Chapter 9
Triangulated Graphs

9.1 Introduction

Triangulated graphs form an important class of graphs. They are a subclass of the class of perfect graphs and contain the class of interval graphs. They possess a wide range of applications. We describe later in this chapter an application of interval graphs in phasing the traffic lights at a road junction.

We begin with the definition of perfect graphs.

9.2 Perfect Graphs

For a simple graph G, we have the following parameters:

$\chi(G)$: The chromatic number of G
$\omega(G)$: The clique number of G (= the order of a maximum clique of G)
$\alpha(G)$: The independence number of G
$\theta(G)$: The clique covering number of G (= the minimum number of cliques of G that cover the vertex set of G).

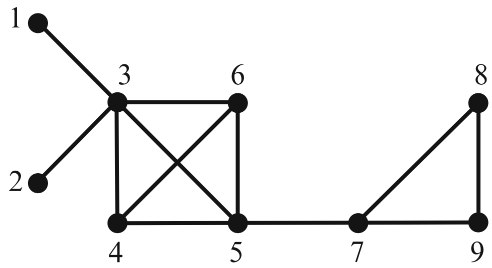
For instance, for the graph G of Fig. 9.1, $\chi(G) = \omega(G) = 4$ and $\alpha(G) = \theta(G) = 4$.

A minimum set of cliques that covers $V(G)$ is $\{\{1\}, \{2\}, \{3, 4, 5, 6\}, \{7, 8, 9\}\}$. In any proper vertex coloring of G, the vertices of any clique must receive distinct colors. Hence it is clear that $\chi(G) \geq \omega(G)$. Further, if A is any independent set of G, any clique of a clique cover of G can contain at most one vertex of A. Hence, to cover the $\alpha(G)$ vertices of a maximum independent set of G, at least $\alpha(G)$ distinct cliques of G are needed. Therefore, $\theta(G) \geq \alpha(G)$.

If G is an odd cycle $C_{2n+1}, n \geq 2$, $\chi(G) = 3, \omega(G) = 2, \theta(G) = n + 1$, and $\alpha(G) = n$. Hence, for such a $G, \chi(G) > \omega(G)$, and $\theta(G) > \alpha(G)$. Moreover,

R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory,*
Universitext, DOI 10.1007/978-1-4614-4529-6_9,

Fig. 9.1 Graph G



$A \subset V(G)$ is an independent set of vertices of G if and only if A induces a clique in G^c. Therefore, for any simple graph G,

$$\chi(G) = \theta(G^c), \text{ and}$$
$$\alpha(G) = \omega(G^c). \tag{9.1}$$

Definition 9.2.1. Let G be a simple graph. Then

(i) G is χ*-perfect* if and only if for every $A \subseteq V(G)$, $\chi(G[A]) = \omega(G[A])$.
(ii) G is α*-perfect* if and only if for every $A \subseteq V(G)$, $\alpha(G[A]) = \theta(G[A])$.

Remark 9.2.2. 1. By (9.1) above, it is clear that a graph is χ-perfect if and only if its complement is α-perfect.
2. Berge [19] conjectured that the concepts of χ-perfectness and α-perfectness are equivalent for any simple graph. This was shown to be true by Lovász [134] (and independently by Fulkerson [69]). This result is often referred to in the literature as the *perfect graph theorem*.

Theorem 9.2.3 (Perfect graph theorem). *For a simple graph G, the following statements are equivalent:*

(i) G is χ-perfect.
(ii) G is α-perfect.
(iii) $\alpha(G[A])\,\omega(G[A]) \geq |A|$ for every $A \subseteq V(G)$.

In view of the perfect graph theorem, there is no need to distinguish between α-perfectness and χ-perfectness; hence, graphs that satisfy any one of these three equivalent conditions can be referred to as merely *perfect graphs*. In particular, this means that a simple graph G is perfect if and only if its complement is perfect. For a proof of the perfect graph theorem, see [76] or [134].

Remark 9.2.4. If G is perfect, by what is mentioned above, G cannot contain an odd hole, that is, an induced odd cycle C_{2n+1}, $n \geq 2$; likewise, by (9.1), G cannot contain an odd antihole, that is, an induced C^c_{2n+1}, $n \geq 2$. Equivalently, if G is perfect, then G can contain neither $C_{2n+1}, n \geq 2$ nor its complement as an induced subgraph. The converse of this result is the celebrated "*strong perfect graph conjecture*" of Berge, settled affirmatively by Chudnovsky et al. [36] (see notes at the end of this chapter).

