Chapter 20
How Might Computer Algebra Systems Change the Role of Algebra in the School Curriculum?

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Abstract Computer Algebra Systems (CAS) are software systems with the capability of symbolic manipulation linked with graphical, numerical, and tabular utilities, and increasingly include interactive symbolic links to spreadsheets and dynamical geometry programs. School classrooms that incorporate CAS allow for new explorations of mathematical invariants, active linking of dynamic representations, engagement with real data, and simulations of real and mathematical relationships. Changes can occur not only in the tasks but also in the modes of interaction among teachers and students, shifting the source of mathematical authority toward the students themselves, and students' and teachers' attention toward more global mathematical perspectives. With CAS a welcome partner in school algebra, different concepts can be emphasized, concepts that are taught can be done so more deeply and in ways clearly connected to technical skills, investigations of procedures can be extended, new attention can be placed on structure, and thinking and reasoning can be inspired. CAS can also create the opportunity to extend some algebraic procedures and introduce and assist exploration of new structures. A result is the enrichment of multiple views of algebra and changing classroom dynamics. Suggestions are offered for future research centred on the use of CAS in school algebra.

Developing an understanding of algebra is central to school mathematics, and the teaching and learning of algebra is receiving increasingly greater attention in a range of national settings. In the USA, for example, the President convened a National Math Panel, a central purpose of which was to provide the best advice on preparing children for the study of algebra. As nations attack the issue of enhancing

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students' understanding of algebra, it becomes important to define what is meant by algebra. Textbooks dedicated to algebra identify topics to be covered, and national, state, and local goals identify the algebraic skills that students must master. But algebra is more than a list of topics. It is also a way of thinking and reasoning. In this chapter, we consider algebra to consist not only of a set of mathematical topics but also ways of thinking.

Just as it is important to examine strategies for improving students' understanding of algebra, it is important to do so in the context of the technological resources available for assisting the learning of algebra. As algebra-specific software becomes increasingly available in school mathematics classrooms, it becomes more and more important to examine the ways in which such software can affect the teaching and learning of algebra and the part that algebra plays in developing students' understanding of mathematics. One configuration of software that is particularly relevant to the learning of algebra is what has come to be known as a Computer Algebra System (CAS). The CAS can, on command, perform symbolic manipulations that often comprise much of a student's algebra skill set. The basic utilities of a CAS are enhanced by the linking capability among its components. A CAS links graphical, numerical, and tabular utilities with that of a symbolic manipulator, and interactive symbolic links to spreadsheets and dynamical geometry programs are becoming a more common part of CAS configurations. Communication among these latter components opens the possibilities for decreasing barriers that have at times separated the study of algebra from the study of other areas of the mathematical sciences. The capability of linking symbolic mathematics capabilities to graphical and dynamic geometry, for example, opens the possibility of symbolic experimentation supplemented by graphical parametric exploration and corroborated through geometrical construction and measurement. Networking with the capacity to collect and display results from a large group of students allows experimentation more easily to become a group project instead of an individual investigation. The ever-increasing possibilities for connections and interactions open the door for an algebra that links traditional notation systems and representations to new ones. The myriad current possibilities for CAS encourage substantial changes in the role of algebra in the school curriculum. This chapter discusses those potential changes.

Brief History of CAS in Mathematics Education

To provide a context, before examining how CAS might affect the role of algebra in the school curriculum, it seems useful to review how the use of technology in mathematics education has evolved. The evolution has been threefold. The type of technology available has evolved, the ways in which that technology is used have evolved, and research and theories about teaching and learning in the context of that technology have evolved. These evolutions have been interdependent with limitations on the available technologies constraining the ways that they could be used, and limitations on uses constraining the field's ability to investigate and explain learning in the context of technology. For each phase of this evolution and each general type of technology, there was initially a time during which there was experimentation with what could be done. This was generally followed by the development of curricula and instructional approaches and investigation of the effects of the technology's use in the consequent range of settings. Finally, often after periods of experimentation and development, theory was developed or expanded to explain the use of that technology. For CAS, the initial work was limited by the platforms on which the CAS was built. Early versions consisted of only symbolic manipulation programs.

Work in use of technology in mathematics education has evolved in the areas of curriculum and instruction. Neither of these foci replaced the others, but the consideration of CAS in each enriched the field's perspective on what was involved with the incorporation of technology in mathematics education. This evolution has been reflected in work with CAS as well, and the development of theories about technology use was accelerated by CAS-related work. Initial curriculum work focussed on development of CAS approaches to algebra by students as exemplified early in small trials of Computer-Intensive Algebra (CIA) (early versions of Fey & Heid, 1995), later in widespread use of CAS calculators in Austrian (Böhm, 2007) and Australian schools (see http://extranet.edfac.unimelb.edu.au/DSME/CAS-CAT/publicationsCASCAT/Publications.shtml#2009 for an extensive publication list related to Australian CAS-CAT work), and finally in the incorporation of CAS work in widely used curricula such as those of the University of Chicago Mathematics Project (Usiskin, 2004). Various configurations were tried in the course of experimentation with CAS in school algebra, ranging from supplements to an entire curriculum. Theory related to instruction has evolved from characterizing the nature of technical work with CAS (Artigue, 2002; Lagrange, 1999), to describing the work methods of students using CAS (Guin & Trouche, 1999; Trouche, 2005a), and to developing theory describing the relationships between the instructor, students, and CAS (Trouche, 2005b). Attention is now turning to the networking and connectedness possibilities with the advent of the TI Navigator for TI-Nspire with CAS (see Roschelle, Vahey, Tatar, Kaput, & Hegedus, 2003, for a discussion of networking and connectedness in mathematics instruction).

In spite of the long history of work with CAS in educational settings, the impact of technology on school mathematics has to date been marginal, and the incorporation of CAS in classrooms has been even slower. Some would attribute this slow movement to the time it takes to implement fully any change (Drijvers & Weigand, 2010). Others would attribute it to the difficulty of making such radical change in the nature of school mathematics or to the difficulties involved in preparing teachers to work effectively with such changes (Zbiek & Hollebrands, 2008). Barriers to incorporation of CAS in school mathematics, however, could also have been related to the nature of the tool itself and its potential uses in school mathematics. The prospect of incorporating CAS as a constant resource in students' algebra experiences has been regarded with trepidation by those who imagined students replacing by-hand facility with symbolic manipulation resulting in a need to depend on technology for transformation of symbolic expressions and equations. They may have suspected that what had been the essence of school algebra would, in

CAS-enabled classrooms, be set aside. On one side, the debate regarding the nature of the change needed in fully integrating the CAS into school mathematics curricula is fuelled by the supposition that a curriculum that does not focus primarily on by-hand symbolic manipulation would deprive students of the insights that could be gained from refined by-hand symbolic manipulation. On the other hand, Dick (1992) pointed out that "to realize the savings in time and to harness the power of computation that a symbolic calculator can provide, students need to pay more, not less, attention to understanding the meaning of the symbols and notation they use" (p. 2).

Throughout its history in school mathematics classrooms, CAS has offered a range of new opportunities for the teaching and learning of algebra and the resultant effects on the nature and depth of mathematical content as well as on the nature of assessment. Researchers have investigated the effects of CAS on the content, teaching, and assessment of school algebra. With constant access to CAS, the nature of tasks, classroom interactions, and views of mathematics could be transformed. Pierce, Stacey, and Wander (2010) illustrated, and richly conveyed, pedagogical opportunities in classrooms that have constant access to CAS (see Figure 20.1). Because of the CAS capacity to execute symbolic procedures rapidly and accurately, time is available to engage students regularly in an expanded range of task types. The symbolic manipulation capacity of the CAS allows for exploration of different mathematical ideas in ways that were either not possible or not feasible without such technological help. These new opportunities involve exploration of mathematical invariants, active linking of dynamic representations, and engagement

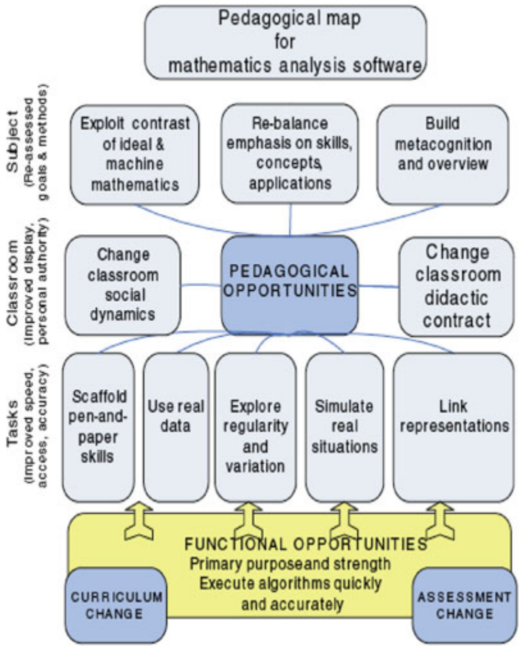


Figure 20.1. Pedagogical opportunities in classrooms that had constant access to CAS (from Pierce, Stacey, & Wander, 2010).

with real data and simulations of real and mathematical relationships. With the welcoming of CAS in school classrooms, changes can occur not only in the tasks but also in the modes of interaction among teachers and students. With powerful tools in students' hands, the source of authority can shift toward the students themselves and teachers and students can engage in a newly defined relationship that includes not only the teacher, the tasks, and the students but also the technology. Students' and teachers' attention can turn toward more global mathematical perspectives, such as recognizing the affordances and constraints of work with technology and maintaining a balance of procedural and conceptual knowledge.

Our examination of literature across the history of CAS in mathematics education suggests three topics that are central to discussions of research, theory, or practice: the interaction of concepts and procedures; new concepts, extended procedures, and structures that can be approached with CAS; and the thinking and reasoning that CAS use inspires or requires. In the following sections, we undertake each of these three topics before we come to terms with the role of algebra in the school curriculum and address associated issues and needed research.

The Role of CAS in Calibrating the Conceptual–Technical Balance of Algebra Instruction

In considering the potential for CAS to affect the role of algebra in the school curriculum, it is the symbolic manipulation capacity of CAS that has drawn the most attention. Initial concern was directed at what was perceived to be the imbalance of procedures and concepts in the algebra curriculum, even though subject matter content may not be readily categorizable into either of these subject matter types. Researchers recognized that often the classroom focus was on procedures with little attention to concepts that would signal when those procedures were called for. They experimented with relegating large parts of the symbolic manipulation to the CAS and concentrating attention on understanding fundamental concepts and when particular symbolic manipulations were appropriate. They investigated whether such a shift in focus would result in atrophy or failure to develop by-hand symbolic manipulation skills, whether such re-balancing could result in a more in-depth development of conceptual understanding, and whether a re-balanced approach would result in improved success in problem solving that required execution of particular procedures.

As described in research syntheses focussed on technology in mathematics instruction (e.g., Heid, 1997; Heid & Blume, 2008), early studies examined the effects of various approaches to using CAS on the balance of mathematical procedures and concepts in the curriculum. Studies by Heid (1984, 1988), Palmiter (1991), and Judson (1990) provided evidence that calculus courses at the collegiate level could be designed to use symbolic calculation programs to foster the development of concepts and understanding regarding when to use particular procedures without harming the development of students' skill at transforming and using symbolic forms. Similar results were obtained in early studies of students' learning of algebra using CIA—see early versions of Fey and Heid (1995)—a functions-based algebra curriculum used at the school and college levels that gave students constant access to some form of CAS (Boers van Oosterum, 1990; Heid, 1992; Heid, Sheets, Matras, & Menasian, 1988; Matras, 1988; O'Callaghan, 1998; Sheets, 1993). Early research on CAS use centred on using symbolic manipulation programs, sometimes supplemented by graphing and spreadsheet programs. In these and other studies of CAS use in algebra instruction (e.g., Hollar & Norwood, 1999; Mayes, 1995), a fairly consistent result was that, in a curriculum that prioritized concepts and applications of algebra, fundamental concepts of algebra could effectively be learned without detriment to symbol manipulation procedures.

