# Chapter 18
# Modelling with Mathematics and Technologies

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**Abstract** This chapter seeks to provide an integrating theoretical framework for understanding the somewhat disparate and disconnected literatures on "modelling" and "technology" in mathematics education research. From a cultural–historical activity theory, neo-Vygtoskian perspective, mathematical modelling must be seen as embedded within an indivisible, molar "whole" unit of "activity." This notion situates "technology"—and mathematics, also—as an essential part or "moment" of the whole activity, alongside other mediational means; thus it can only be fully understood in relation to all the other moments. For instance, we need to understand mathematics and technology in relation to the developmental needs and hence the subjectivity and "personalities" of the learners. But, then, also seeing learning as joint teaching–learning activity implies the necessity of understanding the relation of these also to the teachers, and to the wider institutional and professional and political contexts, invoking curriculum and assessment, pedagogy and teacher development, and so on. Historically, activity has repeatedly fused mathematics and technology, whether in academe or in industry: this provides opportunities, but also problems for mathematics education. We illustrate this perspective through two case studies where the mathematical-technologies are salient (spreadsheets, the number line, and CAS), which implicate some of these wider factors, and which broaden the traditional view of technology in social context.

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## Introduction

Most experienced mathematics educators probably believe they know what is meant by mathematical modelling and how this relates to problem solving, and perhaps even how it situates or is mediated by "technology." Yet, Lesh, and Zawojeski (2007) reported that there was no consensus on this issue among authors and we agree with that reading of the wider literature.

The literature on mathematical modelling is already huge, and is growing in extent, touching on almost the whole of mathematics education and its concerns: epistemology, learning sciences, curriculum, pedagogy, assessment, teacher development, innovation and change, and so on. Several attempts to help the newcomer to this literature must be mentioned. For example, the review by Lesh and Zawojewski (2007) addressed modelling with problem solving, and that by Kaiser and Sriraman (2006), among others, provided an overview and categorization of perspectives on modelling, especially as related to the literature from the International Conference on Teaching Mathematics and its Applications (ICTMA) (Kaiser, Blum, Ferri, & Stillman, 2011).

Blum, Galbraith, Henn, and Niss (2007) set out to present a state-of-the-art review on modelling in mathematics education, but their volume revealed even less convergence, suggesting the diversity of views is ever growing. There are those who see modelling as a new name for Deweyan "inquiry" (Confrey & Maloney, 2007), those from the Freudenthal tradition who see modelling as an emergent, dialectical process (e.g., Gravemeier, Lehrer, van Oers, &Verschaffel, 2002, whose approach is close in spirit to that of this chapter), and others who more or less define modelling "traditionally" through its heuristics and the modelling process, often schematized in a cyclic diagram. Those in this third category are generally guided by modelling as a metacognitive process, as a set of coordinated heuristics in the fashion of Polya (1957), as a tool for categorizing competences and thus assessment of various kinds, as an analytical tool for examining learning, and/or as a guide to teacher intervention. But then there is also a significant literature in the learning sciences, much of which is inspired as we are by cultural historical literatures, including Freudenthal, but also by Vygotskian activity perspectives (typified by authors such as Cobb, van Oers, and Gravemeier).

We will consequently certainly not try here to provide a state-of-the-art summary of mathematical modelling as a whole, but rather begin to develop an integrative, theoretical perspective (with examples and "cases" to help make "sense") that we believe can help conceptualize this field, particularly as regards the topic of this section, that is to say, "technology." Methodologically, because this approach aims for generative insight, it involves "theory and case study" of the phenomenon rather than "sampling and survey." This chapter, in this section of the *Handbook*, will mainly provide theory (with exemplification) while those that follow will likely provide deep "case studies."

We will take a risk here and define a model and modelling in a broad way that builds on a definition put forward by Lesh but also includes most perspectives discussed by Lesh, Blum, Kaiser and others: actually it was inspired by Wartofsky (1979). "A model (or modelling) is a means of seeing a situation (the target domain, sometimes called the 'real') through the lens of another situation (the source domain or 'model,' sometimes the 'mathematics')." Then modelling activity will be "activity" (a concept to be developed below) that involves modelling in a significant way.

Note that this may include all forms of re-presentation, akin to the metaphorical use of language, for instance, and tends to be "two-way", as most mathematical modellers say. Thus, just as the brain can be said to be modelled as a "computer", in computer science the computer is modelled as a brain, and our modern cultural model of computers and brains actually emerges from this two-way dialectic. For an introduction to "cultural models" see Holland and Quinn (1987), and on metaphorical modelling, see Black (1962) and Lakoff and Johnson (1980). This view of course also includes the representation of mathematics by physical models (e.g., counting-beads or the abacus as a model for arithmetic). It even includes much pure mathematical work, even proof, as invoking "modelling" (Hanna & Jahnke, 2007). Importantly, it allows for emergent modelling, and modelling within mathematics, in the sense of those such as Gravemeier and Cobb (see, e.g., Cobb, Yackel, & McClain 2000; Gravemeier, 2007; Gravemeier et al., 2002; Van Oers, 2002) as well as modelling in real problem solving in the continental European and British "trends" (e.g., Blum et al., 2007; Burkhardt, 1981; Pollak, 1969).

Similarly, the term "technology" is often taken for granted and is ill-defined and ill-theorized in the mathematics education literature, though most who address this issue argue that new technologies can be a powerful aid to enriching modelling and provide many examples and innovative approaches in mathematics education. An approach we will find fruitful comes from the analysis of mathematics in the workplace, where mathematics is found embedded or black-boxed in technological artefacts and tools, and mathematical competence may be better described as a form of techno-mathematics or techno-mathematical literacy put forward by Hoyles, Kent, and Noss (e.g., Kent, Guile, Hoyles, & Bakker 2007; Noss, Bakker, Hoyles, & Kent 2007; Noss & Hoyles, 2011).

