Chapter 7
Graph Colorings

7.1 Introduction

Graph theory would not be what it is today if there had been no coloring problems. In fact, a major portion of the 20th-century research in graph theory has its origin in the four-color problem. (See Chap. 8 for details.)

In this chapter, we present the basic results concerning vertex colorings and edge colorings of graphs. We present two important theorems on graph colorings, namely, Brooks' theorem and Vizing's theorem. We also present a brief discussion on "snarks" and Kirkman's schoolgirl problem. In addition, a detailed description of the Mycielskian of a graph is also presented.

7.2 Vertex Colorings

7.2.1 Applications of Graph Coloring

We begin with a practical application of graph coloring known as the *storage problem.* Suppose a university's Department of Chemistry wants to store its chemicals. It is quite probable that some chemicals cause violent reactions when brought together. Such chemicals are *incompatible chemicals*. For safe storage, incompatible chemicals should be kept in distinct rooms. The easiest way to accomplish this is, of course, to store one chemical in each room. But this is certainly not the best way of doing it since we will be using more rooms than are really needed (unless, of course, all the chemicals are mutually incompatible!). So we ask: What is the minimum number of rooms required to store all the chemicals so that in each room only compatible chemicals are stored?

We convert the above storage problem into a problem in graphs. Form a graph $G = (V, E)$ by making V correspond bijectively to the set of available chemicals and making u adjacent to v if and only if the chemicals corresponding to u and

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v are incompatible. Then, any set of compatible chemicals correspond to a set of independent vertices of G. Thus, a safe storing of chemicals corresponds to a partition of V into independent subsets of G. The cardinality of such a minimum partition of V is then the required number of rooms. The minimum cardinality is called the *chromatic number* of the graph G.

Definition 7.2.1. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of independent subsets that partition the vertex set of G. Any such minimum partition is called a *chromatic partition* of $V(G)$.

The storage problem just described is actually a vertex coloring problem of G. A *vertex coloring* of G is a map $f : V \to S$, where S is a set of distinct colors; it is *proper* if adjacent vertices of G receive distinct colors of S. This means that if $uv \in E(G)$, then $f(u) \neq f(v)$. Thus, $\chi(G)$ is the minimum cardinality of S for which there exists a proper vertex coloring of G by colors of S. Clearly, in any proper vertex coloring of G, the vertices that receive the same color are independent. The vertices that receive a particular color make up a *color class*. This allows an equivalent way of defining the chromatic number.

Definition 7.2.2. The *chromatic number* of a graph G is the minimum number of colors needed for a proper vertex coloring of G. G is k-*chromatic* if $\chi(G) = k$.

Definition 7.2.3. A k-*coloring* of a graph G is a vertex coloring of G that uses at most k colors.

Definition 7.2.4. A graph G is said to be k-*colorable* if G admits a *proper* vertex coloring using at most k colors.

In considering the chromatic number of a graph, only the adjacency of vertices is taken into account. Hence, multiple edges and loops may be discarded while considering chromatic numbers, unless needed otherwise. As a consequence, we may restrict ourselves to simple graphs when dealing with (vertex) chromatic numbers.

It is clear that $\chi(K_n) = n$. Further, $\chi(G) = 2$ if and only if G is bipartite having at least one edge. In particular, $\chi(T) = 2$ for any tree T with at least one edge (since any tree is bipartite). Further (see Fig. 7.1),

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases} \tag{7.1}$$

Exercise 2.1. Prove $\chi(G) = 2$ if and only if G is a bipartite graph with at least one edge.

Exercise 2.2. Determine the chromatic number of

(i) The Petersen graph
(ii) Wheel W_n (see Sect. 1.7, Chap. 1)
(iii) The Herschel graph (see Fig. 5.4)
(iv) The Grötzsch graph (see Fig. 7.6)

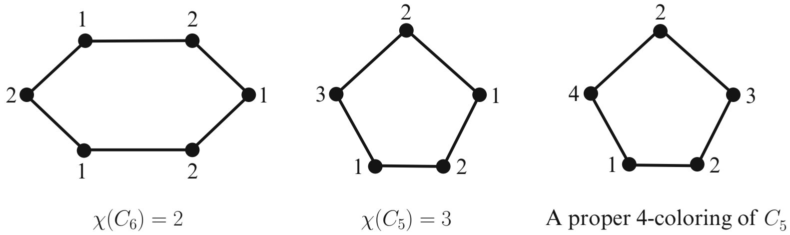


Fig. 7.1 Illustration of proper vertex coloring

We next consider another application of graph coloring. Let S be a set of students. Each student of S is to take a certain number of examinations for which he or she has registered. Undoubtedly, the examination schedule must be such that all students who have registered for a particular examination will take it at the same time.

Let $\mathbb{P}$ be *the set of examinations* and for $p \in \mathbb{P}$, let $S(p)$ be *the set of students* who have to take the examination p. Our aim is to draw up an examination schedule involving only the minimum number of days on the supposition that papers a and b can be given on the same day provided they have no common candidate and that no candidate shall have more than one examination on any day.

Form a graph $G = G(\mathbb{P}, E)$, where $a, b \in \mathbb{P}$ are adjacent if and only if $S(a) \cap S(b) \neq \emptyset$. Then each proper vertex coloring of G yields an examination schedule with the vertices in any color class representing the schedule on a particular day. Thus, $\chi(G)$ gives the minimum number of days required for the examination schedule.

Exercise 2.3. Draw up an examination schedule involving the minimum number of days for the following problem:

Set of students	Examination subjects
S_1	Algebra, real analysis, and topology
S_2	Algebra, operations research, and complex analysis
S_3	Real analysis, functional analysis, and complex analysis
S_4	Algebra, graph theory, and combinatorics
S_5	Combinatorics, topology, and functional analysis
S_6	Operations research, graph theory, and coding theory
S_7	Operations research, graph theory, and number theory
S_8	Algebra, number theory, and coding theory
S_9	Algebra, operations research, and real analysis

Exercise 2.4. If G is k-regular, prove that $\chi(G) \geq \frac{n}{n-k}$.

Theorem 7.2.5 gives upper and lower bounds for the chromatic number of a graph G in terms of its independence number and order.

Theorem 7.2.5. *For any graph G with n vertices and independence number α,*

$$\frac{n}{\alpha} \le \chi \le n - \alpha + 1.$$

Proof. There exists a chromatic partition $\{V_1, V_2, \ldots, V_\chi\}$ of V. Since each V_i is independent, $|V_i| \le \alpha$, $1 \le i \le \chi$. Hence, $n = \sum_{i=1}^{\chi} |V_i| \le \alpha\,\chi$, and this gives the inequality on the left.