Researchers have experimented with using CAS in a variety of curricular configurations, ranging from supplements for an existing curriculum to replacement of all or some of the existing curriculum. For example, the CIA project investigated a completely reconceptualized introductory algebra curriculum. An investigation by Edwards (2001) studied effects of regularly supplementing the traditional algebra curriculum with CAS activities, and Kieran and colleagues (Kieran & Drijvers, 2006; Kieran & Saldanha, 2008) studied the effects of specifically designed CAS activities on students' work with symbolic investigations. It should be noted that each of these studies occurred in the context of a curriculum designed to capitalize on the opportunities provided by the CAS. The question was not whether the incorporation of CAS in and of itself made a difference, but whether the CAS could enable the design of algebra curricula that exemplified particular perspectives on the teaching and learning of algebra. Although these studies gave evidence that a different type of learning could occur in the context of CAS-intensive algebra classrooms, analysis of the specific nature of the learning in those settings was largely unexplored. Not every study resulted in superior performance by the CAS group (e.g., Thomas & Rickhuss, 1992), and it became evident that one of the factors that mattered was the particular way in which CAS was integrated into the curriculum. Developers and mathematics educators became wary of the potential for CAS to obscure the symbolic work and popularized a white box–black box analogy to describe the projected role of CAS in school mathematics (Buchberger, 1989). Soon thereafter, the focus of the debate shifted from the question of what effects CAS would have on understanding of concepts and procedures to the nature of the interactive balance of concepts and skills fostered in CAS-intensive environments.

Analysis of the types of mathematical knowledge involved in use of CAS in school mathematics led to the consideration of *computational transposition*. Computational transposition refers to the formation of additional mathematical knowledge that the use of a particular computational artefact involves (Artigue, 2002; Balacheff, 1994; Hoyles & Noss, 2009). Concerned about the danger of considering technical work and conceptual understanding as separable, French researchers shifted the attention of CAS research to the construct of *technique*, which accentuated the development of integral links between procedures and conceptual reflection (Artigue, 2002; Lagrange, 2003). These researchers pointed out that, within a

CAS-enhanced setting, concepts and techniques are intertwined and embedded within a context. In a landmark book based on the work of this research team, Guin, Ruthven, and Trouche (2005) provided a language to describe how the relationship of user to tool played out in the integration of CAS into school mathematics. Drawing on the field of ergonomics (Vérillon & Rabardel, 1995), the authors of chapters in that book explained that CAS was an artefact that needed to develop into an instrument for teachers and students. They used the phrase *instrumental genesis* to describe the development of an artefact into an instrument, and noted that this genesis involves the transformation of the individual (*instrumentalization*) as well as the transformation of the artefact (*instrumentation*). This attention to the development of the relationship between the CAS and the CAS user accentuated the importance of recognizing that the nature of the use of a tool such as the CAS was not independent of the activity and experience of the user. These constructs hold considerable promise in explaining the range of effects in individual settings and situations for CAS-enhanced instruction.

As Artigue (2002) noted, "any technique, if it has to become more than a mechanically-learned gesture, requires some accompanying theoretical discourse" (p. 261). In the case of tool-assisted procedures, an additional participant in the discourse is the tool itself, and the tool brings with it its own mathematical system. The challenge for students and teachers is to account for the mathematics of the tool as well as the mathematics that students are intended to learn. At the elementary algebra level, for example, the user of a CAS needs to be aware of how the particular CAS being used handles extraneous roots and expressions that are undefined for particular input values.

The question raised by Artigue is how to determine the theoretical discourse needed for adequate student control of the artefact. Hasenbank and Hodgson (2007) suggested that the development of procedural understanding, presumably in the style of technique, can be aided through the implementation of a meta-analytical approach to procedures. They suggest that students engage in a series of questions about their procedural work:

What is the goal of the procedure?
What answer should I expect?
How do I carry out the procedure?
What other procedures could I use?
Why does the procedure work?
How can I verify my answers?
When is this the "best" procedure to use?
What else can I use this procedure to do?

Questions about the role of CAS in developing mathematical knowledge and about the nature of the balance of technical and conceptual understanding has permeated research on CAS-assisted mathematics, yet such research needs both theory that could inform the development of those approaches and venues for trying those different approaches. Empirical advances have been made with the creation and testing of CAS-intensive approaches, and the development of theoretical perspectives and frameworks have refined the field's approach to research on the effects of

CAS-assisted approaches to the learning of algebra. Yet, progress has sometimes been slowed by a general reluctance to welcome CAS into the regular school mathematics curriculum. Nevertheless, the field is positioned to engage in theory-based research with the potential for making significant advances in its understanding of the ways in which CAS can affect the balance and interplay of procedural and conceptual knowledge.

CAS Effect on Changing Emphasis on Concepts, Extending Procedures, and Attending to Structure

Incorporation of CAS in school algebra has the capacity to affect both the content of school algebra and how that content is developed. Different concepts can be emphasized, concepts that are taught can be studied more deeply, investigations of procedures can be extended, and new attention can be placed on structure. In this section we provide illustrations of each of these potential changes.

Changing Treatment of Concepts

The subgroup of the ICMI algebra study that focussed on the use of CAS in algebra learning suggested that one of the crucial questions to ask when considering implementation of CAS was "How does CAS use influence student conceptualization?" (Thomas, Monaghan, & Pierce, 2004, p. 166). One possibility is that CAS offers the opportunity to investigate concepts more deeply and to emphasize concepts that might not otherwise be prominent. In reality, in classrooms where CAS has been used by teachers themselves (rather than by researchers who involved teachers in their work), some research (e.g., Thomas & Hong, 2005b) has suggested that student activity with CAS rarely involves investigating a conceptual idea but is mostly used to obtain procedural answers and check work completed by-hand. This is an example of what Artigue (2002) called "the transmission of the bases of mathematical culture" (p. 246), passing on the socially constructed norm of what constitutes mathematical activity, which has traditionally been primarily by-hand procedural work. In this section we consider some possible activities in which CAS might be used to extend student engagement with mathematical conceptualization.

One of the keys to accessing mathematical concepts with CAS is the set of techniques that is promoted in the classroom. For many teachers these techniques are often perceived and evaluated in terms of their *pragmatic value* (Artigue, 2002), or how much can be efficiently accomplished using them. Artigue (2002) described the *pragmatic value* of techniques as their "productive potential (efficiency, cost, field of validity)" (p. 248) and the *epistemic value* as their contribution "to the understanding of the objects they involve" (p. 248). She stressed that techniques are most often considered and appreciated for their pragmatic value. An example would be the formula for solving quadratic equations, which has high pragmatic value in schools. However, in addition to this value for producing answers, drawing graphs, and other activities, a CAS instrument also has an *epistemic value*; that is, it has the capability to be used to produce knowledge of the object under study and to give rise to new questions that in turn promote new knowledge (Lagrange, 2002, 2003). It is particularly this area of how CAS can assist in construction of knowledge of mathematical concepts that is the subject of this section. We consider three main areas: how the CAS can allow some concepts in the current algebraic content in the curriculum to take on a different emphasis and importance, while emphasizing others that might not otherwise be prominent; how the CAS can create the opportunity to extend some algebraic procedures; and how the CAS can be used to assist exploration of new structures from outside the immediate curriculum.

There are two overarching principles that guide the examples presented here. One is that of using the CAS to assist in generalization. Mason, Graham, and Johnston-Wilder (2005) claim that expressing generality lies at the heart of mathematics and hence "a lesson without the opportunity for learners to express a generality is not in fact a mathematics lesson" (p. 297). They maintain that every page of a textbook should not only contain such opportunities but should clearly signal the need for generalization. This aim lies at the heart of the following examples.

The second principle used here is that, as teachers and researchers, we need to look for ways to use the epistemic value of CAS to improve students' mathematical understanding. Employing it as a "black box" in the context of which the student has little or no idea how the outputs relate to the inputs does little for students' learning of mathematics. In contrast, using the CAS as a tool for investigation can lead students to engage to some extent with the essential core of mathematical thinking. In this manner students will be encouraged to develop both mathematical *ways of thinking* and *ways of understanding* (Harel, 2008).

Delving More Deeply into Concepts

Understanding forms a crucial part of the mathematical experience for a number of fundamental, ubiquitous algebraic concepts. Examples of these concepts are variable, function, expression, and equation. CAS can offer an opportunity to engage with these concepts in a more comprehensive and deeper way than has often been the case.

One manner in which algebraic concepts can be explored more deeply is through a consideration of how they relate to other representations. In this regard Duval (2006) reminded us of two important classes of cognitive activity involving representational transformations (transformations within or between registers or representation systems). Duval designated transformations that happen within the same register as *treatments*, and those that consist of changing a register without changing the object as *conversions*. Although Duval (2006) recommended prioritizing conversions over treatments for those studying mathematical learning, and especially when analyzing student difficulties, CAS environments are capable assistants in both treatments and conversions. Important conceptual aspects arise from relating, through conversions, corresponding elements of conceptual representations. In the context of algebra, the manipulation of expressions or formulas and algebraic solution of equations would be treatments, whereas drawing a graph or producing a table of values for a given algebraic representation of a function would be conversions. CAS environments in which representation systems are linked and interactive are capable of conversion actions in which students need only to choose or enter appropriate commands and then observe the effects of the conversions. Opportunities for student engagement with conversion actions in CAS settings must be carefully crafted. From conversion activity, important aspects of epistemology, and understanding, of a mathematical object can arise, contributing to the goal of helping students attain *versatile thinking* in mathematics (Thomas, 2008a, 2008b), which involves at least three abilities:

- to switch at will in any given representational system between a perception of a particular mathematical entity as a process and the perception of the entity as an object;
- to exploit the power of visual schemas by linking them to relevant logico/analytic schemas; and
- to work seamlessly within and between representations, and to engage in procedural and conceptual interactions with representations.

This third component of the framework for versatile thinking, called *representational versatility* (Thomas, 2008a), incorporates more than Duval's treatments and conversions. The idea of conceptual interactions with representations is one that is highly relevant to CAS use and is exemplified in the following paragraphs.

Algebraic transformations. In a CAS environment the technology can help students to engage with novel (to them) mathematics through conversions. One example of a task that engages students with novel mathematics is the task of asking what algebraic form a function would take when its graph is reflected in the line $y=k$, for some real k. Applying the aforementioned principles by approaching the general through the specific we might ask students to reflect the graph of, say, $y=x^2+3x$ in the line $y=2$. The CAS can be used to draw the graphs (see Figure 20.2). A number of routes and their associated techniques are then possible to attempt to answer the problem. For example, we know that the points of intersection of $y=x^2+3x$ and $y=2$ are invariant under reflection, so we can start by determining these points. Likewise the vertex remains at the same x-value, and this may give ideas for an approach. However, students may develop a strategy involving translating the graph vertically by −2, then reflecting in the x-axis, and then translating vertically by +2. This nicely links the graphical transformations, such as a translation and reflection, with algebraic concepts $f(x)+\mathrm{k}$ and $-f(x)$, and can be accomplished with the CAS (Figure 20.2). The correct answer of $y=-x^2-3x+4$ is seen in Figure 20.3, along with the graph(s) in Figure 20.2 to check that it works.

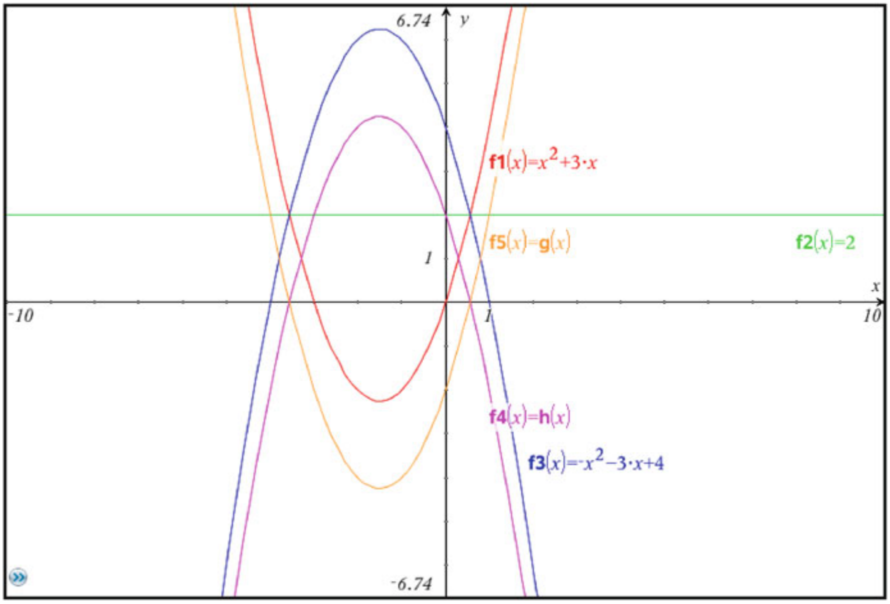


Figure 20.2. Using graphs in CAS to confirm the reflection of function in $y=2$.

