

Calabi–Bernstein Results and Parabolicity of Maximal Surfaces in Lorentzian Product Spaces

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Dedicated to Professor Ángel Ferrández on the occasion of his 60th birthday.

1 Introduction

A *maximal surface* in a 3-dimensional Lorentzian manifold is a space-like surface with zero mean curvature. By *space-like*, we mean that the induced metric on the surface from the Lorentzian metric of the ambient space is a Riemannian metric. The terminology maximal comes from the fact that maximal surfaces locally maximize area among all nearby surfaces having the same boundary [12, 22]. These surfaces have nice physical and mathematical properties.

From a physical point of view, maximal surfaces and, more generally, constant mean curvature space-like hypersurfaces in a Lorentzian space of arbitrary dimension, have a great importance. In fact, these surfaces are used as initial data for solving the Cauchy problem for the Einstein equations on General Relativity (see [31]). Specifically, in 1944 Lichnerowicz [31] proved that the Cauchy problem with initial conditions on a maximal surface is reduced to a first-order linear differential system, and to a second-order nonlinear elliptic differential equation.

From a mathematical point of view, space-like surfaces with constant mean curvature, and in particular maximal surfaces, present interesting Bernstein-type

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properties. A mathematical property or result is said to be of Bernstein type when, under some suitable assumptions, it allows us to conclude some uniqueness results in a given ambient space. In fact, one of the most relevant results in the context of global geometry of space-like surfaces is the well-known Calabi–Bernstein theorem. This result was established in 1970 by Calabi [14], and, in its nonparametric version, it asserts that the only entire maximal graphs in the 3-dimensional Lorentz–Minkowski space, \mathbb{R}_1^3 , are the space-like planes. The Calabi–Bernstein theorem can also be formulated in a parametric version. In that case, it states that the only complete maximal surfaces in the 3-dimensional Lorentz–Minkowski space, \mathbb{R}_1^3 , are the space-like planes.

In Sect. 2 we review the Calabi–Bernstein theorem, and we summarize some of the different extensions and generalizations of it. Specifically, we will sketch three different proofs of the theorem. The first of them is a very neat and simple proof of Romero [39] which only requires the Liouville theorem for harmonic functions on the Euclidean plane \mathbb{R}^2 . The second one is a proof of Alías and Palmer [9] via a duality between minimal graphs in \mathbb{R}^3 and maximal graphs in \mathbb{R}_1^3 . Finally, the third proof is a local approach of Alías and Palmer [10] based on a local integral inequality for the Gaussian curvature of a maximal surface. The reader can consult [7] for a survey on the classical Calabi–Bernstein theorem in \mathbb{R}_1^3 and related results from another point of view.

The aim of the rest of this chapter is to describe some recent results obtained by the authors in [3–6] (see also [2]) for maximal surfaces immersed in a Lorentzian product space of the form $M^2 \times \mathbb{R}_1$ (see Sect. 3 for the details). We will also survey some related results obtained by several authors during these last years.

Our first main result is Theorem 5, which states that any complete maximal surface Σ immersed into a Lorentzian product $M^2 \times \mathbb{R}_1$, where M is a (necessarily complete) Riemannian surface with nonnegative Gaussian curvature, must be totally geodesic. Moreover, if M is non-flat, we conclude that Σ must be a slice $M \times \{t_0\}$, $t_0 \in \mathbb{R}$. Here, by *complete*, it is meant, as usual, that the induced Riemannian metric on Σ from the ambient Lorentzian metric is complete. In Theorem 7 we prove that the same happens if Σ is complete with respect to the metric induced from the Riemannian product $M^2 \times \mathbb{R}$. This allows us to give a nonparametric version of our Calabi–Bernstein result, Theorem 8, where we prove that any entire maximal graph in $M^2 \times \mathbb{R}_1$ must be totally geodesic and that the only entire solutions to the maximal surface equation on any complete, non-flat, Riemannian surface M with nonnegative Gaussian curvature are the constant functions.

It is worth pointing out that, in all the above results, the assumption on the Gaussian curvature of M is necessary as shown by the fact that, when $M = \mathbb{H}^2$ is the hyperbolic plane, there exist examples of complete maximal surfaces in $\mathbb{H}^2 \times \mathbb{R}_1$ which are not totally geodesic, as well as examples of nontrivial entire maximal graphs over \mathbb{H}^2 . We construct these examples in two different ways. Examples 1 and 2 are a consequence of a simple but nice duality result between solutions to the minimal surface equation in a Riemannian product $M^2 \times \mathbb{R}$ and solutions to the maximal surface equation in a Lorentzian product $M^2 \times \mathbb{R}_1$, Theorem 9, that extends [9, Theorem 3]. On the other hand, Examples 3 and 4 are explicit examples of

nontrivial entire maximal graphs in $\mathbb{H}^2 \times \mathbb{R}_1$, which have been found by looking for explicit solutions of the maximal surface equation on \mathbb{H}^2 . Particularly interesting is the fact that the entire maximal graphs given in Examples 2 and 3 are not complete. As is well known, such circumstance cannot occur in the Lorentz–Minkowski space \mathbb{R}_1^3 since, by a result of Cheng and Yau [15], closed surfaces in \mathbb{R}_1^3 with constant mean curvature are necessarily complete.

We also introduce a local approach to our Calabi–Bernstein results, which is based on a parabolicity criterium for maximal surfaces with nonempty smooth boundary in $M^2 \times \mathbb{R}_1$, Theorem 10. In particular, we derive that every maximal graph over a star-like domain $\Omega \subseteq M$ is parabolic. This allows us to give an alternative proof of the nonparametric version of the Calabi–Bernstein result, Theorem 8, for maximal entire graphs in $M^2 \times \mathbb{R}_1$.

Finally, in Theorem 11 we present another local approach based on a local integral inequality for the squared norm of the second fundamental form of a maximal surface in $M^2 \times \mathbb{R}_1$. This approach generalizes a previous result of Alías and Palmer, [10, Theorem 1], where they gave a local integral inequality for the Gaussian curvature of a maximal surface in \mathbb{R}_1^3 . This result only involved the local geometry of the surface and the image of its Gauss map. Under completeness assumption, and as a consequence of Theorem 11, we provide an alternative proof of the parametric version of our Calabi–Bernstein result, Theorem 5, in the case where M^2 is analytic.

2 The Classical Calabi–Bernstein Theorem in \mathbb{R}_1^3

Let \mathbb{R}_1^3 denote the 3-dimensional Lorentz–Minkowski space, that is, the real vector space \mathbb{R}^3 endowed with the Lorentzian metric

$$\langle \cdot, \cdot \rangle = (dx_1)^2 + (dx_2)^2 - (dx_3)^2,$$

where (x_1, x_2, x_3) are the canonical coordinates in \mathbb{R}^3 . A smooth immersion $f : \Sigma^2 \rightarrow \mathbb{R}_1^3$ of a connected surface Σ is said to be a *space-like surface* if the induced metric via f is a Riemannian metric on Σ , which, as usual, is also denoted by $\langle \cdot, \cdot \rangle$. A space-like surface Σ is said to be *complete* if the Riemannian induced metric is a complete metric on Σ .

Let us observe that every space-like surface in \mathbb{R}_1^3 is orientable. In fact, observe that $(0, 0, 1)$ is a unit time-like vector field globally defined on \mathbb{R}_1^3 , which determines a time orientation on \mathbb{R}_1^3 . Therefore, it allows us to choose the unique time-like unit normal field N on Σ which is in the same time orientation as $(0, 0, 1)$, and hence we may assume that Σ is oriented by N . We will refer to N as the *future-directed Gauss map* of the surface Σ . The field N can be regarded as a map $N : \Sigma \rightarrow \mathbb{H}^2$, where \mathbb{H}^2 denotes the 2-dimensional hyperbolic space

$$\mathbb{H}^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}_1^3 \mid \langle x, x \rangle = -1, x_3 \geq 1\}.$$

The image $N(\Sigma)$ will be called the *hyperbolic image* of the space-like surface.

Let $\bar{\nabla}$ and ∇ be the Levi–Civita connections of \mathbb{R}_1^3 and Σ , respectively. Then, the Gauss and Weingarten formulas for Σ in \mathbb{R}_1^3 are given, respectively, by

$$\bar{\nabla}_X Y = \nabla_X Y - \langle AX, Y \rangle N \quad (1)$$

and

$$AX = -\bar{\nabla}_X N, \quad (2)$$

for all tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$, where $A : \mathfrak{X}(\Sigma) \rightarrow \mathfrak{X}(\Sigma)$ stands for the shape operator of Σ in \mathbb{R}_1^3 with respect to its future-directed Gauss map N . The *mean curvature* of the surface associated to A is defined by

$$H = -\frac{1}{2} \text{tr}(A).$$

A space-like surface is said to be *maximal* if H vanishes on Σ , $H \equiv 0$.

As is well known, the Gaussian curvature of the space-like surface Σ , K , is described in terms of the shape operator of Σ by the Gauss equation, [36, Theorem 4.20],

$$K = -\det(A).$$

Besides, if Σ is maximal then

$$A^2 = \frac{1}{2} \|A\|^2 I,$$

where I denotes the identity map on $\mathfrak{X}(\Sigma)$ and $\|A\|^2 = \text{tr}(A^2) = -2\det(A)$. Therefore, the Gauss equation becomes

$$K = \frac{1}{2} \|A\|^2. \quad (3)$$

On the other hand, the Codazzi equation of the surface is given by

$$(\nabla_Y A)X = (\nabla_X A)Y, \quad (4)$$

where $(\nabla_Y A)X = \nabla_Y(AX) - A(\nabla_Y X)$, [36, Corollary 4.34].

Let $f : \Sigma^2 \rightarrow \mathbb{R}_1^3$ be a maximal surface oriented by its future-directed Gauss map N . For each vector $a \in \mathbb{R}_1^3$ we can consider the smooth function on Σ given by $\langle N, a \rangle$. With a straightforward computation we get from Eq. (2) that its gradient is given by

$$\nabla \langle N, a \rangle = -A(a^\top), \quad (5)$$

where a^\top denotes the tangential component of a along Σ ; that is,

$$a^\top = a + \langle N, a \rangle N \in \mathfrak{X}(\Sigma). \quad (6)$$

On the other hand, $\bar{\nabla}_X a = 0$ for any $X \in \mathfrak{X}(\Sigma)$. Thus, from the Gauss and Weingarten formulas (1) and (2) we easily get from Eq. (6) that

$$\nabla_X a^\top = -\langle N, a \rangle AX$$

for every $X \in \mathfrak{X}(\Sigma)$, and using the Codazzi equation (4) we obtain

$$\nabla_X (\nabla \langle N, a \rangle) = -(\nabla_{a^\top} A)X + \langle N, a \rangle A^2 X.$$

Therefore, the Laplacian of $\langle N, a \rangle$ is given by

$$\Delta \langle N, a \rangle = \|A\|^2 \langle N, a \rangle. \quad (7)$$

From Eq. (6) we have $\|a^\top\|^2 = \langle a, a \rangle + \langle N, a \rangle^2$, where for any $X \in \mathfrak{X}(\Sigma)$ $\|X\|$ stands for its norm with respect to the induced metric on Σ from the metric of \mathbb{R}_1^3 . Thus, from Eq. (5) we obtain

$$\|\nabla \langle N, a \rangle\|^2 = \frac{1}{2} \|A\|^2 (\langle N, a \rangle^2 + \langle a, a \rangle). \quad (8)$$

2.1 Space-Like Graphs and the Calabi–Bernstein Theorem

Let $\Omega \subseteq \mathbb{R}^2$ be an open subset of \mathbb{R}^2 , then any smooth function $u = u(x_1, x_2) \in \mathcal{C}^\infty(\Omega)$ determines a *graph* over Ω given by

$$\Sigma(u) = \{(x_1, x_2, u(x_1, x_2)) \mid (x_1, x_2) \in \Omega\} \subset \mathbb{R}_1^3.$$

The metric induced on Ω from the metric of \mathbb{R}_1^3 is given by

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_o - du^2,$$

where $\langle \cdot, \cdot \rangle_o$ stands for the Euclidean metric on \mathbb{R}^2 . Therefore, $\Sigma(u)$ is a space-like surface in \mathbb{R}_1^3 if and only if $|D_o u|_o^2 < 1$ everywhere on Ω , where $D_o u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2})$ denotes the Euclidean gradient of u in Ω and $|D_o u|_o$ its Euclidean norm. On the other hand, it is not difficult to see that the vector field

$$N(x) = \frac{1}{\sqrt{1 - |D_o u(x)|_o^2}} (D_o u(x), 1), \quad x \in \Omega,$$

defines the future-pointing Gauss map of $\Sigma(u)$ and that the shape operator with respect to N is given by

$$AX = -\frac{1}{\sqrt{1 - |D_o u|_o^2}} D_{oX} D_o u - \frac{\langle D_{oX} D_o u, D_o u \rangle_o}{(1 - |D_o u|_o^2)^{3/2}} D_o u$$

for every tangent vector field X on Ω , where D_o also denotes the Levi–Civita connection in Ω with respect to the metric $\langle \cdot, \cdot \rangle_o$. It follows from here that the mean curvature $H(u)$ of a space-like graph $\Sigma(u)$ is given by

$$2H(u) = \text{Div}_o \left(\frac{D_o u}{\sqrt{1 - |D_o u|_o^2}} \right),$$

where Div_o stands for the Euclidean divergence operator on Ω . In particular, $\Sigma(u)$ is a maximal graph if and only if the function u satisfies the following partial differential equation on the domain Ω :

$$\text{Div}_o \left(\frac{D_o u}{\sqrt{1 - |D_o u|_o^2}} \right) = 0, \quad |D_o u|_o^2 < 1. \tag{9}$$

The graph is said to be *entire* when $\Omega = \mathbb{R}^2$.