To prove the inequality on the right, consider a maximum independent set S of α vertices. Then the subsets of $V \backslash S$ of cardinality 1 together with S yield a partition of V into $(n-\alpha)+1$ independent subsets. □

Remark 7.2.6. Unfortunately, none of the above bounds is a good one. For example, if G is the graph obtained by connecting C_{2r} with a disjoint K_{2r} $(r \ge 2)$, by an edge, we have $n = 4r$, $\alpha = r+1$, and $\chi = 2r$, and the above inequalities become $\frac{4r}{r+1} \le 2r \le 3r$. For a simple graph G, the number $\chi^c = \chi^c(G) = \chi(G^c)$, the chromatic number of G^c is the minimum number of subsets in a partition of $V(G)$ into subsets each inducing a complete subgraph of G. Bounds on the sum and product of $\chi(G)$ and $\chi^c(G)$ were obtained by Nordhaus and Gaddum [148] (see also reference [93]), as given in Theorem 7.2.7.

Theorem 7.2.7 (Nordhaus and Gaddum [148]). *For any simple graph G,*

$$2\sqrt{n} \le \chi + \chi^c \le n+1, \text{and} n \le \chi\,\chi^c \le \left(\frac{n+1}{2}\right)^2.$$

Proof. Let $\chi(G) = k$ and let $V_1, V_2, \ldots, V_k$ be the k color classes in a chromatic partition of G. Then $\sum_{i=1}^{k} |V_i| = n$, and so $max_{1 \le i \le k} |V_i| \ge \frac{n}{k}$. Since each V_i is an independent set of G, it induces a complete subgraph in G^c. Hence, $\chi^c \ge \max_{1 \le i \le k} |V_i|$, and so $\chi\,\chi^c = k\,\chi^c \ge k \circ \max_{1 \le i \le k} |V_i| \ge k \circ \frac{n}{k} = n$. Further, since the arithmetic mean of χ and χ^c is greater than or equal to their geometric mean, $\frac{\chi + \chi^c}{2} \ge \sqrt{\chi\,\chi^c} \ge \sqrt{n}$. Hence, $\chi + \chi^c \ge 2\sqrt{n}$. This establishes both the lower bounds.

To show that $\chi + \chi^c \le n+1$, we use induction on n. When $n = 1$, $\chi = \chi^c = 1$, and so we have equality in this case. So assume that $\chi + \chi^c \le (n-1)+1 = n$ for all graphs G having $n-1$ vertices, $n \ge 2$. Let H be any graph with n vertices, and let v be any vertex of H. Then $G = H - v$ is a graph with $n-1$ vertices and $G^c = (H-v)^c = H^c - v$. By the induction assumption, $\chi(G) + \chi(G^c) \le n$.

Now $\chi(H) \le \chi(G) + 1$ and $\chi(H^c) \le \chi(G^c) + 1$. If either $\chi(H) \le \chi(G)$ or $\chi(H^c) \le \chi(G^c)$, then $\chi(H) + \chi(H^c) \le \chi(G) + \chi(G^c) + 1 \le n+1$. Suppose then $\chi(H) = \chi(G) + 1$ and $\chi(H^c) = \chi(G^c) + 1$. $\chi(H) = \chi(G) + 1$ implies that removal of v from H decreases the chromatic number, and hence $d_H(v) \ge \chi(G)$. [If $d_H(v) < \chi(G)$, then in any proper coloring of G with $\chi(G)$ colors at most $\chi(G) - 1$ colors would have been used to color the neighbors of v in G, and hence v can be given one of the left-out colors, and therefore we have a coloring of H with $\chi(G)$ colors. Hence, $\chi(H) = \chi(G)$, a contradiction.] For a similar reason, $\chi(H^c) = \chi(G^c) + 1$ implies that $n - 1 - d_H(v) = d_{H^c}(v) \ge \chi(G^c)$; thus, $\chi(G) + \chi(G^c) \le d_H(v) + n - 1 - d_H(v) = n - 1$. This implies, however, that $\chi(H) + \chi(H^c) = \chi(G) + \chi(G^c) + 2 \le n+1$.

Finally, applying the inequality $\sqrt{\chi \chi^c} \leq \frac{\chi+\chi^c}{2}$, we get $\chi \chi^c \leq (\frac{\chi+\chi^c}{2})^2 \leq (\frac{n+1}{2})^2$. □

Note 7.2.8. Since the publication of Theorem 7.2.7, there had been similar results for other graph parameters (see, for instance, [115] for the domination number γ). All these results have now come to be known as Nordhaus–Gaddum inequalities, with reference to the parameters in question.

Exercise 2.5. For a simple graph G, prove that $\chi(G^c) \geq \alpha(G)$.

Exercise 2.6. Prove $\chi(G) \leq \ell + 1$, where ℓ is the length of a longest path in G. For each positive integer ℓ, give a graph G with chromatic number $\ell + 1$ and in which any longest path has length ℓ.

Exercise 2.7. Which numbers can be chromatic numbers of unicyclic graphs? Draw a unicyclic graph on 15 vertices with $\Delta = 3$ and having each of these numbers as its chromatic number.

Exercise 2.8. If G is connected and $m \leq n$, show that $\chi(G) \leq 3$.

Exercise 2.9. Let G_n be the graph defined by $V(G_n) = \{(i, j) : 1 \leq i < j \leq n\}$, and $E(G_n) = \{\big((i, j)(k, l)\big) : i < j = k < l\}$. Prove

(i) $\omega(G_n) = 2$.
(ii) $\chi(G_n) = \lceil \log_2 n \rceil$. [Note that $\chi(G_n) \to \infty$ as $n \to \infty$.]

Exercise 2.10. Prove that $\chi(G \Box H) = \max(\chi(G), \chi(H))$.

Exercise 2.11. Prove $\chi(G \times H) \leq \min(\chi(G), \chi(H)))$ (A celebrated conjecture of Hedetniemi [104] states that $\chi(G \times H) = \min(\chi(G), \chi(H)))$.

7.3 Critical Graphs

Definition 7.3.1. A graph G is called *critical* if for every proper subgraph H of G, $\chi(H) < \chi(G)$. Equivalently, $\chi(G - e) < \chi(G)$ for each edge e of G. Also, G is k*-critical* if it is k-chromatic and critical.

