Chapter 17
From the Slate to the Web: Technology in the Mathematics Curriculum

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Abstract The employment of physical tools to assist teaching and learning of mathematics did not begin with electronic devices, and has a much longer history than is often recognized. At times, technology has functioned as the inventive embodiment of mathematical ideas, progressing somewhat in step with the evolution of mathematics itself. At other times, technology has entered mathematics from outside, notably from commerce and science. This chapter surveys the evolution and curricular influence of technology in mathematics instruction in the Eastern and Western worlds from ancient times to the present day, with the primary focus being on the last 200 years. Past technology is categorized into tools for information storage, tools for information display, tools for demonstration, and tools for calculation. It is argued that today's computing technology offers teachers and students the potential to move beyond these categories, and to experience mathematics in ways that are different from traditional school mathematics curricula. A window is opened through which mathematics teaching and learning might enter into a new epistemological domain, where knowledge becomes both personal and communal, and in which connective and explorative mathematical knowledge becomes vastly more accessible.

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Introduction

Since the advent of the electronic calculator it has become customary for discussion of "technology" in mathematics education to refer almost exclusively to use of electronic devices. However, this represents a manifestation of historical amnesia. The employment of physical tools to assist teaching and learning of mathematics has a much longer history, and this history provides a valuable perspective on current proposals and debates. At times technology has functioned as the inventive embodiment of mathematical ideas, progressing somewhat in step with the evolution of mathematics itself. But technology also enters mathematics from the larger world outside, notably from commerce and science. Moreover, technological tools used by mathematical practitioners need not translate immediately into mathematics education, and tools useful in an educational setting need have little appeal for professional users of mathematics. Educational use of technology is also subject to overarching educational philosophies prevailing at any given time and place; some would call these fads and fashions. The interactions among technology, mathematics, and education are thus unavoidably complex, and cannot be described by any simple model of historical progress over time.

The historical record suggests that the use of tools always has been inseparable from expressing and doing mathematics. In the ancient Western world the Babylonians carved solutions to geometric problems on small pieces of round clay. Possibly students did these as assessment tasks—for instance to find the length of a diagonal of a square using the square root of two. The ancient artefact depicted in Figure 17.1a might have been the work carried out by such a student. Another Babylonian student may have used a "calculator" to work out a rather complex arithmetic problem. In this case his tool was a counting board made from a slab of stone with groups of markings (parallel lines, semi-circles) on it. The student put pebbles on it to work out his answer. A version of this counting board, which dated

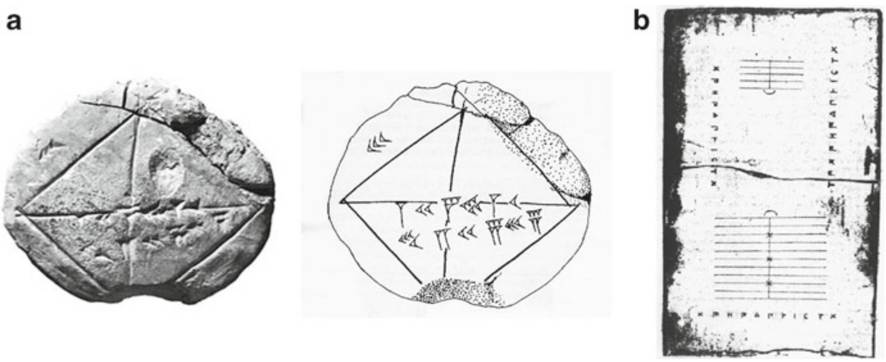


Figure 17.1. (**a**) Mathematical exercise to find diagonal of square, using the square root of 2 [Yale Babylonian Collection http://www.yale.edu/nelc/babylonian.html], (**b**) The Salamis Tablet: The oldest counting board. It is made of marble. Photo from the National Museum of Epigraphy, Athens.

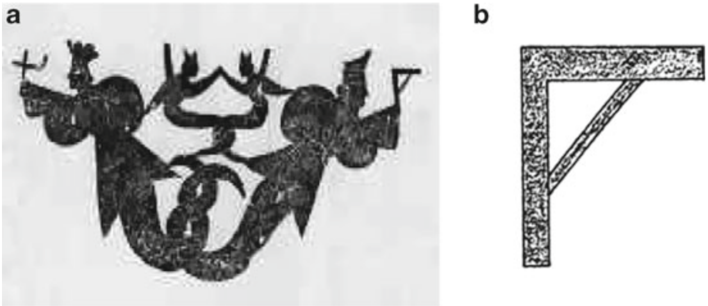


Figure 17.2. (**a**) *Nuwa* (*left*) and *Fuxi* (*right*) with *Nuwa* holding a *gu* and *Fuxi* holding a *ju* [http://sunrise.hk.edu.tw/~planning/sm/images/exect-1/book-j002.JPG]; (**b**) A Chinese set-square.

back to 300 BCE, was found on the Greek island of Salamis in 1846 (Figure 17.1b). In these ancient artefacts mathematics seems to have been embodied and was being preserved under the inventiveness of ancient craft.

Looking to the Eastern world, there was a different type of embodiment. In ancient Chinese mythology, there were demigods *Nuwa* and *Fuxi* who were the progenitors of mankind and shapers of human society. Legends say that *Nuwa* and *Fuxi* invented *guī* (compasses) and *ju* (set-square) to shape the world. On an ancient stone carving found inside a tomb from the East Han dynasty (25 to 220 CE) there is engraved an intertwined image of *Nuwa* and *Fuxi* with *Nuwa* holding a *guī* and *Fuxi* holding a *ju* (Figure 17.2a).

For the ancient Chinese, the basic concept of the world was "heaven is round, earth is square" and there was an ancient motto saying that "without *guiju*, there are no square and circle." This geometrical intuition about the physical world became metaphoric in the human world. The connotative usage of the word *guiju* refers to orderliness according to underlying rules, and even applies to human affairs. Hence, for the Chinese, circle and square were elemental shapes and rules of the universe and they were embodied and symbolized by the tools that produced them. Notice that the two arms of the Chinese set square were not of the same length (Figure 17.2b). This might indicate that the ancient Chinese were already familiar with a Pythagorean-type relation about right-angled triangles. (The Chinese version of Pythagoras' Theorem was *Gougu*: Chapter 9 of the ancient Chinese mathematics treatise *The Nine Chapters*). Thus, behind the design of *ju* there lay an embodiment of a piece of mathematical knowledge. This kind of knowledge mediation, using tools embodying mathematics, was even more deep-seated in another Chinese traditional knowledge system mediated by symbolic visual tools. Ancient Chinese used dot and line pattern diagrams to represent and interpret the phenomenological world. In Figure 17.3 there are three elemental number pattern diagrams that constituted the root of Chinese thought and culture. Chinese used these diagrams (and derivations of them) as coding tools to decipher the hidden laws of the universe.