Of course the key question is whether one can generalize this, both graphically and, more importantly, algebraically. The key idea here is shown in Figure 20.4. Since $g(x)$ is a reflection of $f(x)$ in $y=k$ *every* point of the plane is reflected. Thus for a general point $(x, f(x))$, distance n above the line, $n=f(x)-k$, and so $g(x)=k-n=k-\left(f(x)-k\right)=2k-f(x)$. Hence, the result of reflecting the graph of a continuous, well-behaved function $f(x)$ in the line $y=k$ is to obtain a function $g(x)=2k-f(x)$. For example, the reflection of the graph of $f(x)=x^3-2x$ in the line $y=-1$ gives the graph of the function $g(x)=-2-(x^3-2x)=2x-x^3-2$. Involving students in a few examples with the CAS might serve as a model for them to engage with mathematics at this deeper level.

Equation and equivalence. The constructs of number, symbolic literals, operators, the "=" symbol itself, and the formal equivalence relation, as well as the principles of arithmetic, all contribute to building a deep understanding of equation. However, there is evidence (Godfrey & Thomas, 2008) that many students have a surface structure view of equation (Laborde, 2002), looking at the equation rather than through it (Mason, 1995), and hence failing to integrate the properties of the object with that surface structure (Thomas, 2008a). An example of this provided by Godfrey and Thomas (2008) is the way in which an embodied input–output, procedural or operational view of equation persists for approximately 25% of secondary school students, even when they reach the university level. In addition, charting student progress through the concepts, Godfrey and Thomas (2008) point

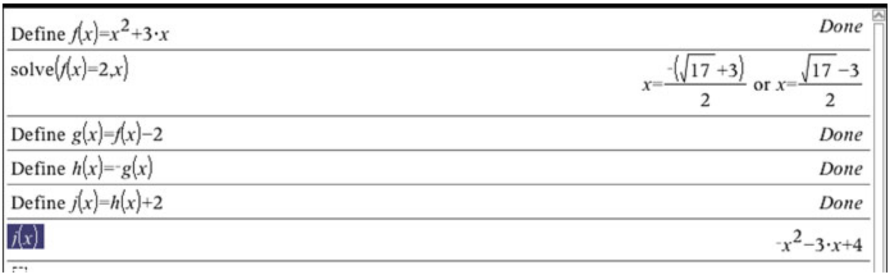


Figure 20.3. Using algebra in CAS to reflect a function in $y = 2$.

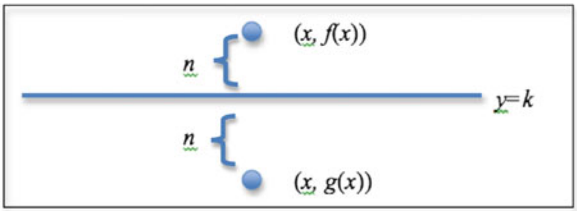


Figure 20.4. Generalizing a reflection in $y = k$.

out that equivalence is not well understood, and that the reflexive, symmetric, and transitive properties forming an equivalence relation are rarely considered in schools, even though they are often assumed.

For example, when solving an equation we may go from $x + 6 = 3x + 1$ to $2x + 1 = 6$, rather than $6 = 2x + 1$, using the symmetric property applied to the *conditional* equation. Or we may reason along the lines that if $y = 2x + 1$ (*identical* equation, defining y), then when $y = 0$ (*conditional* equation), $2x + 1 = 0$ (*conditional* equation), employing the transitive property to do so. Note that *identical* equations are ones that are true for all values of the variable(s) and conditional equations are ones that are true for certain values only. However, we may not explicitly highlight these properties, or the kinds of equations employed, leaving students to abstract these themselves (Godfrey & Thomas, 2008, p. 89).

One study that addressed the issue of CAS use for equivalence, equality, and equation in algebra is that of Kieran and Drijvers (2006). As they comment about equivalence, "On the one hand, equivalence of two expressions relates to the numeric as it reflects the idea of 'equal output values for all input values.' On the other hand, the notion of equivalence of expressions from an algebraic perspective means that the expressions can be rewritten in a common algebraic form" (Kieran & Drijvers, 2006, p. 214). This is another way of describing the proceptual nature of the symbols (Gray & Tall, 1994) as having the dual faces of process (input and output) and object (expression) (Tall, Thomas, Davis, Gray, & Simpson, 2000). As part of Kieran and Drijvers' experiment, 10th-grade students (15-year-old students) considered the equivalence of the expressions in Figure 20.5 and used by-hand techniques to test

CAS technique task

Given expression	Result produced by the Enter button	Result produced by Factor	Result produced by Expand
1. $\dfrac{8x^2-10x-3}{6}$			
2. $\dfrac{(x-3)^2+(x-3)(7x-1)}{4}$			
3. $(3-x)(1-2x)$			
4. $\dfrac{(2x-3)(4x^2-7x-2)}{6x-12}$			

Figure 20.5. Using CAS to consider equivalence of expressions (adapted from Kieran & Drijvers, 2006, p. 216). After they carry out CAS techniques, students compare the results. The purpose is to develop understanding of equivalence.

their conclusions and tried to "reconcile the techniques in the two media." The researchers describe the different techniques arising from each and arrive at several conclusions:

> Two notions of the equivalence of two expressions can be distinguished: an algebraic view as having a common form, and a numerical view on equivalence as having—always, in most cases, or even just in some cases – the same numerical output values. The latter view is related to the previous item, and is reflected in the language issue related to the words equivalent and equal.
>
> … The issue of restrictions on equivalence is an important theoretical aspect of the concept of equivalence. It involves both the particularities of the way the CAS deals with restrictions, and the somewhat strange definition—at least possibly strange in the eyes of the students—of equivalence involving a set of admissible values.
>
> … The relation between solving an equation and the notion of equivalence of expressions, and between restrictions on equivalence and solutions of the equation, could be confusing for students. Both restrictions and solutions have a sense of "exceptions," but in a kind of complementary way. This issue needs coordination…. (p. 220)

The following activity, from Thomas (2009) was designed to assist students to distinguish equivalent equations.

> Which of the following equations have the same solutions? Explain how you worked out your answers and write down reasons for your answers. Use a graphic calculator to help you work out and support your answers with an explanation.

(a) $x^2+x+1=2x^2-x-3$
(b) $x^2+x+5=2x^2-x+1$
(c) $x^2-x+1=2x^2-3x-3$
(d) $x^2+2x+1=2x^2-2x-3$
(e) $2x^2+3x-1=3x^2+x-5$ (p. 153)

Thomas maintained that the theory underpinning this task is to understand the difference between *legitimate transformations* of an equation—those that are mathematically correct and preserve the solutions—and *productive transformations*—those that also move rapidly towards finding the solutions. This distinction is often not understood by students. Linking to the graphical representation can support the students' understanding of the invariance of solutions under legitimate transformations.

Continuity. CAS can also help to use algebraic representations to make concepts such as limits and connecting limits to continuity (and possibly differentiability) more prominent in the curriculum. If we consider, for example the function $f(x)=\begin{cases} x^2 & x\leq 3 \\ x+6 & x>3 \end{cases}$, then the question arises whether the function is continuous at $x=3$. We can define the function piecewise in the CAS using "Define $f(x)$=piecewise(x^2, $x\leq 3$, $x+6$, $x>3$)" and get the CAS to draw the graph of the function (see Figure 20.6). Looking at the left and right limits provides corroborating evidence that the limit exists and is equal to 9, which is also clearly $f(3)$ [which is equal to 3^2]. If the students know about derivatives, and we are beginning to discuss their existence, then getting the CAS to draw the graph of the derived function shows clearly the discontinuity in the derived function at $x=3$. Finding the limits confirms this (see Figure 20.7).

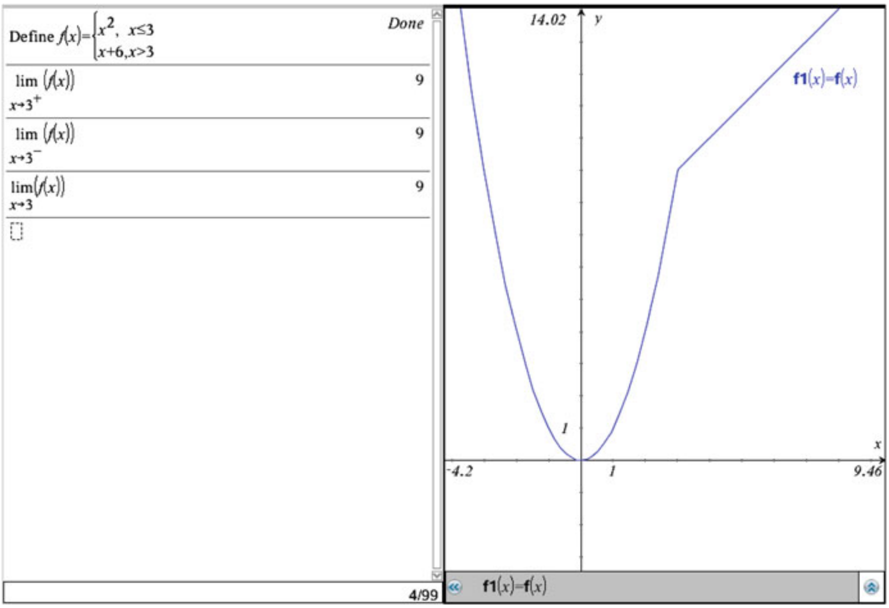


Figure 20.6. Graph of a piecewise-defined function.

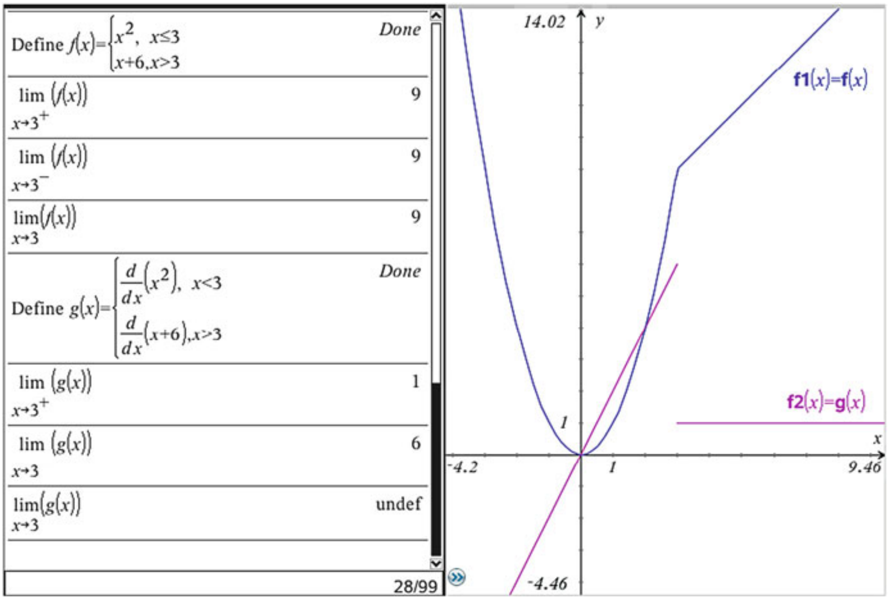


Figure 20.7. Symbolic and graphical confirmation of discontinuity of the derived function at $x=3$.

One area in which the CAS output needs careful scrutiny involves the continuity of functions such as $f(x)=\dfrac{1-x^2}{x^2-2x-3}$. Here the graph (see Figure 20.8) does not show the discontinuity at $x=-1$, although the CAS generates a warning that the "Domain of the result may be larger than the domain of the input." Encouraging students to use the CAS to link representations provides the opportunity for further insight. The table of values shows that the function is not defined at $x=-1$, and this is then confirmed by attempting to generate a value for $f(-1)$. The continuity of other interesting functions can be similarly investigated.

Extending Procedures

In mathematics one of the most important ideas that students need to develop is an understanding that all mathematical processes and constructs have conditions or limitations that influence their use. For example, consideration of the domain of a function is a vital part of its study. One way to build appreciation of this is to extend student knowledge by engaging them in areas of mathematics that lie just beyond their current understanding. In this section we consider some algebraic examples for which CAS may assist with extending procedures to objects beyond those they have experienced or by encouraging generalization of procedures.