An interesting remark on the topology of complete space-like surfaces in \mathbb{R}_1^3 is the fact that they are necessarily entire space-like graphs over the space-like plane $x_3 = 0$, in the sense that the projection $\Pi : \Sigma \rightarrow \mathbb{R}^2$ of Σ onto the plane $x_3 = 0$ is a diffeomorphism. In fact, it is not difficult to see that since Σ is space like, $\Pi : \Sigma \rightarrow \mathbb{R}^2$ is a local diffeomorphism which increases the distance between the Riemannian surfaces Σ and \mathbb{R}^2 , that is, $\Pi^*(\langle \cdot, \cdot \rangle_o) \geq \langle \cdot, \cdot \rangle$. The completeness of Σ implies that $\Pi(\Sigma) = \mathbb{R}^2$ and that Π is a covering map [28, Chap. VIII, Lemma 8.1]. In particular, there exists no compact (without boundary) space-like surface in \mathbb{R}_1^3 . However, it is worth pointing out that no converse of this fact is true in general. More precisely, there exist examples of space like entire graphs in \mathbb{R}_1^3 which are not complete. For instance, let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a real function defined by

$$\phi(x) = \int_0^{|x|} \sqrt{1 - e^{-t}} dt$$

when $|x| \geq 1$, and $\phi(x) = f(x)$ when $|x| < 1$, where $f \in \mathcal{C}^\infty(\mathbb{R})$ is a smooth extension satisfying $f'(x)^2 < 1$ for all $x \in (-1, 1)$. Then, it is not difficult to prove that the entire graph of \mathbb{R}_1^3 given by $x_3 = \phi(x_1)$ defines a space-like surface which is not complete. This fact represents an important difference between the behavior of surfaces in the Euclidean space \mathbb{R}^3 and that of space-like surfaces in the Lorentz–Minkowski space \mathbb{R}_1^3 . Actually, let us recall that every embedded surface in the Euclidean space which is a closed subset in \mathbb{R}^3 is necessarily complete, while there exist examples of complete embedded surfaces in \mathbb{R}^3 which are not closed.

In this context, the classical Calabi–Bernstein theorem asserts in its parametric version that

Theorem 1. *The only complete maximal surfaces in the Lorentz–Minkowski space \mathbb{R}_1^3 are the space-like planes.*

By the comments above, this result can also be stated in a nonparametric version, in terms of entire maximal graphs.

Theorem 2. *The only entire maximal graphs in the Lorentz–Minkowski space \mathbb{R}_1^3 are the space-like planes. That is, the only entire solutions to the maximal surface equation (9) on the Euclidean plane \mathbb{R}^2 are affine functions.*

This theorem was firstly proved by Calabi [14] in 1970, and extended later to the general n -dimensional case by Cheng and Yau [15] as an application of a Simons-type formula in the context of space-like hypersurfaces in a Lorentz ambient space. After that, several authors have approached the Calabi–Bernstein theorem for maximal surfaces ($n = 2$) from different points of view, providing diverse extensions and new proofs of it, both in parametric and nonparametric versions. For instance, Kobayashi introduced in [27] an appropriate Enneper–Weierstrass representation of a maximal surface, providing a new approach to the theorem. McNertey [32] and Estudillo and Romero [18–20] have also considered the study of similar problems in terms of a local complex representation of the surface. We will sketch here three more recent proofs given by Romero [39] and by the second author jointly with Palmer [9, 10]. The choice of these proofs is due to the fact that they have inspired some of the results presented by the authors for maximal surfaces in a Lorentzian product space $M^2 \times \mathbb{R}_1$.

2.2 Romero’s Proof Based on the Liouville Theorem for Harmonic Functions on \mathbb{R}^2

The proof by Romero of the nonparametric version of the classical Calabi–Bernstein theorem is based on the Liouville theorem for harmonic functions on \mathbb{R}^2 . It is inspired in a simple proof of the classical Bernstein theorem given by Chern [16].

Let $\Sigma(u)$ be an entire maximal graph in \mathbb{R}_1^3 over the space-like plane $x_3 = 0$ and consider $a \in \mathbb{R}_1^3$ a light-like vector ($a \neq 0$, $\langle a, a \rangle = 0$) with $\langle N, a \rangle > 0$. From Eqs. (7) and (8), we get

$$\Delta \left(\frac{1}{\langle N, a \rangle} \right) = -\frac{\Delta \langle N, a \rangle}{\langle N, a \rangle^2} + \frac{2 \|\nabla \langle N, a \rangle\|^2}{\langle N, a \rangle^3} = 0,$$

that is, $1/\langle N, a \rangle$ is a positive harmonic function globally defined on Σ .

On the other hand, if we choose $b = (0, 0, -1) \in \mathbb{R}_1^3$ then

$$\langle N, b \rangle = \frac{1}{\sqrt{1 - |D_o u|_o^2}} \geq 1.$$

And, with a similar computation, we obtain

$$\Delta \log(1 + \langle N, b \rangle) = \frac{\Delta \langle N, b \rangle}{\langle N, b \rangle + 1} - \frac{\|\nabla \langle N, b \rangle\|^2}{(\langle N, b \rangle + 1)^2} = K. \quad (10)$$

Let us consider now on $\Sigma(u)$ the conformal metric

$$\hat{g} = (1 + \langle N, b \rangle)^2 g,$$

where for simplicity $g = \langle, \rangle$ is the induced Riemannian metric on $\Sigma(u)$ from \mathbb{R}_1^3 . As is well known, the Gaussian curvature \hat{K} of (Σ, \hat{g}) is given by

$$(1 + \langle N, b \rangle)^2 \hat{K} = K - \Delta \log(1 + \langle N, b \rangle).$$

Thus, we can conclude from Eq. (10) that \hat{g} is a flat metric on $\Sigma(u)$.

On the other hand, observe that for every $X \in \mathfrak{X}(\Sigma(u))$ it holds

$$\begin{aligned} \hat{g}(X, X) &\geq \langle N, b \rangle^2 g(X, X) = \langle N, b \rangle^2 (|X^*|_o^2 - \langle D_o u, X^* \rangle_o^2) \\ &\geq \langle N, b \rangle^2 (1 - |D_o u|_o^2) |X^*|_o^2 = |X^*|_o^2 = \frac{1}{2} (g(X, X) + g'(X, X)) \\ &\geq \frac{1}{2} g'(X, X), \end{aligned}$$

where X^* is the projection of X onto the plane $x_3 = 0$ and g' stands for the Riemannian metric induced in $\Sigma(u)$ from the Euclidean 3-dimensional space \mathbb{R}^3 . This implies that $\hat{L} \geq 1/\sqrt{2}L'$, where \hat{L} and L' denote the length of a curve on $\Sigma(u)$ with respect to the Riemannian metrics \hat{g} and g' , respectively. Moreover, since $\Sigma(u)$ is closed in \mathbb{R}_1^3 the metric g' is complete on $\Sigma(u)$. As a consequence, \hat{g} is also complete on $\Sigma(u)$.

Summing up, $(\Sigma(u), \hat{g})$ is a flat and complete surface. Therefore, using Cartan's theorem we have a global isometry from the Euclidean plane \mathbb{R}^2 onto the entire graph $\Sigma(u)$ endowed with the metric \hat{g} . The invariance of harmonic functions under conformal changes of the metric and this isometry allow us to induce $1/\langle N, a \rangle$ on a positive harmonic function on \mathbb{R}^2 , which must be constant by Liouville theorem. Therefore, the function $\langle N, a \rangle$ is constant on $\Sigma(u)$, which implies by Eq. (7) that the maximal graph is totally geodesic, and so it is a space-like plane.

2.3 Alías and Palmer's Proof Based on a Duality Result

Here we present a duality result given by the second author and Palmer in [9] between minimal entire graphs in the Euclidean space \mathbb{R}^3 , that is, entire graphs with vanishing mean curvature function and maximal entire graphs in the Lorentz–Minkowski space \mathbb{R}_1^3 . Specifically, they show how the nonparametric version of the Calabi–Bernstein theorem in \mathbb{R}_1^3 can be seen as a consequence of the classical Bernstein theorem on minimal surfaces in the Euclidean space \mathbb{R}^3 , and viceversa. Before giving this duality result, let us recall that the classical Bernstein theorem states that the only entire minimal graphs in \mathbb{R}^3 are the planes. Equivalently, the only entire solutions to the minimal surface equation

$$\operatorname{Div}_o \left(\frac{D_o u}{\sqrt{1 + |D_o u|_o^2}} \right) = 0 \tag{11}$$

are affine functions.

Theorem 3 ([9, Theorem 3]). *Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected domain. There exists a non-affine \mathcal{C}^2 solution to the minimal surface equation on Ω (11) if and only if there exists a non-affine \mathcal{C}^2 solution to the maximal surface equation on Ω (9).*

Proof. Assume that u is a non-affine solution of the minimal surface equation (11). Recall that for a vector field X on \mathbb{R}^2 it holds that

$$(\operatorname{Div}_o X)dx_1 \wedge dx_2 = d\omega_{J_o X},$$

where J_o denotes the positive $\pi/2$ -rotation in the plane and $\omega_{J_o X}$ denotes the 1-form on \mathbb{R}^2 which is metrically equivalent to the field $J_o X$, that is, $\omega_{J_o X}$ satisfies

$$\omega_{J_o X}(Y) = \langle J_o X, Y \rangle$$

for every $Y \in \mathfrak{X}(\mathbb{R}^2)$. Then Eq. (11) is equivalent to the fact that $\omega_{J_o U}$ is closed on Ω , where U is the field on Ω given by

$$U = \frac{D_o u}{\sqrt{1 + |D_o u|_o^2}}.$$

Then, since the domain Ω is simply connected, we can write

$$J_o U = D_o w \tag{12}$$

for a certain \mathcal{C}^2 function w on Ω . Since J_o is an isometry, there follows

$$|D_o w|_o^2 = \frac{|D_o u|_o^2}{1 + |D_o u|_o^2} < 1, \tag{13}$$

and also

$$1 + |D_o u|_o^2 = \frac{1}{1 - |D_o w|_o^2}. \tag{14}$$

From Eq. (13) we conclude that w defines a space-like graph. Besides, using that $J_o^2 = -\operatorname{id}$, we obtain from Eqs. (12) and (14) that

$$J_o \left(\frac{D_o w}{\sqrt{1 - |D_o w|_o^2}} \right) = \sqrt{1 + |D_o u|_o^2} J_o(D_o w) = D_o(-u).$$

Thus, the maximal surface equation (9) holds on Ω .

Suppose that the function w that we have obtained is affine, then $D_o w$ is a constant vector. Thus, by Eq. (14) $|D_o u|_o$ is also constant, and so it is $D_o u$ because of Eq. (12), contradicting the assumption that u is non-affine.

A similar argument, starting with a non-affine solution of Eq. (9), produces a non-affine solution of Eq. (11) on Ω . \square

In the particular case where we consider entire graphs, that is when $\Omega = \mathbb{R}^2$, it follows from Theorem 3 the equivalence between the Bernstein theorem in \mathbb{R}^3 and the nonparametric version of the Calabi–Bernstein theorem in \mathbb{R}_1^3 .

2.4 Alías and Palmer's Proof Based on a Local Integral Inequality for the Gaussian Curvature

Finally, we present the proof of the parametric version of the classical Calabi–Bernstein theorem given again by the second author jointly with Palmer [10]. The proof is a consequence of the following local bound for the total curvature of geodesic discs in a maximal surface in \mathbb{R}_1^3 .

