Chapter 14
Developing Mathematics Educators

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Abstract This chapter addresses, from various perspectives, issues associated with teacher education and its development. Several categories of mathematics educators are characterized and their development and roles in the teaching/learning processes are summarized. Cooperation between teachers and researchers as well as the concept of teachers as researchers are discussed from different points of view. The crucial role that observations play at all levels is analyzed and illustrated by two different models of implementation of observations into teachers' and researchers' practice. Throughout the chapter the influence of the research of Guy Brousseau on mathematics education research is recognized.

Introduction

> One of the functions of didactics could be … to contribute to the deceleration of the process of transformation of knowledge into algorithms … To sacrifice to the god of contemporary worship to the so-called efficiency, education follows the path of algorithmic reduction and demathematization. I deeply hope that didactics will be victorious in the battle of this dispossession and dehumanization.
>
> Guy Brousseau, 1989 (translation from French, p. 68)

Our first task, in this reflection upon the development of mathematics educators, is to consider the question: "Who is a mathematics educator?" In fact, different

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answers have been given to this question by different authors. In the first part of this essay, we will discuss different ways of thinking about mathematics education and about mathematics educators, and will establish the crucial concept of "observation" in defining a mathematics educator.

The second part of the chapter will focus on the role of observation in the development of mathematics educators of various kinds. We will show how different vantage points in mathematics education can influence observational schemes and approaches to teaching mathematics. In order to illustrate different aspects of the cooperation of teachers and researchers, we will present two examples of the use of observations in mathematics education research and in the search for phenomena in mathematics. The first will be COREM (Centre for Observation and Research in Mathematics Education—*Centre d'Observation et de Recherche sur l'Enseignement des Mathématiques*), which is an example of successful cooperation of teachers and researchers; the second example will be the Learner's Perspective Study (LPS). These two projects have presented examples of different ways of observing and researching realities in mathematics classrooms. There are other perspectives that have been successfully applied in this field—for instance the ongoing comparative study of teacher education, "Teacher Education and Development Study in Mathematics" (TEDS-M) focusses on the preparation of teachers of mathematics at the primary and lower secondary levels (for more details see teds.educ.msu.edu). We do not intend to provide an exhaustive list of these other examples. Later in the chapter we will consider the role of observation within the increasingly important issues associated with the use of information and communication technologies (ICT) in mathematics education.

As mentioned by Adler, Ball, Krainer, Lin, and Novotná (2005), there is much less written about mathematics teacher educators than about teacher education itself. So in the third part of this chapter we will focus on mathematics teacher educators. The central question is "How does a person become a mathematics educator and/or a mathematics education researcher?" Based on two examples, some important aspects are identified, and these are more deeply discussed in the fourth part of the chapter. This fourth part offers a discussion of the central question of relationships between research and mathematical education, especially in didactics. This discussion will provide a synthesis of the themes covered in the chapter. We will argue that it is important to enhance teachers' didactical cultures without damaging their pedagogical beliefs.

Mathematics Education and Mathematics Educators

In this first part of the chapter, we consider different meanings of the term "mathematics education" and address the question of who mathematics educators are. In fact, various institutions involved in teacher education have their own meanings for "educator" and "mathematics educator," and if we can better understand these different meanings, then we might understand more fully what knowledge is important or required of mathematics educators. That is the main issue for this chapter.

Mathematics Education: Education to Mathematics

Although mathematics is a very old body of knowledge it is always growing. It has a history of having strong relationship with the mastery of vital aspects of reality (quantification, measures, etc.). Furthermore, the development of physics, chemistry, biology, and also economics, etc. has revealed other aspects of mathematics which offer the possibility of secure deductive reasoning. Therefore, mathematics qualifies as a body of knowledge which is universally transmitted inside various societies across the world. In this sense, "mathematics education" can be taken to mean "education to mathematics."

It can be argued that mathematics has a recursive or "Russian doll" structure: a concept that was initially constructed as a tool in order to anticipate the result of an action (e.g., integer as a tool to describe two sets of the same quantity) is considered as an object in another situation (e.g., integer as an already constructed object in the problem "what number must be added to 5 in order to obtain 22?") (Douady, 1991). Brousseau (1997) considered these two aspects as a part of the dialectic between knowledge and knowing. This aspect of mathematics is one of the reasons for the need to learn mathematics at an early stage and to continue learning it over a very long period. Mathematics is at the same time "independent of the world" (Wittgenstein, 1983) and yet something which contributes to the formation of citizens.

Therefore, if we consider "mathematics education" as the social answer to the need to educate people in mathematics, the first meaning of "mathematics educator" is "a person who is in charge of mathematics education." That meaning defines a very large category that includes parents and more generally those adults who are in charge of children's care, and teachers at all levels (from primary to tertiary education). In this chapter we consider all teachers that are in charge of teaching mathematics at any level to be mathematics teachers.

It is well known that many people have opinions, mostly based on observations of their own children, about what mathematics teachers ought to learn in order to improve their teaching. They may in fact be considered as the most basic kind of mathematics educators. However, often their point of observation is very limited, since they implicitly consider their own teaching practices as the central basis for their reflections on the nature of mathematics education.

Mathematics Education: Observing the Learning of Mathematics

Within the development of human sciences, every aspect of human activity may be subject to observation. "Learning mathematics" is therefore a legitimate field of investigation. The elements involved are the subject—child, pupil, person in general—the mathematical knowledge, and the observers of interactions.

Since mathematics is learned by children in their early years, observers of the learning of mathematics are sometimes psychologists, who consider mathematics as a system of "logic." Psychologists generally do not question the mathematics involved (which is considered to be a permanent body of knowledge) and prefer to focus on the development of children in relation to their environment (Piaget, 1985) and in relationship to parents, siblings and early childhood mathematics educators (Bruner, 1966; Vygotsky, 1962).

In school, mathematics is taught and the teachers themselves observe their own students as those students are learning mathematics. The teachers are especially concerned with whether their students are learning what they have been taught. Under certain conditions, teachers may develop further their observations by reflecting on what their students are actually learning. They might also consider which variables are involved in the learning process, and what might be the effects on learning if certain conditions were to be modified.

Since a transmission of mathematics occurs when someone learns mathematics, mathematicians may be interested in observing the learning of mathematics. Such observations would, most likely, be centred on the nature of the mathematical knowledge involved: what does this child know about the mathematics? Is the knowledge that the learner has acquired adequate with respect to my own experience and understanding of mathematics?