Remarks 7.3.2. If $\chi(G) = 1$, then G is either trivial or totally disconnected. Hence, G is 1-critical if and only if G is K_1. Again, $\chi(G) = 2$ implies that G is bipartite and has at least one edge. Hence, G is 2-critical if and only if G is K_2. For an odd cycle C, $\chi(C) = 3$, and if G contains an odd cycle C properly, G cannot be 3-critical.

Exercise 3.1. Prove that any critical graph is connected.

Exercise 3.2. Prove that for any graph G, $\chi(G - v) = \chi(G)$ or $\chi(G) - 1$ for any $v \in V$, and $\chi(G - e) = \chi(G)$ or $\chi(G) - 1$ for any $e \in E$.

Exercise 3.3. Show that if G is k-critical, $\chi(G - v) = \chi(G - e) = k - 1$ for any $v \in V$ and $e \in E$.

Exercise 3.4. [If $\chi(G - e) < \chi(G)$ for any e of G, G is sometimes called *edge-critical*, and if $\chi(G - v) < \chi(G)$ for any vertex v of G, G is called *vertex-critical*.] Show that a nontrivial connected graph is vertex-critical if it is edge-critical. Disprove the converse by a counterexample.

Exercise 3.5. Show that a graph is 3-critical if and only if it is an odd cycle. It is clear that any k-chromatic graph contains a k-critical subgraph. (This is seen by removing vertices and edges in succession, whenever possible, without diminishing the chromatic number.)

Theorem 7.3.3. *If G is k-critical, then $\delta(G) \geq k - 1$.*

Proof. Suppose $\delta(G) \leq k - 2$. Let v be a vertex of minimum degree in G. Since G is k-critical, $\chi(G - v) = \chi(G) - 1 = k - 1$ (see Exercise 3.3). Hence, in any proper $(k - 1)$-coloring of $G - v$, at most $(k - 2)$ colors would have been used to color the neighbors of v in G. Thus, there is at least one color, say c, that is left out of these $k - 1$ colors. If v is given the color c, a proper $(k - 1)$-coloring of G is obtained. This is impossible since G is k-chromatic. Hence, $\delta(G) \geq (k - 1)$. □

Corollary 7.3.4. *For any graph G, $\chi(G) \leq 1 + \Delta(G)$.*

Proof. Let G be a k-chromatic graph, and let H be a k-critical subgraph of G. Then $\chi(H) = \chi(G) = k$. By Theorem 7.3.3, $\delta(H) \geq k - 1$, and hence $k \leq 1 + \delta(H) \leq 1 + \Delta(H) \leq 1 + \Delta(G)$. □

Exercise 3.6. Give another proof of Corollary 7.3.4 by using induction on $n = |V(G)|$.

Exercise 3.7. If $\chi(G) = k$, show that G contains at least k vertices each of degree at least $k - 1$.

Exercise 3.8. Prove or disprove: If G is k-chromatic, then G contains a K_k.

Exercise 3.9. Prove: Any k ($\geq$ 2)-critical graph contains a $(k - 1)$-critical subgraph.

Exercise 3.10. For each of the graphs G of Exercise 2.2, find a critical subgraph H of G with $\chi(H) = \chi(G)$.

Exercise 3.11. Prove that the wheel $W_{2n-1} = C_{2n-1} \vee K_1$ is a 4-critical graph for each $n \geq 2$. Does a similar statement apply to W_{2n}?

Theorem 7.3.5. *In a critical graph G, no vertex cut is a clique.*

Proof. Suppose G is a k-critical graph and S is a vertex cut of G that is a clique of G (i.e., a complete subgraph of G). Let H_i, $1 \leq i \leq r$, be the components of $G \backslash S$, and let $G_i = G[V(H_i) \cup S]$. Then each G_i is a proper subgraph of G and hence admits a proper $(k - 1)$-coloring. Since S is a clique, its vertices must receive distinct colors in any proper $(k - 1)$-coloring of G_i. Hence, by fixing the colors for the vertices of S, and coloring for each i the remaining vertices of G_i so as to give a proper $(k - 1)$-coloring of G_i, we obtain a proper $(k - 1)$-coloring of G. This contradicts the fact that G is k-chromatic (see Fig. 7.2). □

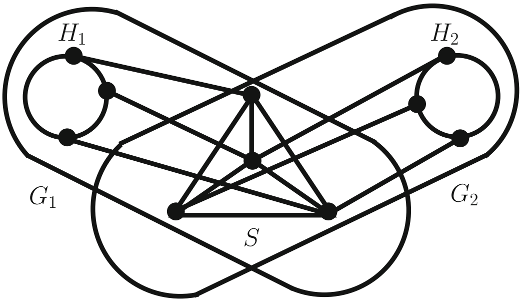


Fig. 7.2 $G[S] \simeq K_4$ $(r = 2)$

Corollary 7.3.6. *Every critical graph is a block.*

Exercise 3.12.* Prove that every k-critical graph is $(k-1)$-edge connected (Dirac [53]).

Exercise 3.13. Show by means of an example that criticality is essential in Exercise 3.12; that is, a k-chromatic graph need not be $(k-1)$-edge connected.

7.3.1 Brooks' Theorem

We next consider *Brooks'* [31] *theorem*. Recall Corollary 7.3.4, which states that $\chi(G) \le 1 + \Delta(G)$. If G is an odd cycle, $\chi(G) = 3 = 1 + 2 = 1 + \Delta(G)$, and if G is a complete graph, say K_k, $\chi(G) = k = 1 + (k-1) = 1 + \Delta(G)$. That these are the only extremal families of graphs for which $\chi(G) = 1 + \Delta(G)$ is the assertion of Brooks' theorem.

Theorem 7.3.7 (Brooks' theorem). *If a connected graph G is neither an odd cycle nor a complete graph, then $\chi(G) \le \Delta(G)$.*

Proof. If $\Delta(G) \le 2$, then G is either a path or a cycle. For a path G (other than K_1 and K_2), and for an even cycle G, $\chi(G) = 2 = \Delta(G)$. According to our assumption, G is not an odd cycle. So let $\Delta(G) \ge 3$.

The proof is by contradiction. Suppose the result is not true. Then there exists a minimal graph G of maximum degree $\Delta(G) = \Delta \ge 3$ such that G is not Δ-colorable, but for any vertex v of G, $G - v$ is Δ-colorable.