9.3 Triangulated Graphs

Definition 9.3.1. A simple graph G is called *triangulated* if every cycle of length at least four in G has a chord, that is, an edge joining two nonconsecutive vertices of the cycle (see Fig. 9.2). For this reason, triangulated graphs are also called *chordal graphs* and sometimes *rigid circuit graphs.*

A graph is *weakly triangulated* if it contains neither a chordless cycle of length at least 5 nor the complement of such a cycle as an induced subgraph. Note that any triangulated graph is weakly triangulated.

Remark 9.3.2. It is clear that the property of a graph being triangulated is hereditary; that is, if G is triangulated, then every induced subgraph of G is also triangulated.

Definition 9.3.3. A vertex v of a graph G is a *simplicial vertex* of G if the closed neighborhood $N_G[v]$ of v in G induces a clique in G.

Example 9.3.4. In Fig. 9.2a, the vertices u_1, u_2, u_3, and u_4 are simplicial, whereas v_1, v_2, v_3, and v_4 are not.

Triangulated graphs can be recognized by the presence of a perfect vertex elimination scheme.

Definition 9.3.5. A *perfect vertex elimination scheme* (or, briefly, a *perfect scheme*) of a graph G is an ordering $\{v_1, v_2, \ldots, v_n\}$ of the vertex set of G in such a way that, for $1 \leq i \leq n$, v_i is a simplicial vertex of the subgraph induced by $\{v_i, v_{i+1}, \ldots, v_n\}$ of G.

Example 9.3.6. For the graph of Fig. 9.2a, $\{u_1, u_2, u_3, u_4, v_4, v_2, v_1, v_3\}$ is a perfect scheme.

Remark 9.3.7. Any vertex of degree 1 is trivially simplicial. Hence, any tree has a perfect vertex elimination scheme. Also, any tree is trivially triangulated. It turns out that these facts can be generalized to assert that any triangulated graph has

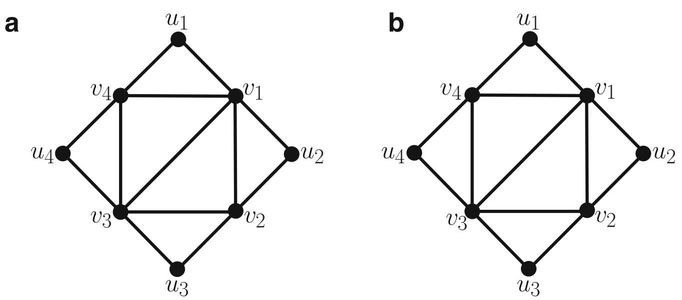


Fig. 9.2 (**a**) Triangulated and (**b**) nontriangulated graphs

a perfect vertex elimination scheme. (Based on this, Fulkerson and Gross [70] gave a "good algorithm" to test for triangulated graphs, namely, repeatedly locate a simplicial vertex and remove it from the graph until there is left out a single vertex and the graph is triangulated, or else at some stage no simplicial vertex exists and the graph is not triangulated.) Before we establish the above result, we need another characterization of triangulated graphs. This result is due to Hajnal and Surányi [89] and also due to Dirac [56].

Lemma 9.3.8. *A graph G is triangulated if and only if every minimal vertex cut of G is a clique.*

Proof. Assume that G is triangulated and that S is a minimal vertex cut of G. Let a and b be vertices in distinct components, say G_A and G_B, respectively, of $G\backslash S$. Now every vertex x of S must be adjacent to some vertex of G_A, since if x is adjacent to no vertex of G_A, then $G\backslash(S\backslash x)$ is disconnected and this would contradict the minimality of S. Similarly, x is adjacent to some vertex of G_B. Hence for any pair $x, y \in S$, there exist paths $P_1 : xa_1 \ldots a_r y$ and $P_2 : xb_1 \ldots b_s y$, with each $a_i \in G_A$ and each $b_j \in G_B$. Let us assume further that the a_i's and b_j's have been so chosen that these x-y paths are of least length. Then $xa_1 \ldots a_r y b_s b_{s-1} \ldots b_1 x$ is a cycle whose length is at least 4, and so it must have a chord. But such a chord cannot be of the form $a_i a_j$ or $b_k b_\ell$ in view of the minimality of the length of P_1 and P_2. Nor can it be $a_i b_j$ for some i and j, as a_i and b_j belong to a distinct component of $G\backslash S$. Hence, it can be only xy. Thus, every pair x, y in S is adjacent, and S is a clique.