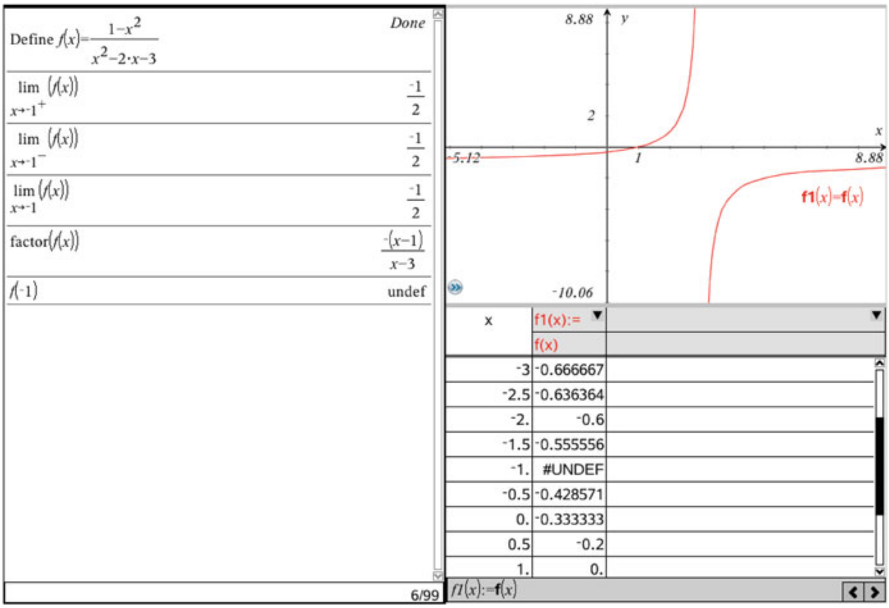


Figure 20.8. Confirmation that the function is not defined at $x = -1$.

Moving toward generalization through extension of factoring. A task used by Kieran and colleagues (Kieran & Drijvers, 2006; Kieran & Saldanha, 2008) considered the use of the factoring command in CAS to get students to move towards a generalization regarding the factorization of $(x^n - 1)$. Students worked in both directions, factoring expressions of the form $(x^j - 1)$, for $j = 2, \ldots 6$, and expanding $(x-1)(x+1)$, $(x-1)(x^2 + x + 1)$, and so on. The outcomes suggested that:

> The notion of complete factorization can come to the fore as soon as students attempt to factor an expression with a non-prime even exponent, such as $x^4 - 1$, according to the general rule [using only a factor of $x - 1$], and are confronted with a CAS factorization that they do not anticipate [e.g., $(x-1)(x+1)(x^2+1)$]. (Kieran & Drijvers, 2006, p. 243)

Thus by-hand techniques are helpful in reconciling these differences. In turn this can elicit further conjectures, such as $(x+1)$ is always a factor of $(x^n - 1)$ for even n, which then requires proof. Kieran and Drijvers proposed that this CAS-based approach led to theoretical development for the students in at least four areas:

1. Resolution of the conflict between by-hand and CAS results led to enhanced theoretical perception of the structure of expressions of the form $(x^n - 1)$.
2. Noticing in CAS output structure that they had not noticed in prior examples.
3. Improved reflection through tentative conjectures based on the examples they generated, and testing the conjectures by means of CAS techniques.
4. Deepening of theoretical thinking involving the coordination and integration of several discrete pieces of theory.

The researchers conclude that technique and theory emerge in mutual interaction, with CAS playing a crucial epistemic role.

Extending polynomial investigations. One general question in engaging students in investigations with CAS is whether activities should start with a general case or not. Since the CAS allows one to consider such cases, for example a cubic $x^3 + ax^2 + bx + c = 0$, it is tempting to make this a starting point. However, there appears to be a stronger case for beginning with specific examples, encouraging students to form conjectures and gradually to motivate them to move their thinking towards the general cases, as seen in the previously described example from the research of Kieran and Drijvers. This again relates to the Task–Technique–Theory (TTT) framework that Kieran and Drijvers (2006) espoused, based on ideas from Artigue (2002) and Lagrange (2002, 2003), namely that it is through the construction of techniques required to perform tasks that the understanding of mathematical objects arises, often through the production of new questions. This deepening of understanding may also arise through reflective comparison of the technique with other techniques (Lagrange, 2003). This is precisely the epistemic role of techniques.

Most school students will at some time be shown the formula for the solutions of a quadratic equation. However, if we are thinking about using CAS to extend what may be considered, then the zeros of a cubic function (or the solutions of a cubic equation) should be a topic for investigation. Careful structuring of the process of considering the Tartaglia-Cardano method of solution may be needed, but this investment would allow for a valuable extension of algebraic thinking and capability. For example, given the cubic equation:

$$x^3 + 3x^2 - 6x + 9 = 0$$

(with some discussion of why the coefficient of x^3 is 1) one could ask how a general method to solve such an equation could be derived (rather than using a black-box approach), and what mathematics would arise from doing so.

Using the CAS we can define the function f such that $f(x) = x^3 + 3x^2 - 6x + 9$. Then our first task is to remove the term in x^2. This can always be done and the resulting production of a depressed cubic is the first fundamental idea in the Tartaglia-Cardano method of solution. This draws nicely on the mathematical idea of composite function, which is usually introduced in school but may often find few applications. Here we want to find a k such that $f(z+k)$ avoids a term in z^2. Students could experiment until they find one that works (see Figure 20.9). Trying other cubics they will be asked to generalize and find a "rule" for a substitution that works. In fact for $f(x) = x^3 + ax^2 + bx + c$, making the substitution $x = z - \frac{a}{3}$ (which can be done relatively easily with the CAS to confirm the generalization) gives

$$f\left(z - \frac{a}{3}\right) = \left(z - \frac{a}{3}\right)^3 + a\left(z - \frac{a}{3}\right)^2 + b\left(z - \frac{a}{3}\right) + c.$$

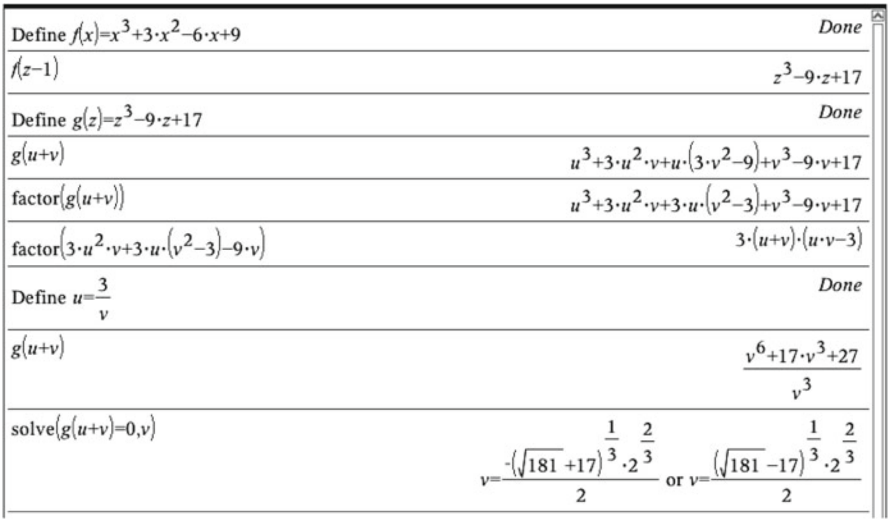


Figure 20.9. TI-Nspire computer screen of the Tartaglia–Cardano method of solving cubic equations.

And this can be seen to result in an equation of the form $z^3+mz+n=0$, as shown in the example in Figure 20.9.

Then we may ask how do we solve this equation? Why is it easier than the original one? Here is where the beauty of the method comes in. If we let $z=u+v$ then, as Figure 20.9 shows, $g(u+v)$ does not, at first sight look very useful, and trying to factor with the CAS does not work. But factoring the terms other than u^3, v^3, and 17 is the key to the method (although seeing why it would be useful requires a leap of insight in the original formulation), since it gives a "nice" factorization. It is this that suggests the idea of setting $uv=3$ to remove these terms (but why?). Doing so we can reduce the cubic to a quadratic and hence find the solution. At each stage of a number of examples the student is encouraged to ask "Is this a special case or will it always happen?" and to find evidence to support their conclusions.

One may ask, why bother to do this when the original cubic can be solved on the CAS in an easy step? We remind the reader who thinks this way of our second principle above. Using CAS to investigate a method such as the one just described will lead students to engage in mathematical thinking and reasoning and will divert attention away from a purely answer-driven approach to mathematics.

Another area whereby known procedures can be extended is that of solving Diophantine equations. Of course, Pythagoras' theorem could be the springboard for this since it is often studied and there are readily accessible integer solutions to $x^2+y^2=z^2$. Although, as has been proved by Andrew Wiles (and as was stated in Fermat's Last Theorem), there are no other integer values of $n>2$ for which any triple (x, y, z) of non-zero integers, gives a solution for $x^n+y^n=z^n$, there are similar looking equations that do have positive integer solutions. One of these, $x^n+y^n=z^{n+1}$,

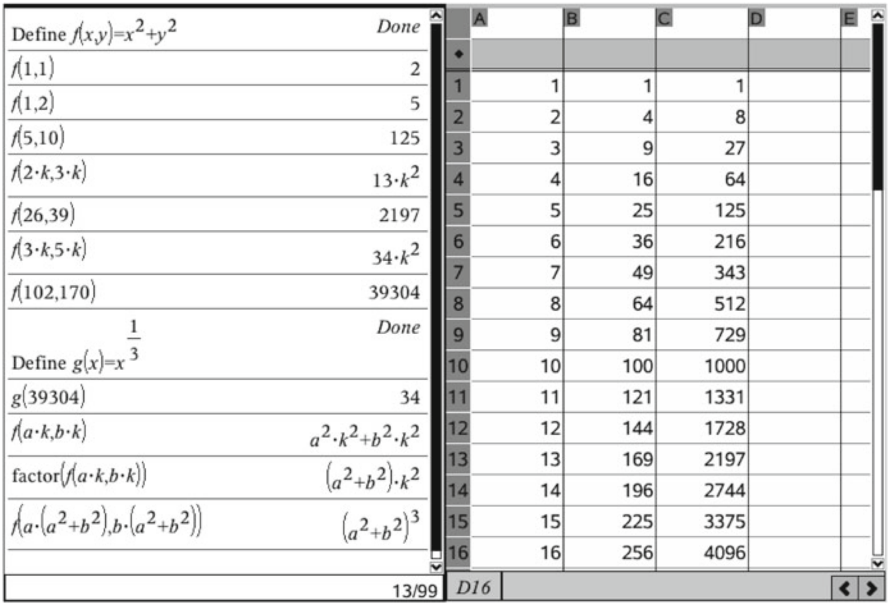


Figure 20.10. CAS screens showing a method of solving $x^2+y^2=z^3$.

which is accessible with CAS, was described by Hoehn (1989). Once again we might start with a particular equation, say $x^2+y^2=z^3$, and ask students to try to find a solution using the CAS. A function of two variables could be defined (see Figure 20.10), introducing a new mathematical construct. After a few trial-and-error attempts using $x=1$ or 2, the use of a spreadsheet with values of n^2 and n^3 could help to find two of the squares that add up to a cube (for example $x=2$, $y=2$ and $z=2$ may be seen immediately). In this way $x=5$ and $y=10$ can also easily be found. Hence, there is at least one solution. If students start to flounder, then some teacher direction could suggest trying something of the form $f(ak, bk)$ for given integers a and b. However, the teacher might aim for this conjecture to come from the class.

In Figure 20.10 we can see examples with $a=2$ and $b=3$, and $a=3$, $b=5$. Now in each case we get an answer of the form ck^2 and since we are looking for something of the form z^3 the idea is to set $c=k$, giving k^3. We soon get some large values and the spreadsheet could be extended to check $\sqrt[3]{39304}$, and so on, or the CAS will do it even better. So now the generalization question comes into play. Will this always work? With the CAS we can try general a and b of course, as seen in Figure 20.10. In this case it still works if we set $a^2+b^2=k$, and the final step shown in the CAS screen shows that this gives z^3, with $z=a^2+b^2$.

The final step of a complete generalization to the solution of $x^n+y^n=z^{n+1}$ is likely to be a step too far for all but the most able school students, but we comment on it here for the sake of completeness and the principle of generalizing results. Figure 20.11 shows an attempt to use the TI-Nspire to apply the same method as above.