We can define technological knowledge broadly as practical or scientific "knowledge of tools, machines, techniques, crafts, systems or methods of organization in order to solve problems" (a Wikipedia definition). Thus, technology includes the instruments, techniques and organisation that often embed mathematics "materially" in tools and methods involved in practical activity. In a sense, the "technology" available in a given context is a combination of the tools and the know-how to use them; these may embed the "ideal" mathematics in various forms, as a pair of compasses embeds the mathematics of "locus" of a circle. We will argue that mathematics in practice is always mediated by such technology, and indeed generally becomes fused with technology through such practice (such an argument was attributed by Vygotsky and others to Spinoza, who suggested that in a deep sense a circle really *is* that which is made by a pair of compasses or the equivalent).

While the literature on modelling and technology has to date emphasized the use of technical *instruments*—usually computer technology—in mathematical modelling, it usually sees the infrastructure including the "forms of organization" in schooling as something separate, a matter of learning and assessment, or pedagogy and teacher education, etc., rather than part of the technology. We will refer to these aspects as part of *educational technology*, or the technology of the industry we call "schooling" or "academe", which the modelling literature has become increasingly concerned with in recent years. In recent volumes of ICTMA proceedings one finds increasing concern for these aspects of modelling: teaching, teacher education, organisation of assessment, etc. These all centrally confront educational technology, the institution of schooling, and even politics of assessment and cultural reproduction.

In this situation, we seek to develop a theoretical perspective integrating modelling and technology in its educational, *essentially social and cultural–historical*, context. We aim thereby to help researchers to see the role of technology and mathematical modelling within activity "as a whole." We try to see how they relate to the development of youth, and to see how they essentially relate to educational institutions and systems in wider contexts. We suspect that the perspective of this chapter might challenge many readers from the field of mathematical modelling. Therefore, we will provide some examples, so as to make our proposed perspective more concrete, and perhaps more palatable.

Thus, if we provoke some to see "modelling" and "technology" in a new, broader theoretical perspective we will have succeeded in our aim. Language and mathematics, for instance, in this view, could be understood as the supreme modelling tools (Bruner, 1960), while "writing/inscribing", "sitting in rows in classrooms and copying the scribe," and later "paper-and-pencil mathematics" were perhaps historically humanity's most important technological evolutions in mathematics education—and still seem even today to be remarkably resilient.

## Cultural–Historical Perspectives on Modelling and Technology

We want to conceive "mathematical modelling" as a kind of "activity", in the activity-theoretical sense. We draw on the revolutionary thinking of Vygotsky—said to be the Mozart of educational psychology—and his followers and contemporaries, especially Leontiev and Bakhtin, and those more modern, such as Cole, Engeström, and Wertsch (see the review by Roth and Lee, 2007). The unit of cultural life is "activity", prototypically that of culturally-historically situated and mediated "human labor." Labor and activity are understood to be constituted socially by a collective of joint actions on "objects", with the goal to produce previously idealized (and so planned, envisaged, initially "ideal") outcomes that fulfil a human "need." The "motive" and the "object" of activity ensure that activity is meaningful, and integrate both emotional and cognitive aspects. Activity is always mediated by the cultural artefacts that have been produced by prior generations of cultural production. Thus, mathematical work is mediated by artefacts that were produced historically by "old" mathematical technologies, and in turn produce new artefacts that embed this mathematical work in new ways. Thus, for example, one sees on the most modern computer screen an icon that looks like a pair of scissors for "cutting", a brush for "pasting," and a pair of compasses for constructing a circle.

We would like to be able to take Vygotsky's legacy—which we will call cultural–historical activity theory or CHAT—for granted. But, although Vygotsky (usually 1978, 1986) is widely cited and "well-known" in the educational literature, even in mathematical modelling literature, this whole corpus of activity theory seems to be often treated somewhat simplistically or superficially, and sometimes degraded to the trivial (there are certainly exceptions such as Bartolini Bussi, van Oers, Cobb and colleagues, etc.). Yes—Vygotsky thought that intellectual functions arise on the social plane first, and the intra-mental plane second; so, yes, the sociality of the classroom is fundamental to learning–teaching activity. But, even if he was inconsistent, so also thought Piaget, if we read his later work on children's development of logic with any care. Yes, Vygotsky explained that internalization was of fundamental importance to development, and revealed some of its essential transformations. But activity theory has much more to offer, especially regarding educational psychology, culture, history, technology, and even modelling.

For Vygotsky, the task was to formulate an educational, social-psychology, along dialectical materialist principles. This is indeed a social or cultural psychology; it invoked Marx at least as the founder of the concept of sociology and social practice in its modern sense. Thus, when Vygotsky referred to scientific concepts (sometimes translated as "academic" concepts) in contrast to "everyday" concepts, he was pointing to the specific cultural–historical, and even institutional conditions in which academe grew. Schools and academies were the source of a specific and very formal-abstract way of practising, talking and thinking that he contrasted with the "everyday" language and work of production and consumption. The leisured classes in academies escaped the immediate concerns of the poor populace (for a fascinating account of leisure and academic cultures, see the new edition of *Crest of the Peacock*, Joseph, 2010). This allowed the academy to engage in lengthy periods of scientific study, to develop and explore formal concepts and codes, and so uncover the scientific essence of things that was not superficially visible or tied to everyday practice and its associated pragmatic language use (see also Bernstein, 2000).

