Global Geodesic Properties of Gödel-type SpaceTimes

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Mathematics Subject Classification (2010): 53C50, 53C22, 58E10.

1 Introduction

Classical critical point theorems and standard Morse theory are directly applicable to functionals bounded from below which satisfy compactness assumptions, such as the Palais–Smale condition (see Sect. 2), and whose critical points have finite Morse index. Unluckily, these tools cannot be applied to many interesting problems involving functionals that are strongly indefinite. For example, geodesics joining two points z_p, z_q on an indefinite semi–Riemannian manifold $(\mathscr{M}, \langle\cdot,\cdot\rangle_L)$ are the critical points of the strongly indefinite C^1 action functional

$$f(z) = \int_0^1 \langle \dot{z}, \dot{z}\rangle_L \mathrm{d}s \qquad (1)$$

defined on the Hilbert manifold Ω of all the H^1–curves joining z_p to z_q in $\mathscr{M}$ (for more details, see Sect. 2). Anyway, starting from the seminal paper [4], in

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M. Sánchez et al. (eds.), *Recent Trends in Lorentzian Geometry*, Springer Proceedings in Mathematics & Statistics 26, DOI 10.1007/978-1-4614-4897-6_7,

some particular settings, and according to the properties of the manifold $\mathscr{M}$ and its indefinite metric $\langle \cdot, \cdot \rangle_L$, the functional f in Eq. (1) has been widely studied by using variational methods, also obtaining sometimes optimal results at least in the Lorentzian case (we refer to the book [22] and to the updated survey paper [12] and references therein). A typical situation occurs when the Lorentzian metric tensor $\langle \cdot, \cdot \rangle_L$ presents symmetries (i.e., Killing vector fields): one gets rid of the negative contributions in the directions of the Killing fields and, by means of some variational principles, it is possible to handle with simpler functionals, which essentially depend only on a Riemannian metric, so that they are bounded from below and satisfy the Palais–Smale condition under reasonable assumptions. This is the case of standard stationary and Gödel-type spacetimes.

Definition 1. A Lorentzian manifold $(\mathscr{M}, \langle \cdot, \cdot \rangle_L)$ is a *standard stationary spacetime* if there exist a smooth, finite–dimensional Riemannian manifold $(\mathscr{M}_0, \langle \cdot, \cdot \rangle_R)$, a vector field δ and a positive smooth function β on $\mathscr{M}_0$ such that $\mathscr{M} = \mathscr{M}_0 \times \mathbb{R}$ and the Lorentz metric (under natural identifications) is

$$\langle \cdot, \cdot \rangle_L \; = \; \langle \cdot, \cdot \rangle_R + 2\langle \delta(x), \cdot \rangle_R \, dt - \beta(x) \, dt^2. \tag{2}$$

When the cross term vanishes ($\delta \equiv 0$), the spacetime is called *standard static*. This is a warped product $\mathscr{M}_0 \times_{\sqrt{\beta}} \mathbb{R}$ with Riemannian base and negative definite fiber.

Recall that every stationary spacetime (i.e., a spacetime admitting a timelike Killing vector field K) is locally a standard stationary one with $K = \partial_t$.

On the other hand, Gödel-type spacetimes are Lorentzian manifolds admitting a pair of commuting Killing vector fields which span a two-dimensional distribution where the metric has index 1 (the causal characters of the Killing vectors could change on the manifold, see [13, Example 5.1]). More precisely, we use the following definition (according to [10]):

Definition 2. A Lorentzian manifold $(\mathscr{M}, \langle \cdot, \cdot \rangle_L)$ is a *Gödel-type spacetime*, briefly *GTS*, if a smooth, finite–dimensional Riemannian manifold $(\mathscr{M}_0, \langle \cdot, \cdot \rangle_R)$ exists such that $\mathscr{M} = \mathscr{M}_0 \times \mathbb{R}^2$ and the metric $\langle \cdot, \cdot \rangle_L$ is described as

$$\langle \cdot, \cdot \rangle_L \; = \; \langle \cdot, \cdot \rangle_R + A(x)dy^2 + 2B(x)dydt - C(x)dt^2, \tag{3}$$

where $x \in \mathscr{M}_0$, the variables (y,t) are the natural coordinates of $\mathbb{R}^2$, and A, B, C are C^1 scalar fields on $\mathscr{M}_0$ satisfying

$$H(x) \; = \; B^2(x) + A(x)C(x) > 0 \quad \text{for all } x \in \mathscr{M}_0. \tag{4}$$

Let us observe that condition (4) implies that metric (3) is Lorentzian. It is also interesting to point out that *GTS* are not necessarily time–orientable (e.g., cf. [13, Remark 1.2]).

In [18] Gödel gives an exact solution of Einstein's field equations with homogeneous perfect fluid distribution, the so-called classical Gödel universe. This spacetime, described in Example 1(e_1) (see also [16, 21] where its geodesic equations are explicitly integrated), has a five dimensional group of isometries, is geodesically complete, and admits closed causal curves (e.g., cf. [19]). In [28], Raychaudhuri and Thakurta start the study of homogeneity properties of GTS investigating homogeneity conditions of a class of cylindrically symmetric metrics; later on, in [29], Rebouças and Tiomno introduce a definition for Gödel metrics in four dimensions and study their homogeneity conditions (see also [6, 15]).

Example 1. The class of GTS depicted in Definition 2 is wide; indeed, this definition covers very different kinds of spacetimes, including some physically relevant examples.