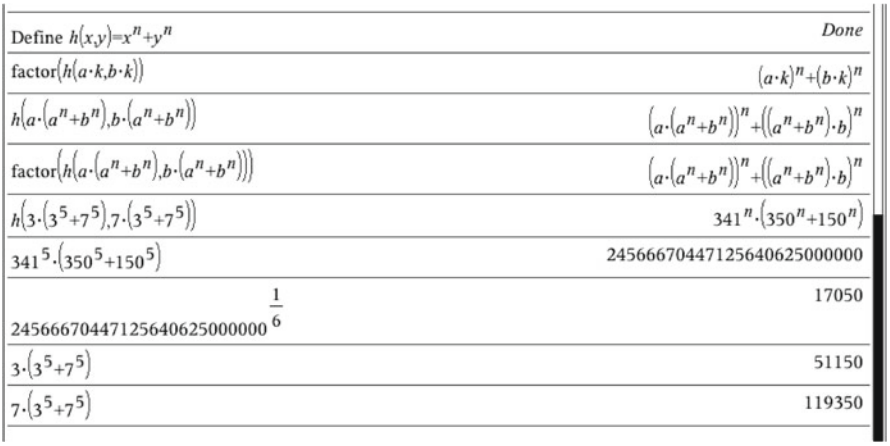


Figure 20.11. CAS screens showing a method of solving $x^n + y^n = z^{n+1}$.

Defining a function $h(x, y) = x^n + y^n$ and considering $h(ak, bk)$ with $k = (a^n + b^n)$ leads to the expression $(a(a^n + b^n))^n + (b(a^n + b^n))^n$, which by hand can readily be seen by an experienced eye to factor to $(a^n + b^n)^n(a^n + b^n)$ and hence equal $(a^n + b^n)^{n+1}$. However, the TI-Nspire program does not seem to be able to cope with this factorization, making this a good example to help the students to see that CAS has its limitations and to realize that they cannot rely on it to do everything for them. Thus, in the above manner, for a given n, we can construct solutions of $x^n + y^n = z^{n+1}$. One example with $n = 5$, $a = 3$ and $b = 7$ is shown in Figure 20.11, where we see evidence that $51150^5 + 119350^5 = 17050^6$.

The previous examples focussed on determining solutions to given equations. Tasks that require the generation of equations with particular features, including given solutions, are another way in which work with polynomial functions might be extended. Relatively early in their experience with factoring polynomials and solving equations, students might be asked the following task, from Böhm (2007):

> Given is a set of solutions L = {3, −1, 1/2}
> Find two equations of degree 5 with L = set of solutions. (p. 3)

Although a CAS Solve command or graphical means could be applied in the hope of determining solutions for an equation of degree 5, the CAS work needed to generate an equation from information about the solutions is not obvious, especially to beginning algebra students. Figure 20.12 shows what we might do as starting points for symbolic, tabular, graphical approaches.

We know other things that are possible or not possible in each approach. For example, the complete symbolic form is $(x-3)(x+1)(x-1/2)(x-\square)(x-\square)=0$ where each box represents one of 3, −1, and 1/2. Choosing one of the solutions for each of the boxes produces an equation that satisfies the conditions.

The question of producing two equations that meet the conditions then allows for generalization at a level appropriate for students. For example, we could see how

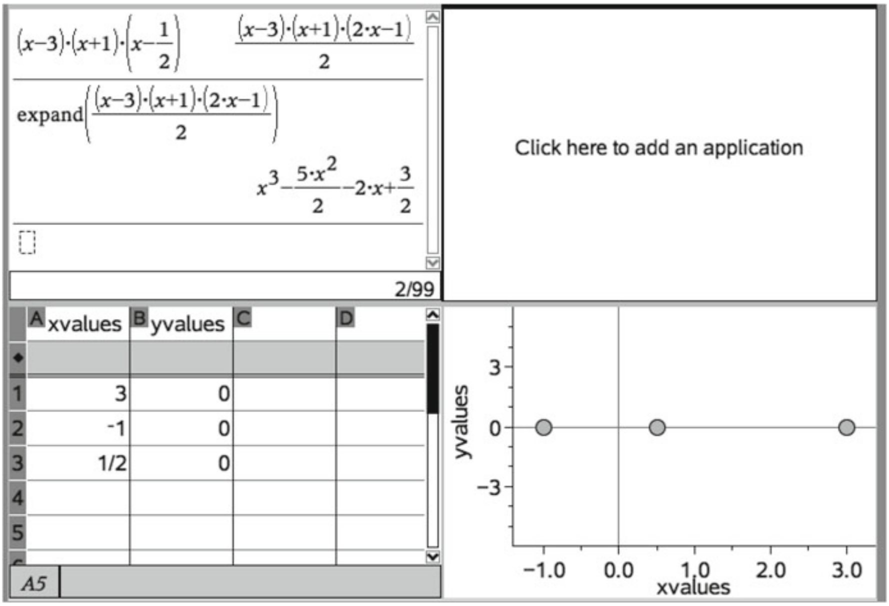


Figure 20.12. Initial symbolic, graphical, and table attempts to produce an equation of degree 5.

many distinct equations are possible when the equation $(x-3)(x+1)(x-1/2)(x-\square)(x-\square)=0$ is expressed in expanded form. The results of testing all nine combinations of two solutions and looking for distinct results could be done with nine CAS Expand commands or, as shown in Figure 20.13, with a CAS-generated table.

The task provides an opportunity for predetermining two or more distinct equations but also to characterize the number and nature of possible equations by reasoning symbolically. Filling both boxes with one of the three solutions yields three distinct quintic expressions. Filling the two boxes with different solutions yields three more distinct quintic expressions. So, there are six possible equations of the form $x^5+bx^4+cx^3+dx^2+ex+f=0$ that satisfy the given conditions.

To this point, an underlying assumption might be that the equation is in the form of a polynomial of degree 5 with leading coefficient 1 set equal to 0. Students familiar with factoring might produce additional equations by using a constant factor with the quintic polynomial. Infinitely many more are possible when any nonzero real number, k, is used as a factor, as in the expression $k(x-3)(x+1)(x-1/2)(x-\square)(x-\square)=0$ or $kx^5+kbx^4+kcx^3+kdx^2+kex+kf=0$.

Graphically, as in Figure 20.14, we could think about the situation in terms of behaviour at each of the three points. If it touches the x-axis at one point, then it must touch without crossing at another point and simply intersect at the third point; there are three ways in which this can happen. If the graph has an inflection point at one point, it simply crosses at the other two, which happens in three ways. If the graph simply crosses at one point, we find it falls into one of the other two cases.

	A box1	B box2	C result	D alternative	E
◆			=f(a[],b[])	=expand(c[])	
1	3	3	(x−3)^3*(x−0.5)*(x+1)	x^5−8.5*x^4+22.*x^3−9.*x^...	one
2	3	-1	(x−3)^2*(x−0.5)*(x+1)^2	x^5−4.5*x^4+13.*x^2+3.*x−...	two
3	3	1/2	(x−3)^2*(x−0.5)^2*(x+1)	x^5−6.*x^4+8.25*x^3+4.75*...	three
4	-1	3	(x−3)^2*(x−0.5)*(x+1)^2	x^5−4.5*x^4+13.*x^2+3.*x−...	repeat
5	-1	-1	(x−3)*(x−0.5)*(x+1)^3	x^5−0.5*x^4−6.*x^3−5.*x^2...	four
6	-1	1/2	(x−3)*(x−0.5)^2*(x+1)^2	x^5−2.*x^4−3.75*x^3+1.75*...	five
7	1/2	3	(x−3)^2*(x−0.5)^2*(x+1)	x^5−6.*x^4+8.25*x^3+4.75*...	repeat
8	1/2	-1	(x−3)*(x−0.5)^2*(x+1)^2	x^5−2.*x^4−3.75*x^3+1.75*...	repeat
9	1/2	1/2	(x−3)*(x−0.5)^3*(x+1)	x^5−3.5*x^4+0.75*x^3+2.87...	six

Figure 20.13. Testing nine symbolic options using a CAS-generated table to determine six distinct results.

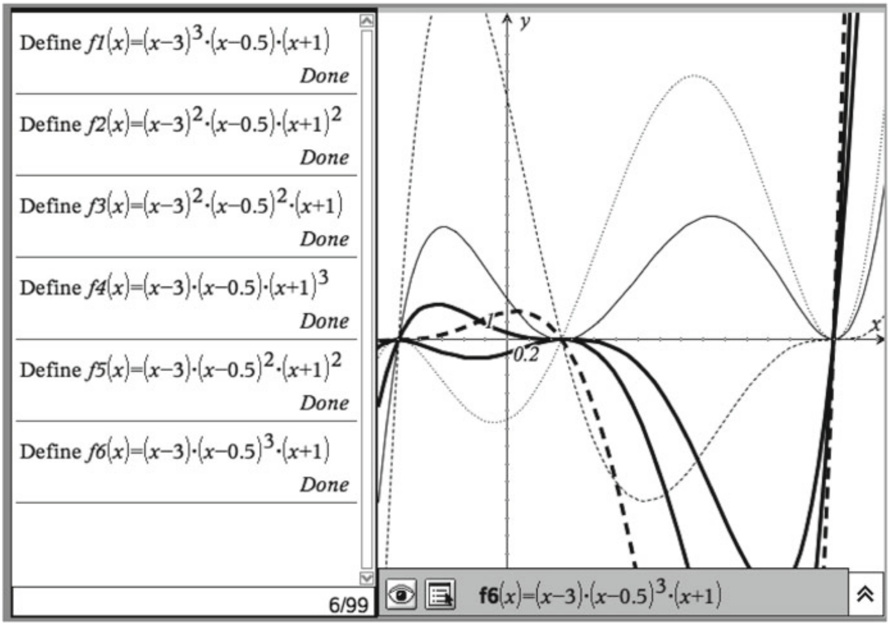


Figure 20.14. Graphs representing quintic functions which lead to six different equations.

As with the symbolic form, we have six general patterns and the graph can draw attention to the meaning of the solution set. Taking amplitude into consideration, we have the effects of the constant factor and infinitely many choices.

Böhm's task requires students to think about characteristics of equations and their solutions. Extending the task with a question about the number and nature of possible equations yields a generalizing experience in elementary algebra.

Exploring "New" Structures

Using CAS there is an opportunity to investigate the structure of other "abstract" algebras where the "rules" or axioms governing the structure of the algebra of generalized arithmetic no longer apply. It can demonstrate that the rules that we take for granted do not extend to all systems. In introducing the following examples we employ some of the appropriate mathematical language describing the structures, although teachers may not want to use this language with students. Some examples are:

1. Students expect $AB = BA$; that is, that multiplication is commutative;
2. Students expect $AB = 0$ if and only if $A = 0$ or $B = 0$, since there are no non-zero divisors of zero.
3. Extending 2 we can see we expect that if $AB - AC = 0$ then $A(B - C) = 0$ and $A = 0$ or $B = C$.

Using CAS it is easy to set up a situation for which this can be investigated. For example we may consider the following 2 by 2 matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix},\quad B = \begin{pmatrix} 3 & -8 \\ 2 & 3 \end{pmatrix},\quad C = \begin{pmatrix} 5 & 2 \\ 1 & -2 \end{pmatrix},\quad D = \begin{pmatrix} 2 & -1 \\ 3 & 3 \end{pmatrix},\quad E = \begin{pmatrix} -4 & -4 \\ 2 & 2 \end{pmatrix}$$

Using a CAS, students can generate the products, AE, BD, DB, AB, and AC, and can find that $BD \neq DB$, $AE = 0$ even though $A \neq 0$ and $E \neq 0$, and $AB = AC$ even though $A \neq 0$ and $B \neq C$.

$$BD = \begin{pmatrix} 3 & -8 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} -18 & -27 \\ 13 & 7 \end{pmatrix}$$

$$DB = \begin{pmatrix} 2 & -1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 3 & -8 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -19 \\ 15 & -15 \end{pmatrix}$$

$$AE = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} -4 & -4 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 3 & -8 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ 21 & -6 \end{pmatrix}$$

$$AC = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ 21 & -6 \end{pmatrix}$$

Then students can be asked to state a conjecture and continue their investigation, possibly considering a proof of it, using, for example, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For instance, they may find that in the ring of 2 by 2 matrices the zero divisors are singular, that is, with determinant 0. Questions arise about whether the order matters for the zero

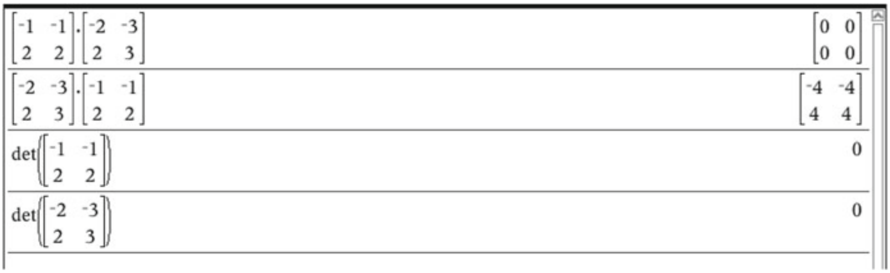


Figure 20.15. TI-Nspire computer screen showing left/right zero divisors with determinant zero.

divisors, and they may find that there are left and right zero divisors (e.g., we can ask whether we can find two non-zero matrices P and Q such that $PQ = 0$ but $QP \neq 0$). Figure 20.15 shows that this is possible.