Luò Shū (The Luò River Writing) and *He Tu* (The River Map) were two different but related arrangements of 1, 2, 3, 4, 5, 6, 7, 8 and 9 using black and white dots.

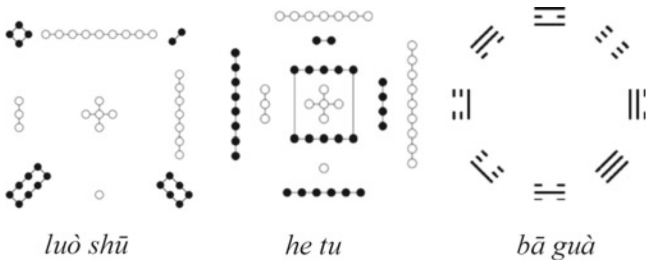


Figure 17.3. Three fundamental symbolic tools that form the basis of Chinese culture.

There were mythical stories about their origins signifying that these patterns were indeed very ancient and sacred. *Luò Shū* is a three-by-three magic square. It has intriguing mathematical properties and has had a deep influence in Chinese culture (Berglund, 1990). *He Tu* is a derivation of *Luò Shū:* it emphasizes the concept of duality (even and odd, *yin* and *yang*). *Bā guà* is the kernel of a binary coding system that classifies natural and human phenomena and is intimately connected to *Luò Shū*. These were the fundamental symbolic tools by which the ancient Chinese derived their concept of the world. They are supposed to embody numerical and geometrical information that guided the development of Chinese civilization. In particular, these diagrams were instrumental in facilitating mathematical calculations to predict occurrences of human affairs and natural phenomena.

The above examples from ancient Babylon and China illustrate that humans invent tools, symbols, and technology that embody mathematics. By this we mean that an object has been created, possibly simple, possibly very complex, which in some sense contains a mathematical idea or procedure. The object is capable of illustrating the idea for an observer, of facilitating the procedure, or of providing some combination of these services. Such tools can in turn endow users of the tools with enhanced ability to deepen their mathematical experiences. Mathematical experience can be thought of as "the discernment of invariant pattern concerning numbers and/or shapes and the re-production or re-presentation of that pattern" (Leung, 2010). Moreover, mathematical concepts are often developed in the process of using tools, whether the tools were designed for mathematical purposes or not. Tools used for the general betterment of social conditions, or for encapsulating features of a cultural worldview, often carry with them indigenous mathematical knowledge. In ancient India (800–500 BCE), notions of geometric shape and measuring techniques emerged in Sanskrit texts on ritual practices, such as prescriptions for constructing fire altars:

> The footprints for the altars were laid out on leveled ground by manipulating cords of various lengths attached to stakes. The manuals described the required manipulations in terse, cryptic phrases—usually prose, although sometimes including verses—called *sūtras* (literally "string" or "rule, instruction"). The measuring cords, called *śulba* or *śulva*, gave their name to this set of texts, the *Śulba-sūtras*, or "Rules of the cord." (Plofker, 2008, p. 17)

The Mayan calendar wheels (1000 BCE) in Central America, based on a vigesimal (base 20) number system, formed a complete philosophy of cyclic time that was believed to guide human destiny (Coe, 1993). The Incas, in the 1400s and 1500s in what is now Peru, used a complex system of knotted strings (*quipus*) as a data collecting and recording device which in effect served as a numerical calculator (Ascher & Ascher, 1997). The Marshall Islanders of the South Pacific used palm ribs and coconut fiber to construct navigation stick charts to represent the behaviour of wave fronts (refraction, reflection and diffraction) as they approach land (Ascher, 2002).

It must be acknowledged that our understanding of the educational practices associated with the above examples is very sparse. We see also from these examples that "technology," if interpreted broadly, can encompass a vast range of human activities, including mathematical notation and language in general. To make our discussion manageable, we therefore define technology in education more narrowly, confining ourselves to physical devices used with the aim of enhancing or amplifying the abilities of the teacher or the student in the mathematics classroom. Thus, although for our purposes we will not count a tool such as logarithms as a technology, the slide rule, a physical device embodying logarithms, will fall under our purview. Electronic devices, and algorithms realized on electronic devices, digital or analog, are also within our scope, inasmuch as there is a physical object involved. In the remainder of this chapter we offer brief histories of several representative devices that have been used in classrooms around the world. To make this survey more relevant to the present day, we furthermore focus mainly on the last 200 years, when mathematics education began to become (haltingly and unevenly across the globe) not merely an acquirement of a small elite, but a mass phenomenon.

We introduce a simple categorization to provide a framework for discussing these tools: tools for information storage, tools for information display, tools for demonstration, and tools for calculation. These categories are admittedly not entirely distinct, and we will see that they become less useful as we move into the electronic era—but they serve well for setting the stage.

Tools of Information Storage

The quintessential information storage tool is the book, which retains a powerful presence in worldwide mathematics education to the present day. The book has a history almost as old as civilization itself, from clay tablets, to the papyrus scroll, to the handwritten codex, to the printed book, and on to the modern e-book (Hobart & Schiffman, 1998). But the history of the mathematics textbook is much shorter, and falls almost entirely within the 200-year window mentioned above, especially if we neglect advanced monographs in favour of books actually used in schools. Certainly, for many centuries individuals learned mathematics independently from books, and likewise tutors used books to teach mathematics to individuals and small-groups, but a new era began with the advent of mass schooling and the mass-produced textbook. These interconnected phenomena did not become prominent until the 19th century in Europe and the Americas, and were materially aided by both political and economic developments. On the political side there was rising support for providing education for a larger proportion of children. On the economic side, there were increasing efficiencies in the production of paper and books, and increasing facilities for transporting goods over long distances, resulting in the ability to manufacture and distribute large numbers of books relatively cheaply (Cordasco, 1976).

When books were scarce, if a class had a book at all it would frequently be the exclusive possession of the teacher. If the class was of any appreciable size this led to the recitation method of teaching, which often meant the teacher simply reading aloud from the book and the pupils attempting, through writing or sheer memorization, to retain what was read, and then to recite it back to the teacher. Notable attempts to scale this system up were made in England and its colonies in the late 18th and early 19th centuries with the so-called monitorial system, in which the teacher would first teach a group of more advanced students, who would in turn teach less advanced students. In mathematics, in particular, the recitation method and the monitorial system primarily supported a curriculum centred on the rote learning of the rudiments of arithmetic (Butts, 1966).

