Chapter 22
Learning with the Use of the Internet

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Abstract In this chapter we discuss how the Internet is interacting with mathematics education. After briefly discussing the rise of the Internet and its impact on education, we suggest that it has the potential to disrupt mathematics teaching and learning. Moving far beyond its used as a data resource, we suggest the Internet will provide on-demand access to mathematics knowledge through the collaborative, multimodal and performative affordances of the media that it supports. We note that such affordances will not come to fruition until pedagogical practices have adapted to the rapid pace of this technological change. We conclude by noting that such fundamental change in the teaching of mathematics does have many obstacles, not least that approximately two-thirds of the world's population does not have sufficient access to the Internet— and in societies where access is available, access to the Internet often remains limited in classroom settings, particularly for students in low socio-economic areas.

Introduction

Imagine a mathematics classroom before the widespread use of the Internet. Mathematics knowledge was the property of teachers and textbooks and mathematics teaching happened in formal classroom settings under the control of teachers and a mandated curriculum. Now imagine a mathematics classroom where students and teachers have constant access to the Internet. What changes might we see?

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Consider a parallel. Imagine society before the widespread use of the Internet. Information was the property of governments and news media and for the most part it was disseminated through their control. Today (in 2012) governments and news media still control and disseminate information, but they no longer have a monopoly. Every person with a cell phone can connect to some aspect of the Internet, not only to access information but also to share information with others who have Internet access. What changes as a result?

The Internet has facilitated the emergence of information sharing that is for the most part beyond the control of governments and traditional news media (Khine & Salleh, 2010). Wikileaks is one example of this, where government records and communications have been made public in unprecedented ways, by individuals posting them on the Internet. Such public sharing of typically secret information adds a level of transparency to government. But there is something else that is at play here that is more than just who controls and disseminates information. Schrage (2001) suggested that the commonly-used label of *information revolution* misses the essence of the paradigm shift due to new media. He suggested that a more accurate description of the paradigm shift is *relationship revolution*. For example, in the case of the Middle East and North Africa, it was the creation and organization of new communities through Internet tools like Facebook and Twitter that played a significant role in challenging existing government structures over the past three years.

Returning now to our initial question of what changes might we see in mathematics classrooms where students and teachers have ready access to the Internet, we can imagine some of the following occurring which in some respects are analogous with the above examples of socio-political developments at large:

- Mathematics knowledge in all its enormity is no longer just the property of teachers and textbooks, nor is it constrained by the communication forms of traditional textbooks. It also exists in publicly available information sites such as Wikipedia and the numerous mathematics education sites that offer textual, multimodal and interactive mathematics content.
- Mathematics teaching is not limited to formal classroom settings. The Internet has become a vast resource of information. For example, a student can search on YouTube for "factoring" and find numerous videos that "teach" mathematics content related to this topic.
- Online mathematics courses have created a new form of "classroom," in which no physical space exists as the classroom. The *new classroom* is a combination of the place where each student-computer is a virtual environment where messages, videos, drawings are posted synchronously or asynchronously. In this sense, the classroom is in the Internet. Thus pedagogical designs need to take into account affordances of the Internet such as collaboration, multimodality and performance (which we discuss later in this chapter).

We suggest that these three fundamental foci within mathematics education—mathematics knowledge, teaching and the context of classrooms—can all undergo, individually and together, radical change with the emergence and use of the Internet. We have noted above some recent (2010–2012) actions in society that most likely would not have occurred in the way they did without the Internet. Events such as these have prompted us to speculate on the impact of the Internet on mathematics education. In doing so, we are mindful that classrooms do function differently from society as a whole, but clearly being an artefact of society there are overlaps. Rather than using the three foci outlined above as our organizing structure for this chapter, we use a structure that incorporates possibilities that are not being practised widely as yet, a structure that the authors believe offers possibilities for mathematics education in the 21st century. Our approach will be based on three key affordances of the Internet: collaboration, multimodality and performance. But we will first start with a general discussion about the Internet.

The Internet

In the *Second International Handbook of Mathematics Education* of this series, Atweh, Clarkson, and Nebres (2003) acknowledged the international nature of mathematics and mathematics education, picking up threads of an argument they had mounted some years earlier (Atweh & Clarkson, 2001). They also detailed some aspects of the impact of globalization on mathematics education which they argued had both advantages and disadvantages, although often it seemed that this multi-pronged process seems overwhelming, unstoppable and often associated with forces that were "impersonal and beyond the control and intentions of any individual or groups of individuals" (Waters, 1995, p. 2). Later, Clarkson (2011) noted that the impact of globalization is not always easy to identify in real time, but often only becomes apparent on reflection. Within this argument, clearly the use and power of the Internet was formidable—both useful and at times overpowering of local initiatives and thinking.

Much of the hardware that is utilized in education was developed for other areas of society. Education is forever playing catch-up. Film, television, audio recording, video and then digital recording, overhead projectors, all of which have been used to varying degrees in schools, were developed first for business, and then later marketed as valuable resources for education. Some, such as video recordings, proved to be useful, but others such as television and film proved far more problematic.

When it comes to information and communication technologies (ICT), again they were invented for business and some for scientific/engineering applications, with education a secondary market. The Internet in particular was originally invented for military purposes. Hence, unlike resources that have from the start been developed for education purposes, these technologies are being utilized as best-fit possibilities in education. It is, therefore, no surprise that there are unexpected occurrences along the way. But the same is true in business. For example, a report from India on the utilization of ICT in micro-businesses shows that it is the cheap digital phones that are the most used and adaptable to that situation, not the far more powerful desk-top computer technology (Ilavarasan & Levy, 2010). Hence, in working through how to use ICT in mathematics education, researchers and curriculum developers should employ investigative techniques that do not lack rigor, but at the same time are designed to capture unexpected outcomes.

It is starting to become difficult to think of schooling, including doing mathematics, without the Internet. The Internet seems to be present when students do work at home, when they communicate with colleagues, and so on. The 2011 worldwide estimate of the number of Internet users was at more than 2.25 billion people, and rising (Internet World Stats, 2011). This is a significant growth since 1995 when there were "only" 16 million Internet users. The popularization of the Internet, which offers new popular and specialized forms of representation and communication of ideas, has an impact on mathematics education.

The Internet and Education

The use of computer technology in schooling has a long history. In the late 1960s and early 1970s enthusiastic teachers found ways to introduce students to the use of computers. This meant collecting hand-punched cards and sending or taking them to some central main-frame computer for processing (Clarkson, 1980). However, the question of whether this technology advanced the quality of teaching and learning for students was never far away. One issue was whether students' performances on assessment tasks increased over time with access to this technology, but this proved to be a very hard and not always productive type of question. It was also recognized at a social level that students needed to know about this technology and its impact, since it was seen to be the start of a revolution in our society.

Throughout the mid-1980s computers themselves began to change. They became smaller and therefore more portable. They became relatively far less expensive and hence, affordable by many more people in many societies. Their power grew exponentially meaning that small laptops could compute faster than the old giant main frames of the 1960s. A laptop now has far more computing power than the computer at Houston, in Texas, that had control of the moon landing in 1969. This rise in computer power allowed the rise of multi-function computers that not only complete mathematical calculations, but also easily handle numerical databases and alpha databases. They also became a facility for playing games. Game playing took off with the interactive screen which allowed for point and click, utilizing high quality graphics, rather than having to remember specific code to type in from the keyboard. When, in the early- to mid-1990s, easy access to the Internet using the World Wide Web (WWW) became available, anyone with a computer and a modem that connected it to the copper wire telephone cable system could have access to virtually unlimited information, and contact anyone who had an email address.