Theorem 4 ([10, Theorem 1]). *Let $f : \Sigma \rightarrow \mathbb{R}_1^3$ be a maximal surface in the Lorentz–Minkowski space. Let p be a point of Σ , and let $R > 0$ be a positive real number such that the geodesic disc of radius R about p satisfies $D(p, R) \subset \subset \Sigma$. Then, for all $0 < r < R$, the total curvature of the geodesic disc $D(p, r)$ of radius r centered at p satisfies*

$$0 \leq \int_{D(p,r)} K dA \leq c_r \frac{L(r)}{r \log(R/r)}, \quad (15)$$

where $L(r)$ denotes the length of $\partial D(p, r)$, the geodesic disc of radius r centered at p , and

$$c_r = \frac{\pi^2}{8} \frac{(1 + \cosh^2 \rho_r)^2}{\cosh \rho_r \arctan(\cosh \rho_r)} > 0.$$

Here, ρ_r denotes the radius of a geodesic disc in \mathbb{H}^2 containing the hyperbolic image of $D(p, r)$.

Before giving the proof of Theorem 4, let us observe how this result implies the parametric version of the classical Calabi–Bernstein theorem, Theorem 1. Indeed, if Σ is complete, for any arbitrary fixed $p \in \Sigma$ and a fixed r , R can approach to infinity in inequality (15), which taking limits gives

$$\int_{D(p,r)} K dA = 0.$$

Taking into account that from Eq. (3) the Gaussian curvature of a maximal surface in \mathbb{R}_1^3 is always nonnegative, this yields $K = 0$ on Σ , and Σ must be a space-like plane.

The proof of Theorem 4 follows from the following (intrinsic) local integral inequality on an analytic Riemannian metric with nonnegative Gaussian curvature.

Lemma 1 ([10, Lemma 3]). *Let Σ be an analytic surface endowed with an analytic Riemannian metric with nonnegative Gaussian curvature $K \geq 0$. Let u be a smooth function on Σ which satisfies*

$$u \Delta u \geq 0$$

on Σ . Then, for $0 < r < R$,

$$\int_{D_r} u \Delta u \leq \frac{2L(r)}{r \log(R/r)} \sup_{D_R} u^2,$$

where D_r denotes the geodesic disc of radius r centered at a fixed point in Σ , $D_r \subset D_R \subset \subset \Sigma$, and $L(r)$ denotes the length of ∂D_r , the geodesic disc of radius r .

The proof of Lemma 1 follows from [8, Lemma 2.1 and inequality (2.4)].

Proof of Theorem 4. Let us assume that the hyperbolic image of $D(p, r)$ is contained in a geodesic disc $\tilde{D}(a, \rho_r)$ in \mathbb{H}^2 of radius ρ_r and centered at the point $a \in \mathbb{H}^2$. That is,

$$N(D(p, r)) \subseteq \tilde{D}(a, \rho_r) = \{x \in \mathbb{H}^2 \mid 1 \leq -\langle x, a \rangle \leq \cosh \rho_r\},$$

so that

$$1 \leq -\langle N(q), a \rangle \leq \cosh \rho_r$$

for all $q \in D(p, r)$.

Observe that, since $f : \Sigma \rightarrow \mathbb{R}_1^3$ is a maximal surface, Σ is an analytic Riemannian surface with nonnegative Gaussian curvature, so we may apply Lemma 1 to an appropriate smooth function u . The idea of the proof is to apply it to the smooth function on Σ given by $u = \arctan(-\langle N, a \rangle)$. From Eqs. (7) and (8) it can be easily seen that

$$\Delta u = -\frac{1}{1 + \langle N, a \rangle^2} \Delta \langle N, a \rangle + \frac{2\langle N, a \rangle}{(1 + \langle N, a \rangle^2)^2} \|\nabla \langle N, a \rangle\|^2 = \frac{-4K \langle N, a \rangle}{(1 + \langle N, a \rangle^2)^2},$$

and therefore

$$u \Delta u = \frac{-4\langle N, a \rangle \arctan(-\langle N, a \rangle)}{(1 + \langle N, a \rangle^2)^2} K = \phi(-\langle N, a \rangle) K \geq 0, \tag{16}$$

where $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\phi(t) = \frac{4t \arctan t}{(1 + t^2)^2}. \tag{17}$$

It is immediate to check that the function $\phi(t)$ is strictly decreasing for $t \geq 1$. As a consequence, for $t \in [1, \cosh \rho_r]$ $\phi(t)$ is bounded from below by

$$\phi(t) \geq \phi(\cosh \rho_r) = \frac{4 \cosh \rho_r \arctan(\cosh \rho_r)}{(1 + \cosh^2 \rho_r)^2}.$$

Hence, at each point $q \in D(p, r)$ we obtain, from Eq. (16),

$$u(q) \Delta u(q) \geq \frac{4 \cosh \rho_r \arctan(\cosh \rho_r)}{(1 + \cosh^2 \rho_r)^2} K(q) \geq 0.$$

Integrating now this inequality over $D(p, r)$, and using Lemma 1, we conclude that

$$0 \leq \frac{4 \cosh \rho_r \arctan(\cosh \rho_r)}{(1 + \cosh^2 \rho_r)^2} \int_{D(p, r)} K dA \leq \int_{D(p, r)} u \Delta u dA \leq \frac{\pi^2}{2} \frac{L(r)}{r \log(R/r)}.$$

The theorem follows easily from this inequality. □

3 Some Preliminaries on Lorentzian Product Spaces

Let $(M^2, \langle \cdot, \cdot \rangle_M)$ be a connected Riemannian surface and consider the topological product space $M^2 \times \mathbb{R}$ endowed with the Lorentzian metric

$$\langle \cdot, \cdot \rangle = \pi_M^*(\langle \cdot, \cdot \rangle_M) - \pi_{\mathbb{R}}^*(dt^2),$$

where π_M and $\pi_{\mathbb{R}}$ denote the projections from $M \times \mathbb{R}$ onto each factor. For simplicity, we will write simply

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_M - dt^2,$$

and we will denote by $M^2 \times \mathbb{R}_1$ the 3-dimensional product manifold $M^2 \times \mathbb{R}$ endowed with that Lorentzian metric.

The basic notions on maximal surfaces in a Lorentzian product space $M^2 \times \mathbb{R}_1$ that we need along this chapter are similar to the preliminaries stated in Sect. 2 for maximal surfaces in \mathbb{R}_1^3 . However, for the sake of completeness, we will present them again in our new context.

A smooth immersion $f : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$ of a connected surface Σ^2 is said to be a *space-like surface* if f induces a Riemannian metric on Σ , which as usual is also denoted by $\langle \cdot, \cdot \rangle$. In that case, since

$$\partial_t = (\partial/\partial t)_{(x,t)}, \quad x \in M, t \in \mathbb{R}$$

is a unitary time-like vector field, globally defined on the ambient spacetime $M^2 \times \mathbb{R}_1$, then there exists a unique unitary time-like normal field N , globally defined on Σ , which is in the same time orientation as ∂_t , so that

$$\langle N, \partial_t \rangle \leq -1 < 0 \quad \text{on } \Sigma.$$

We will refer to N as the *future-pointing Gauss map* of Σ , and we will denote by $\Theta : \Sigma \rightarrow (-\infty, -1]$ the smooth function on Σ given by $\Theta = \langle N, \partial_t \rangle$. Observe that the function Θ measures the hyperbolic angle θ between the future-pointing vector fields N and ∂_t along Σ . Indeed, they are related by $\cosh \theta = -\Theta$.

Let $\bar{\nabla}$ and ∇ denote the Levi-Civita connections in $M^2 \times \mathbb{R}_1$ and Σ , respectively. Then the Gauss and Weingarten formulae for the space-like surface $f : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y - \langle AX, Y \rangle N \tag{18}$$

and

$$AX = -\bar{\nabla}_X N, \tag{19}$$

for any tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$. Here $A : \mathfrak{X}(\Sigma) \rightarrow \mathfrak{X}(\Sigma)$ stands for the *shape operator* (or *second fundamental form*) of Σ with respect to its future-pointing Gauss map N . As is well known, the Gaussian curvature K of the surface Σ is described in terms of A and the curvature of the ambient spacetime by the Gauss equation, which is given by

$$K = \bar{K} - \det A, \quad (20)$$

where \bar{K} denotes the sectional curvature in $M^2 \times \mathbb{R}_1$ of the tangent plane to Σ . On the other hand, it is not difficult to see that \bar{K} can be written in terms of the Gaussian curvature of M as

$$\bar{K} = \kappa_M \left(1 + \|\partial_t^\top\|^2 \right), \quad (21)$$

where, for simplicity, κ_M stands for the Gaussian curvature of M along the surface Σ ; that is, $\kappa_M = K_M \circ \Pi \in \mathcal{C}^\infty(\Sigma)$, where $\Pi = \pi_M \circ f : \Sigma \rightarrow M$ denotes the projection of Σ onto M and K_M is the Gaussian curvature of M . Here and in what follows, if Z is a vector field along the immersion $f : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$, then $Z^\top \in \mathfrak{X}(\Sigma)$ denotes the tangential component of Z along Σ , that is, $Z = Z^\top - \langle N, Z \rangle N$. In particular, $\partial_t^\top = \partial_t + \Theta N$ and then

$$-1 = \|\partial_t^\top\|^2 - \Theta^2. \quad (22)$$

Therefore, Eq. (21) becomes $\bar{K} = \kappa_M \Theta^2$, and the Gauss equation (20) can be written as

$$K = \kappa_M \Theta^2 - \det A.$$

On the other hand, the Codazzi equation of the space-like surface Σ describes the tangent component of $\bar{R}(X, Y)N$ (\bar{R} being the curvature tensor of $M^2 \times \mathbb{R}_1$), for any tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$, in terms of the derivative of the shape operator. It is given by

$$(\bar{R}(X, Y)N)^\top = (\nabla_X A)Y - (\nabla_Y A)X, \quad (23)$$

where $(\nabla_X A)Y = \nabla_X(AY) - A(\nabla_X Y)$. It is not difficult to see that the curvature tensor of $M^2 \times \mathbb{R}_1$ can be written in terms of κ_M , so that the Codazzi equation (23) becomes

$$(\nabla_X A)Y = (\nabla_Y A)X + \kappa_M \Theta \left(\langle X, \partial_t^\top \rangle Y - \langle Y, \partial_t^\top \rangle X \right).$$

For a space-like surface $f : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$, we will call the *height function* of Σ h to the projection of Σ onto \mathbb{R} . That is, $h \in \mathcal{C}^\infty(\Sigma)$ is the smooth function given by $h = \pi_{\mathbb{R}} \circ f$. Observe that the gradient of $\pi_{\mathbb{R}}$ on $M^2 \times \mathbb{R}_1$ is $\bar{\nabla} \pi_{\mathbb{R}} = -\langle \bar{\nabla} \pi_{\mathbb{R}}, \partial_t \rangle \partial_t = -\partial_t$. Therefore, the gradient of h on Σ is

$$\nabla h = (\bar{\nabla} \pi_{\mathbb{R}})^\top = -\partial_t^\top.$$

Observe that from Eq. (22) we get

$$\|\nabla h\|^2 = \Theta^2 - 1. \quad (24)$$

Since ∂_t is parallel on $M^2 \times \mathbb{R}_1$ we have that

$$\bar{\nabla}_X \partial_t = 0 \quad (25)$$

for any $X \in \mathfrak{X}(\Sigma)$. Writing $\partial_t = -\nabla h - \Theta N$ along Σ and using Gauss (18) and Weingarten (19) formulae, we easily get from Eq. (25) that

$$\nabla_X \nabla h = \Theta AX$$

for every $X \in \mathfrak{X}(\Sigma)$. Thus, the Laplacian of Σ of the height function is given by

$$\Delta h = \Theta \operatorname{tr} A = -2H\Theta, \quad (26)$$

where $H = -\frac{1}{2}\operatorname{tr}(A)$ is the mean curvature function of Σ .

On the other hand, we can compute the gradient and the Laplacian of the function Θ in a similar way as we have done in Sect. 2 for the function $\langle N, a \rangle$. We obtain that

$$\nabla \Theta = A \nabla h, \quad (27)$$

and

$$\Delta \Theta = -2\langle \nabla H, \nabla h \rangle + \Theta (\kappa_M(\Theta^2 - 1) + \|A\|^2). \quad (28)$$

Recall that $f : \Sigma \rightarrow M^2 \times \mathbb{R}_1$ is said to be *maximal* if $H \equiv 0$. In this case Eq. (26) implies that h is a harmonic function on Σ . Besides, if Σ is maximal then

$$A^2 = \frac{1}{2}\|A\|^2 I = -\det A I, \quad (29)$$

where I denotes the identity map on $\mathfrak{X}(\Sigma)$. Therefore, the Gauss equation (20) becomes

$$K = \kappa_M \Theta^2 + \frac{1}{2}\|A\|^2. \quad (30)$$

On the other hand, from Eqs. (24), (27) and (29) we also obtain for a maximal surface that

$$\|\nabla \Theta\|^2 = \frac{1}{2}\|A\|^2 \|\nabla h\|^2 = \frac{1}{2}\|A\|^2(\Theta^2 - 1). \quad (31)$$

A space-like surface Σ is said to be a *slice* if its height function is constant. Equivalently, by Eq. (24), if $\Theta \equiv -1$ on Σ . The family of slices constitutes a foliation of $M^2 \times \mathbb{R}_1$ by totally geodesic surfaces.

