Chapter 8
Planarity

8.1 Introduction

The study of planar and nonplanar graphs and, in particular, the several attempts to solve the *four-color conjecture* have contributed a great deal to the growth of graph theory. Actually, these efforts have been instrumental to the development of algebraic, topological, and computational techniques in graph theory.

In this chapter, we present some of the basic results on planar graphs. In particular, the two important characterization theorems for planar graphs, namely, Wagner's theorem (same as the Harary–Tutte theorem) and Kuratowski's theorem, are presented. Moreover, the nonhamiltonicity of the Tutte graph on 46 vertices (see Fig. 8.28 and also the front wrapper) is explained in detail.

8.2 Planar and Nonplanar Graphs

Definition 8.2.1. A graph G is *planar* if there exists a drawing of G in the plane in which no two edges intersect in a point other than a vertex of G, where each edge is a Jordan arc (that is, a simple arc). Such a drawing of a planar graph G is called a *plane representation* of G. In this case, we also say that G has been embedded in the plane. A *plane graph* is a planar graph that has already been embedded in the plane.

Example 8.2.2. There exist planar as well as nonplanar graphs. In Fig. 8.1, a planar graph and two of its plane representations are shown. Note that all trees are planar as also are cycles and wheels. The Petersen graph is nonplanar (a proof of this result is given later in this chapter.).

Before proceeding further, let us recall here the celebrated Jordan curve theorem. If J is any closed Jordan curve in the plane, the complement of J (with respect

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Universitext, DOI 10.1007/978-1-4614-4529-6_8,

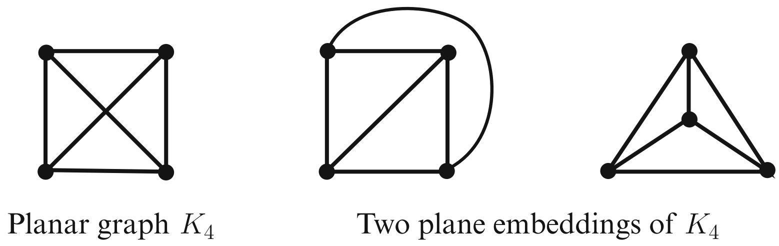
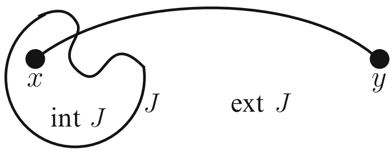


Fig. 8.1 A planar graph with two plane embeddings

Fig. 8.2 Arc connecting point x in int J with point y in ext J



to the plane) is partitioned into two disjoint open connected subsets of the plane, one of which is bounded and the other unbounded. The bounded subset is called the *interior* of J and is denoted by int J. The unbounded subset is called the *exterior* of J and is denoted by ext J. The *Jordan curve theorem* (of topology) states that if J is any closed Jordan curve in the plane, any arc joining a point of int J and a point of ext J must intersect J at some point (see Fig. 8.2) (the proof of this result, although intuitively obvious, is tedious).

Let G be a plane graph. Then the union of the edges (as Jordan arcs) of a cycle C of G form a closed Jordan curve, which we also denote by C. A plane graph G divides the rest of the plane (i.e., plane minus the edges and vertices of G), say π, into one or more faces, which we define below. We define an equivalence relation $\sim$ on π.

Definition 8.2.3. We say that for points A and B of π, $A \sim B$ if and only if there exists a Jordan arc from A to B in π. Clearly, $\sim$ is an equivalence relation on π. The equivalence classes of the above equivalence relation are called the *faces* of G.

Remark 8.2.4. 1. We claim that a connected graph is a tree if and only if it has only one face. Indeed, since there are no cycles in a tree T, the complement of a plane embedding of T in the plane is connected (in the above sense), and hence a tree has only one face. Conversely, it is clear that if a connected plane graph has only one face, then it must be a tree.

2. Any plane graph has exactly one unbounded face. The unbounded face is also referred to as the exterior face of the plane graph. All other faces, if any, are bounded. Figure 8.3 represents a plane graph with seven faces.

The distinction between bounded and unbounded faces of a plane graph is only superfluous, as there exists a plane representation G_1 of a plane graph G in which any specified face of G_1 becomes the unbounded face, as is shown below. (This of

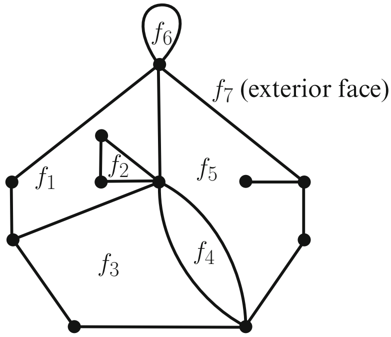


Fig. 8.3 A plane graph with seven faces

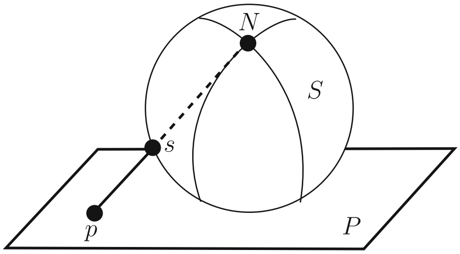


Fig. 8.4 Stereographic projection of the sphere S from N

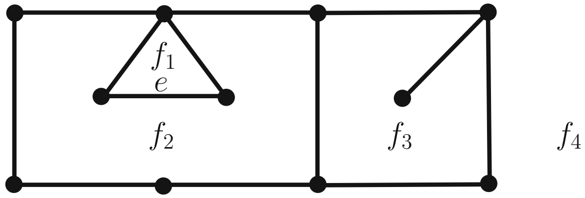
course means that there exists a plane representation of G such that any specified vertex or edge belongs to the unbounded face.) We consider embeddings of a graph on a sphere. A graph is *embeddable on a sphere* S if it can be drawn on the surface of S so that its edges intersect only at its vertices. Such a drawing, if it exists, is called an embedding of G on S. Embeddings on a sphere are called *spherical embeddings*. What we have given here is only a naive definition. For a more rigorous description of spherical embeddings, see [79].

To prove the next theorem, we need to recall the notion of stereographic projection. Let S be a sphere resting on a plane P so that P is a tangent plane to S. Let N be the "north pole," the point on the sphere diametrically opposite the point of contact of S and P. Let the straight line joining N and a point s of $S\backslash\{N\}$ meet P at p. Then the mapping $\eta : S\backslash\{N\} \to P$ defined by $\eta(s) = p$ is called the *stereographic projection* of S from N (see Fig. 8.4).

Theorem 8.2.5. *A graph is planar if and only if it is embeddable on a sphere.*

Proof. Let a graph G be embeddable on a sphere and let G' be a spherical embedding of G. The image of G' under the stereographic projection η of the sphere from a point N of the sphere not on G' is a plane representation of G on P. Conversely, if G'' is a plane embedding of G on a plane P, then the inverse of the stereographic projection of G'' on a sphere touching the plane P gives a spherical embedding of G. $\square$

Fig. 8.5 Plane graph with four faces



Theorem 8.2.6. *(a) Let G be a plane graph and f be a face of G. Then there exists a plane embedding of G in which f is the exterior face.*

(b) Let G be a planar graph. Then G can be embedded in the plane in such a way that any specified vertex (or edge) belongs to the unbounded face of the resulting plane graph.

Proof. (a) Let n be a point of int f. Let $G' = \sigma(G)$ be a spherical embedding of G and let $N = \sigma(n)$. Let η be the stereographic projection of the sphere with N as the north pole. Then the map $\eta\sigma$ (σ followed by η) gives a plane embedding of G that maps f onto the exterior face of the plane representation $(\eta\sigma)(G)$ of G.

(b) Let f be a face containing the specified vertex (respectively, edge) in a plane representation of G. Now, by part (a) of the theorem, there exists a plane embedding of G in which f becomes the exterior face. The specified vertex (respectively, edge) then becomes a vertex (respectively, edge) of the new unbounded face. □

Remark 8.2.7. 1. Let G be a connected plane graph. Each edge of G belongs to one or two faces of G. A cut edge of G belongs to exactly one face, and conversely, if an edge belongs to exactly one face of G, it must be a cut edge of G. An edge of G that is not a cut edge belongs to exactly two faces and conversely.

2. The union of the vertices and edges of G incident with a face f of G is called the *boundary* of f and is denoted by $b(f)$. The vertices and edges of a plane graph G belonging to the boundary of a face of G are said to be *incident* with that face. If G is connected, the boundary of each face is a closed walk in which each cut edge of G is traversed twice. When there are no cut edges, the boundary of each face of G is a closed trail in G. (See, for instance, face f_1 of Fig. 8.3.) However, if G is a disconnected plane graph, then the edges and the vertices incident with the exterior face will not define a trail.

3. The number of edges incident with a face f is defined as the *degree* of f. In counting the degree of a face, a cut edge is counted twice. Thus, each edge of a plane graph G contributes two to the sum of the degrees of the faces. It follows that if $\mathcal{F}$ denotes the set of faces of a plane graph G, then $\sum_{f \in \mathcal{F}} d(f) = 2m(G)$, where $d(f)$ denotes the degree of the face f.

In Fig. 8.5, $d(f_1) = 3$, $d(f_2) = 9$, $d(f_3) = 6$, and $d(f_4) = 8$.

Theorem 8.2.8 connects the planarity of G with the planarity of its blocks.

Theorem 8.2.8. *A graph G is planar if and only if each of its blocks is planar.*

Proof. If G is planar, then each of its blocks is planar, since a subgraph of a planar graph is planar. Conversely, suppose that each block of G is planar. We now use induction on the number of blocks of G to prove the result. Without loss of generality, we assume that G is connected. If G has only one block, then G is planar.