An immense amount of research has focussed on the use of computers (without the Internet). The two ICMI studies (Churchhouse, 1986; Hoyles & Lagrange, 2010) and PME proceedings (e.g., Pinto & Kawasaki, 2010) provide a representative collection of papers on the subject. Interestingly, these collections do not make clear how widespread the use of these computers is in everyday classrooms. This rather fundamental issue of the place and use of computers in everyday school education worldwide is a project that is still to be undertaken.

The popularization of the Internet which offers new popular and specialized forms of representation and communication of ideas has an impact on mathematics education and education in general. DeBell and Chapman (2006) suggested that "children and adolescents commonly use computers for playing games, completing school assignments, word processing, email, and connecting to the Internet. The most frequent online activities for students are using email, playing games, using social network sites, and finding news and product information" (p. 37) (see also Smith & Caruso, 2010).

When it comes to education, Head and Eisenberg (2010) found that "college students use *Wikipedia*. But, they do so knowing its limitation. They use *Wikipedia* just as most of us do—because it is a quick way to get started and it has some, but not deep, credibility" (para. 4). The role of Facebook in education has also been discussed by researchers (Ellison, Steinfield, & Lampe, 2007; Idris & Wang, 2009; Lampe, Ellison, & Steinfield, 2008; Tay, Tan, & Tan, 2009). Selwyn (2007), looking at the cohort of middle-class university students, saw:

> Facebook as being a highly significant but also an unremarkable means of social networking and communication in the everyday lives of the young people. … The Internet has become enmeshed into daily lives and the social interactions of this generation … We have seen how students were using Facebook to communicate with friends in the same house, library or computer lab in an asynchronous and sometimes quasi-synchronous manner. Conversations appeared to skip across Facebook walls, text messaging, MSN and face-to-face contact, leaving the wall postings as just one part of a seamless, multimodal exchange. (p. 17)

The use of short text messages and images through mobile technologies and social network has also become a very popular medium for communication among adolescents and college students (Nanyang Technological University, 2010), and represents a shift away from communication through email. One thing that is consistent about student use of Internet-based resources is an uncertainty about what the next popular mode of communication might be. There is no doubt whatever that when it arrives it will have an impact on education, and mathematics education in particular.

The Internet and Mathematics Education

It is more than 25 years since the interface between information technology (IT) and mathematics education started to become an issue for research. This became more important since personal computers first became available (for a few) in the mid-1980s. Nevertheless it is still not clear in terms of research whether, and if so how, information communication technology (ICT) transforms the teaching and learning of mathematics. It was with the rise of the Internet that the IT changed to ICT. We do know that access to computers is very uneven in schools worldwide. Not surprisingly, ICT is even more unevenly present in education than the presence of computers, since many schools that have computers have limited or no access to the Internet.

Software that allows students to investigate features of functions or geometric figures has become popular in mathematics education conferences, as has the exploration of using spreadsheets in the teaching of algebra. But there is no account, to our knowledge, about how widespread their use is in classrooms (Borba & Villarreal, 2005; Hoyles & Lagrange, 2010; Kieran & Yerushalmy, 2004). There is some suggestion in the literature that the widespread use of scientific and then graphical calculators from the late 1990s led to a reduction in the use of computers in mathematics classrooms compared to their use in other subject areas (Clarkson & Toomey, 2001). We do know they have not been used in international comparative assessments, even though there are movements for international surveys like PISA to introduce computer-based items in their assessment tasks.

We suspect that research on software development in mathematics education has helped to shape mathematics education technology that is available on the Internet, for example in the form of applets. However "could the Internet be fully accepted in (mathematics education)?" is a question posed by Borba (2009). At that point in time he had no comprehensive answer. But it seems that some practices, other than using it as source of reference, have been developed which have the potential to transform the way mathematics is taught and learned.

In the previous section we briefly discussed some of the research related to computers (without the Internet) and mathematics education. We also reviewed very briefly some research in education in general, regarding the use of social networks and other affordances of the Internet to provide learning and to enhance teaching. From this sampling of the research it is clear that ICT, and the Internet, in particular, are changing society, and hence there are radical implications for education, including mathematics education. However, Maltempi and Malheiros (2010) in a survey showed that until 2007 there were few studies published in English text journals, conference proceedings and books about online mathematics education, although they suggested the situation was slightly better in countries like Brazil.

There is a wide variety of free mathematics education resources that students and teachers can use for developing mathematical understanding. For example:

1 The National Council of Teachers of Mathematics (NCTM) maintains the Illuminations Web site (http://www.illuminations.nctm.org) which offers activities, lessons and interactive content for grades K-12.
2 Utah University has developed the National Library of Virtual Manipulatives for mathematics education (http://www.nlvm.usu.edu/en/nav/vlibrary.html).
3 Drexel University runs the Mathematics Forum (http://www.mathforum.org) which offers a bank of math questions and answers, and a free online math help service.
4 Other sites, such as the following, are not run by institutions or professional organizations, but are also of interest for the discussion we will develop in this chapter:
   - http://www.ted.com/talks/salman_khan_let_s_use_video_to_reinvent_education.html, and
   - http://www.wolframalpha.com/.

In addition, Engelbrecht and Harding (2005) identified a number of other online resources that are likely to benefit students: math dictionaries, libraries of puzzles and other enrichment content, online learning or extension material to support face-to-face courses—online material made available by textbook publishers and supplementary notes made available by the teacher—and exploration and demonstration sites with interactive animations.

From the early days when the Internet was beginning to be utilized in classrooms, there were issues in students' learning that were new but still remain on today's research agenda. Gerber, Shuell, and Harlos (1998) noted that when using the Internet "students did not seem to have a clear cut plan for their projects or for locating data prior to using the Internet" (p. 123). They added: "students approached the task of searching in different ways. … [but they] did not search the Internet with a clear plan in mind. … Most of them needed a good deal of scaffolding to focus their searches and find relevant data" (p. 127). A similar comment might also be appropriate for any project which demands students collecting data, whether this involves the Internet or not. But if they are to utilize the Internet, then peculiar issues come into play. Pritchard and Wilson (1999) alluded to this when they noted:

> The Web's very popularity is becoming one of its major weaknesses. To go about looking for data on a particular topic is fairly easy—the difficult part of sifting through the often thousands of documents a search has generated for an article which will contain something which is genuinely helpful or interesting. The fact that the authenticity or veracity of the data or information provided cannot be guaranteed is another failing. (p. 44)

Nevertheless, Herrera (2001) and Engelbrecht and Harding (2005) asserted that the Internet's hands-on environment enables students to see and explore mathematical concepts. Martinovic (2005) suggested that there are a number of potential benefits to students of mathematics using online help sites. According to Martinovic, the Internet

- has a greater potential for students to develop questions that will engage them in a process of self-diagnosis and reflection;
- provides students with answers that may provide models of *thinking through* problems;
- through online help sites offer vicarious benefits even for visitors that do not ask questions, by helping them learn the language of mathematics, how to ask questions, and how to answer them; and
- provides different approaches in answering similar mathematics questions which may help students realize that there is more than one way to solve problems.