Therefore, if we consider mathematics education as the field of observation of the learning of mathematics, the second meaning of "mathematics educator" is "a person who observes mathematics learning." This category of mathematics educators includes teachers, mathematicians, researchers in psychology, and researchers in mathematics education. Psychology, in particular, has had a strong influence on mathematics teacher education. Often, theories from psychology have been assumed to provide satisfactory theoretical backgrounds for mathematics education, with actual mathematics teaching being regarded as an application of such theory. What has been lacking in all of this has been the teaching processes, which include both the didactic transposition that interrogates the mathematics itself (Chevallard, 1985), and some consideration of teachers' attempts to cope with the different resources and constraints within their teaching situations (Margolinas, 2002).

Mathematics Education: Observing the "Learning and Teaching" of Mathematics

As we stated above, the learning of mathematics mainly depends on the teaching of mathematics. The teaching is a conscious attempt to help learners acquire mathematical knowledge. Therefore, another field of investigation might be focussed on "learning and teaching mathematics." The elements involved are: pupil, teacher (in a broad sense: a parent, when deliberately educating his or her child about mathematics, is a teacher; university professors in mathematics are also teachers, etc.), setting, mathematical knowledge, observer.

When considering the "learning and teaching" system we can place the observer as an "outsider," someone who observes the interactions between pupil–teacher–mathematics (see Figure 14.1).

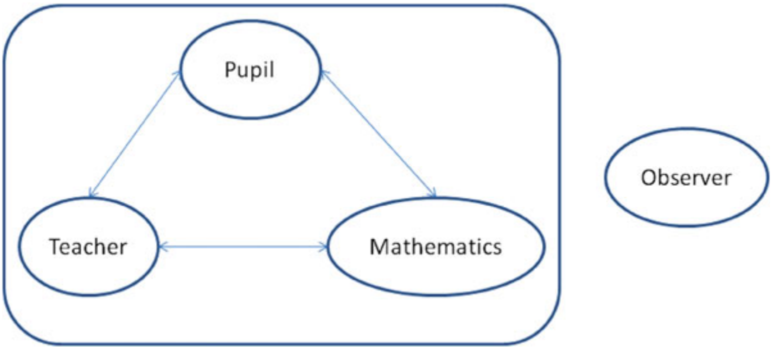

Figure 14.1. An observer outside the interactions between pupil–teacher–mathematics.

The status of the observer has some important consequences. The teacher herself or himself may be an observer, and in this case she or he is a self-observer, which is a difficult vantage point in which to be placed. Or, the "outside" observer may be another teacher, who may be inclined to identify with the teacher. Or the observer may be a mathematician, and in that case will be likely to focus on the knowledge involved and the explicit formulation of this knowledge. The observer could also be a teacher educator who wants to give advice to the teacher about how to cope with the situation, or the teacher's supervisor who has the specific task of evaluating the teacher's effectiveness.

The focus of the observer, then, is partly determined by his or her professional occupation. But the focus can also be determined by the theoretical framework of the observer or the purpose of the observation. This kind of observation may be made by a researcher who wants to increase knowledge about phenomena which occur in the learning and teaching situation. That person might be called a mathematics education researcher.

Therefore, if we consider mathematics education as the field dealing with the observation of the learning and teaching of mathematics, the third meaning of "mathematics educator" is "a person who observes mathematics learning and teaching." This category of mathematics educator can include the teacher (as a self-observer), a teacher educator, a mathematician, the teacher's supervisor, or a mathematics education researcher. What mathematics education and in particular mathematics didactics has stressed is that we need to take into account the whole didactic system (pupil–teacher–mathematics) in order to understand mathematics teaching. It is possible to focus on some of the relations (e.g., pupil–mathematics) but one should not forget the role of the teacher altogether. It is therefore crucial for mathematics teacher education that a scientific field that focusses on the phenomenon that are specific of the *entire* didactic system be developed and that mathematics educators are well informed of its main theoretical perspectives and results.

Observation as an Efficient Tool for the Knowledge Development of Mathematics Educators

What are the sources which assist a mathematics educator's development? They cannot be the same for all categories of mathematics educators mentioned in the first part of this chapter. We now summarize some of the sources associated with the different categories. The summary will not be exhaustive but it will provide some idea of the complexity of the domain.

Student Teachers

When preparing and teaching mathematics lessons, prospective teachers are profoundly influenced by mentor teachers (Cavanagh & Prescott, 2007; Vacc & Bright, 1999). Nathan and Petrosino (2003) point to the intersection between the two knowledge bases, pedagogical and mathematical; they state that preservice teachers with advanced content knowledge in mathematics have the tendency to think beyond their own content expertise when considering their students' possible reactions to the content.

Teachers

Here we draw on the burgeoning research literature on the sources of information concerning the ways teachers influence student thinking and understanding (e.g., Carpenter, Fennema, & Franke, 1996). Kinach (2002) emphasized the importance of a teacher's content knowledge when asking questions of students, anticipating likely responses, and evaluating students' responses. Feiman-Nemser (2001) drew attention to the influence on teachers of the knowledge and experiences of mentors and colleagues. Several other writers have contrasted experienced and novice teachers: when anticipating students' likely mathematical responses, experienced teachers mobilize a number of resources that novices do not have, including their past observations of students learning mathematics and their self-observations of their own teaching (Sherin, 2002). Experience in anticipating responses can help teachers identify and state learning goals embedded in a mathematical task. Research suggests that novice teachers benefit greatly from opportunities to gain experience in this domain (Morris, Hiebert, & Spitzer, 2009).

It is important for teacher educators to understand the ways in which teachers make use of available resources in their everyday teaching practices. An intermediary between research and teaching may become *journals*. In general, journals dealing with mathematics education may be classified into three groups—those aimed at (a) students and non-specialists interested in mathematics; (b) teachers of mathematics;

and (c) mathematics education researchers. When the focus is on mathematics educators, the last two categories are of the special interest.

The objective of many professional development activities is the improvement of teachers' knowledge of mathematics. But teachers often consider content knowledge as being less valuable to them than getting acquainted with the practical ideas for teaching (Wilson & Berne, 1999). Observations offer mathematics educators a wide range of both practical and theoretical information.

In the first part of this chapter we noted that different kinds of observations can be associated with different meanings for the term "mathematics educator." In this part, we show that different kinds of observations are necessary to develop the knowledge of these different kinds of mathematics educators (including mathematics education researchers). We also discuss different structures that have been used for observing mathematics education. We show that different observational vantage points can be somehow connected, even with mathematics education research. Thus, for example, when observing the educational system a researcher may adopt a position of "expert" which is very similar to the position adopted by institutional decision makers. Different vantage points can provoke different types of "observations."

Here we restrict our focus to teachers and researchers as the two main groups of mathematics educators. We show that many activities precipitate observations of different kinds. These may have the same nature and purpose, but are not based on the same knowledge and do not call into play, or monitor, the same set of variables.