(e_1) The Gödel universe (cf. [18]) is a GTS with

$$\mathscr{M}_0 = \mathbb{R}^2, \quad \langle\cdot,\cdot\rangle_R = dx_1^2 + dx_2^2$$

and with coefficients in Eq. (3) given by

$$A(x) = -e^{2\sqrt{2}\omega x_1}/2, \quad B(x) = -e^{\sqrt{2}\omega x_1}, \quad C(x) \equiv 1$$

($\omega > 0$ represents the magnitude of the vorticity of the flow). In [10], by a direct integration of the geodesic equations, it is constructed a geodesic joining each couple of points in $\mathscr{M}$.

(e_2) The Gödel–Synge spacetimes (cf. [31]) are GTS with $\mathscr{M}_0 = \mathbb{R}^2$ and

$$\langle\cdot,\cdot\rangle_L = dx_1^2 + dx_2^2 - g(x_1)dy^2 - 2h(x_1)dydt - dt^2,$$

where g, h are smooth functions of x_1 with $g > 0$. If $2g = h^2$ and $h = e^{x_1}$, this metric reduces to the Gödel classical one.

(e_3) Some Kerr–Schild spacetimes (e.g., cf. [20]) are GTS with again $\mathscr{M}_0 = \mathbb{R}^2$ and

$$\langle\cdot,\cdot\rangle_L = dx_1^2 + dx_2^2 + dy^2 - dt^2 + V(x_1,x_2)(dy+dt)^2,$$

where V is an arbitrary function on $\mathbb{R}^2$. In this case, the coefficients in Eq. (3) are

$$A(x) = 1 + V(x), \quad B(x) = V(x), \quad C(x) = 1 - V(x),$$

and thus, $H(x) \equiv 1$.

(e_4) Some standard stationary spacetimes are GTS with $\mathscr{M} = \mathscr{M}_0 \times \mathbb{R}^2$, being $(\mathscr{M}_0 \times \mathbb{R}, \langle\cdot,\cdot\rangle_R + dy^2)$ the Riemannian part and

$$\langle\cdot,\cdot\rangle_L = \langle\cdot,\cdot\rangle_R + dy^2 + 2\delta(x)dydt - \beta(x)dt^2$$

the stationary metric with $\delta(x,y) \equiv \delta(x) \in \mathbb{R}$ and $\beta(x,y) \equiv \beta(x) > 0$ in $\mathscr{M}_0 \times \mathbb{R}$. Clearly, they are GTS with metric coefficients $A(x) \equiv 1$, $B(x) = \delta(x)$ and $C(x) = \beta(x)$.

Vice versa, some GTS are standard stationary spacetimes when $A(x)C(x) > 0$ on $\mathscr{M}_0$, being standard static if, in addition, $B \equiv 0$. For example, if $A(x) > 0$ on $\mathscr{M}_0$, the spatial part of the stationary spacetime corresponds to $\mathscr{M}_0 \times \mathbb{R}$ equipped with the Riemannian metric $\langle\cdot,\cdot\rangle_R + A(x)dy^2$ (which is complete if so is $\langle\cdot,\cdot\rangle_R$), the vector field becomes $\delta(x,y) = (0, B(x)) \in T\mathscr{M}_0 \times \mathbb{R}$, and the scalar field is $\beta(x,y) = C(x) > 0$ for each $(x,y) \in \mathscr{M}_0 \times \mathbb{R}$.

(e_5) Some examples of general plane fronted waves are also GTS. More precisely, a *general plane fronted wave* is a Lorentzian manifold $\mathscr{M}_0 \times \mathbb{R}^2$ equipped with the metric

$$\langle\cdot,\cdot\rangle_L = \langle\cdot,\cdot\rangle_R + 2dydt + H_0(x,t)dt^2,$$

where $(\mathscr{M}_0, \langle\cdot,\cdot\rangle_R)$ is a Riemannian manifold, (y,t) are the natural coordinates of $\mathbb{R}^2$, and the smooth scalar field H_0 on $\mathscr{M}_0 \times \mathbb{R}$ satisfies $H_0 \not\equiv 0$. Clearly, when $H_0(x,t)$ is autonomous (i.e., it does not depend on t), this spacetime is a GTS. Results on geodesic completeness and connectedness for these spacetimes can be found in [7].

The importance of the spacetimes above justifies the study of global properties such as geodesic connectedness and geodesic completeness. However, one cannot expect to prove general results, as these properties depend strongly on the metric coefficients (see respectively Theorems 2 and 3 and related comments). This dependence is also evident in the study of causality properties for GTS: it is well known that the classical Gödel universe is not chronological and, on the other side, stationary spacetimes can be globally hyperbolic (cf. [30, Corollary 3.4] and [14, Theorem 4.3]).

The chapter is organized as follows. In Sect. 2, we recall some variational principles for geodesics on static spacetimes and GTS. In Sect. 3, we present a new result on geodesic connectedness, and compare it with the previous ones in [2], showing its accuracy by examples. In Sect. 4, we deal with geodesic completeness and state a sufficient condition in order to obtain it. Finally, in the Appendix, we fix some widely known notations about the variational set up.

2 The Variational Principle

According to notations and statements contained in the Appendix, there is a correspondence between geodesics joining two given points z_p, z_q on a semi–Riemannian manifold $(\mathscr{M}, \langle\cdot,\cdot\rangle_L)$ and critical points of the action functional f in Eq. (1) on the Hilbert manifold $\Omega^1(z_p, z_q)$. As already remarked, if $\langle\cdot,\cdot\rangle_L$ is not Riemannian then f is strongly indefinite, but, in some Lorentzian manifolds, this difficulty can be overcome by introducing a new suitable functional.