Claim 1. Let v be any vertex of G. Then in any proper Δ-coloring of $G - v$, all the Δ colors must be used for coloring the neighbors v in G. Otherwise, if some color i is not represented in $N_G(v)$, then v could be colored using i, and this would give a Δ-coloring of G, a contradiction to the choice of G. Thus, G is a Δ-regular graph satisfying Claim 1.

For $v \in V(G)$, let $N(v) = \{v_1, v_2, \ldots, v_\Delta\}$. In a proper Δ-coloring of $G - v = H$, let v_i receive color i, $1 \le i \le \Delta$. For $i \ne j$, let H_{ij} be the subgraph of H induced by the vertices receiving the ith and jth colors.

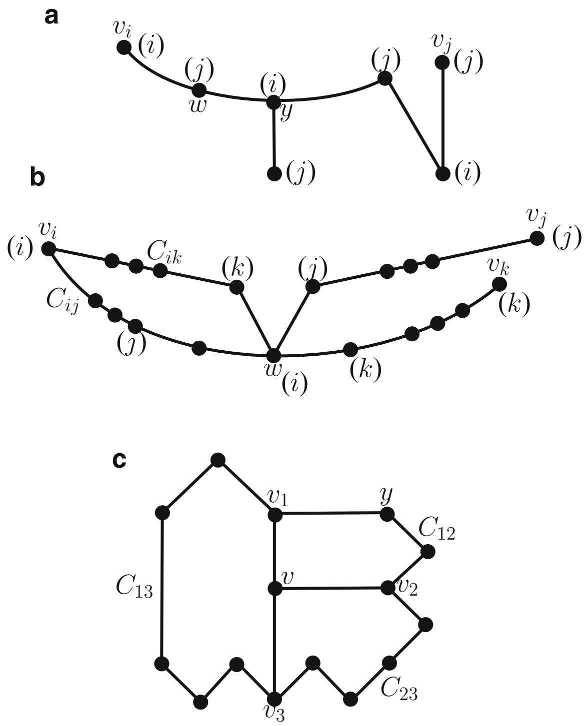


Fig. 7.3 Graphs for proof of Theorem 7.3.7 (The numbers inside the parentheses denote the vertex colors)

Claim 2. v_i and v_j belong to the same component of H_{ij}. Otherwise, the colors i and j can be interchanged in the component of H_{ij} that contains the vertex v_j. Such an interchange of colors once again yields a proper Δ-coloring of H. In this new coloring, both v_i and v_j receive the same color, namely, i, a contradiction to Claim 1. This proves Claim 2.

Claim 3. If C_{ij} is the component of H_{ij} containing v_i and v_j, then C_{ij} is a path in H_{ij}. As before, $N_H(v_i)$ contains exactly one vertex of color j. Further, C_{ij} cannot contain a vertex, say y, of degree at least 3; for, if y is the first such vertex on a $v_i - v_j$ path in C_{ij} that has been colored, say, with i, then at least three neighbors of y in C_{ij} have the color j. Hence, we can recolor y in H with a color different from both i and j, and in this new coloring of H, v_i and v_j would belong to distinct components of H_{ij} (see Fig. 7.3a). (Note that by our choice of y, any $v_i - v_j$ path in H_{ij} must contain y.) But this contradicts Claim 3.

Claim 4. $C_{ij} \cap C_{ik} = \{v_i\}$ for $j \neq k$. Indeed, if $w \in C_{ij} \cap C_{ik}$, $w \neq v_i$, then w is adjacent to two vertices of color j on C_{ij} and two vertices of color k on C_{ik} (see Fig. 7.3b). Again, we can recolor w in H by giving a color different from the colors of the neighbors of w in H. In this new coloring of H, v_i and v_j belong to distinct components of H_{ij}, a contradiction to Claim 2. This completes the proof of Claim 4.

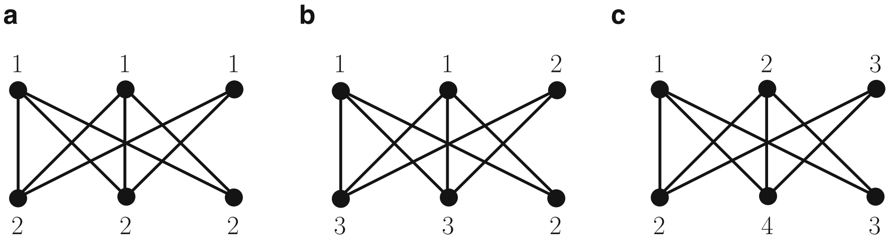


Fig. 7.4 Different colorings of $K_{3,3} - e$

We are now in a position to complete the proof of the theorem. By hypothesis, G is not complete. Hence, G has a vertex v, and a pair of nonadjacent vertices v_1 and v_2 in $N_G(v)$ (see Exercise 5.11, Chap. 1). Then the $v_1 - v_2$ path C_{12} in H_{12} of $H = G - v$ contains a vertex y ($\neq v_2$) adjacent to v_1. Naturally, y would receive color 2. Since $\Delta \geq 3$, by Claim 1, there exists a vertex $v_3 \in N_G(v)$. Now interchange colors 1 and 3 in the path C_{13} of H_{13}. This would result in a new coloring of $H = G - v$. Denote the v_i-v_j path in H under this new coloring by C'_{ij} (see Fig. 7.3c). Then $y \in C'_{23}$ since v_1 receives color 3 in the new coloring (whereas y retains color 2). Also, $y \in C_{12} - v_1 \subset C'_{12}$. Thus, $y \in C'_{23} \cap C'_{12}$. This contradicts Claim 4 (since $y \neq v_2$), and the proof is complete. □

7.3.2 Other Coloring Parameters

There are several other vertex coloring parameters of a graph G. We now mention three of them. Let f be a k-coloring (not necessarily proper) of G, and let $(V_1, V_2, \ldots, V_k)$ be the color classes of G induced by f. Coloring f is *pseudo-complete* if between any two distinct color classes, there is at least one edge of G. f is *complete* if it is pseudocomplete and each V_i, $1 \leq i \leq k$, is an independent set of G. Thus, $\chi(G)$ is the minimum k for which G has a complete k-coloring f.

Definition 7.3.8. The *achromatic number* $\mathrm{a}(G)$ of a graph G is the maximum k for which G has a complete k-coloring.

Definition 7.3.9. The *pseudoachromatic number* $\psi(G)$ of G is the maximum k for which G has a pseudocomplete k-coloring.