Conversely, assume that every minimal vertex cut of G is a clique. Let $axby_1y_2 \ldots y_r a$ be a cycle C of length ≥ 4 in G. If ab were not a chord of C, denote by S a minimal vertex cut that puts a and b in distinct components of $G\backslash S$. Then S must contain x and y_j for some j. By hypothesis, S is a clique, and hence $xy_j \in E(G)$, and xy_j is a chord of C. Thus, G is triangulated. □

Lemma 9.3.9. *Every triangulated graph G has a simplicial vertex. Moreover, if G is not complete, it has two nonadjacent simplicial vertices.*

Proof. The lemma is trivial either if G is complete or if G has just two or three vertices. Assume therefore that G is not complete, so that G has two nonadjacent vertices a and b. Let the result be true for all graphs with fewer vertices than G. Let S be a minimal vertex cut separating a and b, and let G_A and G_B be components of $G\backslash S$ containing a and b, respectively, and with vertex sets A and B, respectively. By the induction hypothesis, it follows that if $G[A \cup S]$ is not complete, it has two nonadjacent simplicial vertices. In this case, since $G[S]$ is complete (refer to Lemma 9.3.8), at least one of the two simplicial vertices must be in A. Such a vertex is then a simplicial vertex of G because none of its neighbors is in any other component of $G\backslash S$. Further, if $G[A \cup S]$ is complete, then any vertex of A is a simplicial vertex of G. In any case, we have a simplicial vertex of G in A. Similarly, we have a simplicial vertex in B. These two vertices are then nonadjacent simplicial vertices of G. □

We are now ready to prove the second characterization theorem of triangulated graphs.

Theorem 9.3.10. *A graph G is triangulated if and only if it has a perfect vertex elimination scheme.*

Proof. The result is obvious for graphs with at most three vertices. So assume that G is a triangulated graph with at least four vertices. Assume that every triangulated graph with fewer vertices than G has a perfect vertex elimination scheme. By Lemma 9.3.9, G has a simplicial vertex v. Then $G \backslash v$ has a perfect vertex elimination scheme. Then v followed by a perfect scheme of $G \backslash v$ gives a perfect scheme of G.

Conversely, assume that G has a perfect scheme, say $\{v_1, v_2, \ldots, v_n\}$. Let C be a cycle of length ≥ 4 in G. Let j be the first suffix with $v_j \in V(C)$. Then $V(C) \subseteq G[\{v_j, v_{j+1}, \ldots, v_n\}]$ and, since v_j is simplicial in $G[\{v_j, v_{j+1}, \ldots, v_n\}]$, the neighbors of v_j in C form a clique in G, and hence C has a chord. Thus, G is triangulated. □

Theorem 9.3.11. *A triangulated graph is perfect.*

Proof. The result is clearly true for triangulated graphs of order at most 4. So assume that G is a triangulated graph of order at least 5. We apply induction. Assume that the theorem is true for all graphs having fewer vertices than G. If G is disconnected, we can consider each component of G individually. So assume that G is connected. By Lemma 9.3.9, G contains a simplicial vertex v. Let u be a vertex adjacent to v in G. Since v is simplicial in G (and so in $G-u$), $\theta(G-u) = \theta(G)$. By the induction hypothesis, $G - u$ is triangulated and therefore perfect and therefore $\theta(G-u) = \alpha(G-u)$. Hence (see Exercise 7.4), $\theta(G) = \theta(G-u) = \alpha(G-u) \leq \alpha(G)$. This together with the fact that $\theta(G) \geq \alpha(G)$ implies that $\theta(G) = \alpha(G)$. The proof is complete since by the induction assumption, for any proper subset A of $V(G)$, the subgraph $G[A]$ is triangulated and therefore perfect. □

9.4 Interval Graphs

One of the special classes of triangulated graphs is the class of interval graphs.

Definition 9.4.1. An *interval graph* G is the intersection graph of a family of intervals of the real line. This means that for each vertex v of G, there corresponds an interval $J(v)$ of the real line such that $uv \in E(G)$ if and only if $J(u) \cap J(v) \neq \emptyset$.

Figure 9.3 displays a graph G and its interval representation.

Remark 9.4.2. 1. Interval graphs occur in a natural manner in various applications. In genetics, the Benzer model [18] deals with the conditions under which two subsets of the fine structure inside a gene overlap. In fact, one can tell when they overlap on the basis of mutation data. Is this overlap information consistent with

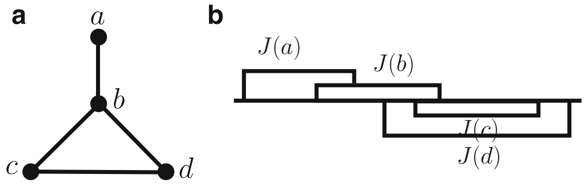


Fig. 9.3 (**a**) Graph G; (**b**) its interval representation

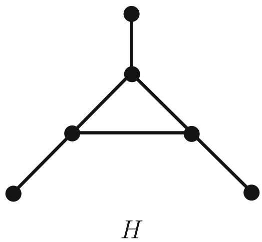


Fig. 9.4 Example of a graph that is not an interval graph

the hypothesis that the fine structure inside the gene is linear? The answer is "yes" if the graph defined by the overlap information is an interval graph.