The kernel of our approach is a variational principle stated in [5, Theorem 2.1] for static Lorentzian manifolds $\mathscr{M} = \mathscr{M}_0 \times \mathbb{R}$, with $\langle \cdot,\cdot \rangle_L$ as in Eq. (2) and $\delta \equiv 0$ (extended to stationary spacetimes in [17, Theorem 2.2], see also [8]). It is based on the fact that $\langle \partial_t, \dot{z} \rangle_L$ is constant along each geodesic z, because of the Killing character of ∂_t. Namely, $z_p = (x_p, t_p)$, $z_q = (x_q, t_q) \in \mathscr{M}$ are connected by a geodesic $\bar{z} = (\bar{x}, \bar{t})$, which is a critical point of the functional f in (1) on $\Omega^1(z_p, z_q) = \Omega^1(x_p, x_q) \times W(t_p, t_q)$, if and only if $\bar{x}$ is a critical point of the functional

$$J(x) \;=\; \frac{1}{2}\int_0^1 \langle \dot{x}, \dot{x} \rangle_R \,\mathrm{d}s - \frac{\Delta_t^2}{2}\left(\int_0^1 \frac{1}{\beta(x)}\,\mathrm{d}s\right)^{-1} \quad \text{on } \Omega^1(x_p, x_q), \tag{5}$$

with $\Delta_t := t_q - t_p$.

Next, let us consider the more general setting of GTS with $\mathscr{M} = \mathscr{M}_0 \times \mathbb{R}^2$ and $\langle \cdot,\cdot \rangle_L$ as in Definition 2. For each $x \in H^1(I, \mathscr{M}_0)$, let us define

$$a(x) = \int_0^1 \frac{A(x)}{H(x)}\,\mathrm{d}s, \;\; b(x) = \int_0^1 \frac{B(x)}{H(x)}\,\mathrm{d}s, \;\; c(x) = \int_0^1 \frac{C(x)}{H(x)}\,\mathrm{d}s, \tag{6}$$

$$\mathscr{L}(x) = b^2(x) + a(x)c(x). \tag{7}$$

As every GTS admits two Killing vector fields ∂_y, ∂_t (not necessarily timelike), an extension of the previous variational principle can be stated (cf. [10, Proposition 2.2]). In this setting, fixing $z_p = (x_p, y_p, t_p)$, $z_q = (x_q, y_q, t_q) \in \mathscr{M}$, with x_p, $x_q \in \mathscr{M}_0$ and (y_p, t_p), $(y_q, t_q) \in \mathbb{R}^2$, we have that $\bar{z} : I \to \mathscr{M}$ is a geodesic joining z_p to z_q in $\mathscr{M}$ if and only if it is a critical point of the action functional (1), with $\langle \cdot,\cdot \rangle_L$ as in Eq. (3), defined on the manifold $\Omega^1(z_p, z_q) = \Omega^1(x_p, x_q) \times W(y_p, y_q) \times W(t_p, t_q)$. Let $x \in \Omega^1(x_p, x_q)$ be such that $\mathscr{L}(x) \neq 0$ (cf. (7)). For all $s \in I$ we define

$$\phi_y(x)(s) := y_p + \frac{\Delta_y\, b(x) - \Delta_t\, c(x)}{\mathscr{L}(x)} \int_0^s \frac{B(x)}{H(x)}\mathrm{d}\sigma + \frac{\Delta_y a(x) + \Delta_t\, b(x)}{\mathscr{L}(x)} \int_0^s \frac{C(x)}{H(x)}\mathrm{d}\sigma,$$

$$\phi_t(x)(s) := t_p - \frac{\Delta_y\, b(x) - \Delta_t\, c(x)}{\mathscr{L}(x)} \int_0^s \frac{A(x)}{H(x)}\mathrm{d}\sigma + \frac{\Delta_y a(x) + \Delta_t\, b(x)}{\mathscr{L}(x)} \int_0^s \frac{B(x)}{H(x)}\mathrm{d}\sigma,$$

with $\Delta_y := y_q - y_p$ and $\Delta_t := t_q - t_p$. Standard arguments imply that the functions ϕ_y and ϕ_t, which go from $\Omega^1(x_p, x_q)$ to $W(y_p, y_q)$ and $W(t_p, t_q)$, respectively, are C^1.

Then, the following proposition holds (see [10, Proposition 2.2]).

Proposition 1. *Let $(\mathscr{M}, \langle \cdot,\cdot \rangle_L)$ be a GTS and x_p, $x_q \in \mathscr{M}_0$ be such that $|\mathscr{L}(x)| > 0$ for all $x \in \Omega^1(x_p, x_q)$. Then, the following statements are equivalent:*

1. *$\bar{z} \in \Omega^1(z_p, z_q)$ is a critical point of the action functional f in Eq. (1);*
2. *setting $\bar{z} = (\bar{x}, \bar{y}, \bar{t})$, the curve $\bar{x} \in \Omega^1(x_p, x_q)$ is a critical point of the C^1 functional*

$$\mathscr{J}(x) = \frac{1}{2}\int_0^1 \langle \dot{x}, \dot{x} \rangle_R \,\mathrm{d}s \;+\; \frac{\Delta_y^2 a(x) + 2\Delta_y \Delta_t b(x) - \Delta_t^2 c(x)}{2\mathscr{L}(x)} \quad \text{on } \Omega^1(x_p, x_q) \tag{8}$$

(see Eqs. (6)–(7)), while $\bar{y} = \phi_y(\bar{x})$, $\bar{t} = \phi_t(\bar{x})$, with ϕ_y, ϕ_t as above.