But, said Vygotsky (1986) and Leontiev ( 1981), let it be noted how this kind of academic study can lead to teaching that is excessively "verbal" and indeed "senseless" to learners. Only by "ascending to the concrete" can these academic concepts become "true", scientific concepts (for more on this theme, see Blunden's preface to Hegel in Wallace, 2008). Only through the resolution of the dialectical contradiction between everyday and academic practices can the truly scientific-yet-practical conceptions (and so new more advanced forms of social practice) emerge. As we perceive the sun "going down" in a glorious blaze of pink and orange over the blackening ocean horizon, we might still conceptualize this experience in its "academic" scientific model, and appreciate that the sun is not moving, but rather the earth is rotating, and that the light from the sun is not changing much, but rather the depth of atmosphere it must penetrate is slowly changing, leading to parts of the spectrum (blue, indigo and violet actually) being more absorbed than when the incidence is normal (thus having a more than usual proportion of red, orange and yellow). Thus, one might integrate subjective, concrete experience of the everyday with academic physics learning of planetary motion and light, and achieve a synthesis of "scientific" analysis and concrete, subjective, embodied grasp of this experience. The subjective experience gives "sense" to the academic theory and concepts; yet, the physics extends experience and potentially allows one to "see" beyond the immediate. Because it penetrates deeper into the objective reality, it tells one that the experience of sunset would be different if one were observing this phenomenon on Mars or the moon; it extends the imagination of reality far beyond the immediate perceptions and surface knowledge of the "everyday."

When the first moon-landers conducted the experiment of dropping a feather and a spanner simultaneously, they knew and we knew, in a scientific, abstract-formal way, that the two should, against all intuitive, everyday experience, fall together. This is why we watched this experiment and perceived this theoretical knowledge with such joy: we "saw" it for the first time and made this scientific knowledge both cognitively and intuitively, practically "true," in Vygotsky's (Hegelian) sense.

This then is what "modelling" means in its most general, scientific activity-theoretical, sense and this implicates what appropriate technology might do for the construction of true, scientific concepts. According to Davydov (1990), mathematics has a special role in this process: mathematics provides the formal language that distances a model theoretically from its everyday content, and allows a domain of investigation where everyday intuition can be helpfully set aside. The scientific essence of a situation or task can thus be investigated without—for the moment—the interference of the surface, and potentially dangerously misleading contents. Thus the mathematical model of the falling spanner/feather may be given by a simple table of data, or a set of related equations or their graphs: $\mathrm{d}V/\mathrm{d}T = g$; $V = gT$; $S = \frac{1}{2}gT^2$, which in turn relate to the similar model for the parallel situation on earth, with the appropriate modification of $g$. But, then, the model works less well here, where we often require a modification such as $\mathrm{d}V/\mathrm{d}T = g - f(V)$ or the like, allowing us—if we have the mathematical technologies to solve such equations—to explain why the feather and spanner fall differently here. Thus mathematical-technologies provide the means for modelling in problem solving in just the way that Vygotsky's highest level of scientific, or "theoretical" thinking specifies, though Vygtosky most often used formal language as the technology of choice in his own examples.

Notice in this developing formulation that the term "mathematics" is here and there substituted by "mathematical-technologies"—we could have said techno-mathematics which is not far off in meaning (Noss et al., 2007). But also in some cases we might say just "mathematics", as if mathematics itself *is* the technology for solving the problem. The danger is that we forget that mathematics is always mediated by the technology, even though in the most extreme case this is, as in Erdos' fine formula for pure mathematical activity, just "paper + pencil + coffee = mathematics." To this we will shortly add the educational technology, which often remains invisible in the accounts of mathematics in schools and universities.

Davydov (1990), in particular, developed the mathematical side of Vygotsky's argument, claiming that the goal of mathematics education should be to teach theoretical thinking to all children as the central goal of schooling. He believed that the gifts that talented mathematicians demonstrated in Krutetskii's (1976) studies were exactly those of good "theoretical thinking" in mathematics, available potentially to all; and Davydov's work went some way to showing this.

An example: Wason's reasoning task has come to be widely known in the psychology literature. It involves deciding which cards to turn over to test a hypothesis. Each card is said to have a number on one side, and a letter on the other side. The hypothesis to be tested is: "Every vowel has an even number on the reverse side." Which cards, out of "*A*," "*D*," "4," and "7", *must* be turned over to check if this hypothesis is true for all these cards? Very few adults, even those with training in mathematics and science, can answer this question as put (though when presented in more obvious everyday contexts its equivalent proves much easier.) Why is this such a difficult problem? One reason, we suspect, is that few apply a mathematical model to the problem. The hypothesis has the form "*X* implies *Y*", and its truth table is the same as Not [*X* and Not (*Y*)] which is always true unless both *X* is true (i.e., the letter is a vowel, e.g., "*A*") and *Y* is false (i.e., the number is not even, e.g., "7"). Those that don't produce such an argument, then, either do not know logic, do not consider mathematical modelling with truth tables relevant to logic, or are not disposed to use this knowledge in such a task—although it must be admitted that the problem can be solved perhaps more easily analogically, especially by those who have been taught empirical scientific methods for testing hypotheses; however, our solution here is the most powerful, formal, mathematical solution to this general class of problems, and arguably underpins the whole scientific logic of empirical hypothesis testing.