Although there are many claims for online learning, those who are teaching such courses have not indicated that the learning of students is without difficulty. Guberman-Glebov, Baruch and Barabash (2003), reflecting on their teaching in this environment, suggested that "students in such a course left on their own, do not manage to make a sufficient progress and need permanent instruction, which renders the distant learning approach (online) in this case time consuming and not efficient" (p. 161). Wadsworth, Husman, Duggan and Pennington (2007) later noted:

> Although students in online courses are implementing many of the same strategies as their counterparts in traditional classrooms, there has been little evidence to show what strategies are most useful in this new environment and how some strategies may translate to a new learning environment. (p. 13)

The role of teacher and the form of teaching when the Internet is utilized to any degree is also of interest. Again there is much in the literature that asserts that the context has changed for the better, but there seems to be little hard research evidence on which these conclusions are based.

Stahl (2009b) called for a new way of teaching when using the Internet because:

> Students learn math best if they are actively involved in discussing math. Explaining their thinking to each other, making their ideas visible, expressing math concepts, teaching peers and contributing proposals are important ways for students to develop deep understanding and real expertise. There are few opportunities for such student-initiated activities in most teacher-led classrooms. (p. 24)

Although the Internet does afford new pedagogical possibilities, "the teacher's role in the use of the Internet is one of significant importance and not to be taken lightly" (Loong & White, 2003; p. 2). As Guberman-Glebov, Baruch, and Barabash (2003) noted:

> The computer and Internet provide some unquestionable advantages as a learning environment, if one learns to use them properly. We assert by that the technology usage is not self-evident for every course and every context, and one needs tools and skills for decision-making as to the choice of teaching methods and strategies involving these techniques. (p. 160)

There is another potential affordance offered by the Internet that may help teachers explore the new roles that are open to them. In a unique way, not available previously, the Internet affords the creation of networks of teachers. Some researchers see it as a venue for developing ideas to improve mathematics teaching. Chinnappan (2006) suggested that "by sharing the problems and concerns of their own school context, teachers can better understand, anticipate, and develop potential solutions to the learning demands of children in their classroom" (p. 357). And yet this assertion hardly needs the Internet for this to occur.

But even in large cities with many schools, teachers have often found it hard to meet and share professionally in a manner that is frequent and continuous over a long period of time. Most inter-school professional meetings of teachers only occur when there is a specific task to be accomplished. Changes may be possible with the Internet.

> Through the Internet, teachers can share expertise, offer one another their ideas on lesson plans and projects, even chat across continents about common problems and interests. Lessons made for one cultural setting may not be suitable for another, but they may still suggest ideas that can be revised and molded for your classroom. (Herrera, 2001, p. 26)

Thus building a professional community of support without having to leave your office, which can meet asynchronistically if necessary, becomes a possibility with the Internet. But how often this occurs, and the gain teachers have from such a community, has not been made at all clear in the research literature.

A critical aspect of teaching is the utilizations of resources that will help develop a useful context for student learning. The traditional resource for mathematics teaching has been the textbook. Unfortunately, many of the resources for mathematics teaching on the Internet essentially are just a reproduction of practices which are based on a paper-and-pencil medium such as downloading books or downloading exercises, a practice that does not take full advantage of possibilities of the Internet (Engelbrecht & Harding, 2005). Herrera (2001) suggested that there are alternatives:

> An in-depth treatment of a topic in this medium can include interactive animation, links to related material, video clips, and opportunities to email experts on the topic. Not all these elements are necessary, and certainly you do not want them included for their "glitz" value, but used properly they enrich the learning experience. (p. 28)

Borba and Villarreal (2005) discussed how there are new forms of communication in an online course taught via chat (see also Beatty & Geiger, 2010). Cazes, Gueudet, Hersant, and Vandebrouck (2006) used the Web to post exercises that they claimed changed the didactical contract in the classroom. However it is not clear whether their exercises just reproduced paper-and-pencil exercises, or whether they took advantage of alternative Internet possibilities. Hoyles et al. (2009), when discussing the Internet, emphasized a notion that they had developed in previous work on microworlds—how *connectivity* within a regular classroom changes the nature of collaboration.

It was recognized some time ago that "using the Internet would allow the children to locate 'real-world' data, and perhaps promote a greater understanding of instances in which one encounters such data, thereby fostering an appreciation for the use of mathematics in the real world" (Gerber et al., 1998, p. 116). The Internet has developed beyond the point where it represents merely a huge accessible database, although that advantage has not changed. Now the availability of dynamic geometry software can transform the types of tasks that can be developed in the classroom (Arzarello & Edwards, 2005; Arzarello, Olivera, Paola, & Robutti, 2002; Ferrara, Pratt, & Robutti, 2006; Laborde, Kynigos, Hollebrands, & Strasser, 2006; Mariotti, 2002; Marrades & Gutierrez, 2000). The relatively new Interactive Whiteboards (IWBs), although used in some countries (e.g., England and the USA) for more than a decade, have only now come to be used in classrooms more widely; they offer exciting opportunities to explore the use of such applications in conjunction with the Internet. Although the use of IWBs have rightly been criticized in general, as well as in mathematics teaching (Clarkson, in press; Zevenbergen & Lerman, 2008), their facility of being able to archive the records of a class's group thinking, including any use made of the Internet, and to display this quickly and easily in subsequent lessons, will be something to watch for the future.

Even with the many advantages of the Internet, there are some issues that are beyond the control of the teacher. For example, there have been critiques of the design and pedagogical quality of online interactive mathematics content. Gadanidis, Sedig, and Liang (2004) noted that designing online mathematical investigations as pedagogical tools is not a simple undertaking. In their opinion many "do not appear to be well designed, neither from a pedagogical nor from an interface design perspective" (p. 294). They suggest that good design becomes possible when mathematics education and human–computer interaction design experts work together, rather than in isolation, simultaneously taking into account pedagogical goals and interface design principles.

Rather than analyzing in detail work such as the above, we have chosen another path. Technologies and modes of communication are rapidly changing, as we have alluded to in earlier sections, making the study of their impact on mathematics education both challenging and exciting. In the next section we discuss some of the themes that appear in the literature regarding the affordances of using the Internet in mathematics education. We have chosen not to report on studies that are predominately text based and/or use rapid response modes aimed mainly at testing students' abilities. Rather, we briefly report on studies that seem to push the boundaries of how the Internet can be used creatively and with worth in mathematics education.

Collaboration, Multimodality and Performance

Collaboration, multimodality and performance are the three new affordances that we have identified and discuss briefly using some case studies in this section. These features are not affordances only of the Internet. But we claim that the Internet transforms them. Hence in one sense they are all objective capabilities of the Internet.

Collaboration has changed with the use of the Internet not only because people who are in different geographic location can interact, but because even when they are face-to-face, collaboration involving the use of the Internet changes its nature.

Multimodality, understood as the combination of different kind of texts, has definitely been changed by the Internet. It is easy to combine video, drawings and music with regular text. Hence with the Internet one is able to bring information to online courses or to face-to-face courses in ways undreamed of in pre-Internet days.

The third subsection deals with performance. We characterize here all kind of performances (such as YouTube videos) that can be found on the Internet that are directly connected to mathematics education.