The different vantage points and interests of teachers and researchers in the observation processes have been studied by authors from different perspectives. Thus, for example, Margolinas, Coulange, and Bessot (2005) focussed on teachers' learning from different situations, and Novotná, Lebethe, Rosen, and Zack (2003) focussed on differences between the roles of teachers and researchers.

In Figure 14.1, the general scheme for observation was presented. But, if we consider the different foci for the two groups of mathematics educators—teachers and researchers—we see substantial differences. Figure 14.2 represents possible perspectives for a teacher, and Figure 14.3 for a researcher.

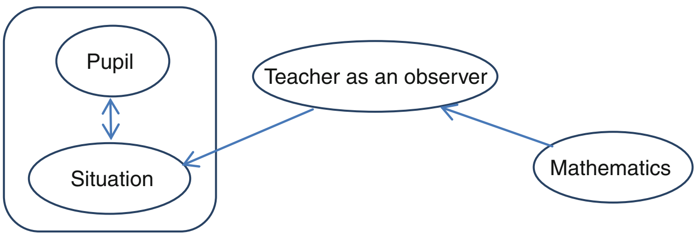

Figure 14.2. Interactions between pupil–teacher–mathematics, from a teacher's vantage point.

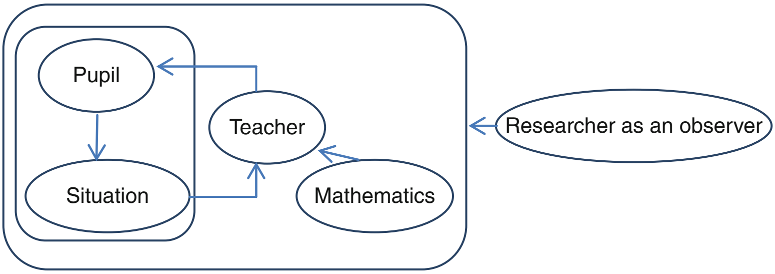

Figure 14.3. Interactions between pupil–teacher–mathematics, from a researcher's vantage point.

The main differences are in the observational role of the observer—the purpose of observation—and in the knowledge that the observer possesses. Although the teacher is interested mainly in a posteriori analysis of the teaching unit (comparison of the lesson plan and a priori analysis with the realities in the classroom and explication of differences among them leading to modifications of the unit design), the researcher's fundamental interest is in discovering general phenomena that influenced the development of the educational situation.

We also wish to focus, here, on mutual relationships between and influences of mathematics educators. We also discuss different structures that have been used in order to observe mathematics education, especially in COREM and in the LPS project.

Differences Between Teachers' and Researchers' Positions in Mathematics Education

The similarities and differences in school and research vantage points and practices were described by Brousseau (2002) in the following terms:

> When I am a *didactician*, the interpretation of every step of teaching begins with a systematic informing, a complex work of the analysis a priori and the confrontation with various aspects of contingency, of observations viewed and rejected later, etc. There is not an evident separation of what is relevant but inadequate, adequate but inadaptable, eligible but inconsistent, as well as transformations of appearance and certainties in falsifiable questions, etc. When I am a *teacher*, I have to take a number of instantaneous decisions in every moment based on the real information got in the same moment. I can use only very few of the subtle conclusions of my work as didactician and I have to fight with starting to pose myself questions which are not compatible with the time that I have and that finally have the chance to be inappropriate for the given moment. I react with my experience, with my knowledge of my pupils, with my knowledge of a teacher of mathematics which I am treating. All these things are not to be known by the didactician.

Differences between the roles of a teacher and a researcher were addressed in a panel session at the 27th annual PME conference held in Honolulu in 2003 (Novotná et al., 2003). We now consider some of the differences.

Teacher as a Researcher

In the text *Navigating Between Research and Practice: Finding My Own Way* (Novotná et al., 2003), Vicki Zack described development as a mathematics educator in the following way:

> My questions emanate from neither theory nor practice alone but from the juxtaposition of the two, and from critical reflection on the intersection between the two (Cochran-Smith & Lytle, 1993, p. 15) in areas which are of intense and enduring interest to me. There is recursiveness in the process, wherein questions are continuously reformulated, extended, re-visited, methods are revised and analysis is on-going. … I recognize the value of practical knowledge, and also respect the place research can hold in informing practice. However, I emphasize the challenge involved in understanding others' ideas. (p. 87)

Although this process in the development of a mathematics educator is individual it has common features. As Zack and Graves (2001) have emphasized, each person appropriates, reworks, re-accentuates while making her or his own way. A fundamental part of this development should be making meaning of the research and associated theoretical issues, and seeing what they might mean for the teacher's work, and for the children, who are making meaning of the mathematics as they work together with their teacher, with their peers in the classroom and, at times, with their parents at home.

Questions which become important for teachers are: How do my students proceed when asked to "prove" that they are correct? What do they consider valid arguments for proving their case and convincing others? What language do they use when presenting their arguments? What kinds of reasoning do they use: inductive, deductive, other? (Novotná et al., 2003).

Cooperation of Teachers and Researchers

The cooperation of teachers and university-based educators (in the following text we refer to them as researchers) in research teams in mathematics education is a broad and relevant topic. In most cases, the focus is on improving the quality of mathematics teaching and learning (see, e.g., Brown & Coles, 2000). Many discussions have been carried out within the last decade about the impact of this type of cooperation in mathematics education (see, e.g., Goos, 2008). Identifying and contrasting the different experiences and knowledge of teachers and researchers have been a focus of investigation in numerous studies (see, for example, contributions by Bennie, Breen, Brown, Hošpesová, Coles, Lebethe, Eddy, Macháčková, Novotná, Pelantová, Poirier, Reid, Rosen, Tichá, Zack in Novotná et al., 2006). Chris Breen (2003) drew attention to the contrasting views on the contributions that teachers are making to the field of mathematics education. Although there is a movement for more teachers to become involved in critical explorations of their practice, through such methods as critical reflection, action research, and lesson studies, some sceptics claim that these activities have done little to add to the body of knowledge in mathematics education.

Despite such controversy, there seems to be little doubt that cooperation within and between communities of practice enriches research in mathematics education. However, the components of this type of cooperation, and how the interactions of these components change teachers' opinions and approaches, are much less investigated. Without paying attention to teacher change, the results of many research activities can seem to be less significant than they actually are.

We now summarize an example of fruitful cooperation between teachers and researchers. The research project was originally designed by Guy Brousseau and Jarmila Novotná, and data collection, analysis and evaluation of the experiment were carried out in cooperation with secondary school teachers in Prague, the Czech Republic.