But let us look a bit closer—we have addressed the notion of scientific conceptions, as this pertains to the advancement of society and culture, but not really its developmental, psychological content in schooling activity. As Engeström (1991) explained, Vygotsky and Leontiev understood that schooling was an artificial institutional activity that always tended towards empty, pre-conceptual, or pseudo-conceptual "verbalism." Yet this emptying of everyday knowledge is also what makes academia essential for the specialist development of academic, scientific concepts. Thus, the social context of school is apparently historically essential, but always dangerous: what is the solution to this contradiction? In practice, the answer to this is that school must always be directed to real, problematic situations. Vygotsky and Leontiev's experiments, and Davydov's curriculum, were always directed to tough problems, just beyond the immediate grasp of the learner, in a zone of proximal development (hereafter "ZPD") where problems required the new conceptual tools or signs that the teacher (or other more advanced peers, or even research and study perhaps) could offer. Much of the best in the mathematical modelling literature and practice over the last half century has been in this mould. Thus, we conclude, new mathematics should be taught in such a zone of proximal development, where the mathematics is necessary for the learner to solve genuinely engaging, problematic, "authentic" and "meaningful" tasks (thankfully terms common in the modelling literature). This, then, is what learning through mathematical modelling should mean.

Technology may allow, however, an expanded ZPD in various ways, as case studies in the literature show. Technological instruments that embed mathematics include calculators of all kinds (from times-tables and Napier's bones to electronic and algebraic calculators and computers) that can make historically-produced mathematics "present" in all kinds of learning–teaching activity. The usual argument is that all learners might then find some task that truly motivates them, but also one that becomes accessible.

Implicit in this view is the consideration of the learner as engaging in "activity", that is defined as joint, collective activity on "objects" with substantial social "motives." In activity theory, schooling is considered to be activity in which the students may engage to please the teacher, to pass examinations, and so on, and so dangerously cut off from socially important and useful adult motives. But if the curriculum is properly directed and managed, the activity has a potential for a more advanced motive: thus, Leontiev explains, a student studying a history book, if told that it is no longer on the syllabus, may throw it aside in disgust—in which case they are clearly motivated by schooling, and examinations. But they may, perhaps, put the book aside reluctantly, or decide to read it anyway, perhaps out of a more developed "interest." In this case Leontiev considers the student to be developing adult motives, interests and capabilities—see Black, Williams, Hernandez-Martinez, Davis, and Wake (2010) for a fuller discussion. The most advanced theoretical thinking which arises in activity, then, is motivated by highly adult motives, to understand the deepest challenges of the scientific and social world. In this view, mathematical modelling is not just "intellectual" but involves social motives, affect, passion, and dispositions to act theoretically on the world.

This, then, is what mathematical modelling means, at least for adolescents (Davydov argues that it remains true for the whole of schooling after the age of seven). Or rather, we argue, this is what it might ideally mean; the implications for educating and developing youth for the school curriculum, and for pedagogy, are quite profound, we think. It involves viewing mathematics as the soft side of technology (in the sense of a semiotic tool) as well as a real theoretical world of its own, but one which is made concrete and material through the use of mathematical-technologies in socially meaningful activity. This view will be recognizable by those regarded as being in the emancipatory, critical trend in mathematical modelling and mathematics education generally. To our knowledge the literature recognizes only one serious critic of this position—Badiou argues that it is the mathematics that is "material" and the "real world" is that of "appearance." We will leave this philosophy to one side—but see Brown (2011).

But then, there are many social and political reasons why this ideal vision may not be realizable or realistic in practice: we discuss some of these below (and see Williams, 2011). We claim only that such an ideal view can provide us with a basis from which to examine and critique practice.

## Reviews of Research on Problem Solving and Modelling from an Activity Perspective

The modern problem-solving literature in mathematics education really began with Polya (1957), and became a researched endeavour in the modern sense with Schoenfeld [see his review, Schoenfeld (1992)]. Research on modelling then followed this pattern: modelling being guided by heuristics that may make applied/real problems accessible, while affective issues arise from the social context and context of curriculum. The whole genre of research and curriculum development in ICTMA conferences has represented this development well. Recent conference proceedings from, say, ICTMA-13 and ICTMA-14, offer a history and bibliography—see, e.g., Kaiser et al. (2011), Lesh, Galbraith, Haines, and Hurford (2010).

Rigorous educational research was slow to catch up with practice, but Schoenfeld's (1992) review of educational research concluded that problem-solving strategies must be made concrete in specific classes of problems to become intelligible, and so of any practical value in problem solving. Very general heuristics were also believed not to be instrumentally useful to problem solvers in the flow of practice, but might be more salient in metacognitive reflection on problem solving with classes of problems. Schoenfeld (after Lampert, 1990) also raised the issue of beliefs about the nature of problem solving, and the hidden curriculum of problem solving: like, for example, the belief that a "mathematics problem" is one that has one answer, one best method, and can usually be completed alone without lengthy working (thus revealing how significant is the institutional aspect of schooling, the educational technology). Bartolini Bussi (1998) raised this also in her development of substantial, culturally-historically based mathematical project practices in classrooms, such as the exploration of perspective in history and art. This approach is typical of many in Italy (such as within Boero's group and that of Arzarello) and elsewhere in ethnomathematics and the history of mathematics traditions. In the case of the Italians, this is usually done explicitly as part of an attempt to make mathematics classrooms social and culturally "mathematical", following a Vygotskian perspective; texts, tools and technologies, often in historical contexts, have an important place.

So, we argue the traditional genre of research may make a crucial mistake in isolating the "modelling processes or heuristics" for research and evaluation, much less teaching: removing processes from the substantive mathematics on the one side and the contexts of practical activity in which they make "sense" on the other, may leave the metacognitive aspect high-and-dry as a new mathematical "verbalism." As we argued in our previous section, the actual mathematics provides a language for theoretical thinking, a crucial "point" of schooling in the development of the learner. But concretely, heuristics like "set up a simple model" may be too general to mean much except through the study of specific mathematical theory on one hand and a space of useful activity contexts on the other. Start with a simple, linear function as a model for a relationship before being more "realistic with a non-linear function" makes lots of sense only when it is attached to practical experience in activity. However, this concretization of the general heuristic of "choosing a simple model first" implies a certain depth of understanding and expertise of the mathematics of functions themselves.

