for r strictly positive, are entirely confined inside the trapped region. On the other side, there is the possibility for a radial null geodesic – an infinite number of geodesics, actually – with support *outside* the trapped region, to be extended up to $t_s(0)$. This feature can be interpreted as a violation of the so-called cosmic censorship conjecture, originally stated by Roger Penrose [15].

It is possible to rephrase the existence problem for such geodesics as an existence problem for the following ODE:

$$\frac{\mathrm{d}t}{\mathrm{d}r} = e^{(\lambda-\nu)(t,r)}, \quad t(0) = t_s(0), \tag{18}$$

but this in principle is not a well-defined Cauchy problem because of the lack of regularity on the initial condition. Therefore, a remarkable property of the apparent horizon may be of help – indeed, $t_h(r)$ is a subsolution of the ODE (18). This fact implies that if there exists a supersolution of Eq. (18), say $t_*(r)$, such that $t_*(r) \leq t_h(r)$ (and $=$ iff $r = 0$), then usual comparison results in ODE theory gives the existence of a family of radial null geodesics emitted from $t_s(0)$ outside the trapped region. Existence problems to Eq. (18) are then linked to solutions to differential inequalities which in principle are easier to check.

Remark 21. One can in principle think at this stage that the existence problem for *nonradial* null geodesics is still left open. However, it can be proved – using again comparison arguments in ODE theory – that if no radial geodesics exist, then nonradial geodesics too cannot be emitted from the singularity.

3 Regular Isotropic Models

It is well known that the system of Einstein field equations (5)–(8) is underdetermined, and then more relations are needed in order to "close" the system. These further relations are usually provided by an equation of state, expressing one (or more) conditions on the stress–energy tensor components. To begin, we will partially fix one of these conditions, specializing to the case of an isotropic fluid, i.e.,

$$p_r(t,r) = p_t(t,r). \tag{19}$$

Moreover, we are going to perform this study in a new coordinate setting. Indeed, we will consider coordinates (a, r, θ, ϕ) where a is given by Eq. (17). In view of Eq. (11) the internal solution will be studied on the set $[0,1] \times [0, r_b] \times \mathbb{S}^2$, and the singularity corresponds to the boundary $a = 0$. It is convenient to introduce the functions $M(a,r) = \frac{2m}{r^3}$,

$$\gamma(a,r) = \frac{p}{\varepsilon}, \tag{20}$$

and $Y(a,r) = R' e^{-\lambda}$. Moreover, we also make the positions

$$w(a,r) = a', \quad z(a,r) = \dot{a}, \tag{21}$$

that will be used to link the new coordinate system to the old one.

Example 31. In this chapter we wish to give some insight into gravitational collapse of isotropic pressure spherical models under some regularity assumption on the solution. As is well known, and as said before, the only inhomogeneous solution with isotropic pressure which is known in full generality is the TBL dust cloud, the one obtained when the pressure vanishes ($p_r = p_t = 0$). The so-called *marginally bound* case of this solution is obtained when $Y \equiv 1$ and is described by $R(t,r) = r(1-k(r)t)^{2/3}$, with $k(r) = 1 - \alpha r^n + o(r^n)$ (and $\alpha > 0$) depending on the mass profile of the solution. The special case $k(r) \equiv$ constant correspond to the homogeneous (in the sense that the singularity curve $t_s(r)$ is constant) Oppenheimer–Snyder model. In the coordinate system (a,r) the solution takes the form

$$w = -\frac{2k'(r)}{3k(r)}\left(\frac{1}{\sqrt{a}} - a\right), \quad z = -\frac{2k(r)}{3\sqrt{a}} \tag{22}$$

$$e^{\lambda} = R' = a + rw = a\left(1 - \frac{2rk'(r)}{3k(r)}\left(\frac{1}{a\sqrt{a}} - 1\right)\right), \quad e^{\nu} = 1. \tag{23}$$

We can see that this solution can be developed in power series with respect to r around $r = 0$, and this will be basically the assumption we will make on the general solution.

Before going on and write the system, we observe from Eq. (23) that the power series in r of λ badly behaves with respect to a, since every coefficient – that is actually a function of a – contains increasing power, diverging terms in a. This suggests the choice of e^{λ} as an unknown, rather than λ itself. For sake of uniformity the same choice will be made for ν. Hereafter, therefore, we introduce two new unknown functions

$$B(a,r) = e^{\lambda}, \quad F(a,r) = e^{\nu}, \tag{24}$$

in place of old variables λ and ν.