Furthermore,

$$\mathscr{J}(x) \;=\; f(x,\phi_y(x),\phi_t(x)) \quad \textit{for all } x \in \Omega^1(x_p,x_q).$$

Thus, the geodesic connectedness problem in the standard static and GTS cases reduces to give conditions on the functionals J in Eq. (5) and $\mathscr{J}$ in Eq. (8), respectively, which allows us to apply the classical critical point theorem below (see [27, Theorem 2.7]).

Theorem 1. *Assume that Ω is a complete Riemannian manifold and F is a C^1 functional on Ω which satisfies the Palais–Smale condition, i.e., any sequence $(x_k)_k \subset \Omega$ such that*

$$(F(x_k))_k \textit{ is bounded} \quad \textit{and} \quad \lim_{k\to+\infty} F'(x_k) = 0,$$

converges in Ω, up to subsequences. Then, if F is bounded from below, it attains its infimum.

Remark 8. In order to obtain a multiplicity result on geodesics joining two fixed points in standard static spacetimes or in GTS, the Ljusternik–Schnirelman theory can be applied to J in Eq. (5) or $\mathscr{J}$ in Eq. (8) whenever the Riemannian part has a "rich topology" (for the static case see [3] and references therein, and for GTS, see [2, 10, 11]).

In order to avoid technicalities, hereafter we assume that $\mathscr{M}_0$ is complete, so that $\Omega^1(x_p,x_q)$ is also complete for each $x_p,x_q \in \mathscr{M}_0$. Moreover, let us recall that a functional F on $\Omega^1(x_p,x_q)$ is *coercive* if

$$F(x) \to +\infty \quad \text{if} \quad \|\dot{x}\| \to +\infty,$$

where $\|\dot{x}\|^2 = \int_0^1 \langle \dot{x},\dot{x}\rangle_R \, \mathrm{d}s$.

The following result holds (cf. [3, Proposition 4.3] and [2, Lemma 5.3]).

Lemma 1. *Let $(\mathscr{M}_0,\langle\cdot,\cdot\rangle_R)$ be a C^3 complete Riemannian manifold and fix two points x_p, x_q in $\mathscr{M}_0$. Then, the following statements hold:*

(a) *if $\mathscr{M} = \mathscr{M}_0 \times \mathbb{R}$ is a static Lorentzian manifold and J in Eq. (5) is coercive on $\Omega^1(x_p,x_q)$, then J satisfies the Palais–Smale condition on $\Omega^1(x_p,x_q)$;*

(b) *if $\mathscr{M} = \mathscr{M}_0 \times \mathbb{R}^2$ is a GTS, $\mathscr{J}$ in Eq. (8) is coercive on $\Omega^1(x_p,x_q)$ and there exists $\nu > 0$ such that*

$$|\mathscr{L}(x)| \;\geq \nu \quad \textit{for all } x \in \Omega^1(x_p,x_q),$$

then $\mathscr{J}$ satisfies the Palais–Smale condition on $\Omega^1(x_p,x_q)$.

Summing up, geodesic connectedness of the mentioned spacetimes is guaranteed by conditions implying the coercivity and lower boundedness of the "Riemannian" functional associated to the problem.

For instance, in the case of J in Eq. (5), these conditions correspond to restrictions on the growth of the (positive) metric coefficient β in Eq. (2): β bounded in the pioneer paper [5] or, more in general, β subquadratic or growing at most quadratically with respect to the distance $d(\cdot,\cdot)$ induced on $\mathscr{M}_0$ by its Riemannian metric $\langle\cdot,\cdot\rangle_R$, i.e., existence of $\lambda \geq 0$, $k \in \mathbb{R}$ and a point $x_0 \in \mathscr{M}_0$ such that

$$0 < \beta(x) \leq \lambda d^2(x,x_0) + k \quad \text{for all } x \in \mathscr{M}_0 \tag{9}$$

(cf. [3, Theorem 1.1] and references therein). Remarkably, this second growth condition on β is optimal, as showed in [3, Sect. 7] by constructing a family of geodesically disconnected static spacetimes with superquadratic, but arbitrarily close to quadratic, coefficients β.

3 Geodesic Connectedness in *GTS*

At a first glance the problem in GTS can be handled in the same manner as in the static case. However, we cannot expect optimality by applying this variational approach. In fact, the classical Gödel universe cannot be studied by our tools, due to the lack of the assumption $\mathscr{L}(x) \neq 0$ on $\Omega^1(x_p,x_q)$ for each couple of points $x_p, x_q \in \mathbb{R}^2$ (cf. Example 1(e_1)). In this section, we state and prove a new theorem on geodesic connectedness for GTS (in addition to the previous ones in [2, 10]), which, even if not optimal, is accurate in the sense described below (see Corollary 1 and Example 2).

Theorem 2. *Let $(\mathscr{M} = \mathscr{M}_0 \times \mathbb{R}^2, \langle\cdot,\cdot\rangle_L)$ be a Gödel-type spacetime such that:*

(h_1) *$(\mathscr{M}_0, \langle\cdot,\cdot\rangle_R)$ is a C^3 complete Riemannian manifold;*
(h_2) *there exists $\nu > 0$ such that $\mathscr{L}(x) \geq \nu > 0$ for all $x \in H^1(I,\mathscr{M}_0)$;*
(h_3) *$m(x) \geq h(x) > 0$ for all $x \in H^1(I,\mathscr{M}_0)$, with $m(x) := \max\{a(x), -c(x)\}$ and*

$$h(x) := \int_0^1 \frac{\mathrm{d}s}{\lambda d^2(x(s),x_0) + k} \qquad \text{for some } \lambda \geq 0,\ k \in \mathbb{R} \text{ and } x_0 \in \mathscr{M}_0.$$

Then, $(\mathscr{M}, \langle\cdot,\cdot\rangle_L)$ is geodesically connected.