2. It is clear that the intervals may be taken as either open or closed.
3. The cycle C_4 is not an interval graph. In fact, if $V(C_4) = \{a, b, c, d\}$ and if ab, bc, cd, and da are the edges of C_4, then $J(a) \cap J(b) \neq \emptyset$, $J(b) \cap J(c) \neq \emptyset$, $J(c) \cap J(d) \neq \emptyset$, and $J(d) \cap J(a) \neq \emptyset$ imply that either $J(a) \cap J(c) \neq \emptyset$ or $J(b) \cap J(d) \neq \emptyset$ [i.e., $ac \in E(G)$ or $bd \in E(G)$], which is not the case. Hence, an interval graph cannot contain C_4 as an induced subgraph. For a similar reason, it can be checked that the graph H of Fig. 9.4 is not an interval graph.

Recall that an *orientation* of a graph G is an assignment of a direction to each edge of G. Hence, an orientation of G converts G into a directed graph. As mentioned in Chap. 2, an orientation is *transitive* if, when (a, b) and (b, c) are arcs in the orientation, then (a, c) is also an arc in the orientation.

Lemma 9.4.3. *If G is an interval graph, G^c has a transitive orientation.*

Proof. Let $J(a)$ denote the interval that represents the vertex a of the interval graph G. Let $ab \in E(G^c)$ and $bc \in E(G^c)$ so that $ab \notin E(G)$ and $bc \notin E(G)$. Hence, $J(a) \cap J(b) = \emptyset$, and $J(b) \cap J(c) = \emptyset$. Now, introduce an orientation for the edges of G^c by orienting an edge xy of G^c from x to y if and only if $J(x)$ lies to the left of $J(y)$. Then $J(a)$ lies to the left of $J(b)$ and $J(b)$ lies to the left of $J(c)$, and therefore $J(a)$ lies to the left of $J(c)$. Hence, whenever (a, b) and (b, c) are arcs in the defined orientation, arc (a, c) also belongs to this orientation. Thus, G^c has a transitive orientation. □

Gilmore and Hoffman [73] have shown that the above two properties (Remark 3 of 9.4.2 and Lemma 9.4.3) characterize interval graphs.

Theorem 9.4.4. *A graph G is an interval graph if and only if G does not contain C_4 as an induced subgraph and G^c admits a transitive orientation.*

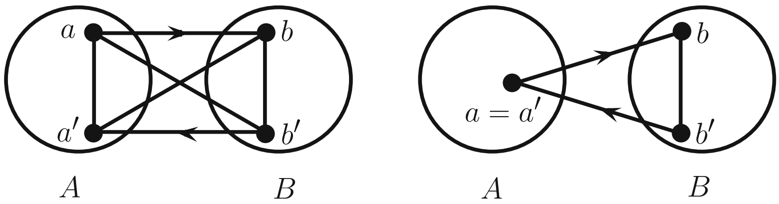


Fig. 9.5 Graph for proof of first condition of Theorem 9.4.4

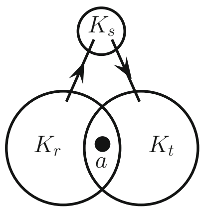


Fig. 9.6 Ordering of maximal cliques

Proof. We have just seen the necessity of these two conditions. We now prove their sufficiency. Assume that G has no induced C_4 and that G^c has a transitive orientation. We look at the set of maximal cliques of G and introduce a linear ordering on it. If A and B are two distinct maximal cliques of G, then for any $a \in A$, there exists $b \in B$ with $ab \notin E(G)$ and therefore $ab \in E(G^c)$. (Otherwise, $G[A \cup B]$ would be a clique of G properly containing both A and B, a contradiction, since A and B are maximal cliques in G.) If ab has the orientation from a to b in the transitive orientation of G^c, we set $A < B$. This ordering is well defined in that if $a' \in A$ and $b' \in B$ with $a'b' \in E(G^c)$, then $a'b'$ must be oriented from a' to b' in G^c (see Fig. 9.5).

To see this, first assume that $a \neq a'$ and $b \neq b'$ and that edge $a'b'$ is oriented from b' to a' in G^c. Then at least one of the edges ab' and $a'b$ must be an edge of G^c. Otherwise, the edges aa', $a'b$, bb', and $b'a$ induce a C_4 in G, a contradiction. Suppose then that $a'b \in E(G^c)$. Then if $a'b$ is oriented from a' to b in G^c, by the transitivity of the orientation in G^c, $b'b \in E(G^c)$, a contradiction. A similar argument applies when ba', ab', or $b'a$ is an oriented arc of G^c. The cases when $a = a'$ or $b = b'$ can also be treated similarly. Thus, if one arc of G^c goes from A to B, then all the arcs between A and B go from A to B in G^c. In this case, we set $A < B$. Since the number of maximal cliques of G is finite, and any two maximal cliques can be ordered by "$<$," we obtain a linear ordering of the set of maximal cliques of G, say, $K_1 < K_2 < \ldots < K_p$.