Example 7.3.10. Figure 7.4 gives (a) a chromatic, (b) an achromatic, and (c) a pseudoachromatic coloring of $K_{3,3} - e$.

It is clear that for any graph G, $\chi(G) \leq \mathrm{a}(G) \leq \psi(G)$.

Exercise 3.14. Let G be a graph and H a subgraph of G. Prove that $\chi(H) \leq \chi(G)$ and $\psi(H) \leq \psi(G)$. Show by means of an example that $\mathrm{a}(H) \leq \mathrm{a}(G)$ need not always be true.

Exercise 3.15. Prove

(i) $\psi(\psi - 1) \leq 2m$.
(ii) $\psi(K_a \vee K_b^c) = a + 1$.

From (ii) deduce that for any graph, $\psi \leq n - \alpha + 1$.

Exercise 3.16. If G has a complete coloring using k colors, prove that $k \leq \frac{1+\sqrt{1+8m}}{2}$. ($m$ = size of G).

Exercise 3.17. Prove that for a complete bipartite graph G, $\mathrm{a}(G) = 2$.

Exercise 3.18. What is the minimum number of edges that a connected graph with pseudoachromatic number ψ can have? Construct one such tree.

Exercise 3.19. If G is a subgraph of H, prove that $\psi(G) \leq \psi(H)$.

Exercise 3.20. Prove: $\psi(K_{n,n}) = n + 1$.

7.3.3 b-Colorings

Definition 7.3.11. A b-*coloring* of a graph G is a proper coloring with the additional property that each color class contains a color-dominating vertex (c.d.v.), that is, a vertex that has a neighbor in all the other color classes. The b-*chromatic number* of G is the largest k such that G has a b-coloring using k colors; it is denoted by $\mathrm{b}(G)$.

The concept of b-coloring was introduced by Irving and Manlove [111].

Exercise 3.21 guarantees the existence of the b-chromatic number for any graph G and shows that $\chi(G) \leq \mathrm{b}(G)$. Note that $\mathrm{b}(K_n) = n$ while $\mathrm{b}(K_{m,n}) = 2$.

Exercise 3.21. Show that the chromatic coloring of a graph G is a b-coloring of G.

Exercise 3.22. Prove that $K_{n,n} - F$, $n \geq 2$, where F is a 1-factor of $K_{n,n}$, has a b-coloring using 2 colors and n colors but none with k colors for any k in $2 < k < n$.

Exercise 3.23. Prove $\mathrm{b}(G) \leq 1 + \Delta(G)$. A better upper bound for $\mathrm{b}(G)$ is given in the next exercise.

Exercise 3.24. Let $d_1 \leq d_2 \leq \ldots \leq d_n$ be the degree sequence of the graph G with vertex set $V = \{v_1, \ldots, v_n\}$, and $d_i = d(v_i)$, $1 \leq i \leq n$. Let $M(G) = \max\{i : d_i \geq i - 1, 1 \leq i \leq n\}$. Prove that $\mathrm{b}(G) \leq M(G)$. Show further that the number of vertices of degree at least $M(G)$ in G is at most $M(G)$.

Exercise 3.25. Let Q_p be the hypercube of dimension p. Prove $\mathrm{b}(Q_1) = \mathrm{b}(Q_2)=2$, and $\mathrm{b}(Q_3) = 4$. [A result of Kouider and Mahéo [125] states that for $p \geq 3$, $\mathrm{b}(Q_p) = p + 1$.]

We complete this section by presenting a result of Kratochvíl, Tuza, and Voigt [126] that characterizes graphs with b-chromatic number 2. Let G be a bipartite graph with bipartition (X, Y). A vertex $x \in X$ (respectively, $y \in Y$) is called a *full vertex* (or a *charismatic vertex*) of X (respectively, Y) if it is adjacent to all the vertices of Y (respectively, X).

Theorem 7.3.12 ([126]). *Let G be a nontrivial connected graph. Then* $\mathrm{b}(G) = 2$ *if and only if G is bipartite and has a full vertex in each part of the bipartition.*

Proof. Suppose G is bipartite and has a full vertex in each part, say $x \in X$ and $y \in Y$. Naturally, in any b-coloring, the color class containing x, say W_1, is a subset of X and that containing y, say W_2, is a subset of Y. If G has a third color class W_3 disjoint from W_1 and W_2, then W_3 must have a c.d.v. adjacent to a vertex of W_1 and a vertex of W_2. This is impossible, as G is bipartite. Therefore, $\mathrm{b}(G) = 2$.

Conversely, let $\mathrm{b}(G) = 2$. Then $\chi(G) = 2$ and therefore G is bipartite. Let (X, Y) be the bipartition of G. Assume that G does not have a full vertex in at least one part, say, X. Let $x_1 \in X$. As x_1 is not a full vertex, there exists a vertex $y_1 \in Y$ to which it is not adjacent. Let X_1 be the maximal subset of X such that $V_1 = X_1 \cup \{y_1\}$ is independent in G. Now choose a new vertex $x_2 \in X \setminus X_1$. Again, as X has no full vertex, we can find a $y_2 \in Y \setminus \{y_1\}$ to which x_2 is not adjacent. Let X_2 be the maximal subset of $X \setminus X_1$ such that $V_2 = X_2 \cup \{y_2\}$ is independent in G. In this way, all the vertices of X would be exhausted and let $V_1, V_2, \ldots, V_k$ be the independent sets thus formed. Also, let Y_0 denote the set of uncovered vertices of Y, if any. Since G is connected, $G \neq \langle V_i \cup V_j \rangle$, and $G \neq \langle V_l \cup Y_0 \rangle$, $i, j, l \in \{1, 2, \ldots, k\}$. Hence, $k \geq 2$ when $Y_0 \neq \emptyset$ and $k \geq 3$ when $Y_0 = \emptyset$. Thus, the partition $V = V_1 \cup V_2 \cup \ldots \cup V_k \cup \{V_{k+1} = Y_0\}$ has at least 3 parts. If each of these parts has a c.d.v., we get a contradiction to the fact that $\mathrm{b}(G) = 2$. If not, assume that the class V_l has no c.d.v. Then for each vertex x of V_l, there exists a color class V_j, $j \neq l$, having no neighbor of x. Then x could be moved to the class V_j. In this way, the vertices in V_l can be moved to the other V_i's without disturbing independence. Let us call the new classes $V_1', V_2', \ldots, V_{l-1}', V_{l+1}', \ldots, V_{k+1}'$. If each of these color classes contains a c.d.v., we get a contradiction as $k \geq 3$. Otherwise, argue as before and reduce the number of color classes. As G is connected, successive reductions should end up in at least three classes, contradicting the hypothesis that $\mathrm{b}(G) = 2$. □

A description of several other coloring parameters can be found in Jensen and Toft [116].