4 A Parametric Version of a Calabi–Bernstein Result

We start by stating the following remarkable property.

Lemma 2 ([5, Lemma 3.1]). *Let M^2 be a Riemannian surface. If $M^2 \times \mathbb{R}_1$ admits a complete space-like surface $f : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$, then M is necessarily complete and the projection $\Pi = \pi_M \circ f : \Sigma \rightarrow M$ is a covering map.*

Proof. The proof follows the ideas of the proof of [11, Lemma 3.1] and is a generalization of the argument given in Sect. 2 for the case of the 3-dimensional Lorentz–Minkowski space \mathbb{R}_1^3 . We briefly sketch it here. Let $f : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$ be a space-like surface and consider $\Pi = \pi_M \circ f : \Sigma \rightarrow M$ its projection on M . It is not difficult to see that $\Pi^*(\langle \cdot, \cdot \rangle_M) \geq \langle \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ stands for the Riemannian metric on Σ induced from the Lorentzian ambient space. This means that Π is a local diffeomorphism which increases the distance between the Riemannian surfaces Σ and M . Then, the proof finishes by recalling that if a map, from a connected complete Riemannian manifold M_1 into another connected Riemannian manifold M_2 of the same dimension, increases the distance, then it is a covering map and M_2 is complete [28, Chap. VIII, Lemma 8.1]. \square

In particular, if $M^2 \times \mathbb{R}_1$ admits a compact space-like surface, then M is necessarily compact (see [11, Proposition 3.2 (i)]. An immediate consequence of Eq. (26) in the compact case is the following.

Proposition 1 ([5, Proposition 3.2]). *Let M^2 be a Riemannian surface. If $f : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$ is a compact space-like surface in $M^2 \times \mathbb{R}_1$ whose mean curvature H does not change sign, then it must be a slice $M \times \{t_0\}$, $t_0 \in \mathbb{R}$. In particular, the only compact maximal surfaces in $M^2 \times \mathbb{R}_1$ are the slices.*

Proof. Observe that, since $\Theta < 0$ and H does not change sign, then Eq. (26) says that the height function h must be either subharmonic or superharmonic on Σ , according to the sign of H . But the compactness of Σ implies that h must be constant, so Σ is a slice. \square

Under completeness assumption, we prove the following parametric version of a Calabi–Bernstein result in $M^2 \times \mathbb{R}_1$.

Theorem 5 ([5, Theorem 3.3]). *Let M^2 be a (necessarily complete) Riemannian surface with nonnegative Gaussian curvature, $K_M \geq 0$. Then, any complete maximal surface Σ^2 in $M^2 \times \mathbb{R}_1$ is totally geodesic. In addition, if $K_M > 0$ at some point on M , then Σ is a slice $M \times \{t_0\}$, $t_0 \in \mathbb{R}$.*

Observe first that the assumptions we have made for the Gaussian curvature of M are necessary. In fact, if $M^2 = \mathbb{R}^2$ is the flat Euclidean plane, then $M^2 \times \mathbb{R}_1 = \mathbb{R}_1^3$ is the 3-dimensional Lorentz–Minkowski space, and any space-like affine plane in \mathbb{R}_1^3 which is not horizontal determines a complete totally geodesic surface which is not a slice. On the other hand, the assumption $K_M \geq 0$ is necessary as shown by the fact that there exist examples of non-totally geodesic complete maximal surfaces in $\mathbb{H}^2 \times \mathbb{R}_1$, where \mathbb{H}^2 is the hyperbolic plane (see Examples 1 and 4 in Sect. 6).

In the proof of Theorem 5, the concept of parabolicity is fundamental.

Definition 1. A Riemannian surface Σ is said to be *parabolic* if any nonpositive subharmonic function on Σ is constant.

In general, it is not easy to decide whether a given surface is or not parabolic. Thus, some parabolicity criteria are usually needed. For our purpose, we find very useful the following classical result due to Ahlfors [1] and Blanc–Fiala–Huber [25].

Proposition 2. *Any complete Riemannian surface Σ with nonnegative Gaussian curvature is parabolic.*

Proof of Theorem 5. Since Σ is maximal and $\kappa_M \geq 0$, Eq. (30) implies that $K \geq 0$ on Σ . Then, as Σ is complete by assumption, from Proposition 2 we deduce that Σ is parabolic.

Recall that $\Theta \leq -1 < 0$. From Eqs. (28) and (31) we can compute

$$\Delta \left(\frac{1}{\Theta} \right) = -\frac{\Delta \Theta}{\Theta^2} + \frac{2\|\nabla \Theta\|^2}{\Theta^3} = -\frac{1}{\Theta} \left(\kappa_M(\Theta^2 - 1) + \frac{\|A\|^2}{\Theta^2} \right) \geq 0. \quad (32)$$

That is, $1/\Theta$ is a negative subharmonic function on the parabolic surface Σ , and hence it must be constant. Then, $\Theta = \Theta_o \leq -1$ is also constant on Σ and, by Eq. (32), we also get that $\|A\|^2 = 0$ and $\kappa_M(\Theta_o^2 - 1) = 0$ on Σ . Therefore, Σ is totally geodesic in $M^2 \times \mathbb{R}_1$ and if $\kappa_M > 0$ at some point on Σ , then it must be $\Theta_o = -1$, so Σ is a slice. Finally, observe that since the projection $\Pi : \Sigma \rightarrow M$ is onto (Lemma 2), then $\kappa_M > 0$ at some point on Σ if and only if $\kappa_M > 0$ at some point on M . \square

It is worth pointing out that, recently, Caballero et al. [13] have generalized Theorem 5 in the context of generalized Robertson–Walker spacetimes, which are defined as follows. Given $(M^2, \langle, \rangle_M)$ a connected Riemannian surface, a *generalized Robertson–Walker spacetime* is defined as the topological product $M^2 \times I$ endowed with the Lorentzian metric

$$\langle, \rangle = \rho^2(t)\langle, \rangle_M - dt^2,$$

where I is an open subset of \mathbb{R} and $\rho \in \mathcal{C}^\infty(I)$ is a positive smooth function known as *warping function*. This Lorentzian manifold is usually denoted by $M_\rho^2 \times I_1$.

It is usual to assume some energy conditions on a generalized Robertson–Walker spacetime or, more generally, in a Lorentzian manifold in order to obtain nice mathematical results in a realistic physical context. In that sense, a Lorentzian manifold obeys the *time-like convergence condition* (TCC) if its Ricci tensor, Ric , satisfies

$$\overline{\text{Ric}}(Z, Z) \geq 0,$$

for all time-like vector Z . The TCC condition is the mathematical way to express that gravity, on average, attracts. A weaker energy condition is the *null convergence condition* (NCC) which reads

$$\overline{\text{Ric}}(Z, Z) \geq 0,$$

for any light-like vector Z . Clearly, a continuity argument shows that TCC implies NCC.

Observe that, in the particular case where the warping function is constant, we recover the Lorentzian product spaces. With a straightforward computation we can check in this case that NCC is equivalent to ask the Gaussian curvature of M to be nonnegative. Taking into account these considerations, the generalization of Caballero, Romero and Rubio of Theorem 5 can be stated as follows.

Theorem 6 ([13, Corollary 5.4 and Theorem 5.9]). *Let $M_\rho^2 \times I_1$ be a generalized Robertson–Walker spacetime obeying NCC. Let Σ be a complete maximal surface in M such that $\kappa_M \geq -\rho'(h)^2$ on Σ , then Σ is totally geodesic. Moreover, if NCC holds strictly at some point of Σ then Σ is a totally geodesic space-like slice.*

The proof of Theorem 6 follows the ideas of the proof of our parametric version of the Calabi–Bernstein result, Theorem 5.

5 A Nonparametric Version of a Calabi–Bernstein Result

We need to expose some notions about graphs in $M^2 \times \mathbb{R}_1$. These are analogous to the ones presented in Sect. 2.1 for graphs in \mathbb{R}_1^3 .

Let $\Omega \subseteq M^2$ be a connected domain. Every smooth function $u \in \mathcal{C}^\infty(\Omega)$ determines a graph over Ω given by $\Sigma(u) = \{(x, u(x)) \mid x \in \Omega\} \subset M^2 \times \mathbb{R}_1$. The metric induced on Ω from the Lorentzian metric on the ambient space via $\Sigma(u)$ is given by

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_M - du^2.$$

Therefore, $\Sigma(u)$ is a space-like surface in $M^2 \times \mathbb{R}_1$ if and only if $|Du|^2 < 1$ on Ω , where Du denotes the gradient of u in Ω and $|Du|$ denotes its norm, both with respect to the original metric $\langle \cdot, \cdot \rangle_M$ on Ω . If $\Sigma(u)$ is a space-like graph over a domain Ω , then the future-pointing Gauss map of $\Sigma(u)$ is given by

$$N(x) = \frac{1}{\sqrt{1 - |Du(x)|^2}} (Du(x) + \partial_t|_{(x, u(x))}), \quad x \in \Omega.$$

We can also compute the mean curvature $H(u)$ of the space-like graph $\Sigma(u)$ with respect to N , and it is given by

$$2H(u) = \text{Div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right),$$

where Div stands for the divergence operator on Ω with respect to the metric $\langle \cdot, \cdot \rangle_M$. In particular, $\Sigma(u)$ is a maximal graph if and only if the function u satisfies the following partial differential equation on the domain Ω , usually known as the maximal surface equation:

$$\text{Div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) = 0, \quad |Du|^2 < 1. \quad (33)$$

A graph is said to be entire if $\Omega = M$. As a direct consequence of Lemma 2, it follows that, when M is a complete Riemannian surface which is simply connected, every complete space-like surface in $M^2 \times \mathbb{R}_1$ is an entire graph. In fact, since M is simply connected, then the projection Π is a diffeomorphism between Σ and M , and hence Σ can be written as the graph over M determined by the function

$u = h \circ \Pi^{-1} \in \mathcal{C}^\infty(M)$. However, as we have pointed out in Sect. 2 for the case $M = \mathbb{R}^2$, an entire space-like graph is not necessarily complete (see also Examples 2 and 3 in Sect. 6 for the case when $M^2 = \mathbb{H}^2$). For that reason, Theorem 5 does not imply in principle that, under the same hypothesis on M , any entire maximal graph in $M^2 \times \mathbb{R}_1$ must be totally geodesic. This is certainly true for entire maximal graphs in \mathbb{R}_1^3 as it states the classical Calabi–Bernstein theorem, for entire maximal graphs in Robertson–Walker spaces of the form $\mathbb{R}_\rho^2 \times \mathbb{R}_1$ under certain assumptions on the warping function [29] and for generalized Robertson–Walker spaces satisfying certain assumptions on the geometry of the fiber M and on the warping function [13]. However, none of these cases can be applied to Lorentzian product spaces. We will see now that, although we cannot establish a similar result for entire maximal graphs in $M^2 \times \mathbb{R}_1$ just as a direct consequence of our Theorem 5, we can obtain it as a consequence of the following result.

Theorem 7 ([5, Theorem 4.1]). *Let M^2 be a (not necessarily complete) Riemannian surface with nonnegative Gaussian curvature, $K_M \geq 0$. Then, any maximal surface Σ^2 in $M^2 \times \mathbb{R}_1$ which is complete with respect to the metric induced from the Riemannian product $M^2 \times \mathbb{R}$ is totally geodesic. In addition, if $K_M > 0$ at some point on Σ , then M is necessarily complete and Σ is a slice.*

In particular, if M^2 is complete and $f : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$ is a space-like surface which is properly immersed in $M^2 \times \mathbb{R}_1$, then the metric induced on Σ from the Riemannian product $M^2 \times \mathbb{R}$ is complete. This happens, for instance, when $\Sigma \subset M^2 \times \mathbb{R}_1$ is a closed embedded maximal surface. Thus, it happens for entire maximal graphs, and it yields the following nonparametric version of our Calabi–Bernstein theorem.

Theorem 8 ([5, Theorem 4.3]). *Let M^2 be a complete Riemannian surface with nonnegative Gaussian curvature, $K_M \geq 0$. Then any entire maximal graph $\Sigma(u)$ in $M^2 \times \mathbb{R}_1$ is totally geodesic. In addition, if $K_M > 0$ at some point on M , then u is constant.*

The examples of Sect. 6 show that the assumption $K_M \geq 0$ is necessary.

Proof of Theorem 7. The proof follows the ideas of Romero’s proof for the classical Calabi–Bernstein theorem for maximal surfaces in \mathbb{R}_1^3 (see Sect. 2.2). We will briefly sketch it here.