We now claim that if a vertex a of G belongs to K_r and K_t, where $K_r < K_t$, then it also belongs to K_s, where $K_r < K_s < K_t$ (see Fig. 9.6).

Suppose $a \notin K_s$. First note that there exists some vertex b in K_s such that b is nonadjacent to a. If not, $K_s \vee \{a\}$ would be a clique properly containing K_s, a contradiction. But then, since $K_r < K_s$, the edge ab of G^c must be oriented from

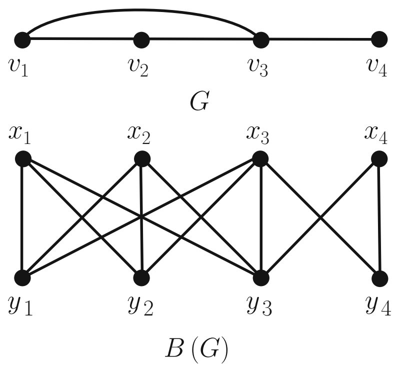


Fig. 9.7 Bipartite graph $B(G)$ of G

a to b. But $a \in K_t$, and this means that $K_t < K_s$, a contradiction. Thus, $a \in K_s$ as well.

In $\{1, 2, \ldots, p\}$, let i be the smallest and j be the greatest numbers such that $a \in K_i$ and $a \in K_j$. We now define the interval $J(a) =$ the closed interval $[i, j]$. Then $J(a) \cap J(b) \neq \emptyset$ if and only if there exists a positive integer k such that $k \in J(a) \cap J(b)$. But this can happen if and only if both a and b are in K_k [i.e., if and only if $ab \in E(G)$]. Thus, G is an interval graph. $\square$

9.5 Bipartite Graph $B(G)$ of a Graph G

Given a graph G, we define the associated bipartite graph $B(G)$ as follows: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Corresponding to $V(G)$, take disjoint sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ and form the bipartite graph $B(G)$ by taking X and Y as sets of the bipartition of the vertex set of $B(G)$. Adjacency in $B(G)$ is defined by setting $x_i y_i \in E(B(G))$ for every i, $1 \leq i \leq n$, and for $i \neq j$, x_i is adjacent to y_j in $B(G)$ if and only if $v_i v_j \in E(G)$ (Fig. 9.7).

Our next theorem relates the chordal nature of a graph G with that of the bipartite graph $B(G)$. Since a bipartite graph has no odd cycles and a 4-cycle of a bipartite graph cannot have a chord, a *bipartite graph* is defined to be *chordal* if each of its cycles of length at least 6 has a chord.

Theorem 9.5.1. *If the bipartite graph $B(G)$ formed out of G is chordal, then G is chordal.*

Proof. Let $C = v_1 v_2 \ldots v_p v_1$ be any cycle of G of length $p \geq 4$. If p is odd, take C' to be the cycle $x_1 y_2 x_3 y_4 \ldots x_p y_p x_1$, while if p is even, take C' to be the cycle $x_1 y_2 x_3 y_4 \ldots x_{p-1} y_p x_p y_1 x_1$ in $B(G)$. As $B(G)$ is chordal and C' is of length at least 6, C' has a chord in $B(G)$. Such a chord can only be of the form $x_i y_j$, where $|i - j| \geq 2$. This means that $v_i v_j$ is a chord of C. Thus, G is chordal. $\square$

9.6 Circular Arc Graphs

Circular arc graphs are similar to interval graphs except that the $J(a)$'s are now taken to be arcs of a particular circle. Consider an interval graph G. Since the number of intervals $J(a)$, $a \in V(G)$, is finite, there are real numbers m and M such that $J(a) \subseteq (m, M)$ for every $a \in V(G)$. Consequently, identification of m and M (i.e., conversion of the closed interval $[m, M]$ into a circle by the identification of m and M) makes G a circular arc graph. Thus, every interval graph is a circular arc graph. Clearly, the converse is not true. However, if there exists a point p on the circle that does not belong to any arc $J(a)$, then the circle can be cut at p and the circular arc graph can be made into an interval graph.