Proof. Let us take any $z_p = (x_p, y_p, t_p)$, $z_q = (x_q, y_q, t_q) \in \mathscr{M}$, with $x_p, x_q \in \mathscr{M}_0$ and (y_p, t_p), $(y_q, t_q) \in \mathbb{R}^2$. From hypothesis (h_2) (in particular $\mathscr{L}(x) \neq 0$), Proposition 1 can be applied, and so the existence of geodesics joining z_p to z_q reduces to find critical points of $\mathscr{J}$ in Eq. (8) on $\Omega^1(x_p,x_q)$. Following the arguments developed in [2, Sect. 5], we have that $\mathscr{J}$ can be written as follows:

$$\mathscr{J}(x) = \frac{1}{2}\|\dot{x}\|^2 - \frac{1}{2}\frac{\Delta_+^2(x)}{\lambda_-(x)} - \frac{1}{2}\frac{\Delta_-^2(x)}{\lambda_+(x)}, \tag{10}$$

where

$$\lambda_{\pm}(x) = \frac{a(x) - c(x) \pm \sqrt{(a(x)+c(x))^2 + 4b(x)^2}}{2}$$

and $\Delta_{\pm}(x)$ are suitable real maps depending also on Δ_y, Δ_t (see Eq. (6) and [2, p. 784]). Since $\mathscr{L}(x) = -\lambda_-(x)\lambda_+(x)$, necessarily, $\lambda_+(x) > 0 > \lambda_-(x)$ for all $x \in \Omega^1(x_p, x_q)$, and thus

$$\mathscr{J}(x) \geq \frac{1}{2}\|\dot{x}\|^2 - \frac{1}{2}\frac{\Delta_-^2(x)}{\lambda_+(x)}.$$

Note also that, by the definition of $m(x)$ in (h_3), we get

$$\lambda_+(x) \geq \frac{a(x) - c(x) + |a(x)+c(x)|}{2} = m(x) > 0.$$

Hence, (h_3) implies

$$\mathscr{J}(x) \geq \frac{1}{2}\|\dot{x}\|^2 - \frac{\Delta_-^2(x)}{2m(x)} \geq \frac{1}{2}\|\dot{x}\|^2 - \frac{\Delta_-^2(x)}{2}(h(x))^{-1} \quad \text{for all } x \in \Omega^1(x_p, x_q).$$

So, from [3, Theorem 1.1], it follows that $\mathscr{J}$ is bounded from below and coercive (cf. Eqs. (5) and (9)). Furthermore, by (h_2) and Lemma 1(b), the functional $\mathscr{J}$ satisfies the Palais–Smale condition. Hence, Theorem 1 can be applied, and a geodesic connecting z_p with z_q is obtained. As z_p, z_q are arbitrary, the thesis follows. □

An immediate consequence of Theorem 2 is the following result concerning some standard stationary spacetimes (cf. Example 1(e_4)).

Corollary 1. *Let* $(\mathscr{M} = \mathscr{M}_0 \times \mathbb{R}^2, \langle\cdot,\cdot\rangle_L)$ *be a standard stationary spacetime with* $\langle\cdot,\cdot\rangle_L = \langle\cdot,\cdot\rangle_R + dy^2 + 2\delta(x)dy\,dt - \beta(x)dt^2$, *where* δ, $\beta : \mathscr{M}_0 \to \mathbb{R}$, $\beta(x) > 0$ *in* $\mathscr{M}_0$. *Assume also that*

(s_1) $(\mathscr{M}_0, \langle\cdot,\cdot\rangle_R)$ *is a* C^3 *complete Riemannian manifold;*
(s_2) *there exist* $\lambda_1, \lambda_2 \geq 0$, $k_1, k_2 \in \mathbb{R}$ *and a point* $x_0 \in \mathscr{M}_0$ *such that*

$$\beta(x) \leq \lambda_1 d^2(x, x_0) + k_1, \quad \delta(x) \leq \lambda_2 d(x, x_0) + k_2 \qquad \text{for all } x \in \mathscr{M}_0.$$

Then, $(\mathscr{M} = \mathscr{M}_0 \times \mathbb{R}^2, \langle\cdot,\cdot\rangle_L)$ *is geodesically connected.*

Proof. As the standard stationary spacetime $(\mathscr{M} = \mathscr{M}_0 \times \mathbb{R}^2, \langle\cdot,\cdot\rangle_L)$ is a GTS with $A(x) \equiv 1$, $B(x) = \delta(x)$, and $C(x) = \beta(x)$, the thesis follows from Theorem 2. □

Notice that Corollary 1 is a particular case of [1, Theorem 1.2] for general standard stationary manifolds $\mathscr{M} = \mathscr{M}_0 \times \mathbb{R}$ with $\langle\cdot,\cdot\rangle_L$ as in Eq. (2), which proof is based on fine estimates involving the metric coefficients. The following example shows the accurate character of this result.

Example 2. Let us consider $\mathbb{R}^3$ endowed with the following family of metrics:

$$\langle\cdot,\cdot\rangle_{L,\varepsilon} = dx^2 + dy^2 - \beta_\varepsilon(x)dt^2, \quad \varepsilon \geq 0,$$

where $(x,y,t) \in \mathbb{R}^3$ and β_ε is a (positive) smooth function on $\mathbb{R}$ such that

$$\begin{cases} \beta_\varepsilon(x) = 1 + |x|^{2+\varepsilon} & \text{if } x \in \mathbb{R} \setminus (-1,1) \\ \beta_\varepsilon([-1,1]) \subset [1,2]. \end{cases}$$

By Corollary 1 (with $\delta \equiv 0$), the spacetime is geodesically connected if ε=0. However, the spacetime is geodesically disconnected for any (and thus, for arbitrarily close to zero) strictly positive ε (see [3, Sect. 7]).