Thinking and Reasoning that CAS Use Inspires or Requires

A striking feature of the examples in the previous section of how CAS allows students to engage with new concepts is the extent to which the mathematical work involves generalization, including generalization of properties, strategies, and other relationships. As Arcavi (1994) observed, CAS is “a tool for understanding, expressing, and communicating generalization, for revealing structure, and for establishing connections and formulating mathematical arguments” (p. 24). The impact of CAS on thinking about connections and formulating arguments can be considered in terms of the objects about which students reason and the tools they employ in their reasoning.

Objects About Which to Reason

Reasoning opportunities with CAS seem to be related to the tool’s multiple representation capacity. We begin with perhaps the most enticing CAS aspect—possibilities in the symbolic register.

Symbolic representations. Arguably the most documented type of CAS-generated opportunity for reasoning about symbols is the resolution of unanticipated symbolic results. Reasoning stems from the need to compare CAS-produced results to by-hand results or to a desired informative equivalent symbolic form. Alonso and colleagues (2001) provided several examples of unexpected results and their use to encourage students to reason about the results and about how they are using CAS.

The duality of reasoning about mathematics and about CAS functions is a common theme in CAS literature.

A related though less frequently mentioned reasoning opportunity is conjecturing and justifying theorems that underlie CAS procedures. Dana-Picard (2007) drew attention to CAS commands that are implementations of theorems that do not typically appear in course syllabi. Her examples include Derive's use of the following theorem when computing $I_n = \int_0^{\pi/2} \sin^n x \, \mathrm{d}x$,

$$\int \mathrm{SIN}(a \cdot x + b)^p \, dx \to -\frac{\mathrm{SIN}(a \cdot x + b)^{p-1} \cdot \mathrm{COS}(a \cdot x + b)}{a \cdot p} + \frac{p-1}{p} \cdot \int \mathrm{SIN}(a \cdot x + b)^{p-2} \, dx$$

(p. 223)

Supported by evidence of student symbolic reasoning, Dana-Picard contended that the user needs to learn new mathematics in order to understand well the CAS process. She referred to these situations as *motivating constraints,* and she contended, despite the connotation of "constraint," that these situations can be used to push the user towards mathematical insight. Her construct of motivating constraint is an addition to Guin and Trouche's (1999) extension of Balacheff's (1994) ideas regarding *internal constraints* of the hardware, *command constraints* of the software, and *organization constraints* of the interface. Dana-Picard's example illustrates how CAS features can motivate identification and justification of theorems beyond the standard syllabi.

Other uses of CAS can help develop student understanding of symbols and symbolic reasoning. Cedillo and Kieran (2003) detail an experiment in which beginning algebra students generated the algebraic code needed for a CAS to produce given numerical patterns (e.g., input numbers 1, 4, 6, 9 with corresponding output numbers 1, 7, 11, 17). Students tested the code and used CAS results to revise it. Results of the study indicate that students developed the notion of "a letter as 'serving to represent any number'" (p. 231). In this case, reasoning about symbols while using CAS was the means by which concepts were developed.

As another example of reasoning about symbols while using CAS, consider the following task from McMullin (2003):

> Use the sequence operation to produce the sequence 3, 6, 9, 12, 15 as many different ways as you can. (p. 268)

Multiple possibilities, including several suggested by McMullin, appear in Figure 20.16. The reasoning for a beginning algebra student that produces each of the options could include simply replicating the terms, attending to a linear pattern, and considering multiples of three—as exemplified in the first three lines of Figure 20.16. Subsequent examples indicate how the task could be differently handled with additional mathematics experience.

Similar to activities used by Cedillo and Kieran, this task engages students' understanding of equivalence through the production of CAS code. The concept

seq(n,n,3,15,3)	{3,6,9,12,15}
seq(n+3,n,0,12,3)	{3,6,9,12,15}
seq(3·n,n,1,5,1)	{3,6,9,12,15}
seq($\left(\sqrt{3\cdot n}\right)^2$,$n$,1,5,1)	{3,6,9,12,15}
seq($\frac{n^2-n}{n-1}$,n,3,15,3)	{3,6,9,12,15}
seq(20−n,n,17,5,-3)	{3,6,9,12,15}
seq(n·sin(0.5·π),n,3,15,3)	{3.,6.,9.,12.,15.}

Figure 20.16. Sequence commands that yield 3, 6, 9, 12, 15.

under consideration in this case is not only sequence but also equivalence. The CAS seq expressions are equivalent because they represent the same finite sequence, although the defining expressions (e.g., n, $3n$, $n\sin(0.5\pi)$, $20-n$) are not necessarily equivalent. These examples underscore the need to understand symbols both as algebraic expressions and as CAS code. They also highlight the importance of distinguishing among the mathematical objects being represented (in this case, sequences and expressions).

Attention to symbolic understanding and the symbolic capacity of CAS foregrounds consideration of symbolic sense. According to Arzarello and Robutti (2010), who built on Arcavi's (1994) notion of symbol sense as they described students working with handheld CAS,

> Students have symbol sense if they are able, for example: to call on symbols in the process of solving a problem and, conversely, to abandon a symbolic treatment for better tools; to recognize the meaning of a symbolic expression; and to sense the different roles symbols can play in different contexts. (p. 720)

Arzarello and Robutti claimed that the symbolic power of a CAS-empowered spreadsheet supports the development of symbol sense in a way that tables of numerical examples cannot. Examples of student work—including the spontaneous use by two students—supported their claim. In generating a table of numerical values for second differences of $y=ax^2+bx+c$ for integer values of x from 0 to 15, students could see a constant numerical second difference (e.g., −4) for a specific quadratic case. However, a table of symbolic results for second differences for x-values of x_0, x_0+h, x_0+2h, …, x_0+15h showed that the constant difference in the general case was $2ah^2$. CAS results made it easier for students to see symbolic patterns and then reason about them.

Reacting to CAS results that are produced in intended or spontaneous ways appears useful in helping students to develop meaning for symbols as they reason with and about these results. Some of the observations in the symbolic register seem to have parallels in other registers. For example the potential of immediate feedback has long been acknowledged in other registers, such as its impact in graphical tasks (e.g., Ruthven, 1990) and geometric environments (e.g., Hillel, Kieran, & Gurtner, 1989). We turn now to consideration of how CAS facility with graphical representations generates opportunities and supports reasoning.

Graphical representations. Graphical reasoning can be an alternative to symbolic reasoning, but connecting graphical and symbolic actions and results is one way in which CAS use provides opportunities that transcend affordances of simpler graphing utilities. For example, recall the reasoning with transformations of functions in the example of reflecting a quadratic about a horizontal line. Students could reason graphically about translating the graph vertically by −2 then reflecting the result in the *x*-axis and then translating that result vertically by +2. Application of this reasoning to the graph as a set of points using three points to generate a quadratic expression connects graphical and symbolic images in a solution that crosses registers.

A second example of integrated graphical and symbolic reasoning involves solving equations by graphical intersection. Such methods generalize to equations for which symbolic methods are not available. Zbiek and Heid (2011) illustrated the reasoning process that draws on characteristics of functions to reason through a solution for $\ln x = 5 \sin x$ that required manipulating graphical images, acknowledging approximate nature of values, and reasoning about the behaviour of the logarithmic and trigonometric functions. Reasoning graphically allows students to expect and identify intersection points beyond those that are produced by a direct solve command (see Figure 20.17a) or that appear in a typical viewing window

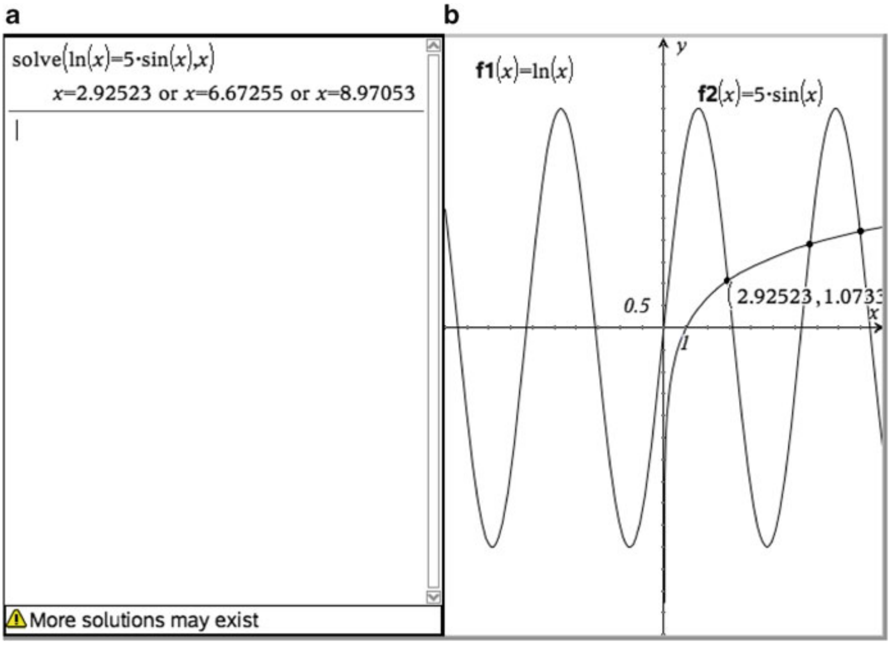


Figure 20.17. Typical direct solve results (**a**) and viewing window image (**b**) suggesting three approximate solutions for $\ln x = 5 \sin x$.

(see Figure 20.17b) and to justify why there is a finite number of solutions. By reasoning about the monotonic behaviour of the logarithmic function in contrast to the bounded values of the sine function, students concluded that, although there are many solutions that they can illustrate by scrolling to see what happens for larger values of x, there are not infinitely many solutions. They also came to terms with the difficulty of representing the solutions in compact symbolic forms due to their non-periodic values. Although it might seem that reasoning about graphs overshadows symbolic reasoning in this example, there are two important elements that symbolic forms offer. First, reasoning about properties of functions requires the symbolic forms. Unlike graphs that provide only approximate values and convey a function relationship for only a subset of a domain, symbolic forms provide the needed specificity for confidence in the argument. Second, examples like this provide opportunities for students to experience instances in which symbolic forms (or graphical forms) fall short as they coordinate among different techniques.

Graphical reasoning related to equation solving might be done not only to identify solutions but also to make sense of how properties of real numbers and properties of equality are used to make sense of steps in symbolic procedures. For example, Zbiek and Heid (2011) assumed a beginning algebra context and use the equation $6x+3=12+3x$ to illustrate how these two types of properties differently affect the values of the two expressions but not the solution of the equation. Figure 20.18a contains a set of steps executed with CAS. The sequence of graph pairs of the members of each equation appears in Figure 20.18b–f.

Figure 20.18b, c shows that application of properties of real number operations does not change the graphs, as it does not change the values of the expressions for any value of x. In contrast, Figures 20.18d–f illustrate that application of properties of equality leave the solutions unchanged but expression values changed. A comparison of these two types of graphical situations illustrates differences as well as the relationship between equivalent expressions (produced by application of properties of real numbers) and equivalent equations (produced by application of properties of equality and properties of real numbers).