Thus the relation of heuristic and mathematics with the context, or contextual range, is also pertinent: modelling in physics in general, and kinematics in particular, is perhaps rather special and even ideal for certain pedagogic purposes. But this is very different from modelling the economy, in which even basic constructs of money supply are disputed. Additionally, this is all crucially sensitive to the technical and cultural tools at hand; the way that processes become objects (reification through automation) has a long history in activity theory itself (Leontiev, 1978) but has entered science and mathematics education through work by Latour (1987) and Sfard (1998, 2008). The dangers involved are that conscious awareness of what is hidden in the black box may become crucial at certain moments—see, for example, the literature on breakdown, but also Sfard's work, and Strässer (2007).

As we also argued, the "context" may provide a societal need, and so a "motive" that allows school study to expand beyond the traditional confines of "schooling" as an activity, because it can provide a social motivation for the student, especially but not solely the adolescent student (see, e.g., Engeström, 1991; Ryan & Williams, 2007). As such, the kind of problem solving or modelling research which isolates heuristics, while making sense in its time, represents a serious limitation in terms of understanding modelling activity within the whole mathematics-educational developmental process. It elevates metacognition but detaches it from the context and the affective (i.e., the motives and emotions).

More recently, Lesh and Zawojeski (2007) similarly summarized the field of research and called for another paradigm shift: based on Lesh and Doerr (2003), they proposed a new way of implementing modelling activity, one which incorporated traditional problem solving but engaged with a broader class of open, engineering- or design-type activities. These invoke complexity, fuzzy problems and can confront instability and inconsistency, which they regard as an essential component of modern life. The argument is that problem solving in practice, as revealed by anthropological studies of situated cognition, for instance, show that real problem solving in practice is unlike the most "realistic" and "authentic" school problems (e.g., Lave, 1988). Furthermore, they suggested that the engagement of students—following the social learning perspective of Lave and Wenger (1991) and Wenger (1998)—requires that students engage in learning via "communities of practice": arguably very difficult to simulate and perhaps impossible to realize in schooling institutions (but see studies in Watson and Winbourne, 2007).

Then there is the Freudenthal tradition which has emerged in (mainly and originally) Dutch schools, influenced by cultural–historical theory: this genre of developmental research makes explicit that the structure of the mathematics at issue is crucial: the point is to provide contexts and problems that are "realistic" (i.e., experientially real to the learners and so engaging) but which "beg to be organized" with the appropriate mathematics to be learnt (Freudenthal, 1983; Streefland, 1991; Treffers, 1987). The emphasis here is most obviously appropriate in the early years, and has the virtue of proven realizability—ecological validity. From our perspective, the notion of "realistic" is about developing "activity" in a schooling context that engages learners: inherent in this is the contradiction inherent in all schooling that tends to get cut off from "life" (Engeström, 1991; Williams, 2011). The question of societal motivation especially during adolescence seems underplayed in the Freudenthal perspective (though there are clear signs in Freudenthal-inspired practice that this has a place, as witness their various texts and materials). The argument laid at the door of socio-cultural theory by Cobb (2007) is worth considering here, i.e., there may indeed be too much "internalisation" and not enough "emergence."

In the next section we will look at a case of modelling with the empty number line in a social context where school mathematics was deployed in trying to understand a workplace mathematical practice (for more examples see Wake, 2007).

## Modelling the Workplace with College Mathematics: An Illustration

In this mini case study we explore the relation between technology, mathematical modelling and education in an expansive setting. The aim is to illustrate modelling-technologies in activity as a whole, in particular how they are both shaped by and are shaping the workplace "knowledge" and the educational experience of the visitor.

Williams and Wake (2007a, 2007b) described an engineer called Dan who was trying to explain a spreadsheet formula to a researcher and two students who were visiting his plant. The formula is designed to compute an estimate of the gas a worker would need to order for the plant to use over the night shift. It is important he gets this right, or as near as possible, since there will be penalty charges from the gas supplier for drawing more or less than the amount ordered. The mysterious formula is shown in Figure 18.1.

The formula is based on a forward projection of how much gas was used (the difference between the 1st and 2nd integrating readings, taken at times which are T2 apart) in the last period of the day before the worker goes off shift, on the assumption that the rate of consumption overnight (a period of time T4) will be the same (a crucial assumption that only became clear later). A simple enough mathematical model … it therefore uses two "readings" to calculate the rate of consumption, then multiplies the rate of consumption by the time period remaining for the shift. Here we have a not untypical mathematical-technology model in daily use, that had been produced quite some time before by Dan, the engineer, and one that is shaped by the history of workplace technology in the sense of its instruments, but also its form of organization (the times of day, etc.). But the formula is so cluttered—by the "everyday" signs that connected the formula to "practice"—that the mathematics, and the theoretical thinking behind it, are opaque to the visiting students (and the research team, and indeed to the workers themselves, and even its author!).

Dan feels obliged to explain: in order to do so he sketches a timeline, an intuitive model but an excellent pedagogical choice (Figure 18.2). He marks in the salient times on the line, then starts to mark the gas readings at each pertinent moment in time; the number line thus emerges in his explanation as a double number line. At this point the light dawns on the researcher (and the reader, perhaps?), that there is

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|---|
| {{{{{$2^{nd}$IntegratingReading – 0600IntegratingReading} +{{{ $2^{nd}$IntegratingReading} – {$1^{st}$IntegratingReading}/T2}*TIME4}}/3.6*CALCV*1000000/29.3071} |

*Figure 18.1.* Dan's formula for estimating the gas needed overnight (adapted from Williams and Wake, 2007a, 2007b).