In order to give a more precise idea of the known results on geodesic connectedness in GTS by applying variational tools, let us review the corresponding results in [2]. In [2, Theorem 4.3], by using the expression Eq. (10) of $\mathscr{J}$, the geodesic connectedness of GTS is proven under assumptions (h_1) and (h_2) in Theorem 2, in addition to the following one:

(h_3') $A(x) - C(x) > 0$ for all $x \in \mathscr{M}_0$ and the (positive) map $\frac{H(x)}{A(x)-C(x)}$ is at most quadratic.

Indeed, these conditions imply that $\mathscr{J}$ is bounded from below and coercive, which allows us to apply Theorem 1 in view of Lemma 1(b).

As an immediate application of this result to Kerr–Schild spacetime (Example 1(e_3)), observe that here $A(x) - C(x) = 2V(x)$, $H(x) \equiv 1$, and $\mathscr{L}(x) \neq 0$ on $H^1(I,\mathscr{M}_0)$. Thus, the geodesic connectedness is ensured if V is strictly positive and $(2V(x))^{-1}$ is at most quadratic.

On the other hand, in [2, Theorem 4.4], we consider the simpler case, where $\mathscr{L}(x) \leq -\nu < 0$ for all $x \in H^1(I,\mathscr{M}_0)$ and $A(x) - C(x) < 0$ for all $x \in \mathscr{M}_0$.

Finally, notice that in [2] the growth assumption involves only the metric coefficients, and not the integrals in Eq. (6). This contrasts with [10, 11], where, in order to get the coercivity of $\mathscr{J}$, it is required that

$$\left|\frac{a(x)}{\mathscr{L}(x)}\right|, \quad \left|\frac{b(x)}{\mathscr{L}(x)}\right|, \quad \left|\frac{c(x)}{\mathscr{L}(x)}\right| \quad \text{are uniformly bounded on } H^1(I,\mathscr{M}_0).$$

Remark 2. Regarding to the case $A \equiv C$ left over in [2], if A (hence C) is always different from zero, then we are in the stationary case (Example 1(e_4)) with $\beta(x) = |A(x)|$.

In general, if $B \equiv 0$ and $H(x) = A(x)C(x) > 0$ with $A(x) > 0$ and $\beta(x) = C(x)$, then we have Example 1(e_4) in the static case. So, $\mathscr{J}(x) \geq J(x)$ on each $\Omega^1(x_p,x_q)$ and the optimal result in [3, Theorem 1.1] can be used. Let us point out that a direct use of (h_3') for the particular case $A \equiv 1$ would give the desired result only for $\beta(x) < 1$.

If $A \equiv C \equiv 0$, then $\mathscr{L}(x) = b^2(x)$ and GTS becomes the more general type of warped product spacetimes, with fiber the two dimensional Lorentz–Minkowski spacetime (see also [10,13] and references therein). In this case, we deal again with a functional as in Eq. (5), and we get global geodesic connectedness for the class of metrics $\langle\cdot,\cdot\rangle_R - 2\delta(x)\,dy\,dt$, where δ is a positive function with at most a quadratic growth (compare with [10, Appendix B]).

Moreover, if $a \equiv c$ on $H^1(I, \mathscr{M}_0)$, then

$$\mathscr{J}(x) \geq \frac{1}{2}\|\dot{x}\|^2 - \frac{\Delta_-^2}{|a(x)|}.$$

Hence, if $A(x) > 0$ in $\mathscr{M}_0$, we obtain geodesic connectedness by assuming that $H(x)/A(x)$ grows at most quadratically in $\mathscr{M}_0$ (cf. Eqs. (5) and (9)).

Remark 3. In [26] Piccione and Tausk generalize the Morse index theorem to semi–Riemannian manifolds admitting a smooth distribution spanned by commuting Killing vector fields and containing a maximal negative distribution for the given metric. So, they obtain Morse relations for standard stationary spacetimes and, when the nondegeneracy condition $|\mathscr{L}(x)| > 0$ holds, for GTS (cf. [26, Theorems 4.6 and 4.8]). In particular, also in our setting, Morse relations hold. In fact, under the assumptions of Theorem 2 or of [2, Theorem 4.3] (where (h_1) and (h_2) hold, while (h_3) is replaced by (h_3')), a formal power series involving the Maslov index and the reduced Maslov index can be stated as in [26, Theorem 4.8] for each pair of non–conjugate points (for this definition, cf., e.g., [25]).

4 Geodesic Completeness

In this section, we establish and prove a result on geodesic completeness for GTS.

Theorem 3. *Let $(\mathscr{M} = \mathscr{M}_0 \times \mathbb{R}^2, \langle\cdot,\cdot\rangle_L)$ be a Gödel-type spacetime such that:*

(c_1) *$(\mathscr{M}_0, \langle\cdot,\cdot\rangle_R)$ is a complete Riemannian manifold;*
(c_2) *there exist $\lambda \geq 0$, $k \in \mathbb{R}$ and a point $x_0 \in \mathscr{M}_0$ such that the map*

$$\mu : x \in \mathscr{M}_0 \mapsto C(x) - A(x) + \sqrt{(A(x)+C(x))^2 + 4B^2(x)} \in \mathbb{R}$$

(which is strictly positive by Eq. (4)) satisfies

$$1/\mu(x) \leq \lambda d^2(x, x_0) + k \quad \text{for all } x \in \mathscr{M}_0. \tag{11}$$

Then, $(\mathscr{M}, \langle\cdot,\cdot\rangle_L)$ is geodesically complete.