Let $f : \Sigma \rightarrow M^2 \times \mathbb{R}_1$ be a maximal surface. For simplicity, we denote by $g = \langle, \rangle$ the Riemannian metric induced on Σ from the Lorentzian product $M^2 \times \mathbb{R}_1$. Since $1 - \Theta \geq 2 \geq 0$, we may introduce on Σ the conformal metric

$$\hat{g} = (1 - \Theta)^2 g.$$

The longest part of the proof is to observe that (Σ, \hat{g}) is a complete surface with nonnegative Gaussian curvature, and this is done just by adapting the proof of Romero to our more general situation. We are not going to develop here all the computations, just recall that, in order to prove that (Σ, \hat{g}) has nonnegative Gaussian curvature, we have to compare the Gaussian curvatures of Σ with respect to the

conformal metrics g and \hat{g} . And the proof of the completeness is done by comparison between the metric \hat{g} and the metric induced on Σ from the Riemannian product $M^2 \times \mathbb{R}$, which is complete by assumption.

Once we know that (Σ, \hat{g}) is a complete surface with nonnegative Gaussian curvature, we conclude that (Σ, \hat{g}) is parabolic by the same classical parabolicity criterium of Ahlfors and Blanc, Fiala and Huber used in the proof of Theorem 5. The Laplacian Δ on Σ with respect to g and the Laplacian $\hat{\Delta}$ on Σ with respect to the conformal metric \hat{g} are related by

$$\Delta = (1 - \Theta)^2 \hat{\Delta},$$

which implies that (Σ, g) is also parabolic. The proof then follows as in the proof of Theorem 5. \square

Remark 1. Observe that Theorem 5 can also be seen as a consequence of Theorem 7, since every complete space-like surface Σ in $M^2 \times \mathbb{R}_1$ is also complete with respect to the metric induced from the Riemannian product $M^2 \times \mathbb{R}$. This follows from the fact that $g' \geq g$, where g and g' stand for the metrics induced on Σ from the Lorentzian and the Riemannian product, respectively.

6 Some Nontrivial Entire Maximal Graphs in $\mathbb{H}^2 \times \mathbb{R}_1$

Let us see in this section examples of complete and noncomplete nontrivial entire maximal graphs in $\mathbb{H}^2 \times \mathbb{R}_1$. To do it, we will consider the half-plane model of the hyperbolic plane \mathbb{H}^2 . That is,

$$\mathbb{H}^2 = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$$

endowed with the complete metric

$$\langle \cdot, \cdot \rangle_{\mathbb{H}^2} = \frac{1}{x_2^2} (dx_1^2 + dx_2^2),$$

which is conformal to the flat Euclidean metric. Observe that given a smooth function $u = u(x) \in \mathcal{C}^\infty(\mathbb{H}^2)$, its hyperbolic gradient Du in \mathbb{H}^2 and its Euclidean gradient $D_o u$ in \mathbb{R}^2 are related by

$$Du(x) = x_2^2 D_o u(x), \quad x = (x_1, x_2), \tag{34}$$

and then

$$|Du(x)|^2 = x_2^2 |D_o u(x)|_o^2. \tag{35}$$

On the other hand, the divergence operators of the hyperbolic metric Div and of the Euclidean metric Div_o are related by

$$\text{Div} = \text{Div}_o - \frac{2}{x_2} dx_2.$$

By Eqs. (34) and (35), this implies that the graph determined by u , $\Sigma(u)$, is a maximal surface in $\mathbb{H}^2 \times \mathbb{R}_1$ if and only if

$$(1 - x_2^2 |D_o u|_o^2) x_2^2 \Delta_o u + x_2^2 (x_2 u_{x_2} |D_o u|_o^2 + x_2^2 Q(u)) = 0 \quad \text{and} \quad x_2^2 |D_o u|_o^2 < 1, \quad (36)$$

where Δ_o stands for the Euclidean Laplacian operator and $Q(u) = u_{x_1}^2 u_{x_1 x_1} + 2u_{x_1} u_{x_2} u_{x_1 x_2} + u_{x_2}^2 u_{x_2 x_2}$.

Analogously, it can be seen that $\Sigma(u)$ determines a minimal graph in $\mathbb{H}^2 \times \mathbb{R}$ if and only if

$$(1 + x_2^2 |D_o u|_o^2) x_2^2 \Delta_o u - x_2^2 (x_2 u_{x_2} |D_o u|_o^2 + x_2^2 Q(u)) = 0. \quad (37)$$

Equations (36) and (37) will help us to construct entire maximal graphs in $\mathbb{H}^2 \times \mathbb{R}_1$ in two different ways: via a duality result between minimal graphs in $\mathbb{H} \times \mathbb{R}$ and maximal graphs in $\mathbb{H} \times \mathbb{R}_1$, and in a direct way by looking for concrete families of solutions of the maximal surface equation (36).

6.1 Duality Between Minimal and Maximal Graphs

In Sect. 2.3 we have presented a duality between minimal graphs in the Euclidean space \mathbb{R}^3 and maximal graphs in the Lorentz–Minkowski space \mathbb{R}_1^3 . Following the ideas of the proof of Alías and Palmer, we observe here that the same duality holds in general between solutions to the minimal surface equation in a Riemannian product space $M \times \mathbb{R}$ and solutions to the maximal surface equation in a Lorentzian product space $M \times \mathbb{R}_1$. First of all, recall that a smooth function u on a connected domain $\Omega \subseteq M^2$ defines a minimal graph $\Sigma(u)$ in $M \times \mathbb{R}$ if and only if u satisfies the following partial differential equation on Ω :

$$\text{Div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0, \quad (38)$$

where, as in Eq. (33), Div and Du stand for the divergence operator and the gradient of u in Ω with respect to the metric $\langle \cdot, \cdot \rangle_M$, respectively. In this context our more general duality result reads as follows.

Theorem 9 ([5, Theorem 5.1]). *Let $\Omega \subseteq M^2$ be a simply connected domain of a Riemannian surface M^2 . There exists a nontrivial solution u to the minimal surface equation on Ω (38) if and only if there exists a nontrivial solution w to the maximal surface equation on Ω (33).*

Here by a *nontrivial solution* we mean a solution with nonparallel gradient. With a straightforward computation, it can be seen that nontrivial solutions to either the minimal or maximal surface equation correspond to non-totally geodesic either minimal or maximal graphs.

Proof. Since Ω is simply connected, it is orientable and can be endowed with a globally defined area form $d\Omega$ and an almost complex structure J . The key of the proof is to observe that, as happened in the case $M = \mathbb{R}^2$, in this more general case it also holds that

$$\operatorname{Div} X d\Omega = d\omega_{JX},$$

where ω_{JX} denotes the 1-form in Ω which is metrically equivalent to the field JX , that is,

$$\omega_{JX}(Y) = \langle JX, Y \rangle_M.$$

From that point, the proof follows exactly as in Sect. 2.3. □

Remark 2. Following the philosophy of our result, Lee [30, Theorem 2] has very recently constructed an analogous duality result between graphs of constant mean curvature H in the Bianchi–Cartan–Vranceanu space $\mathbb{R}^3(\kappa, \tau)$ and space-like graphs of constant mean curvature τ in the Lorentzian Bianchi–Cartan–Vranceanu space $\mathbb{R}_1^3(\kappa, H)$. These spaces are defined as

$$\mathbb{R}^3(\kappa, \tau) = \left(V, \frac{dx_1^2 + dx_2^2}{\left(1 + \frac{\kappa}{4}(x_1^2 + x_2^2)\right)^2} + \left(\tau \frac{x_2 dx_1 - x_1 dx_2}{1 + \frac{\kappa}{4}(x_1^2 + x_2^2)} + dx_3 \right)^2 \right)$$

and

$$\mathbb{R}_1^3(\kappa, \tau) = \left(V, \frac{dx_1^2 + dx_2^2}{\left(1 + \frac{\kappa}{4}(x_1^2 + x_2^2)\right)^2} - \left(\tau \frac{x_2 dx_1 - x_1 dx_2}{1 + \frac{\kappa}{4}(x_1^2 + x_2^2)} + dx_3 \right)^2 \right),$$

where $V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 1 + \frac{\kappa}{4}(x_1^2 + x_2^2) > 0\}$.

In particular, observe that $\mathbb{R}^3(0, 0) = \mathbb{R}^3$, $\mathbb{R}^3(1, 0) = \mathbb{S}^2 \times \mathbb{R}$, $\mathbb{R}^3(-1, 0) = \mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{R}_1^3(0, 0) = \mathbb{R}_1^3$, $\mathbb{R}_1^3(1, 0) = \mathbb{S}^2 \times \mathbb{R}_1$, $\mathbb{R}_1^3(-1, 0) = \mathbb{H}^2 \times \mathbb{R}_1$. Therefore, the classical duality between minimal graphs in \mathbb{R}^3 and maximal graphs in \mathbb{R}_1^3 and the duality of Theorem 9 when $M^2 = \mathbb{S}^2$ or $M^2 = \mathbb{H}^2$ are particular cases of Lee’s duality.

The interest of Theorem 9 resides on the fact that it allows us to construct new solutions to the maximal surface equation from known solutions to the minimal surface equation and viceversa. In particular, as an application of it, we are able to construct counterexamples which show that our Calabi–Bernstein results are no longer true without the assumption $K_M \geq 0$.

It is immediate to check that the functions

$$u(x_1, x_2) = \log(x_1^2 + x_2^2) \tag{39}$$

and

$$u(x_1, x_2) = \frac{x_1}{x_1^2 + x_2^2} \tag{40}$$

satisfy the minimal surface equation (37), so that they define nontrivial entire minimal graphs in the Riemannian product space $\mathbb{H}^2 \times \mathbb{R}$. As far as we know, these examples, which are due to Montaldo and Onnis [34, Examples 2.3 and 2.4] (see also [37]), are the first explicit examples of nontrivial entire minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$. Previous existence results for such minimal graphs have been given in [17, 35].

Example 1 ([5, Example 5.2]). From Theorem 9 and the entire minimal graph defined by the function (39), we know that there exists a smooth function $w \in \mathcal{C}^\infty(\mathbb{H}^2)$ which determines a nontrivial entire maximal graph in $\mathbb{H}^2 \times \mathbb{R}_1$. This shows that the assumption $K_M \geq 0$ in Theorem 5 is necessary. Moreover, we claim that the entire maximal graph determined by w is also complete, in the sense that the induced metric on \mathbb{H}^2 via the graph is complete. To see this, denote by $\langle \cdot, \cdot \rangle$, as usual, the metric given by

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathbb{H}^2} - dw^2.$$

Then, for every $X \in \mathfrak{X}(\mathbb{H}^2)$ we have

$$\langle X, X \rangle = \langle X, X \rangle_{\mathbb{H}^2} - X(w)^2 = \langle X, X \rangle_{\mathbb{H}^2} - \langle X, Dw \rangle_{\mathbb{H}^2}^2,$$

and using the Cauchy–Schwarz inequality we get

$$\langle X, X \rangle \geq \langle X, X \rangle_{\mathbb{H}^2} (1 - |Dw|^2). \quad (41)$$

Because of the duality between $\Sigma(u)$ and $\Sigma(w)$, the norm of the gradients of u and w satisfies the analogous relation to Eq. (13). That is,

$$|Dw|^2 = \frac{|Du|^2}{1 + |Du|^2} = \frac{x_2^2 |D_o u|_o^2}{1 + x_2^2 |D_o u|_o^2}.$$

Therefore, we easily compute from Eq. (39) that

$$1 - |Dw|^2 = \frac{x_1^2 + x_2^2}{x_1^2 + 5x_2^2} \geq \frac{1}{5},$$

which jointly with Eq. (41) implies

$$\langle \cdot, \cdot \rangle \geq \frac{1}{5} \langle \cdot, \cdot \rangle_{\mathbb{H}^2}.$$

Since $\langle \cdot, \cdot \rangle_{\mathbb{H}^2}$ is complete, we conclude that the metric $\langle \cdot, \cdot \rangle$ is also complete on \mathbb{H}^2 , as claimed. This shows that the assumption $K_M \geq 0$ in Theorem 8 is also necessary.

It is even possible to get w explicitly. In fact, in a similar way as Eq. (12), the duality allows us to express the gradient of w in terms of the gradient of u as

$$D_o w = (w_{x_1}, w_{x_2}) = \frac{1}{\sqrt{1 + x_2^2 |D_o u|_o^2}} J_o(D_o u),$$

where, as before, J_o denotes the positive $\pi/2$ -rotation on the plane. Then, we can compute

$$w_{x_1} = -\sqrt{\frac{x_1^2 + x_2^2}{x_1^2 + 5x_2^2}} u_{x_2} = -\frac{2x_2}{\sqrt{(x_1^2 + x_2^2)(x_1^2 + 5x_2^2)}}, \quad (42)$$

and

$$w_{x_2} = \sqrt{\frac{x_1^2 + x_2^2}{x_1^2 + 5x_2^2}} u_{x_1} = \frac{2x_1}{\sqrt{(x_1^2 + x_2^2)(x_1^2 + 5x_2^2)}}. \quad (43)$$

We can explicitly integrate Eqs. (42) and (43) obtaining

$$w(x_1, x_2) = i \frac{2}{\sqrt{5}} F\left(\arcsin\left(\frac{x_1}{x_2}\right), \frac{1}{\sqrt{5}}\right) + c,$$

where c is a real constant, i stands for the imaginary unit and $F(\phi, k)$ stands for the elliptic integral of the first kind with elliptic modulus k and Jacobi amplitude ϕ . See Fig. 1 (left) for a picture of the graph in the case $c = 0$.