The experiment which was designed incorporated the following steps:

- Design of the didactical situations that were intended to change learners' approaches to solving problems.
- Development and implementation of the proposed didactical situations.
- Analysis of the implementation and, based on the experiment results, and reflections on possible modifications.

Even though the primary target group of the research comprised secondary school students, the research provided an opportunity for the participating teachers to develop their professional competences.

The influence of teacher attitudes and teaching has been formulated by Jaworski (2003):

> The action research movement has demonstrated that practitioners doing research into their own practice … learn *in* practice through inquiry and reflection. There is a growing body of research which provides evidence that *outsider* researchers, researching the practice of other practitioners in co-learning partnerships, contribute to knowledge *of* and *in* practice within the communities of which they are a part. (p. 2)

We illustrate changes that were identified among teachers in this collaborative group exercise through the examples of two teachers, who will be referred to as Teacher *A* and Teacher *B* in the following text. The following extracts are from their self-reflections:

Teacher *A*'s reflections.

- ***Experienced a "new" role as a teacher during the adidactical situation.*** The teacher should rather become an observer, moderator of discussions and of the work in the classroom. This role is demanding, and from the perspective of traditional teaching, unusual. When you listen to students during group work and see that they are very close to the solution, it is not easy to answer their question without intervening in their work.
- ***Gained experience in moderating students' discussion.*** I learned to listen and intervene only when it was a must. If I intervene too soon there is a danger that I

divulge to students something that they could find out themselves if I had given them more space.

- ***Gained experience with group work.*** Before the project, I used groups very rarely. I was afraid that I would not succeed in involving all students in the activity, to be able to get all of them actively participating. The experiment showed that with an appropriate choice of activities, this is possible.
- ***Gained experience with the student peer control.*** The teacher is not the only one who can tell students what is correct and what is not. It proved to be more efficient when this evaluation was formulated by the students' own schoolmates.

Teacher *B*'s reflections.

- ***Realized that I tended to underestimate my students' abilities.*** This experiment showed me the conflicts between my expectations and what the students could really do. At the beginning, I was embarrassed that I did not manage to get from them what I wanted, but it motivated me to a deeper reflection on the ways of presenting the stages to students. At present, I find that it is not a negative if students do something differently, because we can all learn from it.
- ***Benefited from gaining feedback from students.*** The experiment made me want to get feedback from the students. Getting feedback should become an integral part of my work as a teacher. Before the project, I could not imagine that more fruitful discussions can took place in mathematics lessons than in lessons for other subjects.
- ***Gained experience in organizing research projects.*** I noticed a shift. In the beginning, I devoted myself solely to organizational items, such as the number of problems, or dividing students into groups. After gaining experience I found that I was attending to more fundamental issues, such as the definition of a mathematical model for a problem, or exploring conceptual links between aspects of the mathematics.

We observed a change in the teachers' perceptions with respect to the use of student problem posing: observing their own students in these situations broadened their knowledge about students learning mathematics. Before participation in the project, teachers were used to assigning problems to students themselves; they saw it mostly as the only appropriate way for managing the teaching/learning process. Their fears had almost certainly been influenced by their own experiences in their own schooling.

Indeed, the project considerably influenced all members of the collaborative group—the teachers as well as the researchers. It was recognized that if the work of the team was to be successful then all the participating persons needed to collaborate fruitfully. The result was changes could be observed, and not only on the teachers' side, or on the students' side, or on the researchers' side. The researchers certainly gained much from the collaboration, and the teachers' inputs helped to consolidate the experimental settings and to analyse the project results.

Interaction Between Observation and the Development of Theory in Mathematical Education: COREM and the Theory of Situations

Brousseau's ideas were successfully implemented at the Jules Michelet School, Talence, France, between 1973 and 2000. The overall project is referred to as COREM, which was created in 1973 with the following objectives (from Salin & Dreslard Nédélec, 1999):

- To conduct research necessary for the advancement of knowledge of mathematics education phenomena.
- To conceive and study new educational situations that will generate better learning of mathematics by pupils.
- To develop in this way a corpus of knowledge necessary for teacher education.

It is important to stress that Jules Michelet School was never an experimental school conceived to improve mathematics teaching or to educate the teachers of this particular school (even if it may have also this result in both cases). The Centre was conceived in order to allow a vast community of researchers to observe the real teaching process in an entire school. The scope was from the beginning a typical scientific project: to understand better didactical phenomena and not to directly implement any innovative teaching. In COREM there was always close collaboration between researchers from the university, teacher educators, elementary school teachers, pupils aged from 3 to 11, school psychologists and students of didactics of mathematics (Novotná et al., 2003). Two major data sets were generated: (a) a longitudinal collection of qualitative and quantitative information about the teaching of mathematics at the elementary level; and (b) records of two types of observations which were destined to assist in the finding and explaining of phenomena of didactics that were relevant to teaching and to research.

Michelet School consisted of 4 kindergarten and 10 elementary school classes. The school was not selective, and pupils came from a very heterogeneous population. The curricula followed in all subjects were those that applied in all other French schools. The teaching staff consisted of "ordinary" teachers without any special training. Their task was to teach, not to do research. They worked in teams, three teachers for two classes. One-third of their working hours were devoted to COREM. This time consisted of four types of activities: (a) coordinating and preparing the ordinary work of the pupils and discussing all the problems of the school (educational, administrative, social, and so on); (b) directly observing the work in the classroom, for research purposes and for normal feedback; (c) participating with the researchers in the design of sessions to be observed and collecting data about the pupils' behaviours in mathematics; and (d) participating in a weekly seminar at which themes selected by the teachers were discussed.

The daily mathematics activities were designed in collaboration with one teacher educator from a Bordeaux institute for teacher education—before 1991 this was called the Ecole Normale, but in 1991 it became the *Institut Universitaire pour la Formation des Maîtres* (IUFM). The teacher educator monitored the mathematics that the students studied during the whole school year. He was expected to make sure that the research program did not compromise the normal educational activities of the school.

There was one important rule in the decision-making processes practised in the team—specifically, in the case of consensus *not* being reached among participants on any issue, the normal teacher would have the final say about what would be done. Detailed analyses of the teaching units were carried out by the whole team, including the teachers.

The observations were of two types:

1. The first type was of observations of sequences prepared by researchers, together with teachers. In this case, the researcher was responsible for elaborating the project's teaching sequences. The researcher presented the project's sequence to the teachers, including the knowledge it was presumed the pupils would attain by the end of the teaching sequences, the problems to be presented to pupils, and a register of the expected pupils' strategies. When the project was accepted by the team, the next step was the elaboration of teaching sequences. The ideal situation was if the teacher was able to accept the scenario of the lesson directly from the project. If this was not the case, other questions were discussed—like, for example: "What vocabulary should be used in each phase and how and when?" "Should the teacher intervene in the pupils' validation of strategies, and if so, how, and when?" "What should be done if pupils do not respond as expected?" "Are the application exercises necessary?" This collective preparation was set out in the form of a written description and was distributed to the observers in advance.