Now suppose that G has k planar blocks and that the result is true for all connected graphs having $(k-1)$ planar blocks. Choose any end block B_0 of G and delete from G all the vertices of B_0 except the unique cut vertex, say v_0, of G in B_0. The resulting connected subgraph G' of G contains $(k-1)$ planar blocks. Hence, by the induction hypothesis, G' is planar. Let $\tilde{G}'$ be a plane embedding of G' such that v_0 belongs to the boundary of the unbounded face, say f' (refer to Theorem 8.2.6). Let $\tilde{B}_0$ be a plane embedding of B_0 in f' so that v_0 is in the boundary of the exterior face of $\tilde{B}_0$. Then (by the identification of v_0 in the two embeddings), $\tilde{G}' \cup \tilde{B}_0$ is a plane embedding of G. □

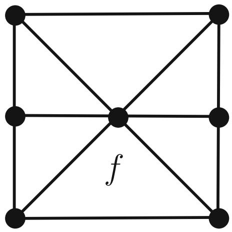
Remark 8.2.9. In testing for the planarity of a graph G, one may delete multiple edges and loops of G, if any. This is so because if a graph H is nonplanar, the removal of loops and parallel edges of H results in a subgraph of H, which is also nonplanar. Also, by Theorem 8.2.8, G can be assumed to be a block and hence 2-connected. If G has a vertex of degree 2, say v_0, and vv_0v' is the path formed by the two edges incident with v_0, contraction of vv_0 and deletion of the multiple edges (if any) thus formed again result in a planar graph. Let G' be the graph obtained from G by performing such contractions successively at vertices of degree 2 and deleting the resulting multiple edges. Then G is planar if and only if G' is planar. From these observations, it is clear that in designing a planarity algorithm (i.e., an algorithm to test planarity), it suffices to consider only 2-connected simple graphs with minimum degree at least 3. (For a planarity algorithm, see [49].)

Exercise 2.1. Show that every graph with at most three cycles is planar.

Exercise 2.2. Find a simple graph G with degree sequence (4, 4, 3, 3, 3, 3) such that

(a) G is planar.
(b) G is nonplanar.

Exercise 2.3. Redraw the following planar graph so that the face f becomes the exterior face.



8.3 Euler Formula and Its Consequences

We have noted that a planar graph may have more than one plane representation (see Fig. 8.1). A natural question that would arise is whether the number of faces is the same in each such representation. The answer to this question is provided by the Euler formula.

Theorem 8.3.1 (Euler formula). *For a connected plane graph G, $n - m + \mathfrak{f} = 2$, where n, m, and $\mathfrak{f}$ denote the number of vertices, edges, and faces of G, respectively.*

Proof. We apply induction on $\mathfrak{f}$.

If $\mathfrak{f} = 1$, then G is a tree and $m = n - 1$. Hence, $n - m + \mathfrak{f} = 2$.

Now assume that the result is true for all plane graphs with $\mathfrak{f} - 1$ faces, $\mathfrak{f} \geq 2$, and suppose that G has $\mathfrak{f}$ faces. Since $\mathfrak{f} \geq 2$, G is not a tree, and hence contains a cycle C. Let e be an edge of C. Then e belongs to exactly two faces, say f_1 and f_2, of G and the deletion of e from G results in the formation of a single face from f_1 and f_2 (see Fig. 8.5). Also, since e is not a cut edge of G, $G - e$ is connected. Further, the number of faces of $G - e$ is $\mathfrak{f} - 1$. So applying induction to $G - e$, we get $n - (m - 1) + (\mathfrak{f} - 1) = 2$, and this implies that $n - m + \mathfrak{f} = 2$. This completes the proof of the theorem. □

Below are some of the consequences of the Euler formula.

Corollary 8.3.2. *All plane embeddings of a given planar graph have the same number of faces.*

Proof. Since $\mathfrak{f} = m - n + 2$, the number of faces depends only on n and m, and not on the particular embedding. □

Corollary 8.3.3. *If G is a simple planar graph with at least three vertices, then $m \leq 3n - 6$.*

Proof. Without loss of generality, we can assume that G is a simple connected plane graph. Since G is simple and $n \geq 3$, each face of G has degree at least 3. Hence, if $\mathcal{F}$ denotes the set of faces of G, $\sum_{f \in \mathcal{F}} d(f) \geq 3\mathfrak{f}$. But $\sum_{f \in \mathcal{F}} d(f) = 2m$. Consequently, $2m \geq 3\mathfrak{f}$, so that $\mathfrak{f} \leq \frac{2m}{3}$.

By the Euler formula, $m = n + \mathfrak{f} - 2$. Now $\mathfrak{f} \leq \frac{2m}{3}$ implies that $m \leq n + \left(\frac{2m}{3}\right) - 2$. This gives $m \leq 3n - 6$. $\square$

The above result is not valid if $n = 1$ or 2. Also, the condition of Corollary 8.3.3 is not sufficient for the planarity of a simple connected graph as the Petersen graph shows. For the Petersen graph, $m = 15$, $n = 10$, and hence $m \leq 3n - 6$, but the graph is not planar (see Corollary 8.3.7 below).

Example 8.3.4. Show that the complement of a simple planar graph with 11 vertices is nonplanar.

Solution. Let G be a simple planar graph with $n(G) = 11$. Since G is planar, $m(G) \leq 3n - 6 = 27$. If G^c were also planar, then $m(G^c) \leq 3n - 6 = 27$. On the one hand, $m(G) + m(G^c) \leq 27 + 27 = 54$, whereas, on the other hand, $m(G) + m(G^c) = m(K_{11}) = \binom{11}{2} = 55$. Hence, we arrive at a contradiction. This contradiction proves that G^c is nonplanar. $\square$

Corollary 8.3.5. *For any simple planar graph G, $\delta(G) \leq 5$.*

Proof. If $n \leq 6$, then $\Delta(G) \leq 5$. Hence $\delta(G) \leq \Delta(G) \leq 5$, proving the result for such graphs. So assume that $n \geq 7$. By Corollary 8.3.3, $m \leq 3n - 6$. Now, $\delta n \leq \sum_{v \in V(G)} d_G(v) = 2m \leq 2(3n - 6) = 6n - 12$. Hence $n(\delta - 6) \leq -12$. Consequently, $\delta - 6$ is negative, implying that $\delta \leq 5$. $\square$

Recall that the *girth* of a graph G is the length of a shortest cycle in G.

Theorem 8.3.6. *If the girth k of a connected plane graph G is at least 3, then $m \leq \frac{k(n-2)}{(k-2)}$.*

Proof. Let $\mathcal{F}$ denote the set of faces and $\mathfrak{f}$, as before, denote the number of faces of G. If $f \in \mathcal{F}$, then $d(f) \geq k$. Since $2m = \sum_{f \in \mathcal{F}} d(f)$, we get $2m \geq k\mathfrak{f}$. By Theorem 8.3.1, $\mathfrak{f} = 2 - n + m$. Hence, $2m \geq k(2 - n + m)$, implying that $m(k-2) \leq k(n-2)$. Thus, $m \leq \frac{k(n-2)}{(k-2)}$. $\square$

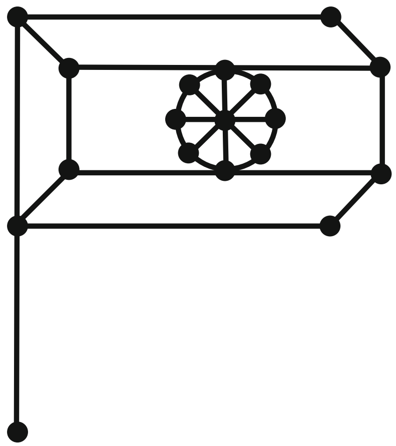
Corollary 8.3.7. *The Petersen graph P is nonplanar.*

Proof. The girth of the Petersen graph P is 5, $n(P) = 10$, and $m(P) = 15$. Hence, if P were planar, $15 \leq \frac{5(10-2)}{5-2}$, which is not true. Hence, P is nonplanar. $\square$

Exercise 3.1. Show that every simple bipartite cubic planar graph contains a C_4.

Exercise 3.2. A nonplanar graph G is called *planar-vertex-critical* if $G - v$ is planar for every vertex v of G. Prove that a planar-vertex-critical graph must be 2-connected.

Exercise 3.3. Verify Euler's formula for the plane graph P.



Exercise 3.4. Let G be a simple plane cubic graph having eight faces. Determine $n(G)$. Draw two such graphs that are nonisomorphic.

Exercise 3.5. Prove that if G is a simple connected planar bipartite graph, then $m \leq 2n - 4$, where $n \geq 3$.

Exercise 3.6. Prove that a simple planar graph (with at least four vertices) has at least four vertices each of degree 5 at most.

Exercise 3.7. If G is a nonplanar graph, show that it has either five vertices of degree at least 4, or six vertices of degree at least 3.

Exercise 3.8. Prove that a simple planar graph with minimum degree at least five contains at least 12 vertices. Give an example of a simple planar graph on 12 vertices with minimum degree 5.

Exercise 3.9. Show that there is no 6-connected planar graph.

Exercise 3.10. Let G be a plane graph of order n and size m in which every face is bounded by a k-cycle. Show that $m = \frac{k(n-2)}{(k-2)}$.

Definition 8.3.8. A graph G is *maximal planar* if G is planar, but for any pair of nonadjacent vertices u and v of G, $G + uv$ is nonplanar.

Remark 8.3.9. Any planar graph is a spanning subgraph of a maximal planar graph. Indeed, if $\tilde{G}$ is a plane embedding of a planar graph G with at least three vertices, and if $e = uv$ is a cut edge of $\tilde{G}$ embedded in a face f of $\tilde{G}$, it is clear that there exists a vertex w on the boundary of f such that the edge uw or vw can be drawn in f so that either $\tilde{G} + (vw)$ or $\tilde{G} + (uw)$ is also a plane graph (see Fig. 8.6a). Further, if C_0 is any cycle bounding a face f_0 of a plane graph H, then edges can be drawn in int C_0 without crossing each other so that f_0 is divided into triangles (see Fig. 8.6b).

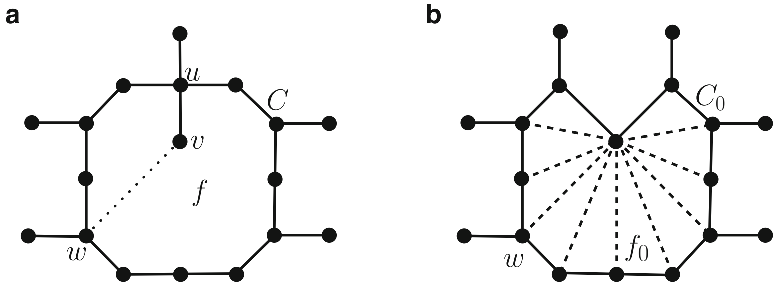


Fig. 8.6 Procedure to get maximal planar graphs

Definition 8.3.10. A *plane triangulation* is a plane graph in which each of its faces is bounded by a triangle. A plane triangulation of a plane graph G is a plane triangulation H such that G is a spanning subgraph of H.