9.7 Exercises

7.1 If e is an edge of a cycle of a triangulated graph G, show that e belongs to a triangle of G.

7.2 What are the simplicial vertices of the triangulated graph of Fig. 9.2a?

7.3 Give a perfect elimination scheme for the triangulated graph of Fig. 9.2a.

7.4 If v is a simplicial vertex of a triangulated graph G, and $vu \in E(G)$, prove that $\theta(G - u) = \theta(G)$.

7.5 Let $t(G)$ denote the smallest positive integer k such that G^k is triangulated. Determine $t(C_n)$, $n \geq 4$.

7.6 Prove G and G^c are triangulated if and only if G does not contain C_4, C_4^c, or C_5 as an induced subgraph. Hence, or otherwise, show that C_n^c, $n \geq 5$ is not triangulated.

7.7 Prove that $L(G)$ is triangulated if and only if every block of G is either K_2 or K_3. Hence, show that the line graph of a tree is triangulated.

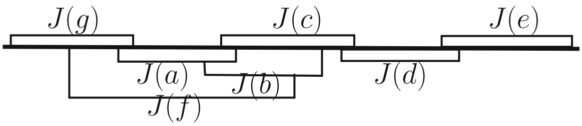
7.8 Let $K(G)$ and $L(G)$ denote, respectively, the clique graph and the line graph of a graph G. [$K(G)$ is defined as the intersection graph of the family of maximal cliques of G; i.e., the vertices of $K(G)$ are the maximal cliques of G, and two vertices of $K(G)$ are adjacent in $K(G)$ if and only if the corresponding maximal cliques of G have a nonempty intersection.] Then prove or disprove

   (i) G is triangulated $\Rightarrow$ $K(G)$ is triangulated
   (ii) $K(G)$ is triangulated $\Rightarrow$ G is triangulated
   (iii) $L(G)$ is triangulated $\Rightarrow$ G is triangulated
   (iv) G is triangulated $\Rightarrow$ $L(G)$ is triangulated

7.9 Show by means of an example that an even power of a triangulated graph need not be triangulated.

7.10 Prove the following by means of a counterexample: G is chordal need not imply that $B(G)$ is chordal.

7.11 Draw the interval graph of the family of intervals below and display a transitive orientation for G^c.



7.12 If G is cubic and if G does not contain an odd cycle of length at least 5 as an induced subgraph, prove that G is perfect. (Hint: Use Brooks' theorem.)
7.13 Show that every bipartite graph is perfect.
7.14 For a bipartite graph G, prove that $\chi(G^c) = \omega(G^c)$.
7.15 Give an example of a triangulated graph that is not an interval graph.
7.16 Give an example of a perfect graph that is not triangulated.
7.17 Show that a 2-connected triangulated graph with at least four vertices is locally connected. Hence, show that a 2-connected triangulated $K_{1,3}$-free graph is Hamiltonian. (See reference [149].)
7.18 Show by means of an example that a 2-connected triangulated graph need not be Hamiltonian.
7.19 Show that the line graph of a 2-edge-connected triangulated graph is Hamiltonian.
7.20 Give an example of a circular arc graph that is not an interval graph.
7.21 * Show that a graph G is perfect if and only if every induced subgraph G' of G contains an independent set that meets all the maximum cliques of G'.
7.22 Let $\{v_1, v_2, \dots, v_n\}$ be a simplicial ordering of the vertices of a chordal graph G. Let

$$d_i = \deg(v_i) \text{ in the subgraph } \langle v_i, v_{i+1}, \dots, v_n \rangle \text{ of } G.$$

Prove that the chromatic polynomial of G is given by $\prod_{i=1}^{n}(t - d_i)$. Hence show that $\chi(G) = \max_{1 \le i \le n}\{1 + d_i\}$. (This shows that the roots of the chromatic polynomial of a chordal graph are nonnegative integers.)

9.8 Phasing of Traffic Lights at a Road Junction

We present an application of interval graphs to the problem of phasing of traffic lights at a road junction. The problem is to install traffic lights at a road junction in such a way that traffic flows smoothly and efficiently at the junction.

We take a specific example and explain how our problem could be tackled. Figure 9.8 displays the various traffic streams, namely, a, b, $\dots$, g, that meet at the Main Guard Gate road junction at Tiruchirappalli, Tamil Nadu (India).

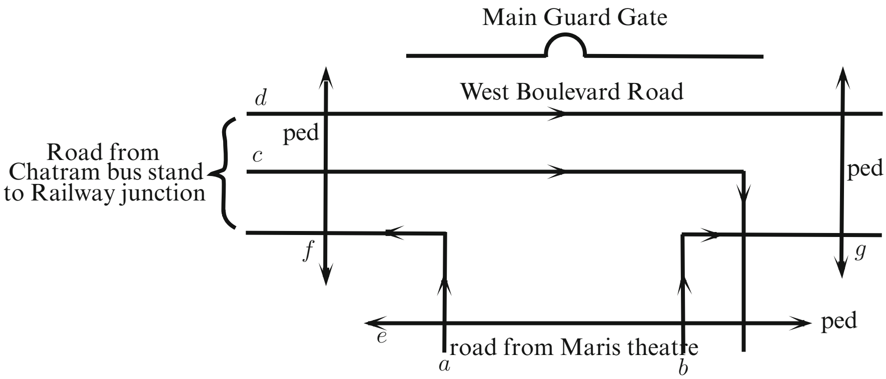
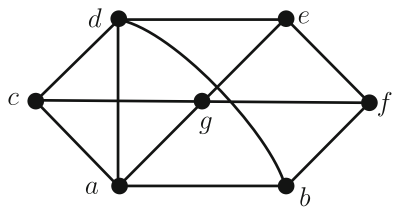