   The teacher was completely responsible for what happened in the classroom. It included the right to make decisions different from those presumed.

   After the planned sequence of events had been carried out, a first analysis of what had transpired occurred immediately. In this analysis, all participants reconstructed as precisely as possible all the events of the session. Analyses proceeded according to a prescribed order: First the teacher summarized, from her or his point of view, what had been good, and what had not been good, and why. The team discussed any issues that arose, and for unusual happenings looked for explanations of why these had occurred. In such a way the observation strategy included the need for involvement. The discussions provided the researcher with a considerable amount of additional information.
2. The second type was of observations of sequences prepared by teachers themselves. Regular *weekly observation of a series of "ordinary"* lessons—that is to say, observations of lessons that had not been prepared with a researcher—served to identify and explain contingent decisions of "all" teachers. The researcher, who was interested in the overall teaching sequences and patterns during a certain period, organized the observations.

   Teachers and researchers were members of one team at least in the preparatory phase. Their roles were different. In the class, the teacher had the responsibility for pupils. Various distortions could happen: for instance, the researcher might not have formulated expectations adequately, or the teacher might not have understood what had been formulated. Sometimes, the teacher had to make important decisions in order to reach the teaching goals.

The successful functioning of COREM depended on the collaboration of all participating persons as well as much administrative and managerial work. Structures and findings were disseminated in various ways; from allowing interested persons to participate in the whole process, to presenting the organization, functioning and results at conferences and symposia in France and abroad. The teaching processes prepared for observation have never been published or given as a model for use in ordinary classroom conditions.

It is important to remark that although the functions expected of teacher and of a researcher differed, these were not differentiated so far as personal status was concerned. In COREM some persons were both teacher and researcher, but never at the same time or for the same activity. The outcomes of these interactions between an entire school and a team of researchers are enormous. The COREM research was recognized as groundbreaking—The quality and uniqueness of Guy Brousseau's work was quickly recognized, and in 2004, he was the first person to be awarded the prestigious Félix Klein medal by the International Commission on Mathematical Instruction. The importance of Brousseau's work is mainly the development of the Theory of Didactical Situation (Brousseau, 1997), that is considered by a great number of researchers as providing a paradigm for mathematics didactics. Further details about mathematics teaching in COREM have been published for the information of interested researchers or teacher educators (Brousseau, & Warfield, 1999; Brousseau, Brousseau & Warfield, 2001, 2002, 2004a, 2004b, 2007, 2008, 2009).

Researchers Observing "Ordinary Classrooms": The Learner's Perspective Study

There exist many papers describing and analysing observations of a single lesson in a single classroom. Undoubtedly, many of these provide important sources of ideas and phenomena. In this section of the chapter we will focus on another type of observation of ordinary classrooms, by researchers—the Learner's Perspective Study (LPS).

LPS methodology has been developed and applied for teaching mathematics in the eighth grade (Clarke, 2001; Clarke, Keitel, & Shimizu, 2006). The main goal of LPS has been to examine classroom practices in a more integrated and comprehensive way than in other international studies. Originally, the project was designed for in-depth analysis of mathematics classrooms in four countries (Australia, Germany, Hong Kong and the USA), but quickly other countries joined the project. In 2006 there were more than 12 countries contributing to the project materials and analyses.

The Learner's Perspective Study was designed to document the processes and events in mathematics classrooms, but not just the obvious set of events that might be recorded on a videotape. A decision was made to determine how the participants construed those events, including their memories and feelings, and the mathematical and social meanings and practices which arose as a consequence of their beliefs and conceptions. The power of the project has been greatly enhanced by the matching of LPS data from different countries.

A series of research questions were formulated in the initial phase of the project. For example: “Is there evidence of a coherent body of student practice(s), and to what extent are these practices culturally-specific?” “To what extent does an individual teacher employ a variety of pedagogical approaches in the course of teaching a lesson sequence?” “What degree of similarity or difference (both locally and internationally) can be found in the learner (and teacher) practices occurring in classrooms?” “To what extent are teacher and learner practices in a mutually supportive relationship?” “To what extent are particular documented teacher and learner practices associated with student constructions of valued social and mathematical meanings?” (Clarke, 1999).

A major characteristic of this study is its documentation of the teaching of a series of lessons instead of just one single lesson. For each participating teacher, documentation includes video from 10 consecutive lessons, obtained through three cameras in the classroom, together with post-lesson video-stimulated interviews. The common database of materials from the participating countries, with access offered to those who contribute to the project, together with their materials, represents a rich source of materials for analyses and comparative studies of classroom practices from both teachers’ and learners’ perspectives.

The materials obtained by LPS methodology serve as a rich source of materials for researchers. But at the same time, they represent extremely important material for teachers themselves. Combining video-recordings, the teacher’s own preparation of the lessons, the real situation in the classroom and the post-lesson interviews with students provides a teacher with huge feedback and impulse for further development of her or his approaches to teaching.

Observation as a Part of Mathematics Teacher Education

The observation of classroom episodes, in both forms—observation of real classrooms or video-recordings of teaching episodes, is an irreplaceable part of teacher education (Stehlíková, 2007). In contrast to experienced teachers, student teachers usually have not obtained enough experience from real classrooms. So, when they observe lessons, their observations have a modified structure, with the mathematics content being separated from the classroom (see Figure 14.4).

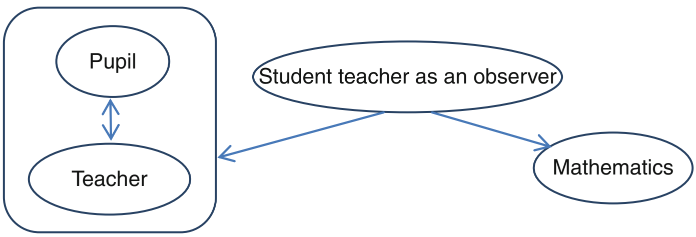

Figure 14.4. Student teacher as an observer.