Remark 8.3.11. Remark 8.3.9 shows that a plane embedding of a simple maximal planar graph is a plane triangulation.

Note that any simple plane graph is a subgraph of a simple maximal plane graph and hence is a spanning subgraph of some plane triangulation. Thus, to any simple plane graph G that is not already a plane triangulation, we can add a set of new edges to obtain a plane triangulation. The set of new edges thus added need not be unique.

Figure 8.7a is a simple plane graph G and Fig. 8.7b is a plane triangulation of G; Fig. 8.7c is a plane triangulation of G isomorphic to the graph of Fig. 8.7b having only straight-line edges. (A result of Fáry [60] states that every simple planar graph has a plane embedding in which each edge is a straight line.)

Exercise 3.11. Embed the 3-cube Q_3 (see Exercise 4.4 of Chap. 5) in a maximal planar graph having the same vertex set as Q_3. Count the number of new edges added.

Exercise 3.12. Prove that for a simple maximal planar graph on $n \geq 3$ vertices, $m = 3n - 6$.

Exercise 3.13. Use Exercise 3.12 to show that for any simple planar graph, $m \leq 3n - 6$.

Exercise 3.14. Show that every plane triangulation of order $n \geq 4$ is 3-connected.

Exercise 3.15. Let G be a maximal planar graph with $n \geq 4$. Let n_i denote the number of vertices of degree i in G. Then prove that $3n_3 + 2n_4 + n_5 = 12 + n_7 + 2n_8 + 3n_9 + 4n_{10} + \ldots$. (Hint: Use the fact that $n = n_3 + n_4 + n_5 + n_6 + \ldots$.)

Exercise 3.16. Generalize the Euler formula for disconnected plane graphs.

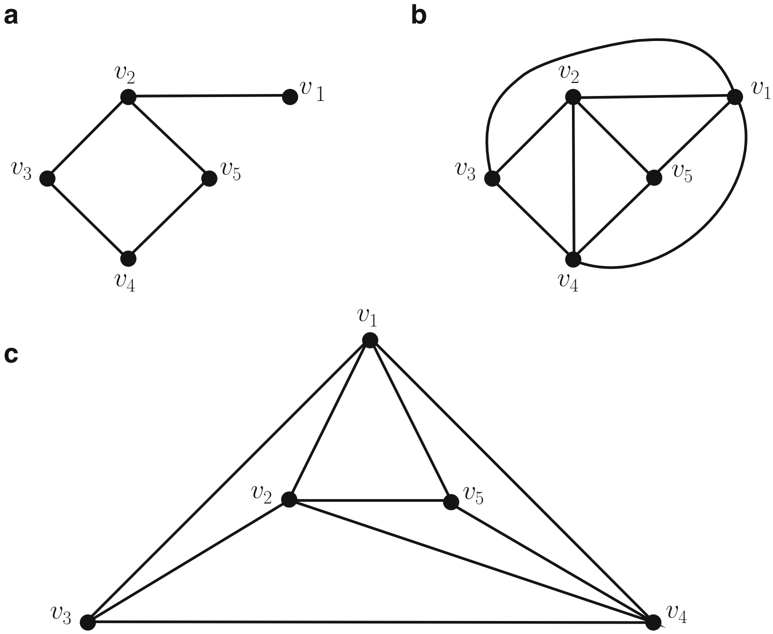


Fig. 8.7 (**a**) Graph G and (**b**), (**c**) are plane triangulations of G

8.4 K_5 and $K_{3,3}$ are Nonplanar Graphs

In this section we prove that K_5 and $K_{3,3}$ are nonplanar. These two graphs are basic in Kuratowski's characterization of planar graphs (see Theorem 8.7.5 given later in this chapter). For this reason, they are often referred to as the two *Kuratowski graphs*.

Theorem 8.4.1. *K_5 is nonplanar.*

First proof. This proof uses the Jordan curve theorem. Assume the contrary, namely, K_5 is planar. Let v_1, v_2, v_3, v_4, and v_5 be the vertices of K_5 in a plane representation of K_5. The cycle $C = v_1v_2v_3v_4v_1$ (as a closed Jordan curve) divides the plane into two faces, namely, the interior and the exterior of C. The vertex v_5 must belong either to int C or to ext C. Suppose that v_5 belongs to int C (a similar proof holds if v_5 belongs to ext C). Draw the edges v_5v_1, v_5v_2, v_5v_3, and v_5v_4 in int C. Now there remain two more edges v_1v_3 and v_2v_4 to be drawn. None of these can be drawn in int C, since it is assumed that K_5 is planar. Thus, v_1v_3 lies in ext C. Then one of v_2 and v_4 belongs to the interior of the closed Jordan curve $C_1 = v_1v_5v_3v_1$ and the other to its exterior (see Fig. 8.8). Hence, v_2v_4 cannot be drawn without violating planarity. $\square$

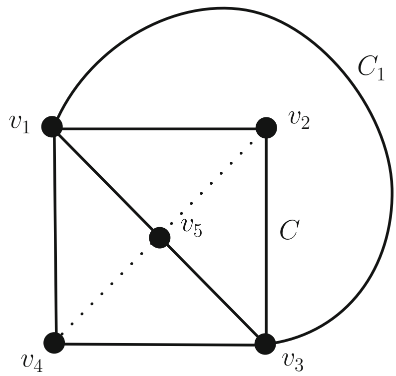


Fig. 8.8 Graph for first proof of Theorem 8.4.1

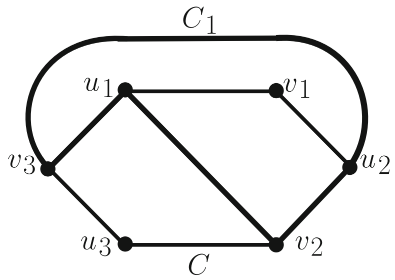


Fig. 8.9 Graph for first proof of Theorem 8.4.3

Remark 8.4.2. The first proof of Theorem 8.4.1 shows that all the edges of K_5 except one can be drawn in the plane without violating planarity. Hence for any edge e of K_5, $K_5 - e$ is planar.

Second proof. If K_5 were planar, it follows from Theorem 8.3.6 that $10 \le \frac{3(5-2)}{(3-2)}$, which is not true. Hence K_5 is nonplanar. $\square$

Theorem 8.4.3. *$K_{3,3}$ is nonplanar.*

First proof. The proof is by the use of the Jordan curve theorem. Suppose that $K_{3,3}$ is planar. Let $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3\}$ be the bipartition of $K_{3,3}$ in a plane representation of the graph. Consider the cycle $C = u_1v_1u_2v_2u_3v_3u_1$. Since the graph is assumed to be planar, the edge u_1v_2 must lie either in the interior of C or in its exterior. For the sake of definiteness, assume that it lies in int C (a similar proof holds if one assumes that the edge u_1v_2 lies in ext C). Two more edges remain to be drawn, namely, u_2v_3 and u_3v_1. None of these can be drawn in int C without crossing the edge u_1v_2. Hence, both of them are to be drawn in ext C. Now draw u_2v_3 in ext C. Then one of v_1 and u_3 belongs to the interior of the closed Jordan curve $C_1 = u_1v_2u_2v_3u_1$ and the other to the exterior of C_1 (see Fig. 8.9). Hence, the edge v_1u_3 cannot be drawn without violating planarity. This shows that $K_{3,3}$ is nonplanar. $\square$

Second proof. Suppose $K_{3,3}$ is planar. Let $\mathfrak{f}$ be the number of faces of $G = K_{3,3}$ in a plane embedding of G and $\mathcal{F}$, the set of faces of G. As the girth of $K_{3,3}$ is 4, we have $m = \frac{1}{2}\sum_{f \in \mathcal{F}} d(f) \geq \frac{4\mathfrak{f}}{2} = 2\mathfrak{f}$. By Theorem 8.3.1, $n - m + \mathfrak{f} = 2$. For $K_{3,3}$, $n = 6$, and $m = 9$. Hence, $\mathfrak{f} = 2 + m - n = 5$. Thus, $9 \geq 2.5 = 10$, a contradiction. □

Exercise 4.1. Give yet another proof of Theorem 8.4.3.

Exercise 4.2. Find the maximum number of edges in a planar complete tripartite graph with each part of size at least 2.

Remark 8.4.4. As in the case of K_5, for any edge e of $K_{3,3}$, $K_{3,3} - e$ is planar. Observe that the graphs K_5 and $K_{3,3}$ have some features in common.

1. Both are regular graphs.
2. The removal of a vertex or an edge from each graph results in a planar graph.
3. Contraction of an edge results in a planar graph.
4. K_5 is a nonplanar graph with the smallest number of vertices, whereas $K_{3,3}$ is a nonplanar graph with the smallest number of edges. (Hence, any nonplanar graph must have at least five vertices and nine edges.)

8.5 Dual of a Plane Graph

Let G be a plane graph. One can form out of G a new graph H in the following way. Corresponding to each face f of G, take a vertex f^* and corresponding to each edge e of G, take an edge e^*. Then edge e^* joins vertices f^* and g^* in H if and only if edge e is common to the boundaries of faces f and g in G. (It is possible that f may be the same as g.) The graph H is then called the *dual* (or more precisely, the geometric dual) of G (see Fig. 8.10). If e is a cut edge of G embedded in face f of G, then e^* is a loop at f^*. H is a planar graph and there exists a natural way of embedding H in the plane. Vertex f^*, corresponding to face f, is placed in face f of G. Edge e^*, joining f^* and g^*, is drawn so that e^* crosses e once and only once and crosses no other edge. This procedure is illustrated in Fig. 8.11. This embedding is the canonical embedding of H. H with this canonical embedding is denoted by G^*. Any two embeddings of H, as described above, are isomorphic.