But with cheaper books came the possibility (though still often not the reality) that not merely the teacher but also many students would have individual access to a textbook. A student with a book could now be asked to read that book both inside and outside of class and to work problems assigned from the book. More sophisticated mathematics instruction for a classroom of pupils was now far more feasible than previously. Thus the rising presence of algebra and geometry in addition to arithmetic in the curriculum of 19th-century schools surely owes a good deal to the proliferation of textbooks. The use of textbooks could also serve to hide problems arising from inadequate teacher preparation. This was certainly the case in the 19th-century USA (Tyack, 1974).

Moreover, the system amplified itself: a greater supply of books produced a greater demand for books, which in turn produced yet more books, and so on. In mathematics this resulted not merely in the creation of individual textbooks, but entire series of textbooks covering the whole range of the curriculum from the lowest grades to the colleges: basic arithmetic to the differential and integral calculus. In Europe and North America by the end of the 19th century there was a well-established textbook industry, and there were specialist authors who became wealthy writing textbooks. In the USA, notable 19th-century authors of mathematics textbooks included Charles Davies, Joseph Ray, and George Wentworth (Kidwell, Ackerberg-Hastings, & Roberts, 2008). Seymour and Davidson (2003) asserted that "until the late 1960s, the textbook was virtually the exclusive curricular and pedagogical approach to the teaching and learning of mathematics in the United States and Canada" (p. 990). A study at the close of the 20th century concluded that in the USA the textbook remained the main source used by mathematics teachers to plan daily classroom instruction (Harel & Wilson, 2011).

One effect of textbook proliferation should be especially noted: the assistance provided to standardization of the curriculum, and the difficulty of dislodging curriculum topics once they were printed in widely distributed textbooks. This is especially striking in the USA, which despite a long tradition of local control of schools, and avoidance of an official national curriculum, rapidly converged on a de facto standard curriculum in mathematics, as a relatively small number of textbooks began to dominate the market. Genuinely innovative mathematics textbooks have never fared well in the US market. Even during the "New Math" era of the1950s and 1960s, supposedly a time of major upheaval, there was substantial continuity in high school textbooks from earlier decades (Dolciani, Berman, & Freilich, 1965; Freilich, Berman, & Johnson, 1952). Many students today have access to textbooks in electronic form, as a supplement to or instead of the traditional paper book. Whether this transition will have a marked effect on the mathematics curriculum is unclear.

Tools of Information Display

The book of course functions as a display device for individuals, as well as a storage device, but with mass education came a pressing need for multiple individuals to view the same display simultaneously. Here the representative tool is the blackboard or chalkboard and its offshoots. Prior to the wall-mounted blackboard, there had been a slow evolution of handheld writing surfaces, culminating in the slate, which could be written on with chalk. In Europe and North America this was often a facet of the recitation method of instruction. The teacher could read a problem from the book and the students could copy and display their solutions on their slates (Burton, 1850; Cajori, 1890).

Prior to the emergence of both the textbook and the blackboard, it was also common practice in many schools in Europe and North America for each student to produce a "copybook" or "cipherbook." Beginning with a collection of blank pages (paper and binding quality could vary widely, depending on economic circumstances) the student would copy out the material spoken aloud by the teacher. In the case of a teacher reading from a printed book this could often mean that the student was almost literally producing a handwritten copy of the book, or the problems from the book. Here again the use of copybooks primarily supported arithmetic instruction, but in some cases this could be fairly elaborate, including square and cube roots and complicated problems from commerce and business. The teacher could periodically inspect the copybooks, so that they could have functioned as what more recent educators would term a "portfolio." But how rigorously 18th- and 19th-century copybooks were evaluated for mathematical correctness is unclear, and some may have been assessed more on aesthetic grounds, such as penmanship (Clements & Ellerton, 2010; Cohen, 1982).

The erasable blackboard, written on with chalk, spread quietly into schools in the early 1800s and was well established by the end of that century (Kidwell et al., 2008). It allowed the teacher to display complicated verbal or pictorial details with far more exactitude than merely reading aloud from a book. Moreover, it allowed students to work out problems on the board themselves, displaying their efforts for both the teacher and other students to see and comment on, thus changing the personal dynamics of the classroom. In mathematics the blackboard worked in conjunction with the textbook to promote the rise of both algebra and geometry in the curriculum.

Blackboards have continued in use in mathematics classrooms to the present time. In many cases the chalkboard has been replaced by the "dry-erase" or "whiteboard," but with no essential change in functionality. The interactive whiteboard, developed in the late 20th century, represents a major innovation, allowing the material displayed on the board to be connected directly to a computer. Opinions vary widely on the value of this technology in the classroom (Smith, Higgins, Wall, & Miller, 2005; Wood & Ashfield, 2008). Tablet personal computers offer similar functionality, including handwriting recognition, whereby the computer is able to interpret handwriting drawn on the screen, not merely type entered via a keyboard (Anderson, 2011).

Another significant classroom display technology is the overhead projector. It came to classrooms in the USA after World War II (Kidwell et al., 2008). Much more than the blackboard, this technology usually remained the exclusive domain of the teacher. It had two primary attractions. First, it allowed the teacher to continue to face the students while displaying materials to them. Second, it allowed the teacher to display elaborate transparencies created before class. For example, a teacher of solid geometry could prepare complicated diagrams with an exactitude that could never be hoped for in hand-drawn diagrams quickly improvized while watched by the students. On the other hand, reliance on prepared slides sometimes encouraged a too rapid succession of material that overloaded the students' ability to assimilate the information presented.

Overhead projectors have continued in use to the present, but in many cases have been superseded by new technologies allowing greater ease of use and a greater range of display functionality. Computer projection systems permit the display of any image, static or moving, available to the host computer, and in particular allow slide shows formerly done via transparencies on an overhead projector to be accomplished via software such as PowerPoint. Another new technology is the document camera (also known as an image presenter or visualizer), which permits any document, or even a three-dimensional object, to be displayed on the overhead screen without any prior preparation of the document or object (Ash, 2009).

Many classrooms in the 21st century provide not only a computer and projector for the teacher but also a computer for each student, networked with the teacher's computer. In some ways this is a return of the handheld slate, with a vast increase in functionality. Its potential for mathematics instruction is just being tapped.