“0600Int” “1stInt”“ 2ndInt”
/ ---------------------------------------------- /----------/ ---------------------------------------- /
T2 TIME4

*Figure 18.2.* The double number line sketch of gas used (*above the line*) and times elapsed between 0600 and the same time next day (*below the line*) (adapted from Williams and Wake, 2007a, 2007b)

an assumption of linearity, and the double number line represents an appropriate ratio model. For us outsiders this linearity was counter-intuitive, as we expected gas consumption might decline when the workers go off shift for the night.

In a later episode the researcher was able to recapitulate the explanation Dan gave in a discussion in which she made sure the students “followed” the argument. The students commented, and we too found this interesting, that the assumption of linearity had not been mentioned by Dan, but they had been left to discover this for themselves. Presumably in his working life this fact of work-process knowledge was too obvious to need explanation. In fact, much mathematics that has been produced historically disappears like this in artefacts and remains hidden from conscious attention there, unexposed until for some reason there is a “breakdown” (e.g., in nursing and drug dosages—see Hoyles, Noss, & Pozzi 2001). The breakdown arose here because of our research “archaeology”—digging up this formula and seeking to understand it.

Despite the workers there present, the key element of the model lies implicit, too obvious in the practice to be spoken of. Thus mathematics, as Strässer (2000, 2007) has pointed out, disappears from conscious attention in the workplace, but actually is hidden everywhere in technological artefacts, in the work process, and of course also within mathematics itself. In activity theory this feature of the automation of processes is known as fossilization, or sometimes crystallization: we see it also in the artefacts of “schooling” (in the curriculum, in assessment, etc.) that make curriculum development and change so difficult.

This explains perhaps the difficulty in motivating mathematics: one can apparently get by “everyday” without any but the most minimal mathematics, until the everyday “breaks down,” the historic mathematical work that went into the production of the everyday is suddenly required to be understood, by someone at any rate. This often involves quite “high-level” mathematics (when the reactor overheats, we call in specialists with “advanced qualifications”); but not always, even in the everyday workplace, we find examples of mathematical work done by workers like Dan.

This kind of breakdown was constructed artificially by a social situation where students and researcher were situated as questioners, and the workers felt obliged to try and explain their systems. Thus, it put a premium on mathematical communication and, indeed pedagogical discourses (informal: worker with team, formal: researcher-teacher with students). In such contexts pedagogical models such as the double number line were, perhaps naturally, prominent. But it might be argued that the model was useful to Dan’s explanation for us because he already used such a model in constructing the formula in the first place. We will never know for sure, of course, but this is plausible and consistent with our theoretical framework: here Dan externalized the "mathematical thought" for our benefit, and the group understanding, insofar as it constituted group understanding, was an emergent property of the group's questions and Dan's—and then again later the researcher's own—explanations.

We argue that this kind of communication is not just "internalisation" by students in a zone of proximal development, but actually is a collective work in which emergence is constituted by internalisations *and* externalisations—in just the sense that Cobb argued is not synergetic with socio-cultural, activity theory (Cobb, 2007).

It is not a coincidence, we argue, that the double number line emerged as a powerful explanatory tool alongside the symbolic mathematics (albeit mediated by the spreadsheet, we call this a "genre" of mathematics, in the linguistic, Bakhtinian sense). As Lakoff and Núñez (2000) argued, the number line itself provides powerful affordances pedagogically in building up mathematics: these types of models are especially powerful when they allow the user to insert their body into the space the model occupies, even if only in imagination. In this case, Dan and the teacher were able to indicate segments of the timeline gesturally, and we assume the students could thereby identify the different points and intervals in time necessary to make sense of the formula.

Finally we note that the social and cultural context in the case seems vital to the students', the researcher's, and the workers' motivation and to their joint sense of the mathematical work as well. Regarded as a pedagogical episode, it broke with all the norms of schooling as an activity. Additionally, even in a narrow sense the spreadsheet formula broke all the norms about appropriate school mathematics and use of advanced technology. Yet, it is consistent with our Vygotskian perspective: making sense of adults' working practices and how mathematics is embedded there, constructing the relation with school mathematics, and perhaps even allowing for some discussion of its peculiar idiosyncrasies (the sorcery of the engineers' mathematics that kept all the other workers, including management, in the dark!). All this seems well suited to our activity perspective on modelling and technology. We argue that this adds a critical social context, an often missing element, to the case for mathematical modelling. Can this kind of expansive learning occur within the confines of schooling?

In the next example the integration of new technology in a manner more consonant with chapters later in this section will be described; and the expansive nature of mathematical-technology for mathematics education is exemplified (Figure 18.2).

## Modelling and Expansive New Technology: Mathematical Technology

This case comes from a year-long study of the use of CAS-enabled technologies (TI *Nspire*) in senior secondary school mathematics classrooms (see Geiger, Faragher, & Goos 2010). The teacher had some experience with CAS but had not used it previously in his teaching. His students had begun to make use of CAS from the beginning of the school year, about 2 months before the vignette outlined below.