The definition of the dual implies that $m(G^*) = m(G)$, $n(G^*) = \mathfrak{f}(G)$, and $d_{G^*}(f^*) = d_G(f)$, where $d_G(f)$ denotes the degree of the face f of G.

From the manner of construction of G^*, it follows that

(i) An edge e of a plane graph G is a cut edge of G if and only if e^* is a loop of G^*, and it is a loop of G if and only if e^* is a cut edge of G^*.
(ii) G^* is connected whether G is connected or not (see graphs G and G^* of Fig. 8.12).

The canonical embedding of the dual of G^* is denoted by G^{**}. It is easy to check that G^{**} is isomorphic to G if and only if G is connected. Graph isomorphism

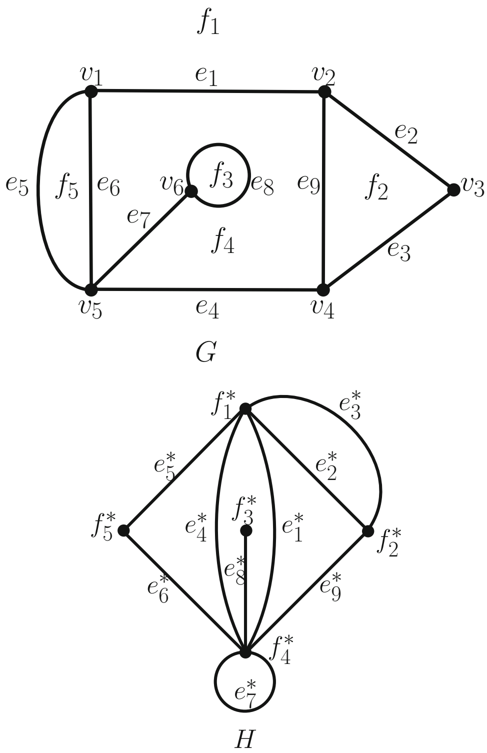


Fig. 8.10 A plane graph G and its dual H

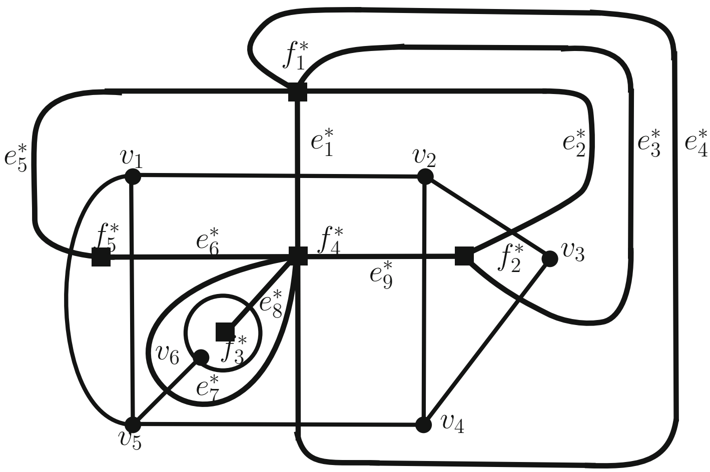
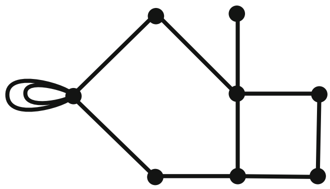


Fig. 8.11 Procedure for drawing the dual graph

does not preserve duality; that is, isomorphic plane graphs may have nonisomorphic duals. The graphs G and H of Fig. 8.13 are isomorphic plane graphs, but $G^* \not\simeq H^*$. G has a face of degree 5, whereas no face of H has degree 5. Hence, G^* has a vertex of degree 5, whereas H^* has no vertex of degree 5. Consequently, $G^* \not\simeq H^*$.

Exercise 5.1. Draw the dual of

(i) The Herschel graph (graph of Fig. 5.4).
(ii) The graph G given below:



G

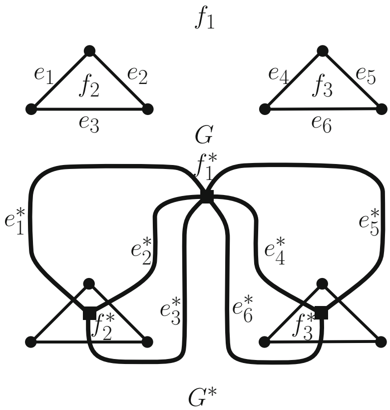


Fig. 8.12 A disconnected graph G and its (connected) dual G^*

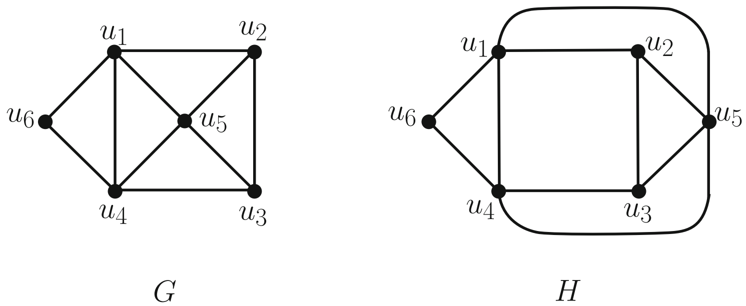


Fig. 8.13 Isomorphic graphs G and H for which $G^* \not\simeq H^*$

Exercise 5.2. A plane graph G is called *self-dual* if $G \simeq G^*$. Prove the following:

(i) All wheels W_n $(n \geq 3)$ are self-dual.
(ii) For a self-dual graph, $2n = m + 2$.

Exercise 5.3. Construct two infinite families of self-dual graphs.

8.6 The Four-Color Theorem and the Heawood Five-Color Theorem

What is the minimum number of colors required to color the world map of countries so that no two countries having a common boundary receive the same color? This simple-looking problem manifested itself into one of the most challenging problems of graph theory, popularly known as the four-color conjecture (4CC).

The geographical map of the countries of the world is a typical example of a plane graph. An assignment of colors to the faces of a plane graph G so that no two faces having a common boundary containing at least one edge receive the same color is a *face coloring* of G. The face-chromatic number $\chi^*(G)$ of a plane graph G is the minimum k for which G has a face coloring using k colors. The problem of coloring a map so that no two adjacent countries receive the same color can thus be transformed into a problem of face coloring of a plane graph G. The face coloring of G is closely related to the vertex coloring of the dual G^* of G. The fact that two faces of G are adjacent in G if and only if the corresponding vertices of G^* are adjacent in G^* shows that G is k-face-colorable if and only if G^* is k-vertex-colorable.

It was young Francis Guthrie who conjectured, while coloring the district map of England, that four colors were sufficient to color the world map so that adjacent countries receive distinct colors. This conjecture was communicated by his brother to De Morgan in 1852. Guthrie's conjecture is equivalent to the statement that any plane graph is 4-face-colorable. The latter statement is equivalent to the conjecture: Every planar graph is 4-vertex-colorable.

After the conjecture was first published in 1852, many attempted to settle it. In the process of settling the conjecture, many equivalent formulations of this conjecture were found. Assaults on the conjecture were made using such varied branches of mathematics as algebra, number theory, and finite geometries. The solution found the light of the day when Appel, Haken, and Koch [8] of the University of Illinois established the validity of the conjecture in 1976 with the aid of computers (see also [6, 7]). The proof includes, among other things, 10^{10} units of operations, amounting to a staggering 1200 hours of computer time on a high-speed computer available at that time.

Although the computer-oriented proof of Appel, Haken, and Koch settled the conjecture in 1976 and has stood the test of time, a theoretical proof of the four-color problem is still to be found.

Even though the solution of the 4CC has been a formidable task, it is rather easy to establish that every planar graph is 6-vertex-colorable.

Theorem 8.6.1. *Every planar graph is 6-vertex-colorable.*

Proof. The proof is by induction on n, the number of vertices of the graph. The result is trivial for planar graphs with at most six vertices. Assume the result for planar graphs with $n-1$, $n \geq 7$, vertices. Let G be a planar graph with n vertices. By Corollary 8.3.5, $\delta(G) \leq 5$, and hence G has a vertex v of degree at most 5. By hypothesis, $G - v$ is 6-vertex-colorable. In any proper 6-vertex coloring of $G - v$, the neighbors of v in G would have used only at most five colors, and hence v can be colored by an unused color. In other words, G is 6-vertex-colorable. □

It involves some ingenious arguments to reduce the upper bound for the chromatic number of a planar graph from 6 to 5. The upper bound 5 was obtained by Heawood [103] as early as 1890.

Theorem 8.6.2 (Heawood's five-color theorem). *Every planar graph is 5-vertex-colorable.*

Proof. The proof is by induction on $n(G) = n$. Without loss of generality, we assume that G is a connected plane graph. If $n \leq 5$, the result is clearly true. Hence, assume that $n \geq 6$ and that any planar graph with fewer than n vertices is 5-vertex-colorable. G being planar, $\delta(G) \leq 5$ by Corollary 8.3.5, and so G contains a vertex v_0 of degree not exceeding 5. By the induction hypothesis, $G - v_0$ is 5-vertex-colorable.

If $d(v_0) \leq 4$, at most four colors would have been used in coloring the neighbors of v_0 in G in a 5-vertex coloring of $G - v_0$. Hence, an unused color can then be assigned to v_0 to yield a proper 5-vertex coloring of G.

If $d(v_0) = 5$, but only four or fewer colors are used to color the neighbors of v_0 in a proper 5-vertex coloring of $G - v_0$, then also an unused color can be assigned to v_0 to yield a proper 5-vertex coloring of G.

Hence assume that the degree of v_0 is 5 and that in every 5-coloring of $G - v_0$, the neighbors of v_0 in G receive five distinct colors. Let v_1, v_2, v_3, v_4, and v_5 be the neighbors of v_0 in a cyclic order in a plane embedding of G. Choose some proper 5-coloring of $G - v_0$ with colors, say, $c_1, c_2, \ldots, c_5$. Let $\{V_1, V_2, \ldots, V_5\}$ be the color partition of $G - v_0$, where the vertices in V_i are colored c_i, $1 \leq i \leq 5$. Assume further that $v_i \in V_i$, $1 \leq i \leq 5$.