Tools of Demonstration

By tools of demonstration we refer to objects to be handled (physically, or, in more recent times, virtually) by either the teacher or the student, with the aim of conveying increased understanding of a concept or procedure. Rather than being tools of education in general, such tools have usually been more unique to mathematics than the tools of information storage and display. However, bringing new demonstration tools into the classroom has often only occurred in conjunction with some larger movement in educational philosophy that has affected more than mathematics alone.

The history of demonstration tools has been strikingly uneven. A few have been deeply imbedded for millennia, while others have come and gone with little trace. We have already noted the important place of the compass in Chinese thought, and it is well known that the classical geometric drawing instruments in the European tradition are the straightedge and the compass (often referred to as a pair of compasses). The Greek mathematician Euclid, in his *Elements* (ca. 300 BCE), gave priority to constructions based on these instruments. Probing the limits of such constructions (squaring the circle, trisecting the angle, etc.) was a spur to mathematical researchers from antiquity to the 19th century. Indeed, although other instruments were often used for various practical purposes, such uses were long considered illegitimate for mathematical demonstration (Knorr, 1986). Since Euclid served as the basis of geometry instruction in Europe and its colonies for centuries, the straightedge and the compass became regular features of this instruction.

In the 17th century, René Descartes, the great French philosopher and mathematician, strenuously challenged the straightedge-compass tradition, and made free use of more complicated mechanisms for geometric constructions. However, this had little influence on education. The discovery of linkages capable of producing exact straight lines in the 1870s produced a brief flurry of interest among mathematicians, and even prompted some to propose a refashioning of geometry education. In 1895 the mathematician G. B. Halsted unsuccessfully called for the Hart inversor (see Figure 17.4) to be a standard part of every elementary geometry course. Such devices have periodically created excitement among mathematics teachers and teacher educators in more recent years, but they have never become more than an enrichment topic (Kidwell et al., 2008).

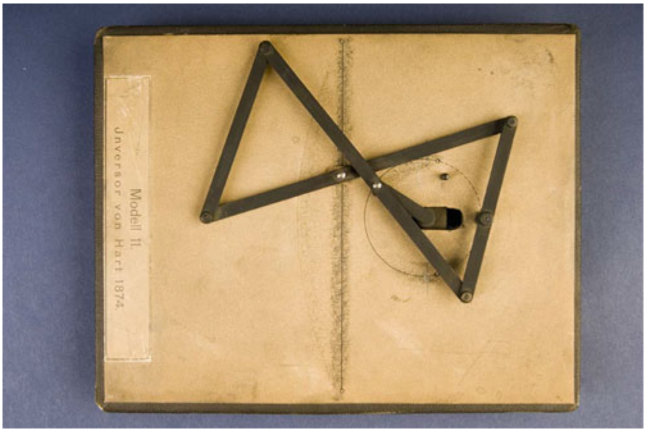


Figure 17.4. The Hart Inversor, a linkage which translates rotary into straight line motion [National Museum of American History collections, gift of Department of Mathematics, University of Michigan. Smithsonian Negative no. 2006–3].

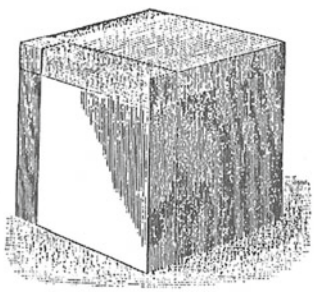
In Europe and North America, there has been a discernable increased use of demonstration tools from the beginning of the 19th century, driven by greater emphasis on using sense data, especially visual, to convey the abstract concepts of mathematics. This has remained a feature, at least in theoretical pronouncements, of much mathematics education to the present day (Bartolini Bussi, Taimina, & Isoda, 2010). The empirical side of the 17th-century scientific revolution appears to have been crucial, with knowledge coming to be understood to depend not only on reason but also on careful sifting of material evidence; induction in addition to deduction.

But although there were some precursors, it was not until the 19th century that this stimulus was widely felt in education. Swiss educator Johann Pestalozzi and his follower Friedrich Froebel were especially influential in bringing material objects into the classroom to be seen or touched by the students. These included objects associated with mathematics, such as geometric solids. Froebel, teaching in Swiss and German towns in the 1830s and 1840s, pioneered the concept of kindergarten for very young children. He recommended organized play with blocks, which would introduce the child to geometric shapes and to arithmetic ideas up to simple fractions. Froebel's ideas spread across Europe and to the USA in the late 19th century (Allen, 1988; Butts, 1966).

One 19th-century educational tool which may have benefited from Froebel's influence was the cube root block, now little remembered. It is based on a method of extracting cube roots based on the binomial expansion of $(a+b)^3$, which can be illustrated with a cube of side $a+b$. (There is a better-known corresponding method for extracting square roots which can be illustrated with a diagram of a square of side $a+b$). Illustrations of this cube can be found in English arithmetic texts from the 17th century (e.g., Recorde, 1632), but it was not until the middle of the 19th century that it became an actual classroom device (see Figure 17.5). With the aim of helping students understand the aforementioned cube root algorithm, scientific

IV. CUBE ROOT BLOCK.

FIG. II.



The extraction of the cube root can be explained most easily by the use of the Cube Root Block. In fact, no person who is unacquainted with Algebra or Geometry can know the reason for this rule without the aid of some such illustration.

From *The Teacher's Guide to Illustration: A Manual to Accompany Holbrook's School Apparatus*, (Hartford, 1857), 34.

Figure 17.5. Illustration of a cube root block.

instrument companies in the USA began to produce and market wooden cube root blocks that could be dissected into constituent parts.

These blocks, for advanced arithmetic students, were often advertized with other classroom objects, such as cones for displaying conic sections, and Froebel's blocks for kindergarten children. Diagrams based on the blocks were a staple of school arithmetic textbooks for many years, but the approach had detractors. The cube root block algorithm never gained any favour with engineers and other users of mathematics for practical purposes, since the efficiency of the algorithm is low compared to other methods, such as logarithms or Newton's method. Moreover, how often did mathematical practitioners even need to compute cube roots? By the 1890s many mathematics educators in the USA were campaigning against cube root extraction, but it persisted in the curriculum well into the 20th century. Cube root blocks were still being sold in the 1920s (Kidwell et al., 2008). Since no studies of the effectiveness of the cube root block as a teaching technique are known, it must be judged a demonstration tool of unclear benefit to support an algorithm of dubious value. Nevertheless for a time it was well ensconced in the curriculum.