Clearly there is overlap between these three issues which we recognize. We are not trying to set out a classification system with these headings. Rather, we are identifying labels through which we can discuss what we believe are affordances arising through the Internet for mathematics education. Before we go into a more detailed discussion of collaboration, multimodality and performance in the following subsections, we note that our own teaching experiences with the Internet have significantly altered our notion of *classroom*. First, all authors have been teaching online courses for at least six years. In online courses, all the interaction, or most of it, takes place in virtual environments. Normally teacher and students never meet face-to-face. Nevertheless, often some of the students from each course have mentioned that they "feel close" even without meeting face-to-face (Borba & Gadanidis, 2008; Borba, Malheiros, & Zulatto, 2010; Engelbrecht & Harding, 2005). The second type of teaching environment that we have experienced for much longer and has relevance to this discussion is that of the *blended learning* environment in which, for the most part, the use of the Internet is combined with face-to-face regular interactions. Lin and Ponte (2008) discussed different ways of how this can help in communities of prospective mathematics teachers. Recently there has been Working Groups on *online teacher education* at PME conferences (Borba & Llinares, 2008; Borba, Llinares, Clay, & Silverman, 2010). Overall, it seems that both online courses and the blended courses seem to suit both continuing teacher education and preservice teacher education programs (Maltempi & Malheiros, 2010).

Clearly, our own experiences of teaching in various ways with the Internet, colours the following discussion. As noted above, many of the practices that involve the use of the Internet are not taking advantage of the changing possibilities that it offers. They are simply mimicking practices of the paper-and-pencil medium. Hence, as we discuss collaboration, multimodality and performance, we will also note some of the reactions from students and explore possibilities for teaching mathematics—such as Math and Science Performance Festival (see http://www.MathFest.ca)—that we believe are offering new perspectives regarding how students and teacher can express their mathematical ideas.

Collaboration

What does online collaboration look like in the case of mathematics education? The two cases we present below illustrate how new technologies can help foster collaboration in online mathematics education settings.

Case 1: "Pass the pen, please". Online mathematics teaching and learning can be in synchronous, asynchronous as well as hybrid environments. In a synchronous environment, all students and the teacher are present using video, text, and/or audio. But how does one explore mathematics in such an environment? Rather than reverting to traditional modes when the instructor simply lectures and the students listen, it is possible for a synchronous environment to provide a shared collaborative workspace, where the teacher and students work together on mathematics problems. One such possibility we call "pass the pen please." The first author has developed and used a platform that allows the screen of any of the participants to be shared with everyone else. For example, we could start by showing a screen of *Geometricks* on our computer. At the same time the class of students, no matter their geographic location, could see the dragging that is performed on a given geometrical construction. To this point there is nothing of real interest. In many ways we as the instructors

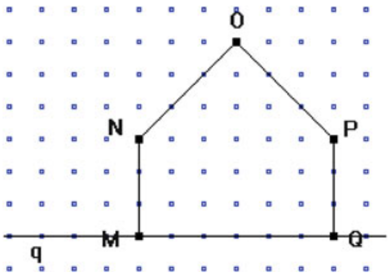

Figure 22.1. Geometricks.

have control of what is happening with the students simply watching. However a special feature of the application, which is important for the theme of collaboration, is the capability to "pass the pen" to another participant who could then add to what was done on the *Geometricks* construction. In this case, technology transforms the nature of the interaction and enables a form of collaborative problem solving to happen (see Borba & Gadanidis, 2008, for more details). This example illustrates how an online environment can support the convergence of different ideas and generate the collective construction of knowledge about geometry.

A particular example involved consideration of symmetry. A *Geometricks* file had already been given to students with the figure MNOPQ (see Figure 22.1) and they were asked to find the symmetric figure, in relation to *axis-q*.

Borba and Zulatto (2010), the professors from the university teaching the courses in which this example arose, report on how they began to learn mathematics from the interaction with the students. That is, once the authors "passed the pen" to the students and let them take the lead in the online activity, both groups, students and professors, became learners in a joint collaborative act.

However this collaborative online mathematics learning environment of "pass the pen, please" involved more than collaboration between teacher and students, and more than collaboration between students. It also involved collaboration between humans and digital mathematics tools. Borba and Villarreal (2005) have developed the theoretical notion of *humans-with-media*, as a means of stressing the idea that knowledge is constructed by collectives which involve humans and different technologies of intelligence such as orality, paper-and-pencil, and ICT (Lévy, 1993).

Hence it is hypothesized that different combinations of teachers, students and technologies result in different kinds of knowledge production. Although we do not at this stage want to make the case that new medium, such as the Internet incorporated into collectives of humans-with-media, enhance student learning, we have evidence that suggests that the Internet is a media that transforms practices of learners and teachers involved with mathematical educational practices. The research group GPIMEM (http://www.rc.unesp.br/gpimem) has documented some of these changes. For example, in online courses that use chat rooms, it is not easy to use mathematical symbolism. Participants have to resort to writing "integral of $2x$ dx" instead of using the normal concise mathematical symbolism for such an expression. Santos (2006) in discussing research into such phenomena suggests that

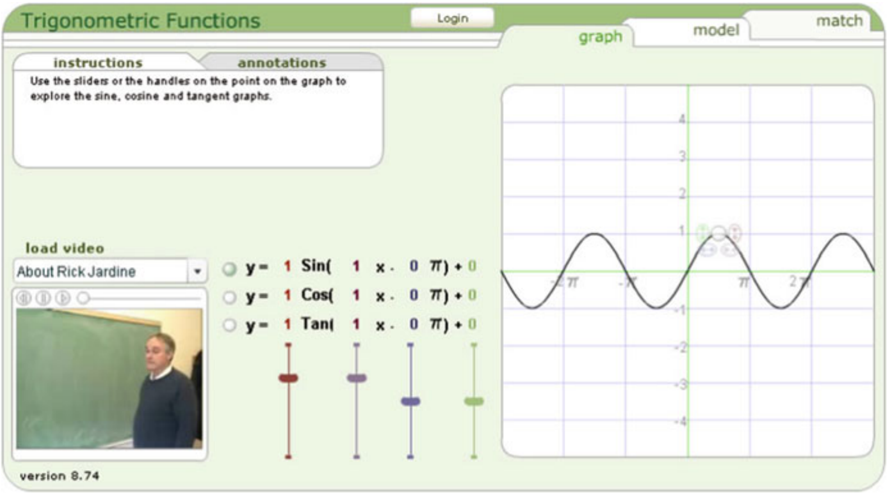

Figure 22.2. Digital Windows into Mathematics.

this change in writing such expressions online may change the nature of students' mathematical thinking.

Case 2: Annotating learning objects. Learning objects are typically viewed as "read-only" interactive content. That is, a user can explore the content but there is typically no method for annotating a particular state with ideas or questions, and then sharing these states and annotations with others. That would be a more difficult to do, as it would require more sophisticated programming and also the use of a database. Through a project called *Digital Windows into Mathematics* (http://www.edu.uwo.ca/dwm) the third author has developed learning objects (see Figure 22.2) that allow for remote collaboration (Gadanidis, Jardine, & Sedig, 2007).

Users have the ability to add their personal annotations (after obtaining a username and a password), and to incorporate personal metadata into the mathematics content. That is, the user can mark-up a learning object using text or freehand drawings and then save these annotations for later reference or for sharing with others. When saving annotations, the current state of the interactive environment is also saved (for example, the current values of the coefficients of the function being plotted, as well as the matching graph, will be saved along with the annotation). Saved annotations may be shared with others. Thus a student can share his/her ideas or questions about a certain state of the learning object, or a teacher may draw student attention to a particular aspect of a concept being explored.

Discussion. Collaboration for the purpose of learning is a prominent goal in mathematics education. Lerman (2000) has noted an emergence of a social-perspective on teaching and learning mathematics, and in particular an emphasis on collaborative learning, in mathematics curriculum documents such as NCTM's (2000) *Principles and Standards for School Mathematics*.