Example 2 ([5, Example 5.3]). Similarly, the nontrivial entire minimal graph in $\mathbb{H}^2 \times \mathbb{R}$ defined by the function (40) gives rise, via Theorem 9, to another nontrivial entire maximal graph in the Lorentzian product $\mathbb{H}^2 \times \mathbb{R}_1$. In contrast to Example 1, this example is not complete. To see this, let $w \in \mathcal{C}^\infty(\mathbb{H}^2)$ stand for the smooth function defining this entire maximal graph, which we denote as usual by $\Sigma(w)$. In an analogous way as in Example 1 we can compute

$$w_{x_1} = -\frac{x_1^2 + x_2^2}{\sqrt{(x_1^2 + x_2^2)^2 + x_2^2}} u_{x_2} = \frac{2x_1 x_2}{(x_1^2 + x_2^2) \sqrt{(x_1^2 + x_2^2)^2 + x_2^2}}, \quad (44)$$

and

$$w_{x_2} = \frac{x_1^2 + x_2^2}{\sqrt{(x_1^2 + x_2^2)^2 + x_2^2}} u_{x_1} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2) \sqrt{(x_1^2 + x_2^2)^2 + x_2^2}}. \quad (45)$$

Let $\alpha : (0, 1) \rightarrow \Sigma(w)$ be the divergent curve in $\Sigma(w)$ given by

$$\alpha(s) = (0, s, w(0, s)).$$

Then $\alpha'(s) = (0, 1, w_{x_2}(0, s))$ and

$$\|\alpha'(s)\|^2 = \frac{1}{s^2} - w_{x_2}(0, s)^2 = \frac{1}{1 + s^2},$$

which implies that α has finite length since

$$\int_0^1 \|\alpha'(s)\| ds = \int_0^1 \frac{ds}{\sqrt{1 + s^2}} = \operatorname{arcsinh}(1) = \log(1 + \sqrt{2}).$$

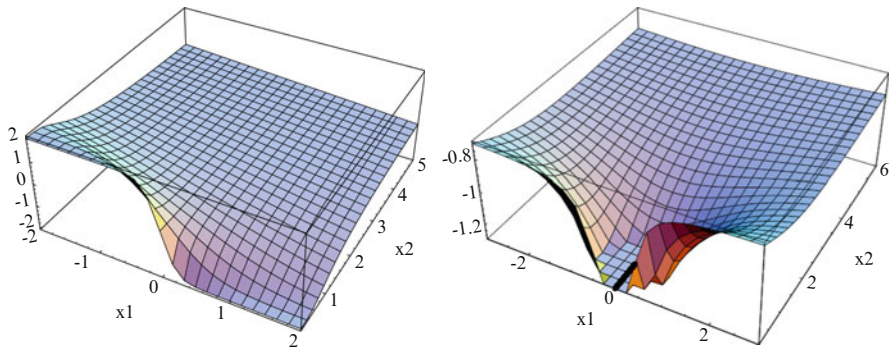


Fig. 1 The complete entire maximal graph of Example 1 and the noncomplete one of Example 2 with its divergent curve of finite length

As a consequence, $\Sigma(w)$ is not complete. Let us recall that such fact cannot occur in the Lorentz–Minkowski space \mathbb{R}_1^3 , since by a result of Cheng and Yau [15], closed surfaces in \mathbb{R}_1^3 with constant mean curvature are necessarily complete.

Again, we can explicitly integrate Eqs. (44) and (45), getting

$$w(x_1, x_2) = \log \left(\frac{x_1^2 + x_2^2}{2 \left(x_2 + \sqrt{x_2^2 + (x_1^2 + x_2^2)^2} \right)} \right) + c,$$

where c is a real constant. See Fig. 1 (right) for a picture of the graph in the case $c = 0$.

Remark 3. Recently, Gálvez and Rosenberg [23] have showed the existence of nontrivial entire minimal graphs in a Riemannian product space $M^2 \times \mathbb{R}$, where M^2 is a complete, simply connected, Riemannian surface with not necessarily constant Gaussian curvature bounded from above by a negative constant. That is, $K_M \leq c < 0$. By our duality result, Theorem 9, we deduce the existence of nontrivial entire maximal graphs in the corresponding Lorentzian product space $M^2 \times \mathbb{R}_1$. Therefore, the nonparametric version of our Calabi–Bernstein theorem is no longer true in such ambient spaces.

6.2 More Examples

Let us show now some explicit entire solutions to the maximal surface equation (36) by solving it for particular types of smooth functions.

Example 3 ([3, Example 3.1]). First, we will look for solutions depending only on one variable. If we consider solutions of the type $u(x_1, x_2) = u(x_1)$, Eq. (36)

reduces to $u''(x_1) = 0$, so u must be of the form $u(x_1, x_2) = ax_1 + b$, $a, b \in \mathbb{R}$. By the space-like condition, we get $x_2^2 u'(x_1)^2 = a^2 x_2^2 < 1$. Thus, as we are looking for entire graphs, the inequality is only satisfied in \mathbb{H}^2 for $a = 0$. Consequently, we only obtain slices.

More interesting is the case where u only depends on the second variable, $u(x_1, x_2) = u(x_2)$. Then, u will determine a maximal surface if and only if it satisfies

$$u''(x_2) + x_2 u'(x_2)^3 = 0 \tag{46}$$

and

$$x_2^2 u'(x_2)^2 < 1. \tag{47}$$

Integrating Eq. (46) we get

$$u(x_1, x_2) = u(x_2) = \log \left(x_2 + \sqrt{a + x_2^2} \right) + b, \quad a, b \in \mathbb{R}, a \geq 0.$$

Moreover, from Eq. (47), u must satisfy

$$x_2^2 u'(x_2)^2 = \frac{x_2^2}{a + x_2^2} < 1,$$

which holds if and only if $a > 0$. Observe that we can assume $b = 0$ up to a translation, which is an isometry of the ambient space. Thus, we have obtained a family of entire maximal graphs, $\Sigma_a(u)$ over \mathbb{H}^2 for all $a > 0$. Let us see that, as in Example 2 these graphs are not complete. In fact, consider $\alpha : (1, \infty) \rightarrow \Sigma_a(u)$ the divergent curve in $\Sigma_a(u)$ given by

$$\alpha(s) = (0, s, u(s)).$$

Then, $\alpha'(s) = (0, 1, u'(s))$ and

$$\|\alpha'(s)\|^2 = \frac{1}{s^2} - \frac{1}{a + s^2} = \frac{a}{s^2(a + s^2)}.$$

Therefore, α has finite length since

$$\int_1^\infty \|\alpha'(s)\| ds = \int_1^\infty \sqrt{\frac{a}{s^2(a + s^2)}} ds = \operatorname{arcsinh}(\sqrt{a}),$$

being $\Sigma_a(u)$ noncomplete. See Fig. 2 (left) for a picture of this graph in the case $a = 1/2$ and $b = 0$.

Example 4 ([3, Example 3.2]). Let us consider now radial solutions to the maximal surface equation. That is, solutions of type

$$u(x_1, x_2) = f(x_1^2 + x_2^2).$$

In this case, Eq. (36) becomes

$$f'(z) + zf''(z) = 0, \quad (48)$$

and

$$4x_2^2 z f'(z)^2 < 1, \quad (49)$$

where $z = x_1^2 + x_2^2 > 0$. From Eq. (48)

$$f(z) = a \log z + b, \quad a, b \in \mathbb{R},$$

and then Eq. (49) results

$$\frac{4a^2 x_2^2}{x_1^2 + x_2^2} < 1$$

in \mathbb{H}^2 , so $a^2 < 1/4$. Therefore, the functions

$$u(x_1, x_2) = a \log(x_1^2 + x_2^2) + b, \quad a, b \in \mathbb{R}, \quad -1/2 < a < 1/2,$$

define a new family of entire maximal graphs in $\mathbb{H}^2 \times \mathbb{R}_1$. We can assume again that, up to an isometry, $b = 0$.

Let us see that these graphs are complete. Observe that, for every $X \in \mathfrak{X}(\mathbb{H}^2)$, we obtain as in Eq. (41)

$$\langle X, X \rangle \geq \langle X, X \rangle_{\mathbb{H}^2} (1 - |Du|^2). \quad (50)$$

As $a^2 < 1/4$, we easily get

$$1 - |Du|^2 = 1 - \frac{4a^2 x_2^2}{x_1^2 + x_2^2} \geq 1 - 4a^2 > 0,$$

which jointly with Eq. (50) implies

$$\langle X, X \rangle \geq (1 - 4a^2) \langle X, X \rangle_{\mathbb{H}^2},$$

so we conclude that \langle, \rangle is complete following a similar argument as the one in Example 1. See Fig. 2 for a picture of this graph when $a = 1/4$ and $b = 0$.

7 Relative Parabolicity of Maximal Surfaces

Along this section, we will consider maximal surfaces with nonempty smooth boundary. For such surfaces, we can give the following definition related to the concept of parabolicity given in Definition 1 in Sect. 4.

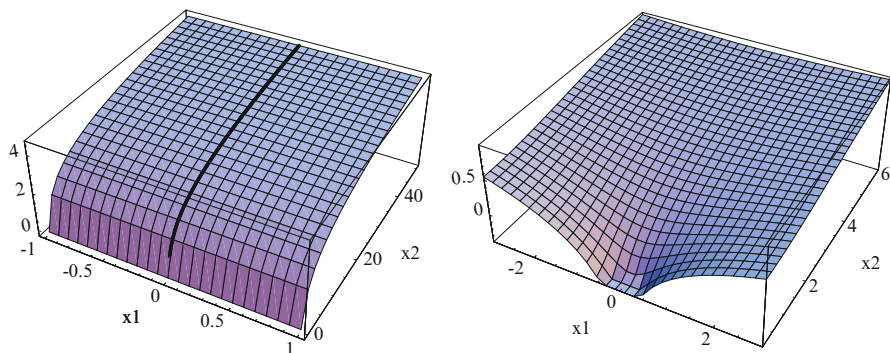


Fig. 2 The noncomplete entire maximal graph of Example 3 with its divergent curve of finite length, and the complete one of Example 4

Definition 2. A Riemannian surface Σ with nonempty smooth boundary, $\partial\Sigma \neq \emptyset$, is said to be *relatively parabolic* if every bounded harmonic function on Σ is determined by its boundary values.

It is interesting to observe that both concepts of parabolicity: parabolicity and relative parabolicity are related in the following way, (see [33] and [24, Theorem 5.1]).

Proposition 3. A Riemannian surface Σ without boundary is parabolic if and only if for every nonempty open set $O \subset \Sigma$ with smooth boundary, $\Sigma \setminus O$ is relatively parabolic.

Example 5. It is well known that \mathbb{R}^2 is a parabolic surface and \mathbb{H}^2 is not parabolic (see for instance [26]). Therefore, by Proposition 3, any open subset with nonempty smooth boundary of \mathbb{R}^2 is a relatively parabolic surface, and it exists at least a nonempty open subset $O \subset \mathbb{H}^2$ with smooth boundary such that $\mathbb{H}^2 \setminus O$ is not relatively parabolic. In fact, it is not difficult to prove, with a similar argument as the one in [26], that any connected, unbounded subset $O \subset \mathbb{H}^2$ with nonempty smooth boundary is not a relatively parabolic surface.

It is not easy to decide again whether a given surface with nonempty smooth boundary is or not relatively parabolic. The following criterium is very useful.

Lemma 3 ([6, Lemma 2]). Let Σ^2 be a Riemannian surface with nonempty smooth boundary, $\partial\Sigma \neq \emptyset$. If there exists a proper continuous function $\psi : \Sigma \rightarrow \mathbb{R}$ which is eventually positive and superharmonic, then Σ is relatively parabolic.

As is usual, by eventually we mean here a property that is satisfied outside a compact set.

The proof of Lemma 3 follows the ideas of the proof of an analogous criterium for proper smooth functions given by Meeks and Pérez [33, 38]. In fact, we only have to observe that the proof of Meeks and Pérez only depends on the minimum

principle for superharmonic functions and that this principle holds for continuous functions (not necessarily smooth).