The end of the 19th century and the beginning of the 20th saw another surge of interest in concrete instructional methods, at both the highest and lowest levels of the curriculum. For advanced instruction this was strongly influenced by a felt need to better align mathematics with science and engineering. In France, the mathematician Émile Borel, concerned that mathematics might lose its place in education due to a public perception that it was useless, called for more practical instruction, including augmenting geometry teaching with surveying exercises. He recommended "laboratories de mathématiques," which would make many connections with physics (Borel, 1904). In the United Kingdom, the engineer John Perry promoted a more concrete and visual approach to mathematics education, helping to break the unquestioned dominance of formal Euclidean geometry in British education. His influence extended to both Japan (where he worked for a time in the 1870s) and the USA (Brock, 1975; Brock & Price, 1980). In the USA, Perry's most prominent disciple was pure mathematician Eliakim Hastings Moore of the University of Chicago, who championed a "laboratory method" of teaching mathematics at both the secondary and college levels. This involved strong emphasis on developing intuition in the student through physical models, weighing and measuring, and drawing on squared paper (an uncommon classroom item up to that time). Moore saw Perry's ideas as helping students aiming to be scientists and engineers, while at the same time supporting future teachers of mathematics and research mathematicians. His curricular program was briefly significant in the USA, but other than an increased use of graphs in algebra instruction, its long-term stimulus was slight (Roberts, 2001).

Moore was also greatly influenced by the German mathematician Felix Klein, who likewise sought to make mathematics education more supportive of engineering. Klein championed the use of geometric models in classroom instruction. This built on a tradition originating in France in the early 19th century, especially with mathematician Gaspard Monge. Models made of plaster, string, wood, and paper were developed in France and Germany. These went beyond the simple solids of Pestalozzi and Froebel to include hyperboloids and other more advanced structures, all the way to objects at the forefront of mathematical research, such as Riemann surfaces. Some of the string models were even dynamic; that is, they could be manipulated to change shape. With Klein's instigation, German models, mainly of plaster, were manufactured and sold worldwide. Colleges and universities in the USA were among the buyers, but there is little evidence to support extensive classroom use of these models; more likely they were treated more as museum pieces. There were also isolated enthusiasts at the secondary school level in the USA, who enjoyed training students to create geometric models, but their effectiveness is very hard to gauge (Committee on Multi-Sensory Aids, 1945; Kidwell et al., 2008).

Meanwhile in Italy, Maria Montessori inherited Froebel's emphasis on teaching young children through tactile experience, buttressing her theories by appealing to more recent developments in psychology and anthropology. She advised that beginning students be given the opportunity to handle objects of various shapes—such as cylinders of varying heights and diameters—continually. Colored cubes and rods were a central feature of her approach to arithmetic. Montessori schools were opened in Italy and Switzerland. After an initially rapid growth of interest in her work in the USA in the 1910s, her influence declined, in part due to criticism from American educational theorists such as William Heard Kilpatrick of Columbia University (Kramer, 1976; Whitescarver & Cossentino, 2008).

The USA experienced a Montessori revival beginning in the 1950s, and this closely coincided with, and perhaps helped to support, renewed interest in both the USA and Europe in using physical objects specifically in teaching mathematics. Other sources of support were found in the work of educational psychologists whose influence extended well beyond mathematics, such as the Swiss, Jean Piaget, and the Russian, L. S. Vygotsky. Among those in the 1960s who helped popularize what came to be called "manipulatives" in mathematics instruction were the Belgian educator Emile-Georges Cuisenaire, the Egyptian-born British educator Caleb Gattegno, and the Hungarian-born educator Zoltan Dienes, who worked in Great Britain, Australia, Canada, and elsewhere (Jeronnez, 1976; Seymour & Davidson, 2003). This period also saw the rise of the "New Math," a conglomeration of curriculum reform programs initially centred in the USA but eventually extending well beyond. Some would see manipulatives such as Cuisenaire rods as incongruous with the emphasis on axiomatics and abstraction characteristic of many of the New Math programs, although Dienes (1960, 1971), for one, saw no contradiction. In any case, the popularity of certain manipulatives to some extent rose and fell with public perceptions of the New Math as a whole. Nevertheless, while New Math programs often experienced severe backlash, the use of manipulatives never went into total eclipse.

The presence of manipulatives in classrooms during the last 50 years is testified to by the fact that the topic has been an active subject of empirical research from the 1960s to the present (Karshmer & Farsi, 2008; McNeil & Jarvin, 2007; Moyer, 2001; Sowell, 1989). This research has painted a mixed picture of the effectiveness of manipulatives. Although some studies have detected very positive effects, others have found that these effects were negated by poor teaching techniques. Some research even suggested that manipulatives could harm students by burdening them with the problem of "dual representation." According to McNeil and Jarvin (2007), "a given manipulative needs to be represented not only as an object in its own right, but also as a symbol of a mathematical concept or procedure" (p. 313).

The computer, especially as connected to the Internet, makes readily available to students and teachers all of the objects mentioned above, and many more, in virtual form. Whether this will prove to have a significantly more positive influence on the mathematics curriculum than physical models that students can hold in their hands remains to be seen. We will note some recent efforts in this direction in the last section of this chapter.

Tools of Calculation

To the consternation of many mathematicians and mathematics educators, calculation is often considered to be synonymous with mathematics by many members of the general public, so these tools naturally loom large in public discussion of mathematics education. Here we briefly discuss the history of three devices—the abacus, the slide rule, and the calculator—that have had a global impact in mathematics education, as it evolved from mechanical to electronic. It should be noted that the slide rule, though intermediate chronologically, is in no sense intermediate conceptually between the abacus and the calculator. This shows the difficulty of imposing any straightforward conception of linear progress in the use of technology in mathematics education.

The abacus. The abacus depicts numbers by means of beads on wires. It apparently evolved from marks in sand or counters on a board. The device seems to have developed somewhere in the eastern Mediterranean world in antiquity, moved east to Asia, then moved back west via Russia into Europe and thence to the Americas. The transmission to Asia is conjectural, and it is possible that it originated there independently. What is clear is that whereas the abacus became a widely used tool of calculation in China and Japan, without a serious competitor until very recent times, it never attained the same level of popularity in this role in Europe and North America. Instead, in the last-named regions, it was primarily confined to use as a demonstration tool for teaching elementary arithmetic to young children.

The Chinese abacus (*suanpan*) appears to have been in substantial use by 1200 and probably much earlier. Transmission to Japan, seems to have occurred via Korea. The Japanese modification of this instrument (called the *soroban*) was in use by 1600 (Smith, 1958). Although the abacus has been a part of education in both Japan and China for centuries, in the decades after World War II major efforts were undertaken in both nations to modernize and formalize this instruction (Hua, 1987; Shibata, 1994). The device has continued to be part of the mathematics curriculum in many East Asian nations to the present day. In Malaysia, for example, although abacus use in schools declined for a time after handheld calculators became widely available, the abacus (*sempoa* in Malay) has more recently experienced an educational resurgence in connection with an increased emphasis on mental arithmetic (Siang, 2007).