Some have suggested that the impact of new media, of which the Internet is an integral part, is less about the information it carries and more about the relationships that can be built. Schrage (2001) suggested that the commonly used label of *information revolution* misses the essence of the paradigm shift due to new media.

> *In reality, viewing these technologies through the lens of "information" is dangerously myopic.* The value of the Internet and the ever-expanding World Wide Web does not live mostly in bits and bytes and bandwidth. To say that the Internet is about "information" is a bit like saying that "cooking" is about oven temperatures; it's technically accurate but fundamentally untrue. (p. 1; original emphasis)

Schrage argued that a more appropriate label is relationship revolution. Hence:

> *The so-called "information revolution" itself is actually, and more accurately, a "relationship revolution."* Anyone trying to get a handle on the dazzling technologies of today and the impact they'll have tomorrow, would be well advised to re-orient their worldview around relationships … *When it comes to the impact of new media, the importance of information is subordinate to the importance of community.* The real value of a medium lies less in the information that it carries than in the communities it creates. (pp. 1–2; original emphasis)

Lankshear and Knobel (2006) argued that the relatively recent "development and mass uptake of digital electronic technologies" represented changes on an "historical scale," which "have been accompanied by the emergence of different (new) ways of thinking about the world and responding to it" (pp. 29–30). These new ways of thinking can be characterized as more "participatory," "collaborative," and "distributed" and less "published," "individuated," "author-centric," or "expert-dominated" (Knobel & Lankshear, 2007, p. 9).

In this same vein, online mathematics learning is beginning to be associated with collaboration, suggesting a definite (which may be causal) relationship between the collaborative affordances of new media and the new emphasis on collaboration in mathematics education. For example, Stahl (2009a) noted:

> We found that participants in virtual math teams spontaneously began to explore their problems together, discussing problem formulations, issues, approaches, proposals and solutions as a group. Moreover, students generally found this interaction highly engaging, stimulating and rewarding. (p. 13)

Likewise, Sarmiento-Klapper (2009) stated:

> In our study of mathematics collaboration online we observe collective creative work as manifested in a wide range of interactions extending from the micro-level co-construction of novel resources for problem solving to the innovative re-use and expansion of ideas and solution strategies across multiple teams. (p. 227)

Another way of approaching the emerging association of online mathematics learning with an increased level of collaboration is to look at an online mathematics course that is taught asynchronously. In such a course, there is no set class time, and the instructor and students can join the course at their convenience. Two aspects of such an asynchronous course may increase online collaboration. First, the instructor needs all students to actively participate online if only to show that they are "present."

In contrast, in a typical face-to-face class (or for that matter a synchronized meeting online), many students do not have to participate actively to be “present.” Second, when students participate online in an asynchronistic manner, chances are that the first person to read and possibly respond to another student’s contribution or to offer assistance to their question will be another student. The teacher-centred communication norms of face-to-face classrooms are disrupted in an online asynchronous environment and there is an increased potential for student-to-student interactions.

A number of researchers have suggested that there are positive implications associated with the collaborative affordances of such an environment. Charles and Shumar (2009) stated:

> The social action that is encouraged is creative and draws upon the participants’ imaginations to see knowledge production as an enjoyable, stimulating activity that is accessible by ordinary people. Understanding how to harness this agentic behavior and to leverage it for scalable, sustainable learning will be a next step for this research. (p. 224)

Sarmiento-Klapper (2009) reported:

> Group remembering and the bridging of interactional discontinuities allowed the teams to expand the referential horizon so that the objects created by themselves or by other teams could be expanded, reconsidered, or challenged. These methods allowed the teams to evolve a sense of collectivity engaged in building new knowledge and made it possible for them to interlink their collaborative interactions with those of other teams. (p. 235)

Cakir, Zemel, and Stahl (2009) also noted the benefits of collaborative online learning:

> The coordination of visual and textual realizations of the mathematical objects that the students co-construct provides a grounding of the algebraic formulas the students jointly derive using the line drawings that they inspect visually together. As the students individualize this experience of group cognition, they can develop the deep understanding of mathematical phenomena that comes from seeing the connections among multiple realizations. (p. 147)

Annetta, Folta, and Klesath (2010) suggested that young people in today’s world

> … are competing and collaborating on a global scale. New technologies, or at least new to education, provide the opportunity to rebuild the collaborative social structures that we have begun to lose in our educational communities. … it is high time to rethink learning. (p. 21)

However, the concept of collaboration in online environments is complex. Issues surrounding the design of online mathematics learning require more research on how best to use and support collaboration. For example, Stahl (2009a) noted that “group size has an enormous impact on the effectiveness of different media” (p. 13). He added that most research on online mathematics learning had focussed on individual learning and commented that “there is not much research on, for instance, math collaboration by different size groups” (p. 13).

Kotsopoulos (2010) noted that when we look closely at student interactions in what appears on the surface to be collaborative learning, we find instances that are “predominantly non-collaborative despite the pedagogical efforts and intentions of the teacher and the task” (p.129). Kotsopoulos identified instances of non-collaboration while students work in groups (in a classroom setting) where “non-collaborative learning sent a message of incompetence and exclusion" to some of the students in the group (p. 138). The author continued:

> [Some] students … received little support from their peers during collaborative learning. Moreover, efforts by these students to collaborate were thwarted by one or more members of the group. The group served to sustain a particular normalized way of collaborating that was exclusionary. (p. 138)

Kortsopoulos concluded that:

> Schools are public places of learning that ought to ensure safe and accessible learning for all students. Consequently, pedagogical strategies should work towards neutralizing the effects of power relations that restrict some learners. (p. 138)

This recent report suggested that in the classroom setting care needs to be taken with assumptions made regarding collaborative learning. It may be seen as a warning that students may not benefit from all online collaborative settings. For example, there, one can find problematic dynamics, such as bullying, occurring in group settings. These dynamics take on new forms in online settings. Cyber bullying is not uncommon among adolescents (Agatston, Kowalski, & Limber, 2007). Weigel, Straughn, and Gardner (2010) drew attention to the possibility that "bullying, which may have been limited to a small cadre of perpetrators and victims, can now spread more quickly and easily to a larger population" (pp. 17–18).

Dewey (1938) noted a long time ago that not all school experiences are educative. Some experiences are mis-educative. Similarly, we cannot assume that online interactions among students are necessarily collaborative in the positive sense. Again we note that this is an issue that needs to be worked through in the relatively new online environment for learning mathematics.

Multimodality

A challenge in teaching and learning mathematics online has been that in its initial manifestation; Internet communication was text-based. Not being able to use graphs and diagrams limited the possible representations of mathematical ideas. Although this problem has not been fully solved, as the support for communication using mathematical symbols and diagrams varies widely among e-learning platforms, the cases below point to developments that help incorporate multimodal elements to online mathematics.

Case 1: "Pass the pen" and *Digital Windows into Mathematics*. The two cases shared in the previous subsection on collaboration are also examples of how multimodal content may be used in online mathematics education. In the case of "pass the pen," the shared, collaborative geometric construction space allowed for communication using text, audio and geometric figures that could be manipulated. In the case of the *Digital Windows into Mathematics* project, the learning objects communicated mathematics ideas using text, diagrams, interactive content and videos of mathematicians talking about the mathematics explored.