Proof. Let $z : [0,T) \to \mathscr{M}$, $z(s) = (x(s), y(s), t(s))$, be an inextendible geodesic. Arguing by contradiction, it is enough to prove that if $T < +\infty$, then the $\langle\cdot,\cdot\rangle_R$–length of $x(s)$ is bounded, and so, z can be extended to T against the maximality assumption (see [25, Lemma 5.8]).

As ∂_y and ∂_t are Killing vector fields, there exist constants $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{cases} A(x)\dot{y} + B(x)\dot{t} \equiv c_1 \\ B(x)\dot{y} - C(x)\dot{t} \equiv c_2 \end{cases} \qquad \text{for all } s \in [0,T), \tag{12}$$

with

$$\mathscr{S}(x) \;=\; \begin{pmatrix} A(x) & B(x) \\ B(x) & -C(x) \end{pmatrix} \tag{13}$$

symmetric matrix with $\det \mathscr{S}(x) = -H(x) < 0$.

Furthermore, as z is a geodesic, there exists a constant $E_z \in \mathbb{R}$ such that

$$\langle \dot{z}, \dot{z}\rangle_L = \langle \dot{x}, \dot{x}\rangle_R + A(x)\dot{y}^2 + 2B(x)\dot{y}\dot{t} - C(x)\dot{t}^2 \equiv E_z \quad \text{for all } s \in [0,T). \tag{14}$$

Thus, by Eqs. (12) and (14) we get

$$\langle \dot{x}, \dot{x}\rangle_R + c_1\dot{y} + c_2\dot{t} = E_z \quad \text{for all } s \in [0,T). \tag{15}$$

On the other hand, by Eqs. (12) and (4) we have

$$\dot{y} = \frac{c_1 C(x) + c_2 B(x)}{H(x)}, \qquad \dot{t} = \frac{c_1 B(x) - c_2 A(x)}{H(x)}.$$

Whence, by Eq. (15) and using the notation $\|\dot{x}\|_R^2 := \langle \dot{x}, \dot{x}\rangle_R$, we get

$$\|\dot{x}\|_R^2 \;=\; E_z + \frac{c_2^2 A(x) - c_1^2 C(x) - 2c_1c_2 B(x)}{H(x)}. \tag{16}$$

Note that the symmetric matrix $\mathscr{S}(x)$ in Eq. (13) admits two (non–null) real eigenvalues

$$\Lambda_\pm(x) = \frac{A(x) - C(x) \pm \sqrt{(A(x)+C(x))^2 + 4B^2(x)}}{2}, \qquad \text{with } \Lambda_+(x) > 0 > \Lambda_-(x).$$

Recall that by standard arguments there exists an orthogonal matrix $Q(x)$ such that

$$Q(x)^T \begin{pmatrix} A(x) & B(x) \\ B(x) & -C(x) \end{pmatrix} Q(x) = \begin{pmatrix} \Lambda_+(x) & 0 \\ 0 & \Lambda_-(x) \end{pmatrix}.$$

Let us denote $(\tilde{c}_1,\tilde{c}_2) = (c_1^2+c_2^2)^{-1/2}(c_1,c_2)$ and $(\tilde{c}_1(x),\tilde{c}_2(x)) = (\tilde{c}_1,\tilde{c}_2)Q(x)$. By definition we have $\mu(x) = -2\Lambda_-(x)$, and, by the orthogonality of $Q(x)$, we have $[\tilde{c}_i(x)]^2 \leq 1$ for $i \in \{1,2\}$. So, we can rewrite Eq. (16) as:

$$
\begin{aligned}
\|\dot{x}\|_R^2 &= E_z + \frac{\begin{pmatrix} c_1 c_2 \end{pmatrix} \begin{pmatrix} A(x) & B(x) \\ B(x) & -C(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}{H(x)} \\
&= E_z + \frac{\begin{pmatrix} c_1 & c_2 \end{pmatrix} Q(x) \begin{pmatrix} \Lambda_+(x) & 0 \\ 0 & \Lambda_-(x) \end{pmatrix} Q(x)^T \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}{H(x)} \\
&= E_z + (c_1^2+c_2^2) \frac{\begin{pmatrix} \tilde{c}_1 & \tilde{c}_2 \end{pmatrix} Q(x) \begin{pmatrix} \Lambda_+(x) & 0 \\ 0 & \Lambda_-(x) \end{pmatrix} Q(x)^T \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix}}{H(x)} \\
&= E_z + (c_1^2+c_2^2) \frac{\begin{pmatrix} \tilde{c}_1(x) & \tilde{c}_2(x) \end{pmatrix} \begin{pmatrix} \Lambda_+(x) & 0 \\ 0 & \Lambda_-(x) \end{pmatrix} \begin{pmatrix} \tilde{c}_1(x) \\ \tilde{c}_2(x) \end{pmatrix}}{-\Lambda_+(x)\Lambda_-(x)} \\
&= E_z - (c_1^2+c_2^2) \left(\frac{[\tilde{c}_1(x)]^2}{\Lambda_-(x)} + \frac{[\tilde{c}_2(x)]^2}{\Lambda_+(x)} \right) \\
&\leq E_z - \frac{c_1^2+c_2^2}{\Lambda_-(x)} = E_z + 2\frac{c_1^2+c_2^2}{\mu(x)}.
\end{aligned}
$$

Thus, by Eq. (11) there exist suitable constants $\bar{\lambda}, \bar{k} > 0$ such that:

$$
\|\dot{x}(s)\|_R \leq \bar{\lambda} d(x(s),x(0)) + \bar{k} \leq \bar{\lambda} \int_0^s \|\dot{x}(r)\|_R \, \mathrm{d}r + \bar{k} \quad \text{for all } s \in [0,T).
$$

In conclusion, we obtain

$$
\log\left(\bar{\lambda} \int_0^s \|\dot{x}(r)\|_R \, \mathrm{d}r + \bar{k} \right) - \log(\bar{k}) \leq \bar{\lambda} s \leq \bar{\lambda} T \quad \text{for all } s \in [0,T)
$$

and then the boundedness of the $\langle \cdot,\cdot \rangle_R$–length of $x(s)$ in $[0,T)$, as required. □

Remark 4. The at most quadratic behavior of the autonomous term $1/\mu$ required for the geodesic completeness of a *GTS* in Theorem 3 is consistent with the (optimal) growth estimates which imply the completeness of the solutions of certain second order differential equations on Riemannian manifolds (see [9]).