Fig. 9.8 Traffic streams at a road junction (ped = pedestrian crossing)

Fig. 9.9 Compatibility graph of Fig. 9.8



Certain traffic streams may be termed "compatible" if their simultaneous flow would not result in any accidents. For instance, in Fig. 9.8, streams a and d are compatible, whereas b and g are not. The phasing of lights should be such that when the green lights are on for two streams, they should be compatible. We suppose that the total time for the completion of green and red lights during one cycle is two minutes.

We form a graph G whose vertex set consists of the traffic streams in question, and we make two vertices of G adjacent if and only if the corresponding streams are compatible. This graph is the compatibility graph corresponding to the problem in question. The compatibility graph of Fig. 9.8 is shown in Fig. 9.9.

We take a circle and assume that its perimeter corresponds to the total cycle period, namely, 120 seconds. We may think that the duration when a given traffic stream gets green light corresponds to an arc of this circle. Hence, two such arcs of the circle can overlap only if the corresponding streams are compatible. The resulting circular arc graph may not be the compatibility graph because we do not demand that two arcs intersect whenever they correspond to compatible flows. (There may be two compatible streams, but they need not get green light at the same time.) However, the intersection graph H of this circular arc graph will be a spanning subgraph of the compatibility graph.

The efficiency of our phasing may be measured by minimizing the total red light time during a traffic cycle, that is, the total waiting time for all the traffic streams

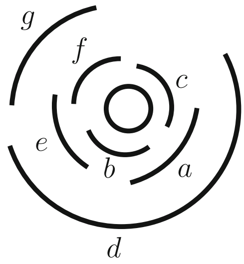


Fig. 9.10 A green light assignment

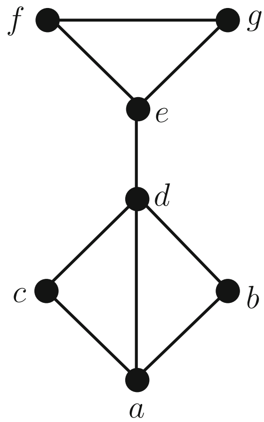


Fig. 9.11 Intersection graph for Fig. 9.10

during a cycle. For the sake of concreteness, we may assume that at the time of starting, all lights are red. This would ensure that H is an interval graph (see the last sentence of Sect. 9.5 on circular arc graphs).

Figure 9.10 gives a feasible green light assignment whose corresponding intersection graph H is given in Fig. 9.11. The maximal cliques of H are $K_1 = \{a, b, d\}$, $K_2 = \{a, c, d\}$, $K_3 = \{d, e\}$, and $K_4 = \{e, f, g\}$. Since H is an interval graph, by Theorem 9.4.4, H^c has a transitive orientation. A transitive orientation of H^c is given in Fig. 9.12.

Since (b, c), (c, e), and (d, f) are arcs of H^c, and since $b \in K_1, c \in K_2$, $d, e \in K_3$, and $f \in K_4$, etc., we have

$$K_1 < K_2 < K_3 < K_4$$

in the consecutive ordering of the maximal cliques of H. Each clique K_i, $1 \le i \le 4$, corresponds to a phase during which all streams in that clique receive green lights. We then start a given traffic stream with green light during the first phase in which it appears, and we keep it green until the last phase in which it appears. Because of the consecutiveness of the ordering of the phases K_i, this gives an arc on the clock circle. In phase 1, traffic streams a, b, and d receive a green light; in phase 2, a, c, and d receive a green light, and so on.