Fernández and López [21, Sect. 4] have recently proved that properly immersed maximal surfaces with nonempty boundary in the Lorentz–Minkowski spacetime \mathbb{R}_1^3 are relatively parabolic if the Lorentzian norm on the maximal surface in \mathbb{R}_1^3 is eventually positive and proper. Motivated by this work, we study some relative parabolicity criteria for maximal surfaces in Lorentzian product spaces. Observe that a natural generalization of the Lorentzian norm on a surface in \mathbb{R}_1^3 to the Lorentzian product $M^2 \times \mathbb{R}_1$ consists in considering the function $\phi = r^2 - h^2$, where the function r measures the distance on the factor M to a fixed point $x_o \in M$

Before describing our main result in this section, we need the following technical observation. Given any function $\hat{\psi} \in \mathcal{C}^\infty(M)$, we can consider its lifting $\bar{\psi} \in \mathcal{C}^\infty(M^2 \times \mathbb{R}_1)$ defined by

$$\bar{\psi}(x, t) = \hat{\psi}(x).$$

Let $f : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$ be a space-like surface. Then, we can also associate to $\hat{\psi} \in \mathcal{C}^\infty(M)$ the function $\psi \in \mathcal{C}^\infty(\Sigma)$ given by $\psi = \bar{\psi} \circ f$. In this context, the Laplacian on Σ of ψ can be expressed in terms of the Laplacian $\bar{\Delta}$ of $\bar{\psi}$ and the differential operators of $\hat{\psi}$ as follows.

Lemma 4 ([6, Lemma 1]). *Along a space-like surface $f : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$ we have that*

$$\Delta \psi = \bar{\Delta} \bar{\psi} + 2H \langle N^*, D\hat{\psi} \rangle_M + D^2 \hat{\psi}(N^*, N^*),$$

where $N^* = \pi_M^*(N) = N + \Theta \partial_t$, and D and D^2 denote the gradient and the Hessian operators on M , respectively.

Proof. Since $\bar{\nabla} \bar{\psi} = \nabla \psi - \langle \bar{\nabla} \bar{\psi}, N \rangle N$, we get from the Gauss (18) and Weingarten (19) formulae that the Hessian operators of $\bar{\psi}$ and ψ satisfy

$$\bar{\nabla}^2 \bar{\psi}(X, X) = \nabla^2 \psi(X, X) + \langle AX, X \rangle \langle \bar{\nabla} \bar{\psi}, N \rangle$$

for every $X \in \mathfrak{X}(\Sigma)$. Therefore, it can be easily seen that

$$\bar{\Delta} \bar{\psi} = \Delta \psi - 2H \langle \bar{\nabla} \bar{\psi}, N \rangle - \bar{\nabla}^2 \bar{\psi}(N, N). \tag{51}$$

Observe now that, as the function $\bar{\psi}$ does not depend on t , then $\bar{\nabla} \bar{\psi}(x, t) = D\hat{\psi}(x)$. Thus, $\bar{\nabla}_N \bar{\nabla} \bar{\psi} = D_{N^*} D\hat{\psi}$ and

$$\bar{\nabla}^2 \bar{\psi}(N, N) = D^2 \hat{\psi}(N^*, N^*),$$

so that Lemma 4 follows directly from Eq. (51). □

Following the notation above, consider the function $\hat{r} : M^2 \rightarrow \mathbb{R}$ defined by $\hat{r}(x) = \text{dist}_M(x, x_o)$, where $x_o \in M$ is a fixed point. Observe that $\hat{r} \in \mathcal{C}^\infty(M)$ almost everywhere. Actually, \hat{r} is smooth on $M^2 \setminus \text{Cut}(x_o)$, where $\text{Cut}(x_o)$ stands for the cut locus of x_o . As is well known, $\dim \text{Cut}(x_o) < 2$ and $\text{Cut}(x_o)$ is a null set. Let

$\bar{r}(x, t) = \hat{r}(x)$ denote the lifting of \hat{r} to $M^2 \times \mathbb{R}_1$, and for a given space-like surface $f : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$, let r stand for the restriction of \bar{r} to Σ , $r = \bar{r} \circ f$. By Lemma 2 Π is a covering map. Therefore, $\dim \Pi^{-1}(\text{Cut}(x_o)) = \dim \text{Cut}(x_o) < 2$ and the function r is smooth almost everywhere in Σ .

Our main result in this section is the following relative parabolicity criterium.

Theorem 10 ([6, Theorem 3]). *Let M^2 be a complete Riemannian surface with nonnegative Gaussian curvature. Consider $f : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$ a maximal surface with nonempty smooth boundary, $\partial \Sigma \neq \emptyset$, and assume that the function $\phi : \Sigma \rightarrow \mathbb{R}$ defined by*

$$\phi(p) = r^2(p) - h^2(p)$$

is eventually positive and proper. Then Σ is relatively parabolic.

It is worth pointing out that the assumption on the nonnegativity of the Gaussian curvature of M is necessary. Actually, let $M^2 = \mathbb{H}^2$ and consider $\Omega \subset \mathbb{H}^2$ an unbounded connected domain with smooth boundary. Then, for a fixed $t_o \in \mathbb{R}$, the slice $\Sigma_{t_o} = \{(x, t_o) \in \mathbb{H}^2 \times \mathbb{R}_1 : x \in \Omega\}$ is a non-relatively parabolic maximal surface in $\mathbb{H}^2 \times \mathbb{R}_1$ on which ϕ is eventually positive and proper.

Proof. Let $a > 1$ and consider $K = \{p \in \Sigma : \phi(p) \leq a\} \subseteq \Sigma$. Since ϕ is eventually positive and proper, K is a compact set. As is well known, relative parabolicity is not affected by adding or removing compact subsets, so that Σ is relatively parabolic if and only if $\Sigma \setminus K$ is relatively parabolic.

The function $\log \phi : \Sigma \setminus K \rightarrow \mathbb{R}$ is a proper positive function on $\Sigma \setminus K$. Therefore, in order to prove that $\Sigma \setminus K$ is relatively parabolic, it suffices to see that $\log \phi$ is superharmonic on the dense subset $\Sigma' \subset \Sigma \setminus K$ where it is smooth. In what follows, we will work on that subset Σ' . From Eqs. (24) and (26) we get

$$\Delta h^2 = 2h\Delta h + 2\|\nabla h\|^2 = 2(\Theta^2 - 1). \tag{52}$$

On the other hand, applying Lemma 4 to $\psi = r^2$ we get

$$\Delta r^2 = \bar{\Delta} \bar{r}^2 + \hat{\nabla}^2 \hat{r}^2(N^*, N^*). \tag{53}$$

As the function \bar{r} does not depend on t , then $\bar{\nabla} \bar{r}(x, t) = D\hat{r}(x)$ and $\bar{\Delta} \bar{r}(x, t) = \hat{\Delta} \hat{r}(x)$, where $\hat{\Delta}$ denotes the Laplacian operator with respect to the metric \langle, \rangle_M . Therefore,

$$\bar{\Delta} \bar{r}^2(x, t) = 2\bar{r}(x, t)\bar{\Delta} \bar{r}(x, t) + 2\|\bar{\nabla} \bar{r}(x, t)\|^2 = 2(\hat{r}(x)\hat{\Delta} \hat{r}(x) + 1),$$

since, as is well known, $\|\bar{\nabla} \bar{r}\|^2 = |D\hat{r}|^2 = 1$. And, after a long but straightforward computation, Eq. (53) becomes

$$\Delta r^2 = 2(r\hat{\Delta} \hat{r}(1 + \langle N^*, \tau \rangle_M^2) + 1 + \langle N^*, D\hat{r} \rangle_M^2), \tag{54}$$

for $\tau \perp_M D\hat{r}$ and $|\tau| = 1$.

Now, from Eqs. (52) and (54) we get that

$$\begin{aligned} \frac{1}{2}\Delta\phi &= \frac{1}{2}\Delta r^2 - \frac{1}{2}\Delta h^2 \\ &= r\hat{\Delta}\hat{r}(1 + \langle N^*, \tau \rangle_M^2) + \langle N^*, D\hat{r} \rangle_M^2 + 2 - \Theta^2. \end{aligned} \quad (55)$$

As M^2 is complete and has nonnegative Gaussian curvature, by the Laplacian comparison theorem we have that $D\hat{r} \leq 1/\hat{r}$, so that

$$r\hat{\Delta}\hat{r} \leq 1$$

on Σ' . Using this in Eq. (55), we obtain that

$$\frac{1}{2}\Delta\phi \leq \|N^*\|^2 + 3 - \Theta^2 = 2, \quad (56)$$

since $\|N^*\|^2 = \Theta^2 - 1$.

On the other hand, $\nabla\phi = 2r\nabla r - 2h\nabla h$, and so we easily obtain

$$\begin{aligned} \|\nabla\phi\|^2 &= 4r^2\|\nabla r\|^2 - 8rh\langle\nabla r, \nabla h\rangle + 4h^2\|\nabla h\|^2 \\ &= 4r^2(1 + \langle\bar{\nabla}\bar{r}, N\rangle^2) + 8rh\Theta\langle\bar{\nabla}\bar{r}, N\rangle + 4h^2(\Theta^2 - 1) \\ &= 4\phi + 4(r\langle\bar{\nabla}\bar{r}, N\rangle + h\Theta)^2 \geq 4\phi. \end{aligned} \quad (57)$$

Therefore, from Eqs. (56) and (57) we finally get

$$\Delta \log \phi = \frac{1}{\phi^2}(\phi\Delta\phi - \|\nabla\phi\|^2) \leq 0,$$

which means that $\log \phi$ is a superharmonic function on Σ' , as we wanted to prove. As a consequence, $\Sigma \setminus K$ is relatively parabolic, so Σ is also relatively parabolic as observed in the beginning of the proof. \square

From the existence relation between parabolicity and relative parabolicity, we can state the following consequence of Theorem 10.

Corollary 1 ([6, Corollary 6]). *Let M^2 be a complete Riemannian surface with nonnegative Gaussian curvature, and let Σ be a maximal surface in $M^2 \times \mathbb{R}_1$ without boundary, $\partial\Sigma = \emptyset$. If the function $\phi = r^2 - h^2$ is eventually positive and proper on Σ , then Σ is parabolic.*

Proof. The proof follows from Proposition 3 and the observation that, for any non-empty open set $O \subset \Sigma$ with smooth boundary, the function ϕ restricted to $\Sigma \setminus O$ is also eventually positive and proper on $\Sigma \setminus O$. Therefore, the maximal surface with boundary $\Sigma \setminus O$ is relatively parabolic by Theorem 10, and Σ is a parabolic surface.

7.1 Relative Parabolicity and Entire Maximal Graphs

Consider $\Omega \subseteq M$ a connected domain and let $x_o \in \text{int}(\Omega)$. We will say that Ω is *star-like with respect to x_o* if for every $x \in \Omega$ there exists a (not necessarily unique) minimizing geodesic segment from x_o to x which is contained in Ω . Obviously, if M is a complete Riemannian surface, then M itself is star-like with respect to any of its points.

Let us see that we can apply the relative parabolicity criterium, Theorem 10, to space-like graphs over a star-like domain.

Proposition 4 ([6, Proposition 3]). *Let M^2 be a complete Riemannian surface and let $\Sigma(u)$ be a space-like graph over a domain Ω which is star-like with respect to some point $x_o \in \text{int}(\Omega)$. Then the function $\phi = r^2 - h^2$ is eventually positive and proper on $\Sigma(u)$.*

Proof. We may assume, without loss of generality, that $u(x_o) = 0$. Since $\Sigma(u)$ is homeomorphic to Ω (via the standard embedding $x \in \Omega \mapsto (x, u(x)) \in \Sigma(u)$), and the thesis of our result is topological, it suffices to prove that the function $\varphi = \pi_M \circ f \circ \phi = \hat{r}^2 - u^2$ is eventually positive and proper on Ω .

Firstly, we will prove that φ is positive for every $x \in \Omega - \{x_o\}$. For a given $x \neq x_o$, consider $\gamma: [0, l] \rightarrow \Omega$ a minimizing geodesic segment such that $\gamma(0) = x_o$, $\gamma(l) = x$ and $l = \text{dist}_M(x_o, x) = \hat{r}(x) > 0$. Let $\alpha(s) = (\gamma(s), u(s)) \in \Sigma(u)$, where $u(s) := u(\gamma(s))$. Since $\Sigma(u)$ is a space-like surface, $\alpha'(s) = (\gamma'(s), u'(s)) \neq (0, 0)$ is a nonvanishing space-like vector, that is,

$$\langle \alpha'(s), \alpha'(s) \rangle = |\gamma'(s)|^2 - u'(s)^2 = 1 - u'(s)^2 > 0.$$

Therefore, $-1 < u'(s) < 1$ for every $0 \leq s \leq l = \hat{r}(x)$, and integrating we get

$$-\hat{r}(x) < u(x) < \hat{r}(x).$$

Consequently, $\varphi(x) > 0$ for every $x \in \Omega$, $x \neq x_o$.