7.4 Homomorphisms and Colorings

Homomorphisms of graphs generalize the concept of graph colorings.

Definition 7.4.1. Let G and H be simple graphs. A *homomorphism* from G to H is a map $f : V(G) \to V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. The map f is an *isomorphism* if f is bijective and $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$.

We write $f : G \to H$ to denote the fact that f is a homomorphism from G to H and write $G \simeq H$ to denote that G is isomorphic to H. If $f : G \to H$ is a graph homomorphism, then $\langle \{f(x) : x \in V(G)\} \rangle$, the subgraph induced in H by the image set $f(V(G))$ is the image of f. If $f(V(G)) = V(H)$, f is an onto-homomorphism. If $f : G \to H$, then for any vertex v of H, $f^{-1}(v)$ is an independent set of G. (If $f^{-1}(v) = \emptyset$, then $f^{-1}(v)$ is an independent subset of $V(G)$, while if $f^{-1}(v)$ contains an edge uw, then $f(u) = v = f(w)$, and hence H has a loop at v, a contradiction to the fact that H is a simple graph).

Lemma 7.4.2. *Let G_1, G_2, and G_3 be graphs and let $f_1 : G_1 \to G_2$, and $f_2 : G_2 \to G_3$ be homomorphisms. Then $f_2 \circ f_1 : G_1 \to G_3$ is also a homomorphism. [Here $(f_2 \circ f_1)(g) = f_2(f_1(g))$.]*

Proof. Follows by direct verification. □

A graph homomorphism is a generalization of graph coloring. Suppose G ia a given graph and there exists a homomorphism $f : G \to K_k$, where k is the least positive integer with this property. Then f is onto and the sets $S_i = \{f^{-1}(v_i) : v_i \in K_k\}$, $1 \le i \le k$, form a partition of $V(G)$. Moreover, between any two sets S_i and S_j, $i \neq j$, there must be an edge of G. Otherwise, $A = S_i \cup S_j$ is an independent set of G, and we can define a homomorphism from G to K_{k-1} by mapping A to the same vertex of K_{k-1}. Thus, $\chi(G) = k$. We state this result as a theorem.

Theorem 7.4.3. *Let G be a simple graph. Suppose there exists a homomorphism $f : G \to K_k$, and let k be the least positive integer with this property. Then $\chi(G) = k$.*

Corollary 7.4.4. *If there exists a homomorphism $f : G \to K_p$, then $\chi(G) \le p$.*

Corollary 7.4.5. *Let $f : G \to H$ be a graph homomorphism. Then $\chi(G) \le \chi(H)$.*

Proof. Let $\chi(H) = k$. Then there exists a homomorphism $g : H \to K_k$. By Lemma 7.4.2, $g \circ f : G \to K_k$ is a homomorphism. Now apply Corollary 7.4.4. □

Example 7.4.6. Let $V_1 = \{u_1, \ldots, u_7\}$ and $V_2 = \{v_1, \ldots, v_5\}$ be the vertex sets of the cycles C_7 and C_5, respectively. Then the map $f(u_1) = v_1$, $f(u_2) = v_2$, $f(u_3) = v_3$, $f(u_4) = v_2$, $f(u_5) = v_3$, $f(u_6) = v_4$, and $f(u_7) = v_5$ is a homomorphism of C_7 to C_5.

7.4.1 Quotient Graphs

Let $f : G \to H$ be a graph homomorphism from G onto H. Let $V(H) = \{v_1, \ldots, v_k\}$, and $S_i = f^{-1}(v_i)$, $1 \le i \le k$. Then no S_i is empty. The quotient graph G/f is defined to be the graph with the sets S_i as its vertices and in which two vertices S_i and S_j are adjacent if $v_i v_j \in E(H)$. This defines a natural isomorphism $\tilde{f} : G/f \simeq H$.

A consequence of the above remarks is the fact that a complete k-coloring of G is just a homomorphism of G onto K_k. Recall that both the chromatic and achromatic colorings are complete colorings. We now establish the coloring interpolation theorem for the complete coloring.

Theorem 7.4.7 (Interpolation theorem for complete coloring). *If a graph G admits a complete k-coloring and a complete l-coloring, then it admits a complete i-coloring for every i between k and l.*

Proof. Let $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_l$ be the color partitions in the two complete colorings. We assume without loss of generality that $k < l$. Clearly, it suffices to construct a complete $(k+1)$-coloring of G. For each $i = 0, 1, 2, \ldots, l$, let $C_i = \bigcup_{1 \le j \le i} B_j$. Let Θ_i denote the partition of $V(G)$ by the nonempty sets of the sequence $B_1, B_2, \ldots, B_i$; $A_1 - C_i, A_2 - C_i, \ldots, A_k - C_i$. The partition Θ_0 has parts $A_1, A_2, \ldots, A_k$; the partition Θ_l has parts $B_1, B_2, \ldots, B_l$ (since $C_l = V(G)$, $A_i - C_l = \emptyset$ for each j). Hence, $G/\Theta_0 \simeq K_k$ and $G/\Theta_l \simeq K_l$. Hence, there must exist a first suffix j, $0 < j \le l$, such that G/Θ_j is not k-colorable. By the choice of j, this implies that G/Θ_j is $(k+1)$-colorable since we can simply color B_j by the $(k+1)$-st color, and hence by Lemma 7.4.2, G is $(k+1)$-colorable. (Just compose the two onto homomorphisms $G \to G/\Theta_j \to K_{k+1}$.) □

Exercise 3.22 shows that an interpolation theorem similar to that of complete coloring does not hold good for the b-coloring.

Exercise 4.1. Let $f : G \to H$ be a graph homomorphism and let $x, y \in V(G)$. Prove $d_H(x, y) \le d_G(x, y)$.

Exercise 4.2. Assume that there exists a homomorphism from G onto C_k, where k is odd. Show that G must contain an odd cycle. Show by means of an example that a similar statement need not hold good if k is even.

Exercise 4.3. Prove that there exists a homomorphism from C_{2l+1} to C_{2k+1} if and only if $l \le k$.

7.5 Triangle-Free Graphs

Definition 7.5.1. A graph G is *triangle-free* if G contains no K_3.