Let G_{ij} be the subgraph of $G - v_0$ induced by $V_i \cup V_j$. Suppose v_i and v_j, $1 \leq i, j \leq 5$, belong to distinct components of G_{ij}. Then the interchange of the colors c_i and c_j in the component of G_{ij} containing v_i would give a recoloring of $G - v_0$ in which only four colors are assigned to the neighbors of v_0. But this is against our assumption. Hence, v_i and v_j must belong to the same component of G_{ij}. Let $P_{i,j}$

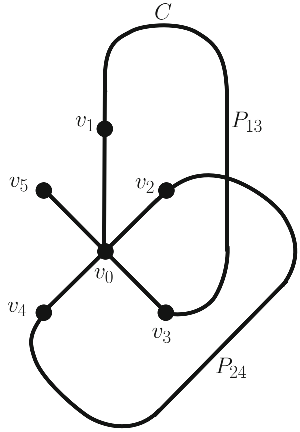


Fig. 8.14 Graph for proof of Theorem 8.6.2

be a v_i-v_j path in G_{ij}. Let C denote the cycle $v_0v_1P_{13}v_3v_0$ in G (Fig. 8.14). Then C separates v_2 and v_4; that is, one of v_2 and v_4 must lie in int C and the other in ext C. In Fig. 8.14, $v_2 \in$ int C and $v_4 \in$ ext C. Then P_{24} must cross C at a vertex of C. But this is clearly impossible since no vertex of C receives either of the colors c_2 and c_4. Hence this possibility cannot arise, and G is 5-vertex-colorable. □

Note that the bound 4 in the inequality $\chi(G) \leq 4$ for planar graphs G is best possible since K_4 is planar and $\chi(K_4) = 4$.

Exercise 6.1. Show that a planar graph G is bipartite if and only if each of its faces is of even degree in any plane embedding of G.

Exercise 6.2. Show that a connected plane graph G is bipartite if and only if G^* is Eulerian. Hence, show that a connected plane graph is 2-face-colorable if and only if it is Eulerian.

Exercise 6.3. Prove that a Hamiltonian plane graph is 4-face-colorable and that its dual is 4-vertex-colorable.

Exercise 6.4. Show that a plane triangulation has a 3-face coloring if and only if it is not K_4. (Hint: Use Brooks' theorem.)

Remark 8.6.3. (Grötzsch): If G is a planar graph that contains no triangle, then G is 3-vertex-colorable.

8.7 Kuratowski's Theorem

Definition 8.7.1. 1. A *subdivision of an edge* $e = uv$ of a graph G is obtained by introducing a new vertex w in e, that is, by replacing the edge $e = uv$ of G by the path uwv of length 2 so that the new vertex w is of degree 2 in the resulting graph (see Fig. 8.15a).

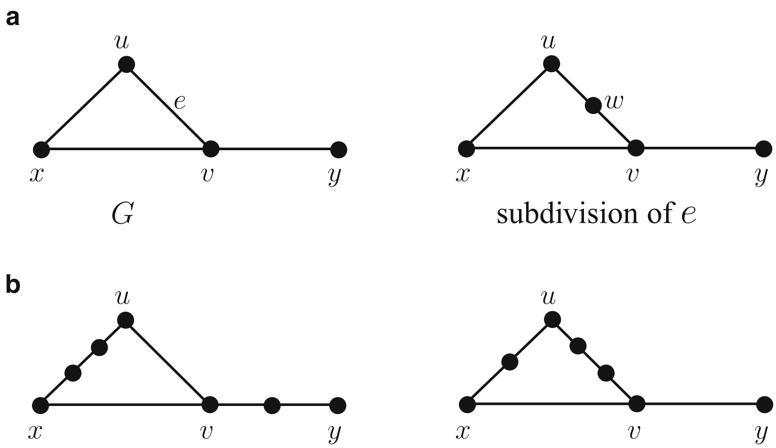


Fig. 8.15 (**a**) Subdivision of edge e of graph G, (**b**) two homeomorphs of graph G

2. A *homeomorph* or a *subdivision of a graph* G is a graph obtained from G by applying a finite number of subdivisions of edges in succession (see Fig. 8.15b). G itself is regarded as a subdivision of G.
3. Two graphs G_1 and G_2 are called *homeomorphic* if they are both homeomorphs of some graph G. Clearly, the graphs of Fig. 8.15b are homeomorphic, even though neither of the two graphs is a homeomorph of the other.

Kuratowski's theorem [129] characterizing planar graphs was one of the major breakthrough results in graph theory of the 20th century. As mentioned earlier, while examining planarity of graphs, we need only consider simple graphs since the presence of loops and multiple edges does not affect the planarity of graphs. Consequently, *a graph is planar if and only if its underlying simple graph is planar.* We therefore consider in this section only (finite) simple graphs. We recall that for any edge e of a graph G, $G - e$ is the subgraph of G obtained by deleting the edge e, whereas $G \circ e$ denotes the contraction of e. We always discard isolated vertices when edges get deleted and remove the new multiple edges when edges get contracted. More generally, for a subgraph H of G, $G \circ H$ denotes the graph obtained by the successive contractions of all the edges of H in G. The resulting graph is independent of the order of contraction. Moreover, if G is planar, then $G \circ e$ is planar; consequently, $G \circ H$ is planar. In other words, if $G \circ H$ is nonplanar for some subgraph H of G, then G is also nonplanar. Further, any two homeomorphic graphs are contractible to the same graph.

Definition 8.7.2. If $G \circ H = K$, we call K a *contraction* of G; we also say that G is *contractible* to K. G is said to be *subcontractible* to K if G has a subgraph H contractible to K. We also refer to this fact by saying that K is a *subcontraction* of G.

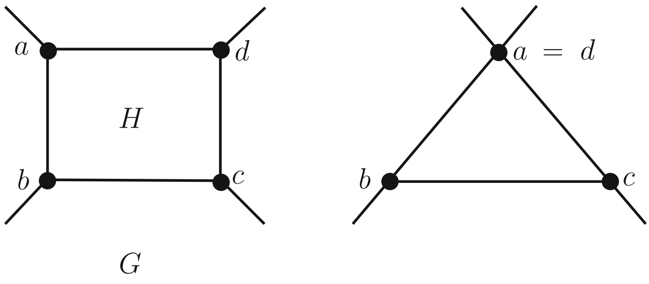


Fig. 8.16 Graph G subcontractible to triangle abc

Example 8.7.3. For instance, in Fig. 8.16, graph G is subcontractible to the triangle abc. (Take H to be the cycle $abcd$ and contract the edge ad in H. By abuse of notation, the new vertex is denoted by a or d.) We note further that if G' is a homeomorph of G, then contraction of one of the edges incident at each vertex of degree 2 in $V(G')\backslash V(G)$ results in a graph homeomorphic to G.

Our first aim is to prove the following result, which was established by Wagner [186] and, independently, by Harary and Tutte [96].

Theorem 8.7.4 ([96,186]). *A graph is planar if and only if it is not subcontractible to K_5 or $K_{3,3}$.*

As a consequence, we establish Kuratowski's characterization theorem for planar graphs.

Theorem 8.7.5 (Kuratowski [129]). *A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.*

The proofs of Theorems 8.7.4 and 8.7.5, as presented here, are due to Fournier [68]. Recall that any subgraph and any contraction of a planar graph are both planar.

Definition 8.7.6. A simple connected nonplanar graph G is *irreducible* if, for each edge e of G, $G \circ e$ is planar.

For instance, both K_5 and $K_{3,3}$ are irreducible.

Proof of theorem 8.7.4. If G has a subgraph G_0 contractible to K_5 or $K_{3,3}$, then since K_5 and $K_{3,3}$ are nonplanar, G_0 and therefore G are nonplanar.

We now prove the converse. Assume that G is a simple connected nonplanar graph. By Theorem 8.2.8, at least one block of G is nonplanar. Hence, assume that G is a simple 2-connected nonplanar graph. We now show that G has a subgraph contractible to K_5 or $K_{3,3}$.

Keep contracting edges of G (and delete the new multiple edges, if any, at each stage of the contraction) until a (2-connected) irreducible (nonplanar) graph H results. Clearly, $\delta(H) \geq 3$. Now, if e and f are any two distinct edges of G, then $(G \circ e) - f = (G - f) \circ e$. Hence, the graph H may as well be obtained by

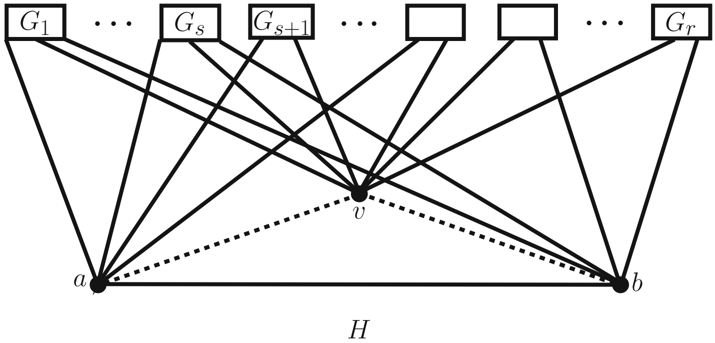


Fig. 8.17 Graph H for case 1 of proof of Theorem 8.7.4

deleting a set (which may be empty) of edges of G, resulting in a subgraph G_0 of G and then contracting a subgraph of G_0. We now complete the proof of the theorem by showing that H has a subgraph K homeomorphic (and hence contractible) to K_5 or $K_{3,3.}$ In this case, G has the subgraph G_0, which is contractible to K_5 or $K_{3,3}$.

Let $e = ab \in E(H)$ and $H' = H - \{a, b\}$. Then H' is connected. If not, $\{a, b\}$ is a vertex cut of H. Let $G_1', \ldots, G_r'$ be the components of H'. As H is irreducible, $H - V(G_r')$ is planar, and there exists a plane embedding of H' in which the edge ab is in the exterior face. As G_r' is planar, G_r' can be embedded in this exterior face of H'. This would make H a planar graph, a contradiction. Thus, H' is connected.]