It remains to prove that φ is proper. Let us define

$$\mathcal{W} := \{(x, t) \in \Omega \times \mathbb{R} : \hat{r}^2(x) - t^2 \geq 0\},$$

and consider on $M^2 \times \mathbb{R}$ the standard Riemannian metric, $\langle \cdot, \cdot \rangle_M + dt^2$. Let us denote by $\text{dist}_+(\cdot, \cdot)$ the distance related to such Riemannian metric. Let us see now that

$$\text{dist}_+((x, t), \partial \mathcal{W}) = \frac{1}{\sqrt{2}} \min\{\hat{r}(x) - t, \hat{r}(x) + t\} = \frac{1}{\sqrt{2}}(\hat{r}(x) - |t|) \quad (58)$$

for every $(x, t) \in \mathcal{W}$. Observe that $\partial \mathcal{W} = \partial \mathcal{W}^+ \cup \partial \mathcal{W}^-$, where

$$\partial \mathcal{W}^+ = \{(x, \hat{r}(x)) : x \in \Omega\} \quad \text{and} \quad \partial \mathcal{W}^- = \{(x, -\hat{r}(x)) : x \in \Omega\}.$$

Therefore,

$$\text{dist}_+((x,t), \partial\mathcal{W}) = \min\{\text{dist}_+((x,t), \partial\mathcal{W}^+), \text{dist}_+((x,t), \partial\mathcal{W}^-)\}.$$

Expression (58) is trivial for $x = x_o$ (and necessarily $t = 0$). For a given $x \neq x_o$, let $\gamma : [0, \hat{r}(x)] \rightarrow \Omega$ be a minimizing geodesic segment such that $\gamma(0) = x_o$ and $\gamma(\hat{r}(x)) = x$. We will compute first $\text{dist}_+((x,t), \partial\mathcal{W}^+)$. Since γ is minimizing, for every $s \in [0, \hat{r}(x)]$ it holds that $\hat{r}(\gamma(s)) = s$, so that $(\gamma(s), s) \in \partial\mathcal{W}^+$ and

$$\text{dist}_+((x,t), (\gamma(s), s))^2 = \text{dist}_M(x, \gamma(s))^2 + (t-s)^2 = (\hat{r}(x) - s)^2 + (t-s)^2.$$

Observe that this expression attains its minimum at $s_o = (\hat{r}(x) + t)/2$, and

$$\text{dist}_+((x,t), (\gamma(s_o), s_o)) = \frac{1}{\sqrt{2}}(\hat{r}(x) - t). \quad (59)$$

We claim that $\text{dist}_+((x,t), \partial\mathcal{W}^+)$ is given by Eq. (59). In fact, for every $y \in \Omega$ we have that

$$\text{dist}_+(x, y) \geq |\text{dist}_M(x_o, x) - \text{dist}_M(x_o, y)| = |\hat{r}(x) - \hat{r}(y)|,$$

and so

$$\begin{aligned} \text{dist}_+((x,t), (y, \hat{r}(y)))^2 &= \text{dist}_+^2(x, y) + (t - \hat{r}(y))^2 \\ &\geq (\hat{r}(x) - \hat{r}(y))^2 + (t - \hat{r}(y))^2 \\ &\geq \min_{s \geq 0} ((\hat{r}(x) - s)^2 + (t - s)^2) = \frac{1}{2}(\hat{r}(x) - t)^2. \end{aligned}$$

Therefore,

$$\text{dist}_+((x,t), \partial\mathcal{W}^+) = \frac{1}{\sqrt{2}}(\hat{r}(x) - t). \quad (60)$$

Analogously, we can see that for $\partial\mathcal{W}^-$ it holds

$$\text{dist}_+((x,t), \partial\mathcal{W}^-) = \frac{1}{\sqrt{2}}(\hat{r}(x) + t). \quad (61)$$

Then Eq. (58) follows from Eqs. (60) and (61).

Let $x \in \Omega$, $x \neq x_o$, and consider again $\gamma : [0, \hat{r}(x)] \rightarrow \Omega$ a minimizing geodesic segment such that $\gamma(0) = x_o$ and $\gamma(\hat{r}(x)) = x$. Let us write $u(s) = u(\gamma(s))$. Then $(\gamma(s), u(s)) \in \mathcal{W}$, and Eq. (58) yields

$$\text{dist}_+((\gamma(s), u(s)), \partial\mathcal{W}) = \frac{1}{\sqrt{2}}(s - |u(s)|).$$

In particular, $\text{dist}_+((\gamma(s), u(s)), \partial\mathcal{W})$ is a positive increasing function for $0 < s \leq \hat{r}(x)$. Thus, if we choose $\delta > 0$ such that the geodesic disc verifies $D_\delta = D(x_o, \delta) \subset \subset \Omega$, then it follows that

$$\text{dist}_+((x, u(x)), \partial \mathcal{W}) \geq \varepsilon > 0 \quad \text{for every } x \in \Omega \setminus D_\delta, \quad (62)$$

where $\varepsilon = \min_{x \in \partial D_\delta} \text{dist}_+((x, u(x)), \partial \mathcal{W}) > 0$.

Finally, we are ready to prove that φ is proper on Ω . Since $\Omega = \overline{D}_\delta \cup (\Omega \setminus D_\delta)$ with \overline{D}_δ compact, it suffices to prove that $\varphi|_{\Omega \setminus D_\delta}$ is proper on $\Omega \setminus D_\delta$. Let $\psi : \Omega \rightarrow \Omega \times \mathbb{R}$ be the standard embedding, $\psi(x) = (x, u(x))$, and let $\bar{\phi} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be the lifting of ϕ to $\Omega \times \mathbb{R}$; that is, $\bar{\phi}(x, t) = \hat{r}^2(x) - t^2$. Observe that ψ is trivially proper. In fact, if $A \subset \Omega \times \mathbb{R}$ is compact, then

$$\psi^{-1}(A) \subset \psi^{-1}(\pi_M(A) \times \pi_{\mathbb{R}}(A)) = \pi_M(A) \cap u^{-1}(\pi_{\mathbb{R}}(A))$$

is also compact. On the other hand, from Eq. (62) we have that

$$\psi(\Omega \setminus D_\delta) \subset \mathcal{U} = \{(x, t) \in \mathcal{W} \mid \text{dist}_+((x, t), \partial \mathcal{W}) \geq \varepsilon\}.$$

Consequently, $\varphi|_{\Omega \setminus D_\delta} = \bar{\phi}|_{\mathcal{U}} \circ \psi|_{\Omega \setminus D_\delta}$. It is easy to see that the map $\psi|_{\Omega \setminus D_\delta}$ is proper. Therefore, it suffices to show that $\bar{\phi}|_{\mathcal{U}} : \mathcal{U} \rightarrow \mathbb{R}$ is proper or, equivalently, that for every $b > 0$, $(\bar{\phi}|_{\mathcal{U}})^{-1}([0, b]) = \mathcal{U} \cap \bar{\phi}^{-1}([0, b])$ is compact. Let $(x, t) \in \mathcal{U} \cap \bar{\phi}^{-1}([0, b])$. Since $(x, t) \in \mathcal{U}$, by Eq. (58) we obtain that

$$|t| \leq \hat{r}(x) - \sqrt{2\varepsilon}.$$

Therefore, since $\bar{\phi}(x, t) \leq b$, we have

$$\hat{r}^2(x) - b \leq t^2 \leq (\hat{r}(x) - \sqrt{2\varepsilon})^2.$$

That is,

$$\hat{r}(x) \leq c := \frac{2\varepsilon^2 + b}{2\sqrt{2\varepsilon}}.$$

This implies that $(\bar{\phi}|_{\mathcal{U}})^{-1}([0, b]) \subset \overline{D}_c \times [\sqrt{2\varepsilon} - c, c - \sqrt{2\varepsilon}]$ is compact, which finishes the proof. \square

As a consequence of Theorem 10 and Proposition 4 we can give an alternative proof of the nonparametric version of our Calabi–Bernstein result, Theorem 8. In fact, if M^2 is complete we can apply Proposition 4 to $\Omega = M$, concluding that the function $\phi = r^2 - h^2$ is eventually positive and proper on $\Sigma(u)$. Therefore, by Corollary 1 we have that $\Sigma(u)$ is parabolic. Then, the proof follows as in the proof of Theorem 5.

8 A Local Estimate for Maximal Surfaces in a Lorentzian Product Space

In this section, we generalize the local approach given by the second author and Palmer, which is described in Sect. 2.3, to the case of maximal surfaces in a Lorentzian product space $M^2 \times \mathbb{R}_1$. Specifically, we prove the following extension of Theorem 4.

Theorem 11 ([4, Theorem 1]). *Let M^2 be an analytic Riemannian surface with nonnegative Gaussian curvature, $\kappa_M \geq 0$, and let $f : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$ be a maximal surface in $M^2 \times \mathbb{R}_1$. Let p be a point of Σ and $R > 0$ a positive real number such that the geodesic disc of radius R centered at p satisfies $D(p, R) \subset \subset \Sigma$. Then, for all $0 < r < R$, it holds that*

$$0 \leq \int_{D(p,r)} \|A\|^2 d\Sigma \leq c_r \frac{L(r)}{r \log(R/r)}, \quad (63)$$

where $L(r)$ denotes the length of the geodesic disc of radius r centered at p and

$$c_r = \frac{\pi^2(1 + \alpha_r^2)^2}{4\alpha_r \arctan \alpha_r} > 0,$$

being $\alpha_r = -\inf_{D(p,r)} \Theta \geq 1$.

Proof. The proof is inspired by the ideas of the proof of Theorem 4. In fact, the key of the proof is to apply Lemma 1 to the smooth function on Σ given by $u = \arctan \Theta$.

Using Eqs. (28) and (31), we can compute

$$\Delta u = \frac{\Delta \Theta}{1 + \Theta^2} - \frac{2\Theta \|\nabla \Theta\|^2}{(1 + \Theta^2)^2} = \frac{2\Theta}{(1 + \Theta^2)^2} \|A\|^2 + \frac{(\Theta^2 - 1)\Theta}{1 + \Theta^2} \kappa_M.$$

Therefore, taking into account that $\Theta \arctan \Theta \geq 0$, $\Theta \leq -1$ and $\kappa_M \geq 0$, we obtain

$$u\Delta u = \frac{2\Theta \arctan \Theta}{(1 + \Theta^2)^2} \|A\|^2 + \frac{(\Theta^2 - 1)\Theta \arctan \Theta}{1 + \Theta^2} \kappa_M \geq \phi(\Theta) \|A\|^2,$$

where, as in Eq. (17)

$$\phi(t) = \frac{2t \arctan t}{(1 + t^2)^2}.$$

Recall that the function $\phi(t)$ is strictly increasing for $t \leq -1$. Therefore, since $-\alpha_r \leq \Theta \leq -1$ on $D(p, r)$ we get

$$\phi(\Theta) \geq \phi(-\alpha_r) = \frac{2\alpha_r \arctan \alpha_r}{(1 + \alpha_r^2)^2} \quad \text{on } D(p, r).$$

From here the proof follows exactly as in Theorem 4. □

Remark 4. In particular, when Σ is complete, the local integral inequality (63) provides an alternative proof of the parametric version of our Calabi–Bernstein theorem, Theorem 5, in the case where M^2 is an analytic surface.

In fact, observe here that if M^2 is analytic, then Σ is also analytic since it is locally a solution of Eq. (33). On the other hand, if Σ is complete, R can approach to infinity in Eq. (63) for a fixed arbitrary $p \in \Sigma$ and a fixed r , which implies

$$\int_{D(p,r)} \|A\|^2 d\Sigma = 0.$$

Therefore, $\|A\|^2 = 0$ and Σ must be totally geodesic. From Eq. (27) we get that $\Theta = \Theta_o \leq -1$ is constant on Σ , and then Eq. (28) implies that, when $K_M > 0$ somewhere in M , it must be $\Theta_o = -1$, so Σ is a slice.

As another application of Theorem 11, at points of a maximal surface where the second fundamental form does not vanish, we are able to estimate the maximum possible geodesic radius in terms of a local positive constant.

Corollary 2 ([4, Corollary 3]). *Let M^2 be an analytic Riemannian surface with nonnegative Gaussian curvature and let $f : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$ be a maximal surface in $M^2 \times \mathbb{R}_1$ which is not totally geodesic. Assume that $p \in \Sigma$ is a point with $\|A\|(p) \neq 0$ and let $r > 0$ be a positive real number such that $D_r = D(p, r) \subset\subset \Sigma$. Then*

$$R \leq r e^{C_r}$$

for every $R > r$ with $D(p, R) \subset\subset \Sigma$, where

$$C_r = \frac{c_r L(r)}{r \int_{D_r} \|A\|^2 d\Sigma} > 0$$

is a local positive constant depending only on the geometry of $f|_{D(p,r)}$.

A similar estimate for stable minimal surfaces in 3-dimensional Riemannian manifolds with nonnegative Ricci curvature was given by Schoen in [40]. See also [8] for another similar estimate given by the second author and Palmer for the case of non-flat space-like surfaces with nonnegative Gaussian curvature and zero mean curvature in a flat 4-dimensional Lorentzian space.

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