Remark 7.5.2. Triangle-free graphs cannot contain a K_k, $k \ge 3$, either. It is obvious that if a graph G contains a clique of size k, then $\chi(G) \ge k$. However, the converse is not true. That is, if the chromatic number of G is large, then G need not contain a clique of large size. The construction of triangle-free k-chromatic graphs, for $k \ge 3$, was raised in the middle of the 20th century. In answer to this question, Mycielski [144] developed an interesting graph transformation known as the *Mycielskian* of a graph.

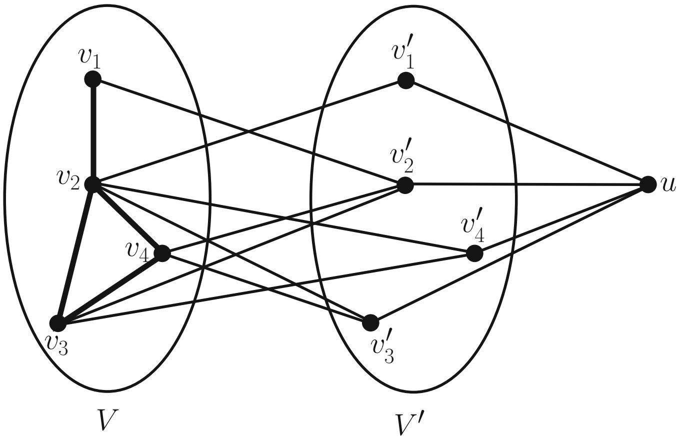


Fig. 7.5 $\mu(K_{1,3}+e)$

Definition 7.5.3. Let G be a finite simple connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *Mycielskian* $\mu(G)$ of G is defined as follows: The vertex set $V(\mu(G))$ of $\mu(G)$ is the disjoint union $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and the edge set of $\mu(G)$ is $E(\mu(G)) = E \cup \{x'y : xy \in E\} \cup \{x'u : x' \in V'\}$.

We denote $V(\mu(G))$ by the triad $\{V, V', u\}$. For $x \in V$, we call $x' \in V'$, the twin of x in $\mu(G)$, and vice versa, and u, the root of $\mu(G)$. Figure 7.5 displays the Mycielskian $\mu(K_{1,3}+e)$.

Remark 7.5.4. The following facts about $\mu(G)$, where G is of order n and size m, are obvious:

(i) $|V(\mu(G))| = 2n + 1$.
(ii) For each $v \in V$, $d_{\mu(G)}(v) = 2d_G(v)$.
(iii) For each $v' \in V'$, $d_{\mu(G)}(v') = d_G(v) + 1$.
(iv) $d_{\mu(G)}(u) = n$.

We now establish some basic results concerning the Mycielskian.

Theorem 7.5.5. $\chi(\mu(G)) = \chi(G) + 1$.

Proof. Assume that $\chi(G) = k$. Consider a proper (vertex) k-coloring c of G using the colors, say, $1, 2, \ldots, k$. We now give a proper $(k+1)$-coloring c' for $\mu(G)$. For $v \in V$, set $c'(v) = c(v)$. For the twin $v' \in V'$, set $c'(v') = c(v)$. For the root u of $\mu(G)$, set $c'(u) = k + 1$. Then c' is a proper coloring for $\mu(G)$ using $k+1$ colors and therefore $\chi(\mu(G)) \le k + 1$. [c' is proper because for any edge xy', $c'(x) = c(x) \neq c(y) = c'(y')$.] We now show that it is actually $k + 1$.

Suppose $\mu(G)$ has a proper k-coloring c'' using the colors $1, 2, \ldots, k$. Assume, without loss of generality, that $c''(u) = 1$. Then for any $v' \in V'$, $c''(v') \neq 1$. Recolor each vertex of V that has been colored by 1 in c'' by the color of its twin under c''.

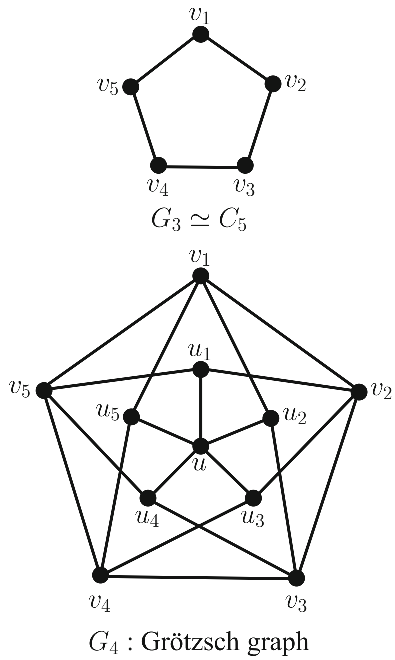


Fig. 7.6 The Grötzsch graph, $\mu(C_5)$

Then this gives a proper coloring of V using the $k-1$ colors $2, 3, \ldots, k$. This is impossible as $\chi(G) = k$. This proves that $\chi(\mu(G)) = k + 1 = \chi(G) + 1$. □

Theorem 7.5.6. *If G triangle-free, then $\mu(G)$ is also triangle-free.*

Proof. Assume that G is triangle-free. If $\mu(G)$ contains a triangle, it can only be of the form vwz', where $v \in V$, $w \in V$, and $z' \in V'$, so that vz' and wz' are edges of $\mu(G)$. This means, by the definition of $\mu(G)$, that vz and wz are edges of G and hence vwz is a triangle in G, a contradiction. □

Theorem 7.5.7 (Mycielski [144]). *For any positive integer p, there exists a triangle-free graph with chromatic number p.*

Proof. For $p = 1, 2$, the result is trivial. [For $p = 1$, take $G = K_1$, and for $p = 2$, take $G = K_2$. For $p = 3$, take $G = \mu(K_2)$. $\mu(K_2) = C_5$ is triangle-free and $\chi(C_5) = 3$.] For $p \geq 3$, by Theorems 7.5.5 and 7.5.6, the iterated Mycielskian $\mu^{p-2}(K_2) = \mu(\mu^{p-3}(K_2))$ is triangle-free and has chromatic number p. □

Remark 7.5.8. The graph $\mu^2(K_2) = \mu(C_5)$ is the Grötzsch graph of Fig. 7.6.

Theorem 7.5.9. *If G is critical, then so is $\mu(G)$.*

Proof. Assume that G is k-critical. Since by Theorem 7.4.5, $\chi(\mu(G)) = k + 1$, we have to show that $\mu(G)$ is $(k + 1)$-critical.