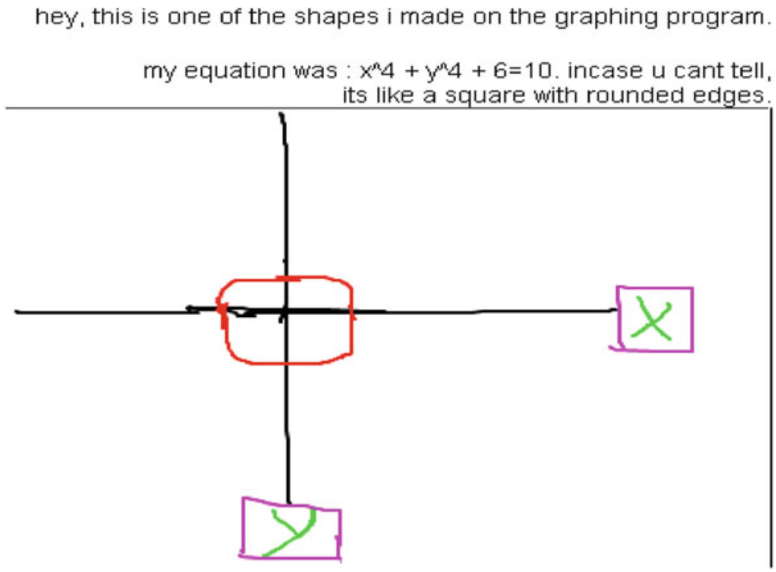

Figure 22.3. Sketch of graph of $x^4+y^4+6=10$.

Case 2: A multimodal online learning platform. For the purpose of offering online mathematics courses, the third author developed a learning platform called *Idea Construction Zone* (Gadanidis, 2007; Gadanidis & Geiger, 2010). This had the following multimodal features:

1. A rich text editor similar to ones used in word processors like *Microsoft Word*;
2. Users can embed the following within postings: video and audio recordings; graphics; *Flash* (swf); diagrams (using the built-in *Draw Tool*); hyperlinks (files and Web pages).

In addition, users have the option of making their posting *Peer Editable*, allowing other users to edit their ideas. Figure 22.3 shows what one Grade 8 student shared in the online discussion environment using the *Draw Tool* about one of the graphs he discovered while exploring an online graphing program.

Gadanidis, Hughes, and Cordy (2011) studied the nature of student learning in a classroom setting where students had ongoing access to the Internet while in a mathematics class and access to an online discussion board between classes using *Idea Construction Zone*. While exploring the graphs that were generated for missing number equations like __+__=10 and __+__=7, they wondered if they could create their own equations that would make the graphs point in a different direction or make the graphs curve. Using function plotters freely available on the Internet they investigated graphs that were well beyond the grades 7–8 mathematics curriculum: polynomial, trigonometric and even implicit, parametric and polar equations.

One example is shown in Figure 22.3. Gadanidis, Hughes, and Cordy (2011) suggested that:

> There was evident energy in the computer lab when students were creating and sharing graphs, as depicted by their eagerness to share ideas within and among groups and their willingness to take up and explore the ideas of others. Students seemed to enjoy working with equations that they initially did not understand, exploring their graphs and trying to make sense of the relationships between the equations and the graphs. Students also used Google and Wikipedia to find information about the various new equations they were encountering. (p. 418)

However, on a less positive note, this study also noted that there was a challenge in maintaining online discussion between classes. Although part of the reason had to do with poor pedagogical planning rather than the affordances of the online environment, this experience drew attention to the fact that classroom use of the Internet is not necessarily simply a positive or a negative. Rather, it also depends on how it is used pedagogically.

Discussion. Some research suggests that the collaborative aspects of online mathematics learning are supported by the multimodal online environments that are becoming increasingly available. For example, Cakir, Zemel, and Stahl (2009) stated:

> Multimodal interaction spaces—which typically bring together two or more synchronous online communication technologies such as text chat and a shared graphical workspace—have been widely used to support collaborative learning activities of small groups. … Engaging in forms of joint activity in such online environments requires group members to use the technological features available to them in methodical ways to make their actions across multiple spaces intelligible to each other and to sustain their joint problem-solving work. (p. 140)

Horstman and Kerr (2010) suggested that multimodal content adds a further level of complexity when designing online learning environments. They stated:

> Perhaps the biggest conceptual transition for e-learning designers is to envision the content and learning objectives through graphical imagery and user interactions rather than by explaining content through text. (p. 196)

Despite the fact that the Internet is increasingly filled with multimodal content, the original text-based communication still persists for many online math courses. Martinovic (2005) noted:

> Text-based communication has little means for presenting graphs, diagrams, and tables. Both tutors and students suffered from an inability to use proper mathematical symbols and sometimes had to put in extra effort to use text editing capabilities for visual presentations. (p. 34)

Because of the original limitations posed by text-only communication, Engelbrecht and Harding (2005) suggested that "at the most basic level of mathematics on the Web is the practice of what has become known as *computerese*, using a text equivalent for formulae such as *sqrt*(*x*) for the square root function" (p. 237). Clearly this formulation was needed in the early days of the Web, but nevertheless it did build another layer of complexity for communicating mathematics.

However, despite the *computerese* limitations of mathematics communication on the Internet, online communication is generally becoming increasingly multimodal in nature. This stands in contrast with many school-based experiences, especially in mathematics, which continue to rely on discourses that are monomodal or bimodal (in cases where diagrams or graphs are employed). Kress and van Leeuwen (2001) pointed out that in a digital environment "meaning is made in many different ways, always, in the many different modes and media which are co-present in a communicational ensemble" (p. 111).

The shift from text-based communication to multimodal communication is not simply a quantitative change. It is not just a case of having more communication modes. It can be seen as a qualitative shift, analogous to the change that occurred when we moved from an oral to a print culture. In the case of mathematics, we are seeing an emergence of online resources that combine text, symbols, animation, interactivity and videos. Such communication, which mirrors what young people are expecting in their overall Web-based interactions, will also be needed in their online mathematics experiences. Much is still to be done in researching this development.

Performance

Kress and van Leeuwen (2001) and Hughes (2008) noted that the multimodal nature of new media offers performative affordances. This is evident in the multimedia authoring tools used to create online content, such as *Flash*, which often use performance metaphors in their programming environments. For example, one programs on what is referred to as the *stage*, one uses *scenes* to organize *actors* or *objects* and their relationships, and one controls the performance using *scripts*. The Web as a performative medium is evident in the popularity of portals like YouTube. Hughes suggested that new media that has infused the Web draws us into performative relationships with and representations of our *content*. To use new media is to, in part, adopt a performative paradigm. Below we present two cases of Internet-based mathematics performance.

Case 1: Performing new images of mathematicians. The images of mathematicians performed in the media are typically narrow and negative. Picker and Berry (2000) have found that mathematicians are essentially invisible for students, and students rely on stereotypical images from media for their images of mathematicians. How might the Internet be a venue for offering students new views of mathematicians? The *Windows into Elementary Mathematics Project* of the Fields Institute (Gadanidis, 2010; Gadanidis & Scucuglia, 2010) uses new media tools to make mathematicians visible and offers a more positive image of mathematics and mathematicians (see Figure 22.4). In the videos, mathematicians spoke of their feelings about mathematics. Lindi Wahl stated that "One of the things that I really love about … mathematics … is that I'm creating something new all the time." Peter Taylor talked about choosing "the problems I do based on beauty."