In China and Japan the beads move on vertical wires, but the version of the abacus that became common in Russia featured horizontal wires. This would prove advantageous for using it as a display device for young children, since the teacher could hold the abacus up in front of the class and the beads would remain in place. It was used in Russia for early education until recent decades. The French mathematicians Jean Victor Poncelet encountered the abacus while imprisoned in Russia following Napoleon's invasion of 1812 and introduced it to France on his return. It spread widely across France as a teaching tool in the 19th century (Gouzévitch & Gouzévitch, 1998; Régnier, 2003).

A similar teaching device began to appear in the USA in the 1820s, likely inspired at least in part by the French version. Here it meshed well with the Pestalozzian object-teaching philosophy that was gaining in popularity, and by the 1830s it was being sold under various names, including "numeral frame," by companies catering to the growing education market. These teaching abaci were not without detractors, however, some of whom felt they might even stifle the imagination of the child. They remained as a tool for only the youngest learners of arithmetic (Kidwell et al., 2008). In more recent years, some educators (e.g., Ameis, 2003), apparently reacting to the perceived success of Asian students in mathematics, have advocated more use of the Asian abacus in Western schools.

The slide rule. The slide rule was a direct embodiment of the theory of logarithms pioneered by Scottish mathematician John Napier and English mathematician Henry Briggs in the early 1600s. By marking two straightedges with logarithmic scales and sliding one with respect to the other it was possible to calculate approximate answers to multiplication problems quickly. Even more complicated problems could be handled with sufficient ingenuity, although the fact that the slide rule was an analog instrument meant that it always provided only approximate answers, and thus was not appropriate for most business applications of mathematics or for accounting. Variations involving circular rules were also possible, and both possibilities had been explored by the middle of the 17th century in England. These slide rules were slowly improved over the next century, and became a tool used by engineers, such as James Watt, in the UK. By the early 1800s they had spread to the European continent and to the USA (von Jezierski, 2000).

It was not until the late 19th century that the slide rule became an educational tool, beginning first with colleges featuring an engineering curriculum, such as Rensselaer Polytechnic, the US Military Academy, and the Massachusetts Institute of Technology. In the early 20th century the slide rule began to filter down into the secondary schools, helped by the movement to establish mathematical "laboratories" which emphasized the mathematics of measurement and applications to the physical sciences. Instrument makers were selling slide rules to the high school market by the 1920s and some were also selling oversized models that could be displayed in front of a classroom for all students to see. The slide rule remained a recognized feature, although in most cases not a central one, of many mathematics and science classrooms until the advent of cheap electronic calculators in the 1970s (Kidwell et al., 2008).

The calculator. Unlike the slide-rule, the calculator is fundamentally a digital instrument, which seems to have given it a decided advantage in achieving a place in mathematics instruction. Its fate in the classroom is still being written. European development of mechanical calculators dates from the 17th century, with such notable mathematicians as Pascal and Leibniz prominently involved (Goldstine, 1972). But it was not until the middle of the 19th century that industrial processes were sufficiently advanced to allow construction of calculating devices on a commercial basis, both in Europe and the USA. By the 1920s they had become a standard feature of many office settings. But it appears that it was not until after World War II that they received much consideration as educational devices. In the 1950s there was some minor experimentation in classrooms with mechanical calculators, or mechanical calculators with electrical assistance, but the size of these machines made them inconvenient as personal devices (Kidwell et al., 2008).

The major breakthrough occurred in the 1970s, with the arrival of inexpensive, fully electronic calculators. Initially these calculators were still relatively bulky, and were able to perform little beyond the familiar four operations of arithmetic. But by the 1980s calculators had become readily portable, and were able to compute trigonometric and other transcendental functions and to display graphs, thus far surpassing the functionality of mechanical calculators and slide rules. Classroom use became practical, and although very uneven, soon became widespread enough to create disputes between enthusiasts and detractors. Calculators greatly increased the range of feasible problems that could be given to students, but concern was expressed about the effect on basic arithmetic skills, and doubts were raised about the readiness of teachers to use calculators effectively (Kelly, 2003; Waits & Demana, 2000). By the mid-1990s computer algebra systems (CAS) were available on hand-held devices, leading to further debate. Now, in the 21st century, although the generic name persists, high-end devices referred to as "calculators" in fact provide a huge range of information storage, information display, and demonstration capabilities, in addition to pure calculation (Aldon, 2010; Trouche, 2005). Some controversy has persisted, but in recent years the use of calculators has been increasing around the world in secondary and elementary schools, and at the college level as well.

The Virtual World: The Potential of 21st-Century Technology for Mathematics Education

During the past two decades, pedagogical theories in mathematics education, such as instrumental genesis and semiotic mediation, have placed tools, artefacts, and technology at the centre stage of discussion on mathematics knowledge acquisition (see, e.g., Artigue, 2002; Bartolini Bussi & Mariotti, 2008). Studying the pedagogical potential of technology is a major research field of study in mathematics education (see, e.g., Blume & Heid, 2008; Heid & Blume, 2008). The question arises, regarding the plethora of electronic devices now available to mathematics teachers and students, and the evident integration of these devices into what appears is becoming a comprehensive technology platform: is this something fundamentally new for mathematics education or does it merely provide the means for delivering the services of the older technologies more quickly and efficiently? It would certainly appear that the distinctions made earlier in this chapter among classes of technologies are increasingly irrelevant. The computer can function simultaneously as an information storage device, an information display device, a demonstration device, a super calculator, and much more. In the remainder of this chapter we describe some indications that the new technology environment does indeed provide unprecedented opportunities.

Tools from the past are far from irrelevant to the new environment, since the Web can function as a window to access information on historical mathematical tools instantly. This provides the potential to construct mathematical knowledge via simultaneous attention to the multifarious facets in the evolution of that knowledge, as reflected in the tools, thereby creating a virtual thematic museum of mathematical artefacts. One could, if one wished, virtually go back in time, by constraining students to use only the tools available in a certain era in a specific geographic locale. This powerful capability for integrating history, pedagogy and mathematics opens a vast range of intriguing possibilities in conceptualizing the mathematics curriculum.