It is often the case that before a student teacher observes a class she or he is asked to focus on certain features of the lesson that will be observed. Such foci could be any of the following:

Input

Scaffolding
Advance organizers and outlines
Dual code model (verbal and non-verbal representational systems)
Multiple verbal representations
Inductive approaches to learning
Textual support
Graphic organizers

Learner Differences

Varying methods according to the learners' age
Multidimensional model: something intellectual, plus something emotional
Additional time and support during writing assignments
Multiple-abilities treatment (sharing responsibilities)

Learner Processes

Strategy training (cognitive): Teaching the learners how to learn
Strategy training (social): Group-worthy tasks, cooperative strategies, peer support

Output

Support for communication
Norms of collaboration and cooperation (turn-taking; rotating roles: facilitator, materials manager, reporter, harmonizer; status treatment to equalize participation)
Inclusion of similar components in every lesson/series of lessons
Explicit evaluation criteria

The requirement of focussed observations from student teachers has obvious advantages for the development of future teachers. Student teachers will meet, and learn to recognize, a variety of teaching strategies during their study. During the observations, non-experienced student teachers will be expected to develop and interpret their theoretical knowledge and skills, linking it to real and relevant situations (Santagata, Zannoni, & Stigler, 2007).

Mathematics Educators and ICT

The use of computers in mathematics is a very up-to-date topic—see Chapter 17. Computers have become tools of motivation, and can foster comprehensible interdisciplinary links between mathematics and other subjects. However, the use of computers in teaching asks for new approaches to exposition and to mathematical content (Artigue, 2002). This might be one of the reasons why recent studies in mathematics education show that, despite many national and international actions aiming at integrating ICT into mathematics classrooms, such integration in schools remains underdeveloped.

There are several reasons for the discrepancy—ranging from the huge diversity of ICT resources (Lagrange, 2011) to the lack of experience among teachers, at all levels, in using technology in mathematics lessons. A vital part of the knowledge of mathematics educators, indeed of teacher educators, is knowledge of potential, advantages and dangers of inclusion of activities using ICT into teaching (Jančařík & Novotná, 2011).

There are many projects, seminars and conferences dealing with this topic. As a recent example, aspects of Working Group 15 ("Technologies and Resources in Mathematics Education") at the Seventh Congress of the European Society for Research in Mathematics Education (CERME 7), held in Poland in 2011, is considered. A common focus of several contributions in the Working Group was on the challenges that teachers encounter when teaching mathematics supported by ICT for developing mathematical understanding and skills. Teaching with ICT is a complex activity, requiring insight in the subject, knowledge of the ICT tools, and understanding of pupils' thinking (Fuglestad, 2011). Shulman (1986) introduced the term pedagogical content knowledge, PCK, to denote the intersection of pedagogical and content knowledge in order to consider the complex interaction between pedagogy and subject content. Mishra and Koehler (2006) extended Shulman's model to include technology and introduced the term technology pedagogical content knowledge, TPACK; Figure 14.5 (retrieved from http://tpack.org/tpck/index.php?title=TPCK_-_Technological_Pedagogical_Content_Knowledge) is a scheme indicating several areas of knowledge. Using ICT effectively in teaching requires more than just learning to handle the computers with software and other digital tools.

But what are the implications of TPACK for teacher-education programs? How can this specialized pedagogical content knowledge be best developed? When a student teacher observes a lesson within a technology-rich environment, what should she or he observe? That question, and many other like questions in the area of mathematics education and ICT, urgently need attention.

Mathematics Educators in the Position of Teacher Educators

Obviously, it would be disappointing if the results of mathematics education research did not have important implications for theory and practice in mathematics education. On the other hand, researchers often have no direct access to teachers, and vice versa; therefore, mathematics educators, viewed as a specific body of teachers (they teach teachers, and in that sense they are teacher educators) form an extremely important category influencing a great deal the spreading of theoretical

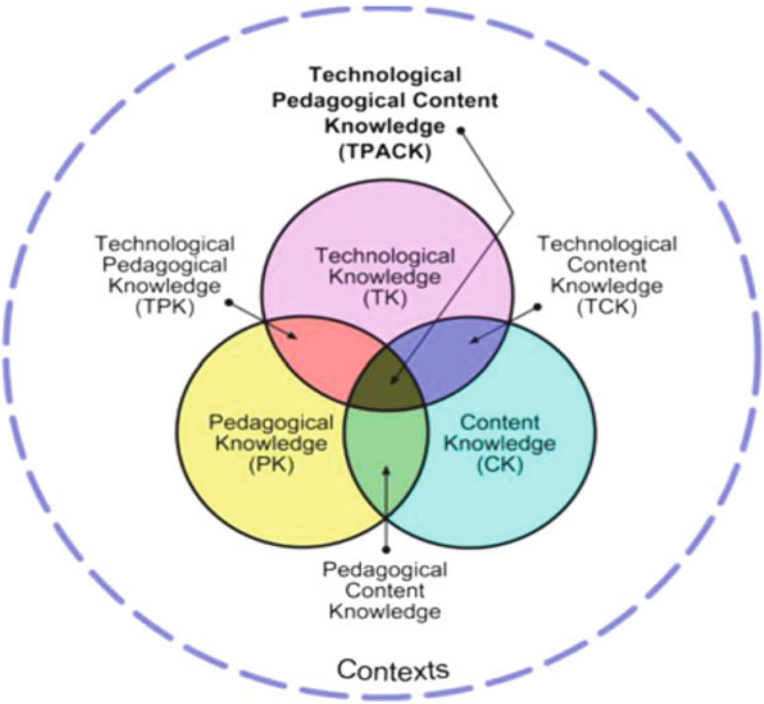

Figure 14.5. Technology pedagogical content knowledge—TPACK (Mishra & Koehler, 2006).

knowledge in the domain of mathematics education. In this part of the chapter we consider issues surrounding the development of mathematics teacher educators.

How Does Someone Become a Mathematics Teacher Educator?

There is no well-defined and unique pathway for becoming a teacher educator. Some teacher educators were originally teachers in schools and took up appointments in teacher-education institutions after years of classroom practice. Others became teacher educators immediately after completing their PhD studies (or even during their PhD studies). In cases where the PhD is completely set inside the field of education research, it has been possible to become a teacher educator without having had much experience teaching in schools. Others were originally mathematics specialists who became mathematics teacher educators without any special training in relation to psychological, pedagogical and didactical issues. The following question arises: What *basic* requirements should we expect of someone who wants to become a teacher educator (regardless of what we understand by the term "good" teacher educator), with respect to mathematics, pedagogy, psychology and mathematics education? This is a complex question, already dealt with in many papers and discussions.

It is generally accepted that mastering mathematics itself is not sufficient for successfully teaching it at any level (see, e.g., Nieto, 1996). In teacher education, it is necessary to determine the balance between the following components:

Specific knowledge. Four main areas are identified:

- Knowledge of mathematics (mathematical concepts and procedures, methodology, relationships with other areas);
- Psychological–pedagogical knowledge (general aspects of teaching/learning processes, getting to know students, planning and management of lessons, curriculum creation, knowledge of teaching contexts);
- Knowledge of learning/teaching mathematics (learning/teaching strategies for specific topics, curricular and pedagogical materials); and
- Knowledge, beliefs and attitudes towards mathematics.