With the above positions, Einstein field equations in (a,r) coordinates become (subscript denotes partial derivative)

$$3M + M_r r + wrM_a - \varepsilon a^2(wr + a) = 0, \tag{25}$$

$$M_a + \gamma\varepsilon a^2 = 0, \tag{26}$$

$$(\gamma+1)\varepsilon(wr+a)Y_a + Y\left[(\gamma\varepsilon)_r + w(\gamma\varepsilon)_a\right]r = 0, \tag{27}$$

$$r(F_r + wF_a)Y - (wr+a)Y_aF = 0, \tag{28}$$

Moreover, since $wr + a = R' = YB$, and using Eq. (4), we have the following two functional dependencies:

$$z = -F\left(\frac{M}{a} + \frac{Y^2-1}{r^2}\right)^{1/2},$$

$$w = \frac{YB - a}{r}. \tag{29}$$

Finally, an equation is needed to express a compatibility property between w and z introduced in Eq. (21), i.e., $\dot{w} = z'$, that in the (a,r) coordinate system reads as

$$z_r + wz_a - zw_a = 0. \tag{30}$$

As outlined above we now make our last assumption: that B, F, Y, ε and M are solutions of Eqs. (25), (27), (28), and (30) that are C^n with respect to r in $[0, r_b]$, $\forall a \in]0,1]$. Then

$$M(a,r) = \sum_{i=0}^{n} M_i(a) r^i + o(r^n),$$

for n is sufficiently large, and analogously for A, B, F, and ε. The $o(r)$ above and hereafter must be intended as a function of both (a,r), of course.

We stress that the remaining unknown functions γ, p, w, and z can be expressed using the functional dependencies (20), (26), (29) and (29). With the above ansatz, the equations can be expanded in order to recover relations for the Taylor coefficients of the unknown functions. The higher degree in regularity – with respect to r – the more accurate information we can get, and an iterative scheme can be found out that determines the whole solution up to $M_0(a)$, $Y_2(a)$ and $B_i(a)$, $i \geq 1$. In particular, for example, since $\varepsilon_0(a) = \frac{3M_0(a)}{a^3}$ then from Eq. (26), we see that the choice of $M_0(a)$ determines the leading power of γ:

$$\gamma_0(a) = -\frac{aM_0'(a)}{3M_0(a)}.$$

A further specialization – in addition to Eq. (19) – can be used to prescribe each of the remaining functions up to a constant, that corresponds to the freedom to choose the initial data for the evolution.

The information about leading order terms of the unknown functions can be used to check compatibility with the assumptions stated in Sect. 2.1 in a right neighborhood of $r = 0$, and to study the behavior of the apparent horizon and the radial null geodesic possibly emitted by the singularity $a = 0$, using methods already sketched in Sect. 2.2 opportunely adapted to this new coordinate setting.

4 Qualitative Results and Discussion

In this section, we review the qualitative results that can be inferred from the form of the coefficients of the solution for different choices of the free functions corresponding to significant isotropic fluid models.

4.1 Barotropic Fluids with a Linear Equation of State

This models corresponds to the case $\gamma = \frac{p}{\varepsilon} = \mu \in \mathbb{R}$. We restrict $\mu \in [-1,1]$ to satisfy the dec (9).

This equation of state fixes all free functions except $Y_2(a)$ – for this reason, we choose to test these models with the family $Y_2(a) = y_2 a^{\alpha}$, with y_2, α real parameters. The structure of the singularity for these fluids is qualitatively much different from the TBL case, and the main reason for that is that the apparent horizon drastically changes behavior depending on the values of α and μ [8]. In some case we have a horizon "of a first type" which has a behavior similar to the TBL models leading to a black hole, and indeed here, the singularity is completely hidden inside the trapped region and is invisible to faraway observers. But there are also situations "of a second type" emerging for these fluids – occurring for negative pressure only, unless one does not violate some of the conditions expressed in Sect. 2.1 as no shell-crossing singularity formation (13) – where the horizon lets the central singularity be naked; this kind of horizon appearing here does *not* have a corresponding limit $\mu \to 0$, as the previous one.

These features make the singular boundary for these fluids qualitatively different from the geometry of dust collapse. Although central naked singularities are potentially occurring in both cases, perturbing the horizon of a TBL solution – which there gives rise to either a naked singularity or to a black hole, depending on the data of the problem – produces here a horizon of the first type, which always leads to a black hole. The horizon of the second type, possibly leading to a central naked singularity, is a distinctive feature of the $\mu \neq 0$ models.

This bifurcating behavior may be explained by observing that when $\mu \neq 0$, some of the functions determined by the procedure discussed in the previous section are not well defined in the limit $\mu \to 0$. The naked singularity formation in dust collapse can thus be seen as an unstable phenomenon with respect to (regular) linear perturbation of the equation of state and in addition, under the regularity hypotheses made in this chapter, these fluids cannot be seen at all as a proper perturbation of the case $\mu = 0$.

Finally, to make the whole picture even more complex, when $\mu \leq -\frac{1}{3}$, there are choices – always generic, except the case $\mu = \frac{1}{3}$ – of the free functions that lead to absence of horizon. This means that the whole singularity curve – not only the central ones, then – can be globally naked, which is consistent with the analysis by Cooperstock et al. [3]. Let us remark that, if we imposed the strong energy condition – that in case of isotropic fluids correspond to assume $\varepsilon + 3p \geq 0$ and $\varepsilon + p \geq 0$ – instead of the dominant, this case would have been excluded, except the special case $\mu = -\frac{1}{3}$ that, as said before, produces a globally naked singularity for a nongeneric choice of the free functions.