Research into integrating the history of mathematical tools with the school mathematics curriculum, by having students visit and study historical mathematical tools via present day accessible technology, has been carried out in teacher education and in mathematics classrooms (Bartolini Bussi et al., 2010; Maschietto & Trouche, 2010). On the one hand, this can assist students to acquire mathematical understanding in a techno-cultural context, which raises the relevance of school mathematics as a part of social development. On the other hand, students can re-visit and re-think (even re-conceptualize) familiar mathematical concepts in an old-meets-new context. This simultaneity may bring about awareness of invariants that constitute the core of abstract mathematical concepts. This looking back to *re*-interpret and *re*-present the mathematics embodied in historical tools somewhat echoes Hans Freudenthal's (1991) idea of mathematization, in which mathematical concepts are re-invented using tools that are more powerful than our predecessors possessed. According to Freudenthal, "children should repeat the learning process of mankind, not as it factually took place but rather as it would have been done if people in the past had known a bit more of what we know now" (p. 48).

There have been substantial recent efforts to study classroom use of historically significant tools, both as originally conceived and in a digital form. Maschietto and Trouche (2010) have revisited the idea of the mathematics laboratory in classroom practice, explicitly citing Borel's early 20th-century proposal. They studied the use of both "old" technology (the mechanical calculator of Blaise Pascal, the abacus) and "new" technology (networked electronic calculators) in such laboratory situations, while exploring notions of good contexts and good teaching practices. Cornell University (USA) has digitized and enhanced its collection of kinematic models, in what they call the Kinematical Models for Design Digital Library (KMDDL). These models (including linkages generating straight lines, mentioned earlier in this chapter), were originally created as physical models in the 1870s by the German engineer Franz Reuleaux. At Cornell they are being used to teach the mathematics of machine design. In the 1990s at the Centre for Research on International Cooperation in Educational Development (CRICED) at Tsukuba University (Japan), there was a rebirth of interest in using mechanical instruments in mathematics instruction, facilitated by LEGO blocks and dynamic geometry software. The project has also made use of e-textbooks to weave together historical books and interactive dynamic simulations. And the University of Modena (Italy) has established a Laboratory of Mathematical Machines, which provides digitizations of familiar mathematical instruments, including the compass. Dynamic simulations are available on the Web as a source for teaching and learning activities with prospective mathematics teachers (Bartolini Bussi et al., 2010).

There are several key research questions for this historical pedagogy. How can this re-invention process be best realized in a pedagogic process? Will the re-invention embody "more" or "less" mathematical knowledge? How can this pedagogical perspective be integrated into the curriculum? We illustrate and discuss an example in geometry.

As just noted, the Laboratory of Mathematics of the University of Modena in Italy holds a large collection of replicated mechanical "geometrical machines" from different historical periods—where by geometrical machine is meant a tool that forces a point to follow a trajectory or to be transformed according to a given law (Bartolini Bussi & Maschietto, 2008). These geometrical machines were re-constructed based on old scientific and technical literature, and after experimentation on their possible pedagogical potential. In the Museum's Web site, beside the pictures of some of the replicated geometrical machines, there are corresponding virtual animations, constructed by dynamic geometry software, showing what the machines do. Such a parallel representation is depicted in Figure 17.6, which shows a replica of a Scheiner pantograph, a device invented in Germany in 1603 by Christoph Scheiner for making a scaled copy of a given figure.

This juxtaposition of old and new technology (wooden craft and virtual craft) provides a good context for implementing historic re-invention pedagogy in the mathematics classroom. Figure 17.7 has four equal rods hinged by adjustable pivots at A, B, C and P with OA = AP and PC = P′C = AB. It is fastened by a pivot at O. Placing a pencil at P (or P′) to trace a figure, a dilated image is obtained at P′ (or P). Note that APCB is a parallelogram, O, P and P′ are collinear, and OP′/OP = OB/OA = constant. Antonini and Martignone (2011) have studied the didactical potential of the pantograph in proof and argumentation for geometrical transformations.

In a mathematics classroom, students can construct a make-shift pantograph using geometry sticks, appropriate fastening pivots and writing implements (Figure 17.8), which can be used as an explorative tool to investigate the geometry of similarity (homothety/dilation). The pivot points of this tool can be readily adjusted, which enables students to access, easily, different ratio variations between the sides of parallelogram ABCP. Since all pivot points are free, students can choose

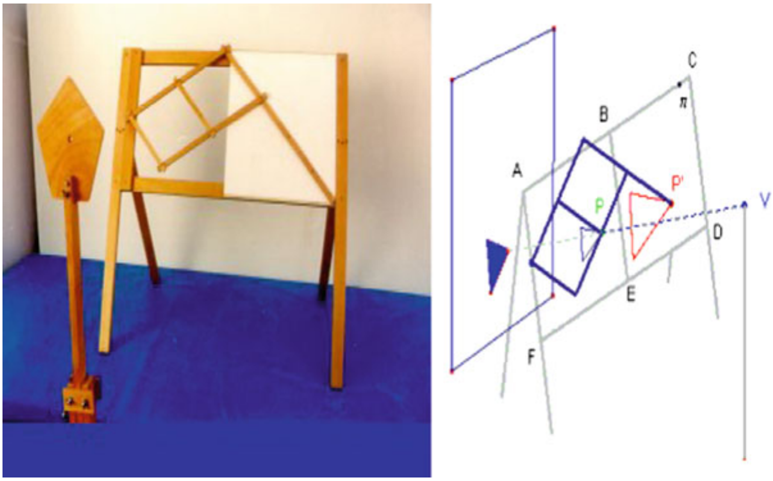


Figure 17.6. A wooden replica of a Scheiner pantograph and a dynamic animation of how it works [Source: http://www.museo.unimo.it/theatrum/macchine_00lab.htm].

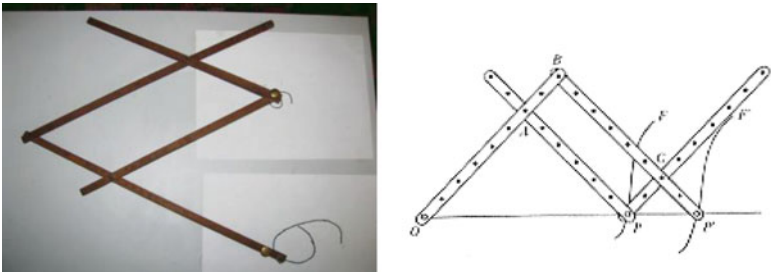


Figure 17.7. A modern day pantograph [Sources: http://www.isaacwunderwood.com/gallery2/displayimage.php?album=4&pos=0 http://www.datavis.ca/milestones/index.php?group=1600s].