Appendix

Taking a connected, finite–dimensional semi–Riemannian manifold $(\mathscr{M}, g)$, let $H^1(I,\mathscr{M})$ be the associated Sobolev space for some auxiliar Riemannian metric on $\mathscr{M}$. Then, $H^1(I,\mathscr{M})$ is equipped with a structure of infinite–dimensional manifold modelled on the Hilbert space $H^1(I,\mathbb{R}^n)$. For any $z \in H^1(I,\mathscr{M})$, the tangent space of $H^1(I,\mathscr{M})$ at z can be written as follows:

$$T_z H^1(I,\mathscr{M}) = \{\zeta \in H^1(I,T\mathscr{M}) : \zeta(s) \in T_{z(s)}\mathscr{M} \text{ for all } s \in I\},$$

where $T\mathscr{M}$ is the tangent bundle of $\mathscr{M}$.

If $\mathscr{M}$ splits globally in the product of two semi–Riemannian manifolds $\mathscr{M}_1$ and $\mathscr{M}_2$, i.e. $\mathscr{M} = \mathscr{M}_1 \times \mathscr{M}_2$, then

$$H^1(I,\mathscr{M}) \equiv H^1(I,\mathscr{M}_1) \times H^1(I,\mathscr{M}_2)$$

and $T_z H^1(I,\mathscr{M}) \equiv T_{z_1} H^1(I,\mathscr{M}_1) \times T_{z_2} H^1(I,\mathscr{M}_2)$ for all $z = (z_1,z_2) \in \mathscr{M}$.

On the other hand, if $(\mathscr{M}_0, \langle\cdot,\cdot\rangle_R)$ is a C^3 complete Riemannian manifold, it can be smoothly and isometrically embedded in a Euclidean space $\mathbb{R}^N$ (see [24]); moreover such embedding can be chosen closed (see [23]) and this is used in the proof of Lemma 1. Hence, $H^1(I,\mathscr{M}_0)$ is a closed submanifold of the Hilbert space $H^1(I,\mathbb{R}^N)$. In this case, we denote by $d(\cdot,\cdot)$ the distance induced on $\mathscr{M}_0$ by its Riemannian metric $\langle\cdot,\cdot\rangle_R$, i.e.,

$$d(x_p,x_q) \;:=\; \inf\left\{\int_a^b \sqrt{\langle \dot{x},\dot{x}\rangle_R}\, \mathrm{d}s :\; x \in A_{x_p,x_q}\right\},$$

where $x \in A_{x_p,x_q}$ if $x : [a,b] \to \mathscr{M}_0$ is any piecewise smooth curve in $\mathscr{M}_0$ joining $x_p, x_q \in \mathscr{M}_0$.

Taking z_p, $z_q \in \mathscr{M}$, let us consider

$$\Omega^1(z_p,z_q) = \{z \in H^1(I,\mathscr{M}) : z(0) = z_p,\; z(1) = z_q\},$$

which is a submanifold of $H^1(I,\mathscr{M})$, complete if $\mathscr{M}$ is complete and having tangent space described as

$$T_z\Omega^1(z_p,z_q) = \{\zeta \in T_z H^1(I,\mathscr{M}) : \zeta(0) = 0 = \zeta(1)\} \quad \text{at any } z \in \Omega^1(z_p,z_q).$$

Moreover, for any l_p, $l_q \in \mathbb{R}$, let us denote

$$W(l_p,l_q) = \{l \in H^1(I,\mathbb{R}) : l(0) = l_p\;,\; l(1) = l_q\}.$$

Clearly,

$$W(l_p,l_q) = H^1_0(I,\mathbb{R}) + l_{pq},$$

with $H^1_0(I,\mathbb{R}) = \{l \in H^1(I,\mathbb{R}) : l(0) = 0 = l(1)\}$, $l_{pq} : s \in I \mapsto (1-s)l_p + s l_q \in \mathbb{R}$. Hence, $W(l_p,l_q)$ is a closed affine submanifold of the Hilbert space $H^1(I,\mathbb{R})$ with tangent space

$$T_l W(l_p,l_q) = H^1_0(I,\mathbb{R}) \quad \text{for every } l \in W(l_p,l_q).$$

Acknowledgements The authors of this chapter acknowledge the partial support of the Spanish Grants with FEDER funds MTM2010-18099 (MICINN). Furthermore, R. Bartolo and A.M. Candela acknowledge also the partial support of M.I.U.R. Research Project PRIN2009 "Metodi Variazionali e Topologici nello Studio di Fenomeni Nonlineari" and of the G.N.A.M.P.A. Research Project 2011 "Analisi Geometrica sulle Varietà di Lorentz ed Applicazioni alla Relatività Generale"; J.L. Flores acknowledges also the partial support of the Regional J. Andalucía Grant P09-FQM-4496, with FEDER funds.

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