CAS-supported reasoning across graphical and symbolic domains can target aspects of student understanding other than equation solving and problem solving. Kidron (2010) shared an example of a discussion of resolving a definition of horizontal asymptote in a calculus course. Nathalie, who previously offered examples and rules but not a definition for asymptote, was asked what an asymptote is. The college calculus student then worked through a specially designed set of tasks to challenge her concept image of asymptote. Kidron described how Nathalie's understanding progressed beyond her initial notion of asymptote as "some kind of a line" such that the "function tends to it—not touching it, but approaching it." Tasks provided instances in which a graph intersected a horizontal asymptote and in which there were infinitely many such intersections. As a result, Nathalie revised her concept definition to acknowledge that "'tending to' is not only when the graph of the function looks like a line which approaches steadily the asymptote, but when the value of the function at infinity equals some number, approaches some specific value." From this example, we suggest that tasks that challenge concept images

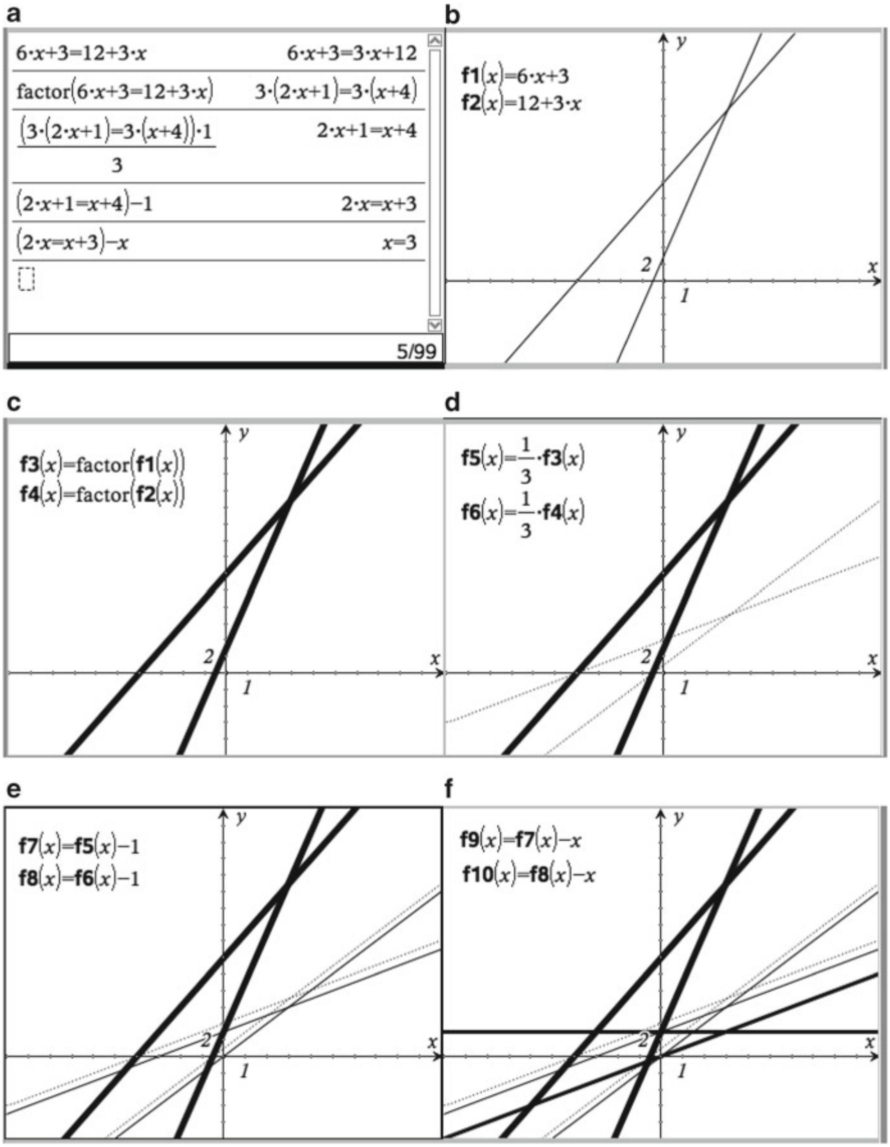


Figure 20.18. Symbolic (**a**) and graphical (**b–f**) representations of steps in solving $6x+3=12+3x$.

through the use of graphical representations can help students develop and understand rich, symbolically stated definitions in addition to common and generalized symbolic solution methods. Reasoning supports symbol sense while capitalizing on CAS multiple representation capacity in developing techniques.

Tools for Reasoning

Although CAS, with its symbolic emphasis and multiple representation capacity, has potential as a tool for reasoning, recent technology developments raise new questions. Use of the previously mentioned CAS-generated tables whose elements can be symbolic algebraic expressions is one way in which students have expedient ways to generate multiple instances within and across registers.

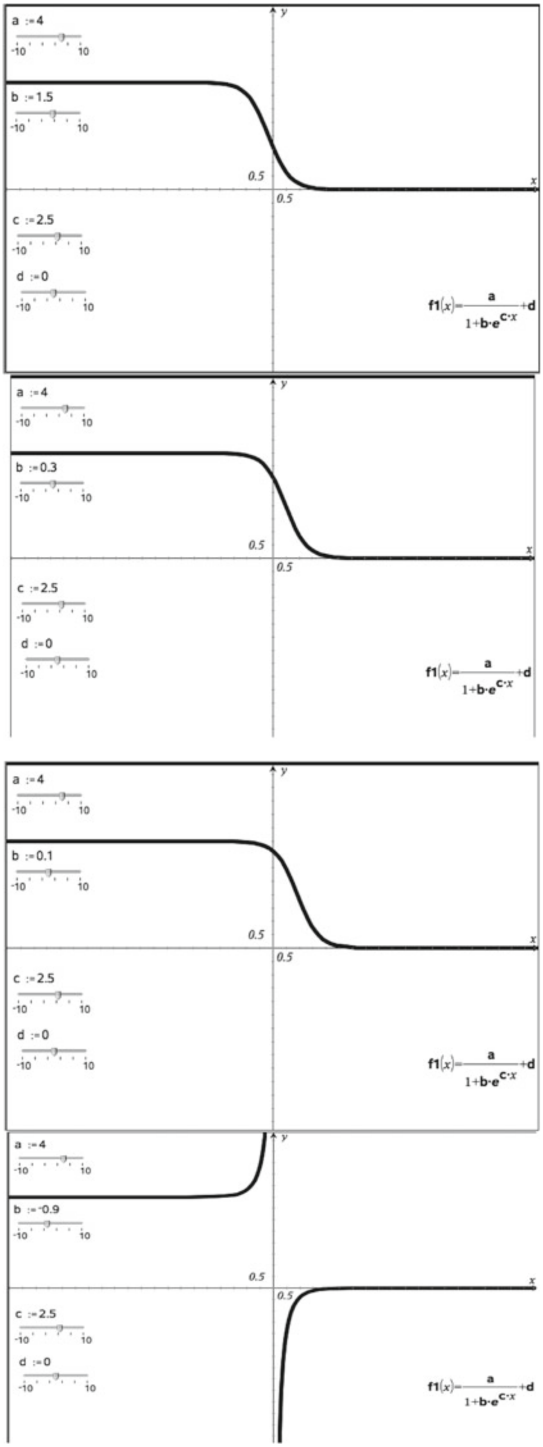
Dynamically linked representations. CAS environments feature not only multiple representations but also dynamically linked representations in ways that allow users to progress quickly through multiple examples by clicking or dragging an element of one representation and seeing corresponding changes in other registers. Scholars working outside of CAS environments (e.g., Hegedus & Kaput, 2007) have emphasized the potential of dynamically linked representations to allow students to see how a phenomenon in one representation might not be apparent in another. Duncan (2010) indicated that teachers believe that linked dynamic representations provide students with evidence to support their reasoning. As Kieran (2007) noted, research on effects of controlled change on dynamically linked representation is an underdeveloped research domain.

Relating both dynamically linked representations and reasoning about results come into play as users can generate multiple values of a parameter by manipulating a "slider." Zbiek and Heid (2001) provided an example with a task that was initially developed in a dynamic geometry setting and was subsequently moved to a CAS slider environment. Students used sliders to explore the family of functions generally represented by $f(x) = a/(1+be^{cx})+d$, where a, b, c, and d are real numbers. When students dragged a slider to change the value of b (as represented by the sequence of graphs in Figure 20.19), they observed a sudden "break" in the graph. The surprise was not as striking when produced with static selection of particular values for the parameter in the absence of a slider. Spurred by the sudden event in the dynamic setting, students reasoned symbolically to justify why such a break would occur. Although empirical research is not extensive, dynamically linked representations have promise as tools to elicit and support reasoning that links the symbolic register to other registers.

Integrated technology environments. Dynamic elements underlie questions that might be leading CAS-focussed researchers to work in broader technology environments. Lagrange and Chiappini (2007) describe the work of two research groups with digital tools that blend CAS with other dynamic elements. A promising feature of one of the artefacts, Cassyopée,[1] is its inclusion of geometry and a connection of algebra to other domains. The integrated or linked nature of representations with current CAS leads to the question of how one reasons within and across different representations. Lagrange and Gelis (2008) describe two lesson

[1] Cassyopée is the spelling used in the referenced paper.

Figure 20.19. Sequence of graphs representing dragging slider to change the value of b in $f(x) = a/(1+be^{cx})+d$.



sets from the Casyopée project, a project involved in adapting or altering CAS to allow students a way to access mathematical symbols. The lessons target difficulties that students have with function ideas (e.g., notation, covariation, linked representations). CAS allows for geometrical calculations and parameter manipulation and supports conjecturing and proving, allowing symbolic work to go with graphical work. Lagrange and Gelis not only find the CAS connection to dynamic geometry in Casyopée important but they also note that a notepad feature—which is a communication medium rather than a mathematical one—allows users to give an account of their work, which is particularly useful for proof work.

As illustrated in these last instances, current research on the nature and potential of CAS is now conceptualized in terms of broader technology environments. Given the evolution of CAS technology, we question what to call tools that include CAS capability among a more extensive suite of tools. Holton, Thomas, and Harradine (2009) use *collection of technologies* (COT) rather than CAS to label calculators and computer software with symbolic manipulation in addition to other capabilities. Pierce and Stacey (2010) refer to calculators or computer software that perform algorithms necessary to execute routine procedures from any branch of mathematics, including but not limited to algebra, as *mathematics analysis software* (MAS). The examples of reasoning in CAS environments that appear in the literature suggest the potential of COT or MAS to support reasoning across registers about algebraic entities and their counterparts in other areas of mathematics.

Role of Algebra in the School Curriculum

We described three foci central to CAS research, theory, and practice: the interaction of concepts and skills, the concepts that can be approached with CAS, and the thinking and reasoning that CAS inspires or requires. With these themes from the literature and issues around teachers and other factors as background, we turn to the question of how CAS change the role of algebra in the school curriculum. Multiple perspectives, approaches, and conceptions of algebra are represented in the literature, including algebra as: generalization (Lee, 1996; Mason, 1996), a study of function (Chazan & Yerushalmy, 2003; Fey & Heid, 1995; Heid, 1996; Mayes, 2001; Yerushalmy & Chazan, 2002), a problem-solving tool (Bednarz & Janvier, 1996; Rojano, 1996), a study of structure (Cuoco, 2002), and a modelling tool (Nemirovsky, 1996).

Introducing CAS into algebra seems to have a direct effect on a functions approach to algebra. Multiple and now dynamically linked symbolic forms, graphs, and tables facilitate the study of functions. The ease of sliders and other tools to study parameter effects facilitates exploration of function families. Most CAS work, like the examples previously reported, involves functions and clearly enriches a functions approach to algebra. However, CAS also enriches other views of school algebra. The capability to construct and alter different symbolic expressions yields modelling possibilities. The ability to build and manipulate complex expressions and the new concepts introduced encourage generalization. Symbolic results to interpret and control provide a venue for algebra as a study of structure. In short, CAS allows each of the views of algebra that we have identified to be enriched.

Many of the examples we have provided have focussed on school mathematics that is likely beyond the capability of beginning algebra students. However, entire curricula have been constructed for beginning algebra students based on the premise of availability of CAS. The aforementioned CIA (Fey & Heid, 1995) curriculum is an example. In the case of the CIA curriculum, integration of CAS allowed the development of a curriculum that took as its central theme the construct of function. For example, solutions of linear equations were taken as the input value, x, for the point of intersection of the functions defined by $f(x)$ and $g(x)$. Equations in two unknowns were viewed as statements about the relationship between two functions of two variables. [See Heid, 1996, for results regarding student learning in the context of the CIA curriculum.] Through attention to blended concepts and procedures, techniques, and new concepts, CAS supports more seamless thinking across arithmetic, algebra, and calculus. Newer CAS-inclusive technologies allow other areas of mathematics, such as geometry and data analysis, to be more closely tied to the symbolic power of algebra. The impact of CAS on the role of algebra in the school curriculum seems to be as a means to make symbolic work more prevalent as students blend procedures and old and new concepts and reason symbolically across the mathematics curriculum and within the sciences.