Case 1. H' has a cut vertex v. Let $G_1, G_2, \ldots, G_r$ ($r \geq 2$) be the components of $H' - \{v\}$, and let $G_1, G_2, \ldots, G_s$, $0 \leq s \leq r$, be those components that are connected to both a and b. (see Fig. 8.17). If $r > s$, then each of $G_{s+1}, \ldots, G_r$ is connected to only one of a or b. Assume that G_r is connected to b and not to a. From the plane representation of $G \circ (G_{s+1} \cup \ldots \cup G_r)$, the contraction of G obtained by contracting the edges of $G_{s+1}, \ldots, G_r$, we can obtain a plane representation of H' (see Fig. 8.17). [In fact, if G_r is contracted to the vertex w_r, then as the subgraph $A_r = \langle v, b, v(G_r) \rangle$ of H' is planar, the pair of edges $\{vw_r, w_r b\}$ can be replaced by the planar subgraph A_r and so on.] Hence this case cannot arise. Consequently, $r = s$. If $r = s = 2$, the plane embeddings of $H' \circ G_1$ and $H' \circ G_2$ yield a plane embedding of H', a contradiction (see Fig. 8.18). Consequently, $r = s \geq 3$. In this case, H' contains a homeomorph of $K_{3,3}$ (see Fig. 8.19), with $\{w_1, w_2, w_3\,;\, a, b, v\}$ being the vertex set of $K_{3,3}$. (Other possibilities for w_1, w_2, w_3 will also yield a homeomorph of $K_{3,3}$.)

Case 2. H' is 2-connected. Then H' contains a cycle C of length at least 3. Consider a plane embedding of $H \circ e$ (where $e = ab$, as above). If c denotes the new vertex to which a and b get contracted, $(H \circ e) - c = H'$. We may therefore suppose without loss of generality that c is in the interior of the cycle C in the plane embedding of $H \circ e$.

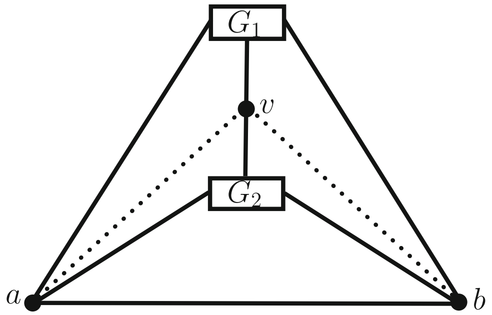


Fig. 8.18 Plane embedding for case 1 of proof of Theorem 8.7.4

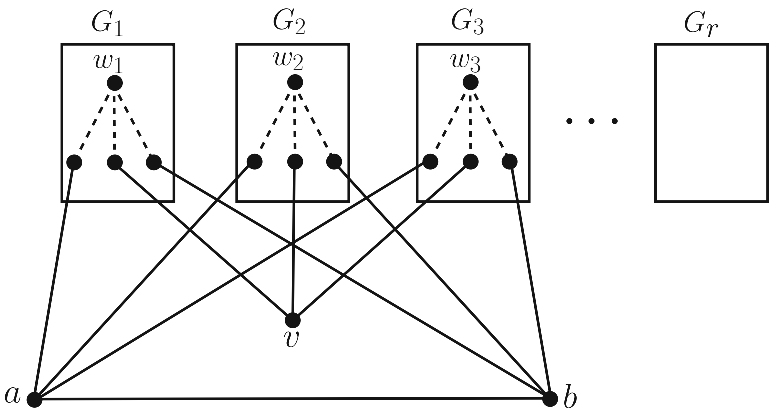


Fig. 8.19 Homeomorph for case 1 of proof of Theorem 8.7.4

Now, the edges of $H \circ e$ incident to c arise out of edges of H incident to a or b. There arise three possibilities with reference to the positions of the edges of $H \circ e$ incident to c relative to the cycle C.

(i) Suppose the edges incident to c occur so that the edges incident to a and the edges incident to b in H are consecutive around c in a plane embedding of $H \circ e$, as shown in Fig. 8.20a. Since H is a minimal nonplanar graph, the paths from c to C can only be single edges. Then the plane representation of $H \circ e$ gives a plane representation of H, as in Fig. 8.20b, a contradiction. So this possibility cannot arise.

(ii) Suppose there are three edges of $H \circ e$ incident with c, with each edge corresponding to a pair of edges of H, one incident to a and the other to b, as in Fig. 8.21a. Then H contains a subgraph contractible to K_5, as shown in Fig. 8.21b.

We are now left with only one more possibility.

(iii) There are four edges of $H \circ e$ incident to c, and they arise alternately out of edges incident to a and b in H, as in Fig. 8.22a. Then there arises in H

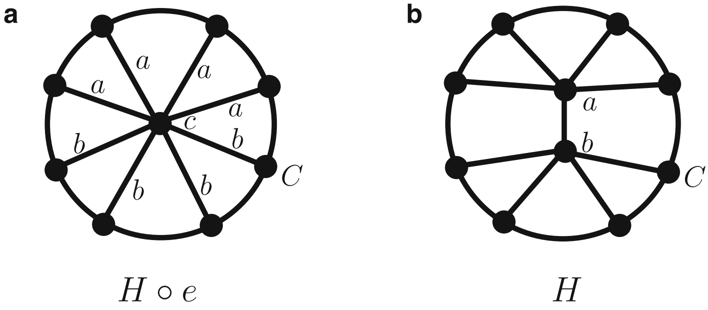


Fig. 8.20 First configuration for case 2 of proof of Theorem 8.7.4. Edges incident to a and b are marked a and b, respectively

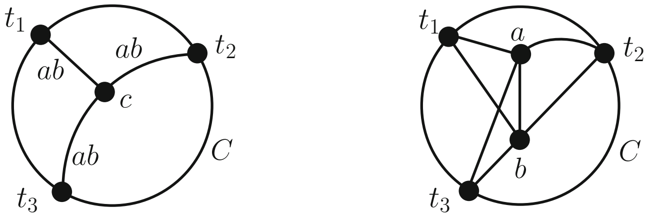


Fig. 8.21 Second configuration for case 2 of proof of Theorem 8.7.4. Edges incident to both a and b are marked ab

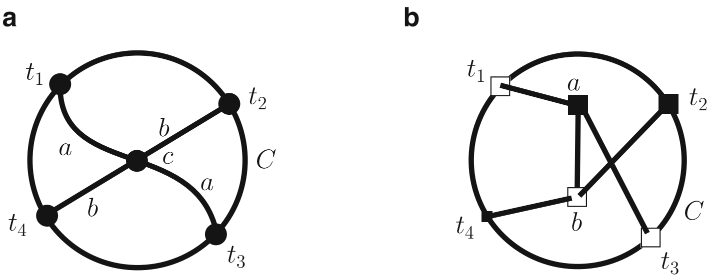


Fig. 8.22 Third configuration for case 2 of proof of Theorem 8.7.4

a homeomorph of $K_{3,3}$, as shown in Fig. 8.22b. The sets $X = \{a, t_2, t_4\}$ and $Y = \{b, t_1, t_3\}$ are the sets of the bipartition of this homeomorph of $K_{3,3}$. □

We now proceed to prove Theorem 8.7.5.

Proof of theorem 8.7.5. The "sufficiency" part of the proof is trivial. If G contains a homeomorph of either K_5 or $K_{3,3}$, G is certainly nonplanar, since a homeomorph of a planar graph is planar.

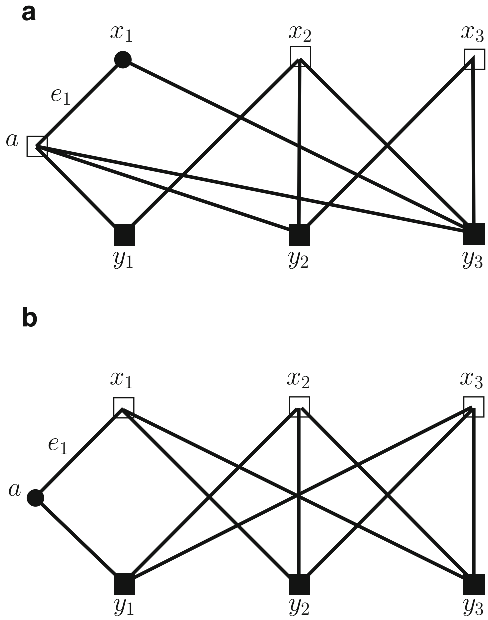


Fig. 8.23 Graphs for proof of Theorem 8.7.5

Assume that G is connected and nonplanar. Remove edges from G one after another until we get an edge-minimal connected nonplanar subgraph G_0 of G; that is, G_0 is nonplanar and for any edge e of G, $G_0 - e$ is planar. Now contract the edges in G_0 incident with vertices of degree at most 2 in some order. Let us denote the resulting graph by G_0'. Then G_0' is nonplanar, whereas $G_0' - e$ is planar for any edge e of G_0', and the minimum degree of G_0' is at least 3. We now have to show that G_0' contains a homeomorph of K_5 or $K_{3,3}$.

By Theorem 8.7.4, G_0' is subcontractible to K_5 or $K_{3,3}$. This means that G_0' contains a subgraph H that is contractible to K_5 or $K_{3,3}$. As $G_0' - e$ is planar for any edge e of G_0', $G_0' = H$. Thus, G_0' itself is contractible to K_5 or $K_{3,3}$. If G_0' is either K_5 or $K_{3,3}$, we are done. Assume now that G_0' is neither K_5 nor $K_{3,3}$. Let $e_1, e_2, \ldots, e_r$ be the edges of G_0', when contracted in order, that result in a K_5 or $K_{3,3}$.

First, let us assume that $r = 1$, so that $G_0' \circ e_1$ is either K_5 or $K_{3,3}$. Suppose that $G_0' \circ e_1 = K_{3,3}$ with $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ as the partite sets of vertices. Suppose that x_1 is the vertex obtained by identifying the ends of e_1. We may then take $e_1 = x_1 a$ (by abuse of notation), where a is a vertex distinct from the x_i's and y_j's (Fig. 8.23a). If a is adjacent to all of y_1, y_2 and y_3, then $\{a, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ form a bipartition of a $K_{3,3}$ in G_0'. If a is adjacent to only one or two of $\{y_1, y_2, y_3\}$ (Fig. 8.23b), then again G_0' contains a homeomorph of $K_{3,3}$.