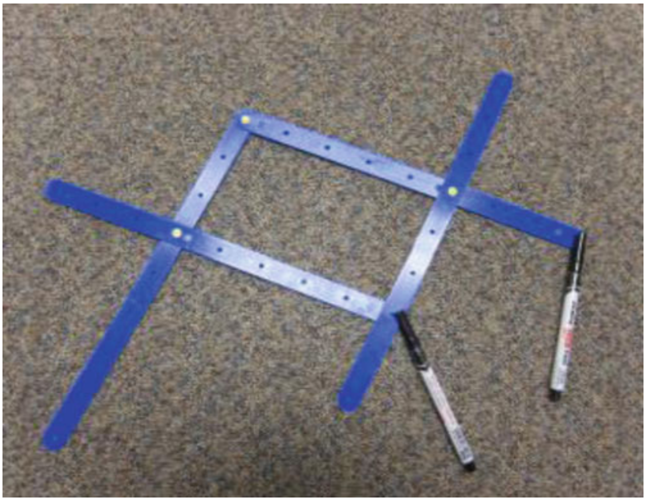


Figure 17.8. A classroom make-shift pantograph constructed using geometry sticks.

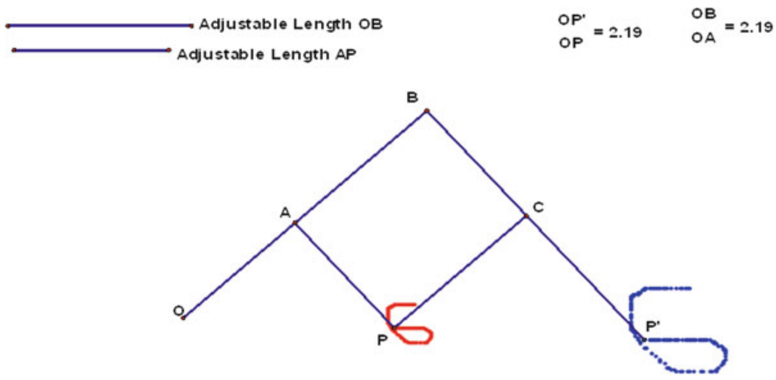


Figure 17.9. A dynamic geometry pantograph constructed in Sketchpad™.

which one to be the fastened one, and which to be where the pens are. Furthermore, the shape of ABCP can be changed to other shapes. These different degrees of freedom of the tool open up a vast pedagogic space for teachers and students.

For more advanced lessons, students can construct dynamic geometry pantographs and use the construction activity to explore the mathematics that can be embodied in a pantograph—an example of which is depicted in Figure 17.9.

For this virtual pantograph, the lengths of OB and AP are adjustable variables and points O and P are free. These features facilitate students experiencing the variation that this virtual tool can offer, providing opportunities for them to discover geometrical properties (Leung, 2008). We have here an example of old and new technologies meeting together in the mathematics curriculum, enabling meaningful mathematics teaching and learning. Such examples suggest that by utilizing the multi-functional nature of the computer, and the connectivity power of the evolving virtual technology, mathematics pedagogy could take on a new paradigm that supports connective and explorative knowledge building in a powerful way. By "connective knowledge building" is meant the ability of teachers and students to (re)construct mathematical knowledge connectively and collectively, and in particular, through the idea of "webbing." Webbing refers to "the presence of a structure that learners can draw upon and reconstruct for support—in ways that they choose as appropriate for their struggle to construct meaning for some mathematics" (Noss & Hoyles, 1996, p. 108).

Thus, webbing can be interpreted as an affordance in the virtual world to facilitate mathematics pedagogy, where connective structures that empower mathematical experience can be built by teachers and students, utilizing multi-functional tools present in the virtual environment. As Web technology advances in terms of speed, accessibility and information content, one can easily surf the Web to connect to information on mathematical artefacts, ancient or new, like those described earlier in this chapter.

The virtual platform can be designed to collect students' perception of mathematical concepts, thus forming a "knowledge database" that serves as a source to connect students' different ways of understanding. This collective understanding via a virtual environment can then be used pedagogically for developing mathematical concepts in the classroom. Leung and Lee (Lee, Wong, & Leung, 2006; Leung & Lee, 2008) have been conducting research on such a platform in an ambient dynamic geometry environment to categorize visually students' perceptions of geometrical concepts. This kind of platform may be extended to become a virtual forum (or community of practice) where teachers and students co-construct mathematical knowledge and even formulate curriculum decisions.

By explorative knowledge building is meant students engaging in explorative activities in specific virtual environments like spreadsheets, dynamic geometry software, computer algebra systems, and other purpose driven software that support mathematics knowledge construction. Students are empowered in these environments to develop tool instrumentation schemes, to discern mathematical patterns and to develop situated discourses. In this connection, Leung (2011) has proposed a framework of *techno-pedagogic task design* that aims to organize and capture trajectories of learning in a technology-rich pedagogical environment by a sequence of progressively inclusive epistemic modes: establishing practice mode, critical discernment mode, and situated discourse mode. This technology-dependent cognitive sequence can empower learners to see mathematics in situated abstract ways and hence enlighten their understanding of traditional mathematics by providing alternative passages to mathematical knowledge (Leung, 2011).

Using the mathematics knowledge embodied in computing technology, teachers and students can potentially experience mathematics in ways that are different from traditional school mathematics curricula. A window is opened through which mathematics teaching and learning might enter into a new epistemological domain, where knowledge becomes both personal and communal, and in which connective and explorative mathematical knowledge becomes vastly more accessible.

How soon or how fully this vast potential might be utilized for mathematics education is a difficult question. The historical examples given earlier in this chapter suggest that we should be cautious about predicting revolutionary changes. Moreover, it is entirely possible that the most profound effects will come not from explicit efforts to design technologies for mathematics education, but rather from the side effects of technologies adopted by the wider society. This has certainly been the case with the book, which did not originate as a special tool of mathematics education, but became ubiquitous both inside and outside mathematics classrooms. And while there is little inherently mathematical about the blackboard, its influence on the mathematics curriculum has been substantial. The computer, with its offshoots and allied technologies, represents an especially intriguing case, and a huge challenge for those who attempt to forecast the future. The computer surely does explicitly embody mathematical concepts and processes (e.g., base two arithmetic), but it does not follow that the primary applications in education will come from this direction, especially as the computer is such a versatile device. As we have indicated, mathematics educators are proposing exciting pedagogical innovations based on the newest technologies, but meanwhile the pace of technological evolution may be changing the overall place of mathematics within education and within society in ways that we cannot yet foresee.

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