Issues Related to Implementation of CAS

In this section we briefly address some of the issues that may arise when teachers consider implementation of CAS in their classroom. These include unfavourable attitudes of students, their parents, and society in general regarding the use of CAS calculators in mathematics teaching; the influence of external assessment practice on CAS use; the problems inherent in integration of CAS into current practice; and especially, the attitude and capabilities of the teachers themselves and the changing dynamics of the didactic contract when CAS is present. This last issue covers a number of aspects that must converge to enable the kinds of conceptual use of CAS previously described.

One issue with regard to CAS use relates to student attitudes, which in turn may tend to reflect those of parents and of society in general. The common misconception that use of any calculator is detrimental to the acquisition of mathematical skills appears widespread and persistent. A number of studies have demonstrated that a significant minority of students show some resistance to CAS use, often because they are satisfied with by-hand methods, or believe that this is the only proper way to do mathematics (Ball & Stacey, 2005; Pierce, Herbert, & Giri, 2004; Stewart, 2005). In a study of university students using computer-based CAS, Stewart, Thomas, and Hannah (2005) categorized student attitudes toward CAS, describing one group whose members are openly opposed to computers and believe strongly in the superiority of by-hand work for doing and understanding mathematics. They also described students who use CAS primarily for checking by-hand answers, a practice that has also been noticed among school students (Stewart & Thomas, 2005; Thomas & Hong, 2004, 2005b).

Researchers have identified a number of factors that influence teacher adoption and implementation of technology in mathematics teaching. These include, for example, previous experience in using technology, time, opportunities to learn, professional development, access to technology, availability of classroom teaching materials, support from colleagues and school administration, pressures of curriculum and assessment requirements, and technical support (Forgasz, 2006a; Goos, 2005; Thomas, 2006). Hence, although teachers may acknowledge that technology such as CAS may be used to improve students' learning, many teachers perceive a variety of barriers to the use of the technology (Pierce & Ball, 2009). Forgasz (2006a) lists access to computers and/or computer laboratories as the most prevalent inhibiting factor, with lack of professional development and technical problems, including lack of technical support next. Thomas (2006) agrees, citing availability of technology as the major issue, followed by a lack of resources, training, and confidence. There is also some evidence that a teacher's personal beliefs, values, and attitudes related to mathematics and technology, what Schoenfeld calls *orientations* (Schoenfeld, 2008, 2011) could influence perspectives on obstacles to CAS use. Positive orientations include a strong belief in the value of technology in learning mathematics, confidence in using technology to teach, enjoyment of technology, and an openness to personal learning (Forgasz, 2006a; Hong & Thomas, 2006; Pierce, Stacey, & Wander, 2010; Thomas & Hong, 2005a). Schoenfeld (2011) holds that the teachers' orientations not only shape the goals that they set but also the priority attached to the goals. Schoenfeld further posits that, once the teacher has oriented herself and set goals for the current situation, she then decides on the direction necessary to achieve the goals, and calls on the resources, including technology, to meet them. Goals can emerge in the process of teaching, and Monaghan (2004) claims that the presence of technology can influence goals that emerge during a lesson. Once the goals have been set decisions are made in order to meet them, and it is the quality of this decision making that affects how successful a teacher is in attaining the goals. Since the whole process is underpinned by teacher beliefs as a major part of their orientations, there is a need to focus on what teachers believe about technology use, and how this may change over time (Lagrange et al., 2003). Whereas beliefs are generally stable, and so attempts to influence them have to be long term, appropriate, targeted professional development may be able to shift beliefs about technology, leading to more positive use, as has been noted in other areas (Paterson, Thomas, & Taylor, 2011).

The pressure teachers are under to have their students perform well on external assessment has a strong influence on what they do, or do not do, in the classroom. Many feel that there is a time burden associated with adding technology to their already overcrowded lessons. This perspective is unlikely to change unless CAS use in examinations is sanctioned by educational authorities. Two issues that come to the fore with regard to using CAS in examinations are, first, the effect on what is actually being assessed, given the capability of the calculators, and second, the perceived lack of equality of access caused by the cost of handheld CAS. The latter was reported by Thomas and colleagues (2008) to be of only minor concern to teachers surveyed in New Zealand, but the same research showed that the former does worry teachers. There has been research on the use of CAS in examinations, much of it emerging from Victoria, Australia, where VCE Mathematical Methods (CAS), a CAS-permitted examination, has been in place for some years. The research from Victoria suggests that CAS scaffolds students, helping them engage with extended response analysis examination questions and achieve relatively good success (Evans, Norton, & Leigh-Lancaster, 2005; Norton, Leigh-Lancaster, Jones, & Evans, 2007). In addition there is support for the claim that students who use CAS develop at least the same level of skills as those who use graphic calculators, countering the loss of skills argument. However, to achieve this positive outcome, Ball and Stacey (2004, 2005) concluded that since new mathematical practices and processes of learning emerge when CAS is employed, communicating this to students requires active participation of teachers and a different curriculum emphasis. One aspect of this is the rubric RIPA (*Reasons–Inputs–Plan–(some) Answers*) proposed as a guide for teaching students how to record their solutions when they use CAS. The integration of CAS in the curriculum, including assessment practice, is a crucial issue impinging on CAS use. Research by Oates (2004, 2009), although focussed on tertiary mathematics, pointed out the need for a refined taxonomy to describe what is really meant by such a technology-integrated curriculum.

To use CAS in teaching to its full potential requires a particular set of skills and attitudes on the part of teachers, and so addressing teacher-related issues is crucial. One of these is that while many teachers claim to support the use of technology in their teaching (Forgasz, 2006a; Thomas, 2006) the degree and type of use in the classroom are variable (Zbiek & Hollebrands, 2008). There is also a sizeable minority of teachers who are either not convinced of its value (Forgasz, 2006b) or actively oppose its use (Thomas, Hong, Bosley, & delos Santos, 2008). This latter study reported that 60.5% of teachers disagreed with the statement that "All types of calculators should be allowed in examinations," with only 21.7% in favour, and that 27% of teachers thought that using calculators can be detrimental to student understanding of mathematics. There are many intrinsic factors that may influence a teacher's decision to use (or not to use) technology. These include their orientations; their instrumental genesis of the tools (Artigue, 2002; Guin & Trouche, 1999; Rabardel, 1995; Vérillon & Rabardel, 1995); their perceptions of the nature of mathematical knowledge and how it should be learned (Zbiek & Hollebrands, 2008); their mathematical content knowledge; and their mathematical knowledge for teaching (Ball, Hill, & Bass, 2005; Hill & Ball, 2004; Zbiek, Heid, Blume, & Dick, 2007), which includes Shulman's pedagogical content knowledge (PCK) (Shulman, 1986). PCK refers to understanding not only the mathematical ideas in a particular topic but also how these relate to the principles and techniques required to teach and learn the topic, including appropriate structuring of content and relevant classroom discourse and activities.

Considering these factors led Thomas (Hong & Thomas, 2006; Thomas, 2009; Thomas & Chinnappan, 2008; Thomas & Hong, 2005b) to propose the notion of *pedagogical technology knowledge* (PTK) as a useful way to think about what teachers need in order to use technology, such as CAS, when teaching mathematics. He also suggests that the level of a teacher's PTK may be a key driver of CAS use. The teacher development of PTK for mathematics involves adding a number of attributes to mathematical PCK. The most important of these, enabled by a strong mathematical content knowledge, is a shift in focus, from seeing the technology as simply something added to the teaching of mathematics to putting the mathematics at the centre of activity, and asking how the CAS can enable students to understand the mathematical concepts better. To attain this may require a change in orientations with regard to mathematics and CAS technology. Hence, the affective domain is also involved, with personal confidence in teaching with CAS one dimension of PTK (Thomas et al., 2008). Another aspect of PTK is instrumental genesis of CAS (comprising both instrumentation and instrumentalization), by which CAS tools are transformed into epistemic instruments. Guin and Trouche (1999) argue that instrumental genesis and conceptualization should occur concurrently in the classroom, and, in order for this to happen, teachers need to have developed their PTK sufficiently to be able to focus CAS activity on specific mathematical conceptions, such as those suggested in this chapter. It seems reasonable that teachers who have strong PTK are likely to feel comfortable in accessing CAS when designing mathematical learning experiences. Pierce, Stacey, and Wander (2010) report that initially teachers principally regarded the CAS as a tool for doing, rather than exploring mathematics. However, they believe that this may change as teachers grow in confidence and skills with the CAS. According to Pierce (2005) a teacher who can discern strategic use of CAS and model its effective use to students will make qualitative progress in technology use. One way in which strong PTK may influence teachers is in the use of CAS to mediate student learning through development and use of innovative mathematical tasks and approaches (Clark-Wilson, 2010). In turn, teacher privileging of the technology (Kendal & Stacey, 1999, 2001) has been shown to have a positive impact on students' uptake of technology in exploring mathematics.

How can teachers be assisted to develop PTK further? One critical element in the promotion of teacher PTK, which might lead to improved use of CAS for development of activities that encourage conceptual thinking, is focussed preservice training and inservice professional development (PD) of mathematics teachers (Fitzallen, 2005; Forgasz, 2006b). One suggestion by Goos and Bennison (2005) for improving PD is to employ online discussion by teachers to build a community of practice. It also appears that giving teachers personal experience of using CAS in their own classroom as a component of PD may help them develop their PTK (Ball & Stacey, 2006).

Even when teachers have a high level of PTK, studies show that there are issues involving the didactic contract that arises in classrooms when technology is introduced. Monaghan (2004) suggests that there is no common structure for teacher–student interactions in CAS classrooms, and this can lead to a disconnect between students and teachers with regard to the didactic contract (Pierce, Stacey, & Wander, 2010). While both students and teacher agree that the teacher has a responsibility to teach technology skills, students may see these skills as the main point of the lesson, while teachers view the lesson as primarily about teaching mathematics. An example of how things may change is seen in Duncan's (2010) study, in which teachers recognized that when using CAS they changed the didactic contract, moving from a general class teaching style to greater use of student investigation and discussion. It has also been shown that CAS technology can play a role in the conceptualization of mathematical models rather than simply being a tool that is used to solve a mathematical problem after it has been abstracted, and this can also provoke a change in student–student and student–teacher interactions (Geiger, Faragher, Redmond, & Lowe, 2008). In the light of these and other influences on classroom dynamics and relationships there is likely to be a need for negotiation to adapt didactic contracts.

Needed Research

As we examined the empirical and theoretical literature on the use of CAS, we found promising strands of research. We also realized that there is much yet to be learned about how the incorporation of CAS can affect the teaching and learning of school algebra. We end with a few suggestions for what we see as promising directions for future research centred on the use of CAS in school algebra.

Each of these suggestions requires developing school settings in which CAS technologies are welcome and available. In these environments, we need to know more about how CAS can affect the ways in which students reason about mathematics:

What does research across COT or MAS suggest about student reasoning, such as the role of representations and moving across registers?

How does use of dynamically linked representations motivate reasoning, facilitate reasoning, and contribute to the development of a capacity to reason?

How does prolonged experience with CAS (COT or MAS) affect how students understand and use algebraic symbols?

How can CAS be used to influence student conceptualization? What factors can improve the epistemic value of CAS?

Are there long-term conceptual benefits from CAS use? If so what are they?

We need to know more about instructors and instructional strategies in CAS-present classrooms.

Can we improve the student construction of CAS-related schemes through classroom presentation and discussion of techniques, and, if so, how?

What is the relationship between teacher confidence and pedagogical technology knowledge (PTK)? Along what trajectories does PTK develop? Can PTK be validly and reliably measured, and, if so, how?

How does the introduction of CAS change student–student and student–teacher interactions? Can these changes be captured by descriptions of the didactic contract?

We need to know more about CAS-intensive mathematics curricula.

What does it mean to have a CAS-integrated curriculum? What would it look like? How can we describe what is really meant by a CAS-integrated curriculum at any level?

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