Next, let us assume that $G_0' \circ e_1 = K_5$ with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$. Suppose that v_1 is the vertex obtained by identifying the ends of e_1. As before, we may take $e_1 = v_1 a$, where $a \notin \{v_1, v_2, v_3, v_4, v_5\}$. If a is adjacent to

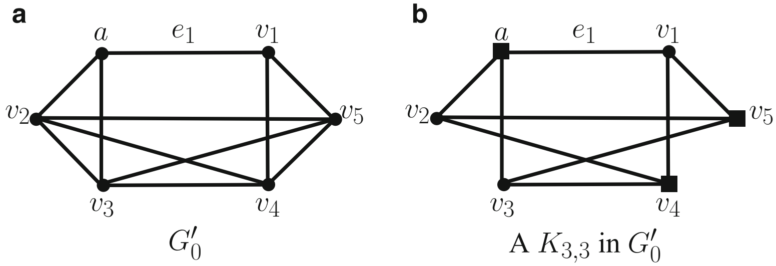
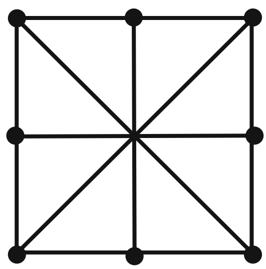


Fig. 8.24 Graphs for proof of Theorem 8.7.5

all of $\{v_2, v_3, v_4, v_5\}$, then $G_0' - v_1$ is a K_5, contradiction to the fact that any proper subgraph of G_0' is planar. If a is adjacent to only three of $\{v_2, v_3, v_4, v_5\}$, say v_2, v_3, and v_4, then the edge-induced subgraph of G_0' induced by the edges $av_1, av_2, av_3, av_4, v_1v_5, v_2v_3, v_2v_4, v_2v_5, v_3v_4, v_3v_5$, and v_4v_5 is a homeomorph of K_5. In this case, G_0' also contains a homeomorph of $K_{3,3}$. Since $d_{G_0'}(v_1) \geq 3$, v_1 is adjacent to at least one of v_2, v_3, and v_4, say v_2. Then the edge-induced subgraph of G_0' induced by the edges in $\{av_1, av_3, av_4, v_1v_2, v_2v_3, v_2v_4, v_1v_5, v_3v_5, v_4v_5\}$ is a $K_{3,3}$, with $\{a, v_4, v_5\}$ and $\{v_1, v_2, v_3\}$ forming the bipartition. We now consider the case when a is adjacent to only two of v_2, v_3, v_4 and v_5, say v_2 and v_3. Then, necessarily, v_1 is adjacent to v_4 and v_5 (since on contraction of the edge v_1a, v_1 is adjacent to v_2, v_3, v_4, and v_5). In this case G_0' also contains a $K_{3,3}$ (see Fig. 8.24b). Finally, the case when a is adjacent to at most one of v_2, v_3, v_4, and v_5 cannot arise since the degree of a is at least 3 in G_0'. Thus, in any case, we have proved that when $r = 1$, G_0' contains a homeomorph of $K_{3,3}$. The result can now easily be seen to be true by induction on r. Indeed, if $H_2 = H_1 \circ e$ and H_2 contains a homeomorph of $K_{3,3}$, then H_1 contains a homeomorph of $K_{3,3}$. □

The nonplanarity of the Petersen graph (Fig. 8.25a) can be established by showing that it is contractible to K_5 (see Fig. 8.25b) or by showing that it contains a homeomorph of $K_{3,3}$ (see Fig. 8.25c).

Exercise 7.1. Prove that the following graph is nonplanar.



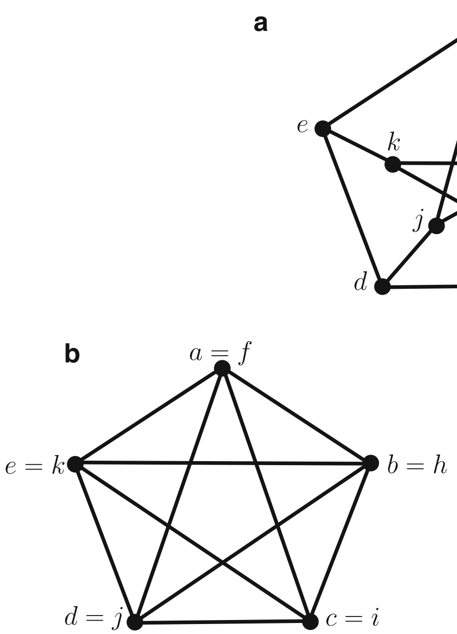


Fig. 8.25 Nonplanarity of the Petersen graph. (**a**) The Petersen graph P, (**b**) contraction of P to K_5, (**c**) A subdivision of $K_{3,3}$ in P

8.8 Hamiltonian Plane Graphs

An elegant necessary condition for a plane graph to be Hamiltonian was given by Grinberg [78].

Theorem 8.8.1. *Let G be a loopless plane graph having a Hamilton cycle C. Then $\sum_{i=2}^{n}(i-2)(\phi_i'-\phi_i'') = 0$, where ϕ_i' and ϕ_i'' are the numbers of faces of G of degree i contained in int C and ext C, respectively.*

Proof. Let E' and E'' denote the sets of edges of G contained in int C and ext C, respectively, and let $|E'| = m'$ and $|E''| = m''$. Then int C contains exactly $m' + 1$ faces (see Fig. 8.26), and so

$$\sum_{i=2}^{n} \phi_i' = m' + 1. \tag{8.1}$$

(Since G is loopless, $\phi_1' = \phi_1'' = 0$.)

Moreover, each edge in int C is on the boundary of exactly two faces in int C, and each edge of C is on the boundary of exactly one face in int C. Hence, counting the edges of all the faces in int C, we get

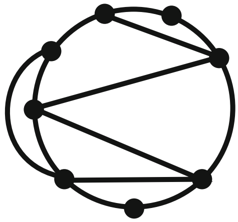


Fig. 8.26 Graph for proof of Theorem 8.8.1

$$\sum_{i=2}^{n} i\phi'_i = 2m' + n. \tag{8.2}$$

Eliminating m' from (8.1) and (8.2), we get

$$\sum_{i=2}^{n} (i-2)\phi'_i = n - 2. \tag{8.3}$$

Similarly,

$$\sum_{i=2}^{n} (i-2)\phi''_i = n - 2. \tag{8.4}$$

Equations (8.3) and (8.4) give the required result. □

Grinberg's condition is quite useful in that by applying this result, many plane graphs can easily be shown to be non-Hamiltonian by establishing that they do not satisfy the condition.

Example 8.8.2. The Herschel graph G of Fig. 5.4 is non-Hamiltonian.

Proof. G has nine faces and all the faces are of degree 4. Hence, if G were Hamiltonian, we must have $2(\phi'_4 - \phi''_4) = 0$. This means that $\phi'_4 = \phi''_4$. This is impossible, since $\phi'_4 + \phi''_4 =$ (number of faces of degree 4 in G) $= 9$ is odd. Hence, G is non-Hamiltonian. (In fact, it is the smallest planar non-Hamiltonian 3-connected graph.)

Exercise 8.1. Does there exist a plane Hamiltonian graph with faces of degrees 5, 7, and 8, and with just one face of degree 7?

Exercise 8.2. Prove that the Grinberg graph given in Fig. 8.27 is non-Hamiltonian.

8.9 Tait Coloring

In an attempt to solve the four-color problem, Tait considered edge colorings of 2-edge-connected cubic planar graphs. He conjectured that every such graph was 3-edge colorable. Indeed, he could prove that his conjecture was equivalent to the

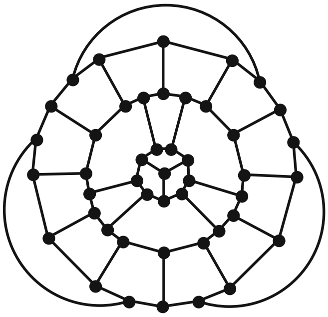


Fig. 8.27 The Grinberg graph

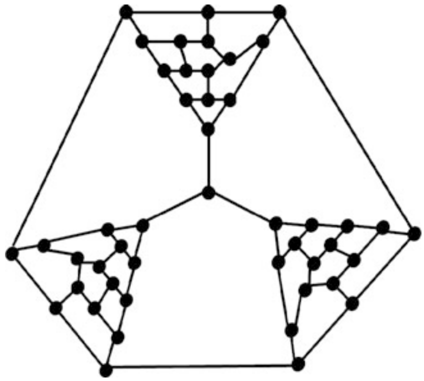


Fig. 8.28 The Tutte graph

four-color problem (see Theorem 8.9.1). Tait did this in 1880. He even went to the extent of giving a "proof" of the four-color theorem using this result. Unfortunately, Tait's proof was based on the wrong assumption that any 2-edge-connected cubic planar graph is Hamiltonian. A counterexample to his assumption was given by Tutte in 1946 (65 years later). The graph given by Tutte is the graph of Fig. 8.28. It is a non-Hamiltonian cubic 3-connected (and therefore 3-edge-connected; see Theorem 3.3.4) planar graph. Tutte used ad hoc techniques to prove this result. (The Grinberg condition does not establish this result.)

We indicate below the proof of the fact that the Tutte graph of Fig. 8.28 is non-Hamiltonian. The graphs G_1 to G_5 mentioned below are shown in Fig. 8.29.