Practical skills. These components are only general; they do not answer the basic question about the content and extent of knowledge required from teachers.

Teachers who become teacher educators have the experience of practice but usually lack any theoretical background. Mathematicians who wish to become teacher educators often have a tendency to overlook the importance of pedagogical–psychological components and prefer to focus on the deep and precise knowledge of the subject content; from their perspective, issues associated with the depth and extent of the mathematics to be mastered are crucially important, and other matters are much less important.

Until recently, little was known about the professional learning or development of mathematics teacher educators (Llinares & Krainer, 2006). As Chapman (2008) reported, even in cases where mathematics teacher educators have researched their own practice, not much is known about their learning, for example, how they reflected to gain self-understanding, what practical knowledge they acquired, and how this knowledge had an impact, or is likely to have an impact, on their future behaviour in working with students.

Despite the views of some sceptics, the importance of theoretical perspectives on the learning and development of university-based mathematics teacher educators is well recognized by the International Group of Psychology in Mathematics Education (IGPME). This topic arose from interactions between PME conference participants, and editors and authors of a special issue on "Teacher Change" of the *Journal of Mathematics Teacher Education*. The learning and development of mathematics teacher educators were explored in a PME discussion group in 2010, and in a PME working session in 2011 (see Goos, Brown, Chapman, & Novotná, 2010, 2011).

Instead of striving to identify a general framework, which could be a fruitless task, an example of a teacher becoming a teacher educator is presented. The following written statement was prepared by a teacher from South Africa who described the difficulties she had after she took steps to become a teacher educator (quoted in Novotná et al., 2003).

> As a teacher educator teaching teachers, my practice has often been constructed for me. Course content is sometimes prescribed and so have been the models of delivery.

During the last 2 years I have found myself strangled and twisted in a thread of tension. The Department of Education embarked on a national strategy to train and equip mathematics, science and technology teachers. They developed a five-year programme to train a substantial amount of educators in each of our provinces. The programme targeted Intermediate Phase (Grade 4 to 6) and Senior Phase (Grade 7 to 9) to ensure an early and solid foundation for learners at higher levels. The intention is that teachers will emerge with an Advanced Certificate in Education (ACE). The National Education Department set out the following outcomes for the programme and for the institutions that would deliver the programme:

- A progressive through-put of well-trained mathematics, science and technology educators per province, who can:
  - demonstrate competence and confidence in classroom practice;
  - assess teaching and learning in line with curriculum stipulations;
  - demonstrate understanding of policy imperatives impacting on teacher development; and
  - become professionally qualified educators with an ACE. (p. 78)

The course attempted to integrate theory and practice but at a very superficial level. My concerns were that as teacher educators:

- We need to think very carefully about what kind of theory is most useful and how we should teach this theory so that teachers can use it to deepen their understanding of educational processes.
- We also need to consider the educative roles of experience.
- And, how exactly should theory and practice be related when the Education authorities want well-trained maths educators?

Theories will die if they remain disconnected from me (my practice) and my practice would be lifeless if not inspired by theory.

My experience with practice has included researching my own practice. To distil the tensions I embarked on a research process that allowed me to probe my own assumptions and to investigate how these influenced the ACE course. I tried to pay attention to the voices of some of my students from the course so that this knowledge could be shared with colleagues with the possibility of reshaping the ACE programme and contributing to our understanding of professional development and teacher education. The purpose of the research was to find out from the teachers what it meant to be a mathematics teacher in their everyday, lived situations.

I do have a slight problem. I am not sure about the role that generalizability will play in the research. At this stage I remain undecided whether to use the stories (the teachers' and mine) to reflect further on the ways that individuals and institutions construct courses in teacher education in South Africa (pp. 79-80).

Theory and practice can exist separately and they can belong to the same world.

People do not stay neatly in a role: at times, setting aside the role of practitioner or of theorist. The educational theorist is a practitioner of education (a teacher); at times the teacher (as educational practitioner) is a theorist (Carr, 1995). (p. 83)

Who Teaches Mathematics Educators? How Does Research Contribute to Mathematics Education?

In the previous parts of the chapter we tried to find answers to the following questions: "Who is a mathematics educator?," "What are the most common paths for becoming a mathematics educator?," and "What is the main role of observations in mathematics educators' work and in mathematics education generally?" We have seen that, to a great extent, mathematics education is determined by "mathematics educators." The category of "mathematics educators" includes all the individuals, regardless of their status, who contribute either intentionally or non-intentionally to establishing or transforming the relationship of a subject with situations that may be modelled by mathematics. This is the place where mathematics education takes place, because knowledge of mathematics is always manifested as an expression of this relationship. But it is also the place of their establishing. As Wittgenstein (1980) stated: "Teach it to us and you established it" (p. 381).

But immediately, the paradox that Marx posed in his third thesis on Feuerbach appears: *Who will teach the educators?* Although Marx never really answered his question, Morin (1999) proposed a number of paths including that of "providing a culture that allows organizing knowledge" (p. 118). This path is promising because in fact, it allows the incorporation of the question of knowledge and its transition in the domain of educational policy and more largely in the culture: the set of ways of reacting, thinking or doing, proper to nations and communities. It is linked with considering this question in the set of strongly diverse dimensions: historical, epistemological, political, etc. These dimensions determine, but not mechanically, what pupils learn and the ways that they learn it.

In fact, although mathematics can be considered as universal, the kinds of mathematical experience pupils gain, are diverse, set in different contexts and periods, influenced by educational style (Sarrazy, 2002; Sarrazy & Novotná, 2005). Although it is possible to include questions related to mathematics education to broader discussion on education and educational policy, we can also study the specific modalities of contribution of research in social sciences and more particularly of didactics of mathematics to mathematics education. That will be our focus in the following discussion, which is a follow-up to the previous parts of the chapter. It provides a more general, more philosophical reflection on mathematics education, mathematics teachers and the education of mathematics educators. The ideas presented show the variety of possible approaches and sources. The discussion is based on the notion of didactical situation as that was introduced by Guy Brousseau in the Theory of Didactical Situations in mathematics (Brousseau, 1997).

A Necessary But Not Exclusive Specificity

From the end of the 1960s the theory of didactical situations (Brousseau, 1997) asked for mathematics education and the sciences of education to be seen in a new way. Didactical problems needed to be specific for the considered domain of education. Learning mathematics has no relationship with, for example, learning to cook or learning to play football! We will not focus on that aspect, which is largely consensual today. But if in their practice, mathematics educators (in the large sense) have no room for manoeuvre for mathematics, this room considerably increases if they examine the situations for communicating them.