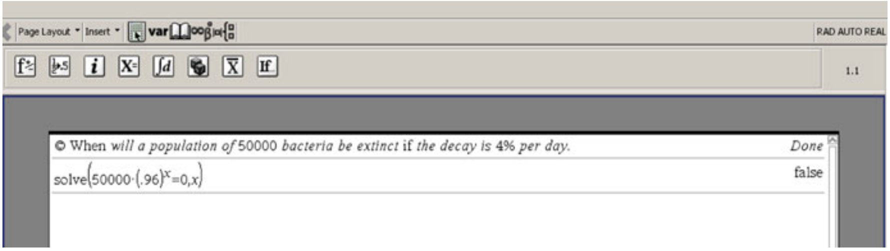


*Figure 18.3.* The CAS responds to a request to "solve" $50{,}000 \times (0.96)^x = 0$: "false."

The students were working on the following question: "When will a population of 50,000 bacteria become extinct if the decay rate is 4% per day?"

One pair of students developed an initial exponential model for the population $y$ at any time $x$ days after the initial population calculation: $y = 50{,}000 \times (0.96)^x$. They then equated the model to zero in order to represent the point at which the bacteria would be extinct, with the intention of using CAS to solve this equation. When they entered the equation into their CAS calculator, however, it unexpectedly responded "*false*" (see Figure 18.3).

The students thought this response was a result of a mistake with the syntax of their command. When they asked their teacher for help, he confirmed their syntax was correct, but said they should "think harder" about their assumptions. Eventually, when he realized that the students were making no progress, the teacher directed the problem to the whole class and one student commented: "You can't have an exponential equal to zero." This resulted in a whole-class discussion of the assumption that "extinction" should be represented by a population equal to zero. It was decided to modify the original assumption by representing extinction as "any number less than one." Students then used their CAS calculators to solve this resulting equation and obtain a numerical solution.

In a follow-up interview, directly after the lesson, the researcher asked the teacher (Teacher 1) about the episode.

Researcher: I saw an element of what we just talked about today when conflict was generated by an interpretation of the question about bacteria. Students developed an equation and then, because no bacteria were left, they equated it to zero. The calculator responded with a false message. In some ways you could think it was a distraction and that the procedure didn't work; some kids might just give up. But on the other hand, what it provoked in your class was an opportunity to discuss. "Did you push the wrong buttons? Oh, you think you did—let's look at the maths. Well your maths is right! Do you understand why it couldn't be? Let's talk about the assumption."

Teacher 1: Simon was one of those, he said—"No way you could get that to equal zero," without necessarily understanding why. Not that he couldn't solve it when it equalled zero, it was that concept he couldn't see; that population couldn't become zero.

Researcher: Yes, they didn't need CAS to understand that, they just understood it because they knew their maths well enough.

Teacher 1: Yeah we actually use the CAS to create the confrontation.

In this episode the teacher exploited the "confrontation" created by the CAS output to promote productive interaction among the class and develop a broader understanding of the role of assumptions in the mathematical modelling process.

In a later focus group interview, all teachers who participated in the project confirmed that similarly productive discussion arose from instances where technology produced unexpected, problematic results or responses. This is seen in the following transcript where Teacher 1 commented on events during the lesson on the decay of bacteria.

Teacher 1: It was pretty obvious to me why it didn't work but I deliberately made a point of that with a student to see what their reaction would be. And it was a case of pretty much what I expected. That they just grasped this new technology *Nspire* and were so wrapped up in it that they believed it could do everything and they didn't have to think too much. And so suddenly, when it didn't work, it took a fair amount of prompting to get them to actually go back and think about the mathematics that they were trying to do and why it did not give a result.

Researcher: … Interestingly you didn't just go over and tell them what to do. You just looked at it and said the syntax is all right—go and have a think about it. And they did for quite a while, and I don't know if anyone sorted it out. They may have but they didn't say. You then brought it back to the whole class and said, "What's gone wrong here?" Someone eventually said that you can't have an exponential equal to zero. What happened out of that—you might want to fill in more—is that there was quite a protracted discussion about what happened. Extinction is zero isn't it? So there is a little bit of a conflict between the way students think about it mathematically and the way it works in context. The context implies zero but there are other answers that could still make it work. So, you have to do this bit of a fudge and say the equation has to be equal to anything less than one—if it is a bacteria.

Teacher1: Even if the kids were solving that by traditional methods, they would still need to have that discussion. It was an issue with CAS that they were just expecting an instant answer and they didn't want to go and think about what was really going on.

Researcher: What is it about CAS-enable(d) technologies that would be different to ordinary technology, in this instance?

Teacher 1: I'll just reiterate and say with CAS that kids are looking for the quick solution, the immediately obvious without looking at what is underlying the discussions and the decisions that they are making. And they assume—like I did—that the machine can handle it.

In this discussion, the teacher identified a "blackbox" use of CAS (Drijvers, 2003) as the source of the impasse that the students experienced when attempting to determine when the bacteria would become extinct. Interestingly, then, students' expectation of technology's ability to produce "an answer" can potentially undermine any expected benefits of technology making challenging problems more accessible. As the teacher noted, students would have had to think carefully about their assumptions regarding extinction, whether or not they used CAS technology to tackle this task. A traditional approach might have led equally problematically to the logarithm of zero. What matters most is how the teacher responded. Such instances can be used to the advantage of students' learning if the teacher has the disposition, mathematical expertise, technological competence and confidence to manage such serendipitous opportunities.

The CAS black box here may usefully be thought of as a "mathematical-technology" which was instrumental in their modelling of bacteria-decay; but the zero value in the model here causes a "breakdown," a problematic, one which required the black box to be re-opened. As such we can argue the students had a problem in their zone of proximal development. When the students tried to enter an illegal value, the machine's response could be diagnosed as either a technology breakdown or a mathematical breakdown—the technology and the mathematics were here "fused"! They initially opted for a technology breakdown, that they had the wrong "code/syntax." The teacher said "think again/harder," because he saw the mathematical, conceptual issue, and this helped create a zone of proximal development from which, through joint exploration, there emerged a way forward, arguably a solution.