It is easy to check that there is no Hamilton cycle in the graph G_1 containing both of the edges e_1 and e_2. Now, if there is a Hamilton cycle in G_2 containing both of the edges e_1' and e_2', then there will be a Hamilton cycle in G_1 containing e_1 and e_2. Hence there is no Hamilton cycle in G_2 containing e_1' and e_2'. In $G_3 - e'$, u and w are vertices of degree 2. Hence if $G_3 - e'$ were Hamiltonian, then in any Hamilton cycle of $G_3 - e'$, both the edges incident to u as well as both the edges incident to w must be consecutive. This would imply that G_2 has a Hamilton cycle containing e_1' and e_2', which is not the case. Consequently, any Hamilton cycle of G_3 must contain the edge e'. It follows that there exists no Hamilton path from x to y in $G_3 - w$. A redrawing of $G_3 - w$ is the graph G_4. It is called the "Tutte triangle." The Tutte

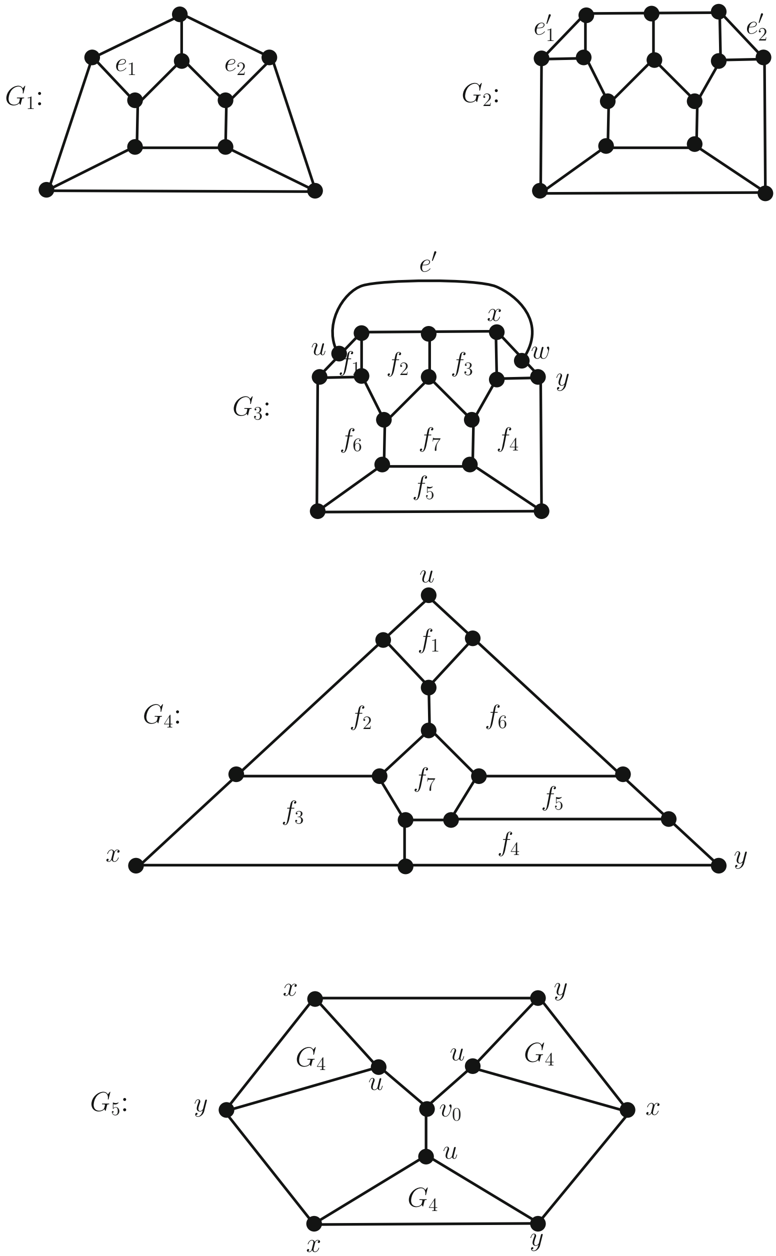


Fig. 8.29 Graphs G_1 to G_5

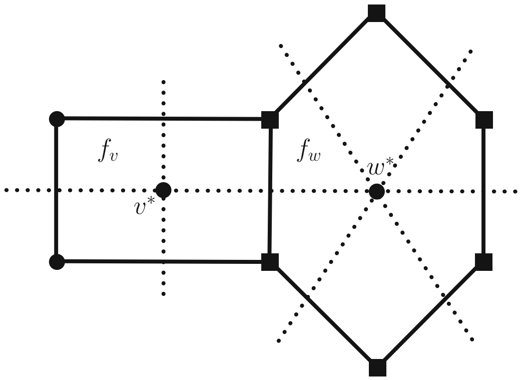


Fig. 8.30 Graph for proof of (i) ⇒ (ii) in Theorem 8.9.1

graph (Fig. 8.28) contains three copies of G_4 together with a vertex v_0. It has been redrawn as graph G_5 of Fig. 8.29. Suppose G_5 is Hamiltonian with a Hamilton cycle C. If we describe C starting from v_0, it is clear that C must visit each copy of G_4 exactly once. Hence, if C enters a copy of G_4, it must exit that copy through x or y after visiting all the other vertices of that copy. But this means that there exists a Hamilton path from y to x (or from x to y) in G_4, a contradiction. Thus, the Tutte graph G_5 is non-Hamiltonian.

We now give the proof of Tait's result. Recall that by Vizing–Gupta's theorem (Theorem 7.5.5), every simple cubic graph has chromatic index 3 or 4. A 3-edge coloring of a cubic planar graph is often called a *Tait coloring*.

Theorem 8.9.1. *The following statements are equivalent:*

(i) All plane graphs are 4-vertex-colorable.
(ii) All plane graphs are 4-face-colorable.
(iii) All simple 2-edge-connected cubic planar graphs are 3-edge-colorable (i.e., Tait colorable).

Proof. (i) ⇒ (ii). Let G be a plane graph. Let G^* be the dual of G (see Sect. 8.4). Then, since G^* is a plane graph, it is 4-vertex-colorable. If v^* is a vertex of G^*, and f_v is the face of G corresponding to v^*, assign to f_v the color of v^* in a 4-vertex coloring of G^*. Then, by the definition of G^*, it is clear that adjacent faces of G will receive distinct colors. (See Fig. 8.30, in which f_v and f_w receive the colors of v^* and w^*, respectively.) Thus, G is 4-face-colorable.

(ii) ⇒ (iii). Let G be a plane embedding of a 2-edge-connected cubic planar graph. By assumption, G is 4-face-colorable. Denote the four colors by $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, the elements of the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$. If e is an edge of G that separates the faces, say f_1 and f_2, color e with the color given by the sum (in $\mathbb{Z}_2 \times \mathbb{Z}_2$) of the colors of f_1 and f_2. Since G has no cut edge, each edge is the common boundary of exactly

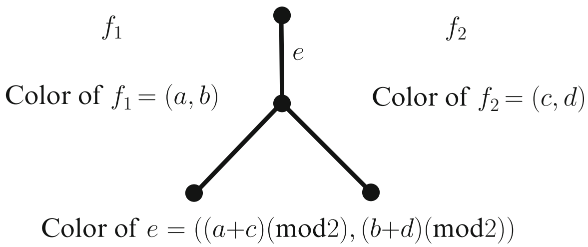


Fig. 8.31 Graph for proof of (ii) ⇒ (iii) in Theorem 8.9.1

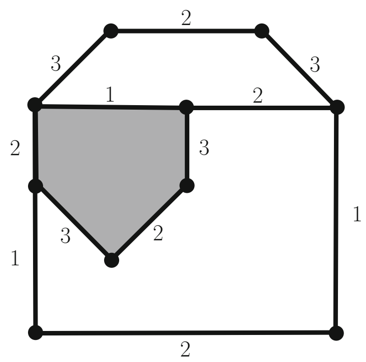


Fig. 8.32 Graph for proof of (iii) ⇒ (i) for Theorem 8.9.1

two faces of G. This gives a 3-edge coloring of G using the colors $(1, 0)$, $(0, 1)$, and $(1, 1)$, since the sum of any two distinct elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not $(0, 0)$ (see Fig. 8.31).

(iii) ⇒ (i) Let G be a planar graph. We want to show that G is 4-vertex-colorable. We may assume without loss of generality that G is simple. Let $\tilde{G}$ be a plane embedding of G. Then $\tilde{G}$ is a spanning subgraph of a plane triangulation T, (see Sect. 8.2), and hence it suffices to prove that T is 4-vertex-colorable.

Let T^* be the dual of T. Then T^* is a 2-edge-connected cubic plane graph. By our assumption, T^* is 3-edge-colorable using, for example, the colors c_1, c_2, and c_3. Since T^* is cubic, each of the above three colors is represented at each vertex of T^*. Let T^*_{ij} be the edge subgraph of T^* induced by the edges of T^* which have been colored using the colors c_i and c_j. Then T^*_{ij} is a disjoint union of even cycles, and thus it is 2-face-colorable. But each face of T^* is the intersection of a face of T^*_{12} and a face of T^*_{23} (see Fig. 8.32). Now the 2-face colorings of T^*_{12} and T^*_{23} induce a 4-face coloring of T^* if we assign to each face of T^* the (unordered) pair of colors assigned to the faces whose intersection is f. Since $T^* = T^*_{12} \cup T^*_{23}$, this defines a proper 4-face coloring of T^*. Thus, $\chi(G) = \chi(\tilde{G}) \leq \chi(T) = \chi^*(T^*) \leq 4$, and G is 4-vertex-colorable. (Recall that $\chi^*(T^*)$ is the face-chromatic number of T^*.) □

Exercise 9.1. Exhibit a 3-edge coloring for the Tutte graph (see Fig. 8.28).

Notes

The proof of Heawood's theorem uses arguments based on paths in which the vertices are colored alternately by two colors. Such paths are called "Kempe chains" after Kempe [121], who first used such chains in his "proof" of the 4CC. Even though Kempe's proof went wrong, his idea of using Kempe chains and switching the colors in such chains had been effectively exploited by Heawood [103] in proving his five-color theorem (Theorem 8.6.2) for planar graphs, as well as by Appel, Haken, and Koch [8] in settling the 4CC. As the reader might notice, the same technique had been employed in the proof of Brooks' theorem (Theorem 7.3.7). Chronologically, Francis Guthrie conceived the four-color theorem in 1852 (if not earlier). Kempe's purported "proof" of the 4CC was given in 1879, and the mistake in his proof was pointed out by Heawood in 1890. The Appel–Haken–Koch proof of the 4CC was first announced in 1976. Between 1879 and 1976, graph theory witnessed an unprecedented growth along with the methods to tackle the 4CC. The reader who is interested in getting a detailed account of the four-color problem may consult Ore [152] and Kainen and Saaty [120].

Even though the Tutte graph of Fig. 8.28 shows that not every cubic 3-connected planar graph is Hamiltonian, Tutte himself showed that every 4-connected planar graph is Hamiltonian [180].