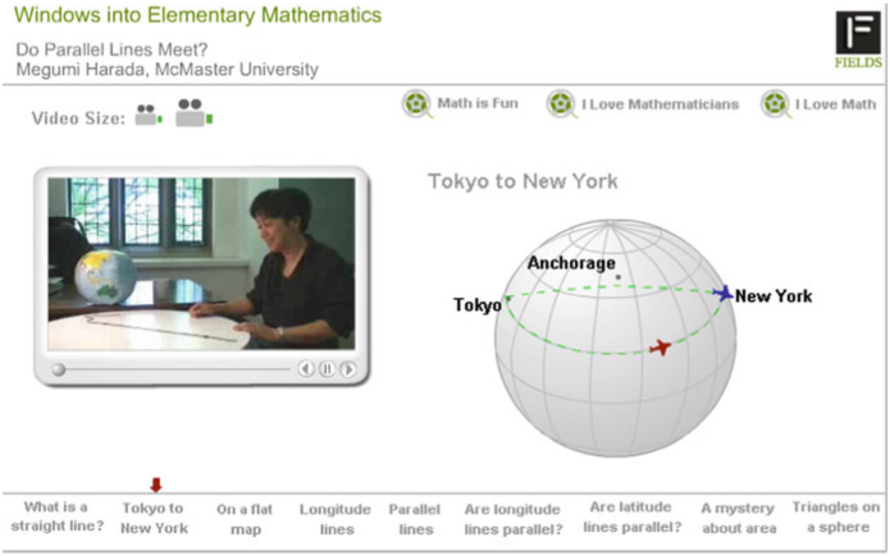

Figure 22.4. Windows into elementary mathematics.

> When one is doing mathematical biology, there are a lot of things to pay attention to, and there are a lot of papers to read, and a lot of ideas to think about, but the things I choose to work on, and the things I give to my graduate students, are the things where the structure fills me with a sense of beauty, where the aesthetics speak to me and lead me on.

Megumi Harada noted that:

> I love mathematicians. I can say that without any doubt that the math students were the most fun to be around, and I think it's because, as a group, mathematicians love what they do more than many, many other groups of people I know.

This online resource disrupts stereotypical images of mathematics as cold and abstract (Ernest, 1996) and views mathematics as a fun, interesting, imaginative, aesthetic and fully human activity (Sinclair, 2001; Sinclair, Pimm, & Higginson, 2006; Upitis, Phillips, & Higginson, 1997).

There is a little evidence that the new images of mathematicians do have some effect. The third author teaches fully online math-for-teachers courses for teachers who self-identify as "fearing or disliking mathematics." In these courses, teachers explore some of the mathematics problems explored by the mathematicians in the *Windows into Elementary Mathematics* project discussed above, and also view the video interviews with the mathematicians. It is interesting that teachers with initial negative outlooks towards mathematics end up making unsolicited positive comments about the mathematicians. For example, here are two teacher comments about mathematician Lindi Wahl:

> It is evident that she truly loves her job. She enjoys the challenge of creating brand new formulas to explain concepts. She loves collaborating with others who are specialists in their respective fields.

I love the way she talks about math! It’s great to hear someone talk so passionately about it for once!

This engagement of teachers who “fear or dislike” mathematics with mathematicians who are passionate about their subject, and the resulting positive impact on teacher attitudes, has been made possible by the Internet.

Case 2: Performing classroom mathematics. In traditional mathematics classrooms, students communicate their ideas to fellow students and to their teacher. It is rare that the classroom mathematics experience spills over beyond the classroom walls. Our informal surveys of students and parents have suggested that when a student is asked “What did you do in math today?” the typical response is “Nothing” or “I don’t know.” In some of our work we have been exploring the idea of students as performance mathematicians, where the audience for their learning is expanded to include students in other classes, family and friends, and the wider world through the use of the Internet (Gadanidis & Borba, 2008; Gadanidis, Hughes, & Borba, 2008). An example of this is available at http://www.edu.uwo.ca/mpc/bigideas/bbw (see Figure 22.5). Here, a Grade 2 teacher relates the experience of his students:

(a) Scripting dialogues of mathematics conversations they might have at home when someone asks, “What did you do in math today?,”
(b) Performing their mathematics learning for a Grade 7 class,
(c) Performing their learning as a song and music video posted on the Internet, at the *Math and Science Performance Festival* (see http://www.MathFest.ca).

Another example of a performance from the online *Math and Science Performance Festival* in Canada that has been supported by the Fields Institute, the Imperial Oil Foundation and the Canadian Mathematical Society is *Now I’m a Trapezoid* (available at http://www.edu.uwo.ca/mathscene/geometry/geo1.html). This is a

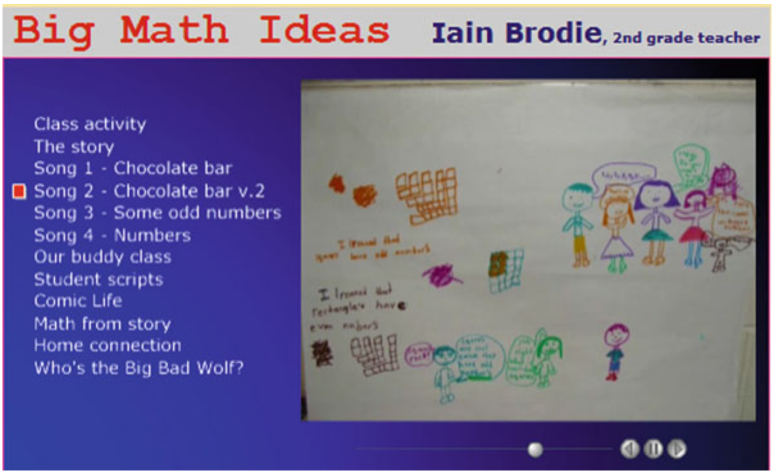

Figure 22.5. Students as performance mathematicians.

Figure 22.6. A performance from the *Math and Science Performance Festival.*

song by a fifth-grade student about a triangle that has lost its head. Saddened by this loss, the triangle laments that it is now a trapezoid (see Figure 22.6). The creation of such performances involves pedagogical shifts for mathematics teachers, putting a greater emphasis on mathematics communication through the arts and mathematics communication for a public audience.

Such pedagogical shifts are supported by the assertion of Gadanidis and Borba (2008) and Gadanidis, Hughes, and Borba (2008) that students might be viewed as *performance mathematicians* and that a performance (as in the Arts) lens might be useful in framing the teaching, learning and doing of mathematics, especially in a technology-rich setting. Such a lens helps us see and judge mathematics activity as we would see and judge a film. For example, if a mathematics activity was to be judged as we might judge a film, then Gadanidis and Borba (2008), using the work of Boorstin (1990), suggested that it would "work" if it offered us opportunities to experience the following pleasures:

- the pleasure of seeing the new and the wonderful in mathematics;
- the pleasure of being surprised mathematically;
- the pleasure of feeling emotional moments in doing and learning mathematics; and
- the pleasure of sensing mathematical beauty.

Discussion. Borba and Villarreal (2005) suggested that humans-with-media form a collective where new media serves to disrupt and reorganize human thinking. Likewise, Lévy (1993) saw technology not simply as a tool used by humans, but rather as an integral component of a *cognitive ecology* of the humans-with-technology. He added that technologies *condition* thinking. Can we imagine *what might be* if students and teachers, through their immersive experiences with performative affordances of new media, were *conditioned* to think about learning and teaching in performative ways? Lévy (1998) also claimed "as humans we never think alone or without tools. Institutions, languages, sign systems, technologies of communication, representation, and recording all form our cognitive activities in a profound manner" (p. 121).