This first aspect will be quickly illustrated by an anecdote. Two doctoral students were assistants in a big school in Rome; both of them were good mathematicians. The first was a perfectionist and for his lessons he always chose problems whose success was delicate and strongly clear for his pupils. The second was disordered and had no so clear and explicit vision of what he wanted his pupils to develop; he taught something because he found it interesting and useful. Despite that, the examination results of the second were regularly much better than these of the first one. A possible explanation could be that the perfect organization of the first one's teaching from the perspective of mathematics did not leave any space for interrogation with his pupils, whereas pupils of the second had to find for themselves relationships between diverse problems that looked to be entirely independent.

Fully finished mathematics (rules, algorithms, theorems etc.) might be thought of as dead mathematics. A big part of the work of teachers consists in creating specific conditions of their "resurrection" for pupils. For doing it, they do not have any other choice than to create situations enabling them to show their pupils the use, interest, and meaning of mathematics. The reason is that the concept of situations, their managing, their organization, their evaluation, their regulation, etc. have fundamentally one specific dimension. They are of an immense complexity, taking into account their multiple determinations, conscious or not, that lay stress on the structures, declared or effective functions and the dynamics of these situations: observations, evaluations, regulations. These determinants are situated at various levels of organization according to excessively complex modes of relations—political, epistemological, pedagogical, scientific, etc.—that create an ideological framework that is relatively influential in its effects. It is very difficult, if not impossible, not only to build hierarchies of the forms of determination but also to evaluate their pertinence and their course of action for mathematics education. The reason is that the theory of situations allowed isolating (in the sense of Stengers, 1995) the didactical dimension of pedagogical, social, psychological, anthropological etc. aspects; it allowed making efforts and having success in modelling properties and conditions, specific for mathematics, of pupils' interactions with the environment and thus contributing to the emergence of the didactics of mathematics. We believe that one of the conditions of mathematics education development is certainly the identification of its non-limiting specificity; this specificity is proper to the epistemology of mathematics but narrowly linked with anthropological dimensions that are not specific for mathematics but nevertheless necessary for understanding social (economical, statistical etc.) use of mathematics.

Education and Mathematical Education

It is banal to say that mathematical education does not focus merely on creating mathematicians or on communicating mathematics that is useful for social and domestic life. It is less banal to say, as many mathematicians—Bertrand Russell, for example—have said, that mathematics contributes to the creation of citizenship in its way of being in the world and of taking it into consideration.

Besides, it is sufficient to compare, in the diachrony and synchrony, forms of teaching, curricula, the roles of mathematicians in the social and school selection, for taking into account the extreme diversity of the conception of mathematics education. An equally important diversity can be found in the conceptions of mathematics by mathematicians themselves. It is not to be accepted that mathematics education could be placed under the control of one discipline or trend only. Specialists of the discipline and of its education enrich democratic discussions about the social, school and more largely political uses of mathematics from the perspective of their science. In the same way, one could imagine that researchers could clarify political decisions by their capacity for anticipating the consequences of certain political measures to the conditions of their dissemination. Unfortunately, we can confirm without much risk that the legitimate care for rationalization, efficiency and equity of education leads to the exponential development of evaluation; moreover, individualism as it appeared in the 1980s and the 1990s, together with liberalism had more impact on the ways of disseminating mathematics and mathematical culture than results of research in mathematics education accumulated during the last 40 years.

For example, Nichols and Berliner (2005) have clearly demonstrated the serious impact of evaluation policy on all the levels of the educational system. In the USA, the *No Child Left Behind* legislation envisages sanctions against teachers and institutions that do not reach the level required on mandatory high-stake tests. This policy has had serious consequences:

1. The growth of discrimination by the closing of schools in the poorest environments.
2. Teachers being forced to operate in untenable pedagogical and social environments.
3. The weakest pupils becoming frustrated, which can result in their exclusion.
4. The important development of corruption within social relationships (e.g., result fiddling).

Over two decades ago, Brousseau (1989) explained how, in such situations:

> Teachers are led to leave the objectives of high taxonomical levels for the benefit of objectives of a low level: learning algorithms and isolated facts. Each of these measures grows the teaching/learning time and presents cumulative difficulties: metadidactical shifts, repetitions and individualization swallow the collective educative time, fragmentation of knowledge cuts the comprehension and the field of its utilization, etc. This degrading form of lessons was developed since the trivialization of tests, first for the tools of information and soon as the tool for the management of educational policy. In this system, the measures of failure are a priori denounced as unsupportable and designated responsible are pupils and particularly teachers. Against all reasons, present methods are disapproved, opposed to others that are said to be forgotten, and declared better against any proof, but only for justifying the accusation of general incompetence. (Quoted in Sarrazy, 2009, p. 13)

Didactical Culture and Social Anticipation

Should education result in a "full head" or a "head well done"? Should we look for a good mastery of algorithms or allow pupils to be creative and use algorithms in new situations? This recurrent and often counterproductive debate not only has scientific overtones, it is also political because it poses questions about the type of men and women who are to be formed. If these two intentions appear together, they appear in a paradoxical relationship. In fact, the more pupils are sure of the efficiency of an algorithm, the less they authorize themselves to invent other uses than those they met originally. Like a disciple to whom a teacher shows the moon, they see the finger.

This is the place of mathematics education, between the academic dimension of knowledge and mathematical activity. The theory of didactical situations is born from the theorization and scientific study of conditions that allow exceeding this paradox. Although its recognition among the scientific community is manifest, its dissemination and use in teacher education remain strongly limited. Should we regret it?

What are the consequences for teacher and mathematics educators' education? Teacher education appears as an important lever enabling teachers to step out from the discussion between the "full head" and the "head well done." We think that it would be desirable to expand teachers' didactical culture significantly but we would make a mistake if we push them to expel their pedagogical ideas. It would be a serious mistake because teachers, as well as pupils, need a certainty and illusion at the same time. Researchers in didactics of mathematics, whose agreement on ideas is far from being unified, contribute to clarifying conditions enabling the creation of knowledge that is new for the pupil (that does not depend on the pupil but on the culture). Pedagogues are responsible for fostering such conditions under which pupils have a chance of active participation in the adventure that nobody else can experience for them, the adventure of reinventing the world by her or his activity. Pupils can hardly be expected to produce anything new unless they have had some direct experience of this process. Fostering discussions on the definition of educational policies, of clarification of the possible consequences of certain political decisions would be of much benefit for research in mathematics education in general and for teacher education.

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