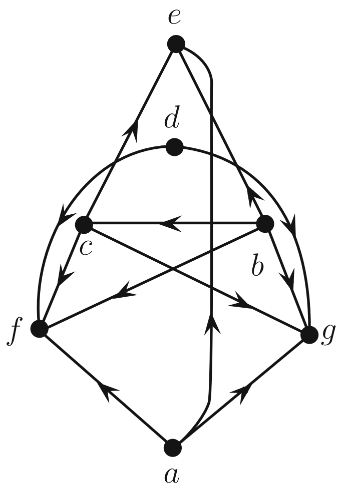


Fig. 9.12 Transitive orientation of H^c

Suppose we assign to each phase K_i a duration d_i. Our aim is to determine the d_i's (≥ 0) so that the total waiting time is minimum. Further, we may assume that the minimum green light time for any stream is 20 seconds. Traffic stream a gets a red light when the phases K_3 and K_4 receive a green light. Hence, a's total red light time is $d_3 + d_4$. Similarly, the total red light times of traffic streams b, c, d, e, f, and g, respectively, are $d_2 + d_3 + d_4$; $d_1 + d_3 + d_4$; d_4; $d_1 + d_2$; $d_1 + d_2 + d_3$; and $d_1 + d_2 + d_3$. Therefore, the total red light time of all the streams in one cycle is $Z = 4d_1 + 4d_2 + 4d_3 + 3d_4$. Our aim is to minimize Z subject to $d_i \geq 0$; $1 \leq i \leq 4$, and $d_1 + d_2 \geq 20$; $d_1 \geq 20$, $d_2 \geq 20$, $d_1 + d_2 + d_3 \geq 20$, $d_3 + d_4 \geq 20$, $d_4 \geq 20$, $d_3 \geq 0$ and $d_1 + d_2 + d_3 + d_4 = 120$. (The condition $d_1 + d_2 \geq 20$ signifies that the green light time that stream a receives, namely, the sum of the green light times of phases K_1 and K_2, is at least 20. A similar reasoning applies to the other inequalities. The last condition gives the total cycle time.) An optimal solution to this problem is $d_1 = 80$, $d_2 = 20$, $d_3 = 0$, and $d_4 = 20$ and min $Z = 480$ (in seconds). But this is not the end of our problem. There are other possible circular arc graphs. Figures 9.13a,b give another feasible green light arrangement and its corresponding intersection graph. With respect to this graph, min $Z = 500$ seconds. Thus, we have to exhaust all possible circular arc graphs and then take the least of all the minima thus obtained. The phasing that corresponds to this least value would then be the best phasing of the traffic lights. (For the above particular problem, this minimum value is 480 seconds.)

Notes

Exercise 7.9 shows that an even power of a triangulated graph need not be triangulated. However, an odd power of a triangulated graph is triangulated [11].

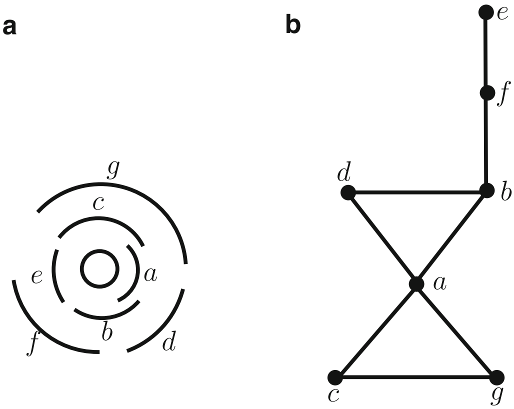
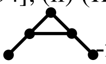


Fig. 9.13 (**a**) Another green light arrangement; (**b**) corresponding intersection graph

Moreover, if G^k is triangulated, then so is G^{k+2} [130], and consequently, if G and G^2 are triangulated, then so are all the powers of G.

A simple graph G is called *Berge* if it contains neither an odd cycle of length at least 5 nor its complement as an induced subgraph. The *strong perfect graph conjecture* asserted that a graph G is perfect if it is Berge. This conjecture was proposed by Claude Berge in 1960 and was settled affirmatively by Maria Chudnovsky, Neil Robertson, Paul Seymour, and Robin Thomas in 2002 [36]. The authors show that every Berge graph is in one of the four classes of perfect graphs—basic, 2-join, M-join, and balanced skew partition. Earlier the conjecture was proved to be true for several classes of graphs: (i) $K_{1,3}$-free graphs [154]; (ii) $(K_4 - e)$-free graphs [156]; (iii) K_4-free graphs [178]; (iv) bull-free, that is, -free graphs [40] (v) triangulated graphs (see Theorem 9.3.11); (vi) weakly triangulated graphs [102] and so on.

Perfect graphs were first discovered by Berge in 1958–1959. Their importance is both theoretical (because of their bearing on graph coloring problems) and practical (because of their applications to perfect communication channels, operations research, optimization of municipal services, etc.).

Four books that give a very good account of perfect graphs are references [19, 21, 20, 76]. In addition to the classes of perfect graphs mentioned above, there are also other known classes of perfect graphs, for instance, wing-triangulated graphs and, more generally, strict quasi-parity graphs. For details, see reference [107]. Our discussion on the phasing of traffic lights is based on Roberts [166], which also contains some other applications of perfect graphs.