Pineau (2005) suggested that "[t]he claim that teaching is a performance is at once self-evident and oxymoronic" (p. 15). However, as a theoretical claim, it is highly problematic. Pineau maintained that the typical interpretations of teaching-as-performance as (a) *teacher-as-actor* and (b) *teacher-as-artist* are weak, as the former reduced teaching to "teaching like an actor," and the latter equated it with "intuition, instinct, and innate creativity" (pp. 18–21). As an alternative, Pineau raised issues of power and authority and saw performance as political struggle and resistance.

Performance as a form of political struggle and resistance has been the centre-piece of the work of Boal (1985), namely his book *Theatre of the Oppressed*. In one of Boal's Forum Theatre performances, a person in poverty shopped for groceries and was confronted by the cashier as he did not have the money with which to pay for the food his family needed to survive. As the play unfolded, members of the audience (spect-actors) could at any time replace an actor and navigate the play in directions they deemed to be appropriate. There were at least two important things *at play* in such a performance. First, the common script of "shop, pay, take home" was disrupted. A second important thing at play was the agency of the audience. A spect-actor had the same right as the actor to be a part of the play.

Viewing students as performance mathematicians helps disrupt the traditional hierarchy of knowledge and authority in the mathematics classroom. Internet-based performances of mathematics help bring to public light the mathematical thinking of students themselves, who have traditionally been silenced outside the confines of mathematics classrooms. Just as importantly, seeing public performances of student mathematics raises the question, "What makes for a good mathematical performance?"

Boorstin (1990) identified three pleasures that we derive from performances such as at the movies: (a) the new and the surprising; (b) emotional moments; and (c) visceral sensations. It is interesting that Norman (2004) stated that his principles for technological design "bear perfect correspondence" (p. 123) to the principles of what make movies work, identified by Boorstin. These principles have been used in Canada and in Brazil to research how they might be used as a basis for pedagogical design in mathematics education and how they might help us see teachers and students as performance mathematicians (Gadanidis, Borba, Hughes, & Scucuglia, 2010).

Our focus on performance in mathematics parallel our immersive work with Internet-based new media. Although we cannot make a strong claim of effect, anecdotal records of our experience suggests that the performative affordance of Internet-based media helped influence and support our thinking; or, as Borba and Villarreal (2005) suggested, disrupt and reorganize our thinking in this direction.

Final Reflections

> The most recent information will be easily and directly available through online databases and the World Wide Web. Students will be able to participate in deterritorialized virtual conferences, where the best researchers in the field will be present. The primary role of education will no longer be the distribution of knowledge that can now be obtained more

> efficiently by other means. It will help provoke learning and thinking. Education will become a driving force of the collective intelligence for which it is responsible. It will focus on managing and monitoring learning: encouraging people to exchange knowledge, relational and symbolic mediation, personalized guidance for apprenticeship programs, and so forth. (Lévy, 2001, p. 151)

Philosophers such as Pierre Lévy have made several predictions about the world with the Internet. In the above quote, from a book originally published in French in 1997, Lévy foresaw some of the transformations powered by the Internet that have already occurred, such as the availability of databases with almost any information. It is still not quite clear how this will transform education overall as the Internet shapes more and more of our world. It is not quite clear either what consequences it will have for schooling. As already noted in this chapter, traditionally teachers and books were the main source of information for students. School could be seen as the place where information would possibly become knowledge, collective knowledge. As the Internet plays an increasing role in education, including mathematics education, classrooms will be transformed or "dissolved" in the Internet (Borba, 2009).

However at present it is not clear how widespread the use of the Internet in schooling has become. But with its ever-growing presence, critical questions arise. It is fair to say that most of what is asked in mathematical examinations is easily found with the help of a search device on the Internet. How schools deal with this issue, given that all students have been born into the Internet-world, is still an open question. Will textbooks and regular lectures disappear, or just continue to be replicated online, as authors such as Engelbrecht and Harding (2005) have documented? It is too early to know.

We have tried to show how some practices are already being developed, merging arts, and particularly performance, in a way that students can post their work with little expense and can change the usual way they participate in mathematical studies. But the change in places of teacher and students is not the only result of the participation of the Internet in the production of mathematical knowledge in schools. Multimodality seems to be another key word. Students have the possibility of expressing mathematics using simulators, animations and pictures, combined with usual text and mathematical formulas. We still do not know the place that this kind of activity will have in regular schooling. The observation by Castells (2009) is worth noting, as he reminded us that advances in communication systems can not only generate possibilities but also create problems:

> Each one of the components of the great communication transformation represents *the expression of the social relationships, ultimately power relationships that underlie the evolution of the multimodal communication system*. This is most apparent in the persistence of the digital divide between countries and within countries, depending on their consumer power and their level of communication infrastructure. … Even with growing access to the Internet and to wireless communication, abysmal inequality in broadband access and educational gaps in the ability to operate a digital culture tend to reproduce and amplify the class, ethnic, race, age, and gender structures of social domination between countries and within countries. (p. 57; original emphasis)

Although, in this chapter, we have tried to show possibilities of the use of the Internet, we have also hinted at various points the disproportionate spread of

Internet access. As in early 2011, still two-thirds of the world's population does not have access to the Internet. Hence although the Internet is accessible by two billion people, this also means that it is not available for between four and five billion people! In this sense the Internet can be seen as a double-edged sword: opening possibilities for some, but increasing the gap between those who have access and those who do not. In this sense, the Internet creates a new educational divide in the world. There are now the "haves" and "have nots" related to their educational access to the Internet.

Different countries have come up with different policies to include all or most of its citizens, in a time that having an electronic address seems to be as important as having a street address. But this divide is not just in terms of countries. Castells (2001) predicted that the Internet could increase the creation of a *fourth world* in many big cities. He developed the idea that the old division between first, second and third world was being modified and that we could actually have all the different worlds in almost every country. New York would have areas with high Internet access and others with low or not at all. This would coincide with the first and the fourth World respectively, in terms of economic power. We suspect this is happening and it does have implications for mathematics education.

This requires public policies that help all to be able to take advantage of such technology. The *market forces* on their own can take too long to reach the "do-not-have-Internet" since they are for the most part the ones with very little economic power. In addition, just as one aspect of Internet technology seems to become popular, another quickly and sometimes unexpectedly takes its place. For example, although most adults in developing countries continue to rely on the use of email for person-to-person communication, many students are keeping track with friends using social networking sites, such as Facebook or Twitter. All of this rapid and unpredictability makes the adoption of current Internet technology a daunting task. Nevertheless, as we have noted in this chapter, these developments, with all their hopes and confusions, do have an impact on mathematics education.

The case studies we have presented in this chapter represent not what is typical mathematics learning in today's classrooms but what might be possible. Will the Internet help transform mathematics education and enhance student learning? Past experiences with *new* technologies (such as television) indicate that the promises that they held for enhancing student learning were not fulfilled, at least not on a broad scale. Will it be different with the Internet?

We finish the chapter with one dimension of the changes brought by the Internet that has only been noted in passing but could have profound ramifications for mathematics education: assessment. The Internet has brought multimodality, which we noted may transform the nature of how we express mathematical ideas. If that occurs, what will this do to the manner in which we assess mathematics in the future? Furthermore, if most students ultimately have access to the Internet, and most answers for most mathematical problems are published on the Internet, what then becomes a challenging mathematical problem with which we can assess students' knowledge? Again, we have no answers. Nevertheless, elaborating problems that, as yet, have no answers may make us think more clearly about the potential worlds that may open before us. We will have to wait a few more years yet before we can see clearly the place the Internet will occupy in the educational scenario, including within mathematics education.

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