In contrast to the previous workplace case, this case revealed a naturally-occurring breakdown moment in a classroom, caused apparently by the mathematical-technology which declined to cooperate with the students and give them a solution to the equation: $50{,}000 \times (0.96)^x = 0$. It is interesting that the students' first thought was to question their own CAS technical competence, and this is probably quite general (cf., dividing 1 by zero and getting "error" on a numeric calculator, or sketching $y = \sin(x)$ on a graphics calculator and getting a straight line through the origin).

Teacher 1, who happened to have acquired a reasonable technical mastery of CAS, was able to see that the syntax is valid, but also had the mathematical competence to see a mathematical reason for CAS's resistance. He was thus competent to diagnose this as a moment to "think again." It seems the students and the teacher reached opposite diagnoses: the students looked to a technical fault, the teacher to a mathematical fault, and between the two there was "joint" problem-solving activity.

But actually things were a little more complex: the mathematics here was arguably not "wrong," in that the equation itself has no answer (except, perhaps, infinity). Rather, it was the mathematical modelling of the real situation that was problematic. The teacher persuaded the class—in what was (in the above account) called "whole-class discussion"—that a more sensible estimate would have been obtained by finding the time at which the model would predict a population value of one (or less) which resolved the problematic for the time being (getting a number that would be more satisfying than "infinity").

But actually, even this was questionable: one might rather ask whether the model was valid as a description of what happens to a single bacterium, and in what sense the problem of "extinction" is a "real problem" for which we need to formulate a model "fit for the purpose." A critical mathematics educator might like to run with this broader issue, and consider the purpose of such population models and the problems they can usefully address. Often, arguably, such exponential models arise at a population level (large number of particles/bacteria) of what is thought of as a probabilistic model at the micro-level (actually the probability of a bacterium dying or radioactive particle decaying in a given time is modelled as $p=0.04t$) and so when a large population becomes small one needs properly perhaps to switch back to the probabilistic model, predicting a range of time over which the last particles are likely to decay (and then maybe the time for the last few particles to decay becomes a Poisson approximation to the binomial). But whether an analyst is pushed to such model refinements really depends on the "real problem"—which is not specified in this case. A satisfactory "critical" endpoint to the class discussion might best have been "why do they want to know?" or "what really is the real problem at issue?"

The point here is that the fusion of mathematics with technology generated a problematic which was not entirely technical, not entirely mathematical, not entirely contextual, but an amalgam of all three. As such, the activity of "mathematical modelling with technology" can be trebly rich in complexity, i.e., when it is three-dimensional (mathematics, technology, activity-or-problem-context). In this case it was not just the pupils/learners who were challenged, and this case shows how such "joint problem solving" or "joint study" can become joint activity of learning and teaching, and maybe even research.

We noted here the demands this kind of work places on the teacher, to which we could add also strains on the curriculum and assessment, and the school organisation. In the ZPD both the students and the teacher were working hard at the problem from two distinct points of view; that of learning (and of engaging with the teacher) on the one hand and that of teaching (and modelling the learner) on the other (see Roth & Radford, 2011). This is truly a joint activity.

We explain this by suggesting that what is involved in "breakdown" is not so much the mathematics but the breakdown of modelling with mathematical-technology in context. Activity theory insists that "activity" is an indissoluble whole, and that any change in or neglect of one of its "moments" implies a change in all the other moments and transformation of the whole: thus an apparently innocent change in the "tools" may induce a treacherous change in the mathematics, in the subject's consciousness (the teacher's and learners' perceptions of the mathematics) and the relations and norms of behaviour in the activity system (the educational technology, curriculum/assessment, etc.) In complex systems of activity, small changes in one apparently "distant" moment can induce treacherous hurricanes downstream.

## Conclusion

We have argued that mathematical modelling should ideally be conceived as adding "theoretical thinking" to real, practical problem-solving activity, and that this should have developmental consequences for students. We have used this ideal conceptualisation to situate modelling and technology within a Vygtoskyan, CHAT theoretical frame, and thus to criticize—or at least to develop a perspective from which to criticize—previous and contemporary research and practice.

It also provides a vantage point from which to see mathematics itself as a reflexive "tertiary" modelling artefact (Wartofsky, 1979, also adopted and developed by Cole, 1996), and hence as a problem-solving technology itself. We have suggested the term mathematical-technology to remind ourselves that activity tends to fuse the two in practice, and often in black boxes, and how these can provide expansive opportunities at breakdown moments. We argued that mathematics inevitably, as part of productive activity, appears alongside and even fused with, technologies in the solution of problems, producing new objects (that also may in turn hide mathematics) as outcomes. These mathematical-technological objects typically become instrumental in their turn, and provide new tools for future actions, which tend to new breakdowns. This cultural cycle fuses and re-fuses mathematics with technology, perhaps helping to solve but also causing contradictions and problems in new contexts of activity.

Particularly powerful new technologies have arisen lately (many described in the following chapters) which expand the language of mathematics, and allow learners wider scope for theoretical thinking and modelling in practice. Potentially, these may allow a wider appreciation of theoretical thinking in practical work than has been common previously, in part through the breakdowns and problems they introduce into activity. But we must not ignore the wider social context which also mediates change in educational technology, and which so often has provided the key obstacles to progress. We have only begun to touch on these here, hinting at the demands that working with mathematical-technology make of teachers and researchers, and so implicitly curriculum, assessment, and educational technology generally.

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