4.2 Generalized Chaplygin Gas

The above discussion of the γ-constant case suggests that the role of pressure is crucial in the singularity trapping process. Then one can think about models with a softer equation of state, and see whether naked singularity appears as it did in dust case. An example can be provided by a generalization of Chaplygin fluids, where the equation of state is given by

$$p = \mu \varepsilon^{-\alpha}, \quad \alpha \geq 0, \mu \in \mathbb{R}. \tag{31}$$

In these models, when the collapse ends into a singularity, the divergence of the energy density provides an upper bound for the (absolute value of the) pressure p. For the sake of simplicity here we report on cases $\alpha = 0$ and $\alpha = 1$ only, stressing the fact that for this class of models, the dust solution is recovered in the limit $\mu \to 0$.

In the first case, corresponding to nonzero (yet bounded) pressures in the approach to the singularity, it is seen that the horizon forms and completely covers central and noncentral singularities. Then, getting bounded pressures seems not enough to retain naked singularities, and indeed the situation dramatically changes when $\alpha = 1$. Here, the horizon still covers noncentral singularities, but the central singularity can be naked in a generic way, regardless of the pressure sign.

It clearly appears from the above reviewed cases that the pressure highly influences the qualitative behavior of the singularity and therefore, the causal structure of the collapsing model. In the linear case $p = \mu \varepsilon$ the pressure diverges with the energy density in the approach to the singularity, which results to be hidden by the apparent horizon – when the latter forms. These models also contain some interesting cases where the horizon does not even form and then the singular boundary is globally naked.

This intricate picture becomes simpler when the equation of state is perturbed in such a way that the pressure goes to zero as the energy diverges – here, these models *are* proper dust perturbations, since TBL solutions are recovered in the limit $\mu \to 0$, and in fact a central naked singularity may take place. The case when pressure remain bounded but also bounded away from zero near the singular boundary yet presents some uncertainty – preliminary studies applied to the parametric equation of state $p = e^{-1/\rho}$, $\varepsilon = \rho\, e^{-1/\rho}$ show that naked singularity may appear in the center of symmetry of the system as the energy diverges ($\rho \to +\infty$), unlike the $\alpha = 0$ Chaplygin model.

The boundedness of pressure near the singular boundary then seems a key ingredient to produce counterexamples to cosmic censorship. As is well known (see, e.g., [6] and references therein), many examples are known in literature of anisotropic models with naked singularities, where both tangential and radial pressures diverge – although in different manners – and then isotropy places a further constraint to naked singularities. But this constraint is far from simplifying the geometry of the spacetime, at least when the pressure diverges near the singularity – as seen for the linear equation of state. Of course, to get a complete picture one should be able to prove convergence theorems for the series of the unknown functions of the system, which is basically related to a global existence–uniqueness theorem for the Einstein field equations, and this would in principle cut out some of the examples above reviewed.

References

1. Choptuik, M.W.: Phys. Rev. Lett. **70**, 9 (1993)
2. Christodoulou, D.: Commun. Math. Phys. **93**, 171 (1984)
3. Cooperstock, F.I., Jhingan, S., Joshi, P.S., Singh, T.P.: Class. Quant. Grav. **14**, 2195 (1997)
4. Fayos, F., Torres, R.: Class. Quantum Grav. **28**, 215023 (2011)
5. Giambò, R., Giannoni, F., Magli, G., Piccione, P.: Commun. Math. Phys. **235**(3), 545 (2003)
6. Giambò, R., Giannoni, F., Magli, G., Piccione, P.: Class. Quantum Grav. **20**, L75 (2003)
7. Harada, T.: Phys. Rev. D **58**, 104015 (1998)
8. Iguchi, H., Harada T., Mena, F.C.: Class. Quantum Grav. **22**, 841 (2005)
9. Joshi, P.S., Malafarina, D., Saraykar, R.V.: *Int. J. Mod. Phys. D*, arxiv:1107.3749 (2011)
10. Joshi, P.S., Singh, T.P.: Class. Quant. Grav. **13**, 559 (1996)
11. Neilsen D.W., Choptuik, M.W.: Class. Quantum Grav. **17**, 761 (2000)
12. Nolan, B.C.: Class. Quantum Grav. **20**, 575 (2003)
13. Ori, A., Piran, T.: Phys. Rev. D **42**(4), 1068 (1990)
14. Papapetrou A., Hamoui, A.: Ann. Inst. H Poincaré **6**, 343 (1967)
15. Penrose, R.: R. Nuovo Cim. **1**, 252 (1969); reprinted in Gen. Rel. Grav. **34**, 1141 (2002)
16. Snajdr, M.: Class. Quantum Grav. **23**, 3333 (2006)