Start with a $(k+1)$-coloring c with colors $1, 2, \ldots, k+1$ of $\mu(G)$ with vertex set $\{V, V', u\}$.

We first show that $\chi(\mu(G) - u) = k$. Without loss of generality, assume that $c(u) = 1$. Then 1 is not represented in V'. Let S be the set of vertices receiving the color 1 in V under c. Recolor each vertex v of S by the color of its twin $v' \in V'$. This gives a proper coloring of $\mu(G)-u$ using k colors and hence $\chi(\mu(G)-u) = k$. [Recall that adjacency of v and w in G implies adjacency of vw' and $v'w$ in $\mu(G)$.]

Next remove a vertex v' of V' from $\mu(G)$. Without loss of generality, assume that $c(u) = 1$ and $c(v') = 2$. Now recolor the vertices of $G - v$ by the $k-1$ colors $3, \ldots, k, k+1$ (this is possible as G is k-critical) and recolor the vertices of $V' - v'$, if necessary, by the colors of their twins in $V - v$. Also, give color 1 to v. This coloring of $\mu(G) - v'$ misses the color 2 and gives a proper k-coloring to $\mu(G) - v'$.

Lastly, we give a k-coloring to $\mu(G) - v$, $v \in V$. Color the vertices of $G - v$ by $1, 2, \ldots, k-1$ so that the resulting coloring of $G - v$ is proper. Let A be the subset of $G - v$ whose vertices have received color 1 in this new coloring and $A' \subset V'$ denote the set of twins of the vertices in A. Now color the vertices of $(V' \backslash A') - v'$ by the colors of their twins in G, the vertices of $A' \cup \{v'\}$ by color k, and u by color 1. This coloring is a proper coloring of $\mu(G) - v$, which misses the color $k+1$ in the list $\{1, 2, \ldots, k+1\}$. Thus, $\mu(G)$ is $(k+1)$-critical. □

Remark 7.5.10. Apply Theorem 7.5.12 to observe that for each $k \geq 1$, there exists a k-critical triangle-free graph. Not every k-critical graph is triangle-free; for example, the complete graph K_k $(k \geq 3)$ is $k-$critical but is not triangle-free.

Lemma 7.5.11. *Let $f : G \to H$ be a graph isomorphism of G onto H. Then $f(N_G(x)) = N_H(f(x))$. Further, $G - x \simeq H - f(x)$, and $G - N_G[x] \simeq H - N_H[f(x)]$ under the restriction maps of f to the respective domains.*

Proof. The proof follows from the definition of graph isomorphism. □

Theorem 7.5.12 ([13]). *For connected graphs G and H, $\mu(G) \simeq \mu(H)$ if and only if $G \simeq H$.*

Proof. If $G \simeq H$, then trivially $\mu(G) \simeq \mu(H)$. So assume that G and H are connected and that $\mu(G) \simeq \mu(H)$. When $n = 2$ or 3, the result is trivial. So assume that $n \geq 4$. If G is of order n, then $\mu(G)$ and $\mu(H)$ are both of order $2n+1$, and so H is also of order n. Let $f : \mu(G) \to \mu(H)$ be the given isomorphism, where $V(\mu(G))$ and $V(\mu(H))$ are given by the triads (V_1, V_1', u_1) and (V_2, V_2', u_2), respectively.

We look at the possible images of the root u_1 of $\mu(G)$ under f. Both u_1 and u_2 are vertices of degree n. If $f(u_1) = u_2$, then by Lemma 7.5.11, $G = \mu(G) - N[u_1] \simeq \mu(H) - N[u_2] = H$.

Next we claim that $f(u_1) \notin V_2$. Suppose $f(u_1) \in V_2$. Since $d_{\mu(H)}(f(u_1)) = d_{\mu(G)}(u_1) = n$, it follows from the definition of the Mycielskian that in $\mu(H)$, $\frac{n}{2}$ neighbors of $f(u_1)$ belong to V_2 while another $\frac{n}{2}$ neighbors (the twins) belong to V_2'. (This forces n to be even.) These n neighbors of $f(u_1)$ form an independent subset of $\mu(H)$. Then $H' = \mu(H) - N_{\mu(H)}[f(u_1)] \simeq \mu(G) - N_{\mu(G)}[u_1] = G$.

Now if $x \in V_2$ is adjacent to $f(u_1)$ in $\mu(H)$, then x is adjacent to $f(u_1)'$, the twin of $f(u_1)$ belonging to V_2' in $\mu(H)$. Further, $d_{H'}(f(u_1)') = 1 = d_G(v)$, where $v \in V_1$ (the vertex set of G) corresponds to $f(u_1)'$ in $\mu(H)$. But then $d_{\mu(G)}(v) = 2$, while $d_{\mu(H)}(f(u)') = \frac{n}{2} + 1 > 2$, as $n \geq 4$. Hence, this case cannot arise.

Finally, suppose that $f(u_1) \in V_2'$. Set $f(u_1) = y'$. Then y, the twin of y' in $\mu(H)$, belongs to V_2. As $d_{\mu(G)}(u_1) = n$, $d_{\mu(H)}(y') = n$. The vertex y' has $n-1$ neighbors in V_2, say, $x_1, x_2, \ldots, x_{n-1}$. Then $N_H(y) = \{x_1, x_2, \ldots, x_{n-1}\}$, and hence y is also adjacent to $x_1', x_2', \ldots, x_{n-1}'$ in V_2'. Further, as $N_{\mu(G)}(u_1)$ is independent, $N_{\mu(H)}(y')$ is also independent. Therefore, $H =$ star $K_{1,n-1}$ consisting of the edges $yx_1, yx_2, \ldots, yx_{n-1}$. Moreover, $G = \mu(G) - N[u_1] \simeq \mu(H) - N[y'] =$ star $K_{1,n-1}$ consisting of the edges $yx_1', yx_2', \ldots, yx_{n-1}'$. Thus, $G \simeq K_{1,n-1} \simeq H$. □

7.6 Edge Colorings of Graphs

7.6.1 The Timetable Problem

Suppose in a school there are r teachers, $T_1, T_2, \ldots, T_r$, and s classes, $C_1, C_2, \ldots, C_s$. Each teacher T_i is expected to teach the class C_j for p_{ij} periods. It is clear that during any particular period, no more than one teacher can handle a particular class and no more than one class can be engaged by any teacher. Our aim is to draw up a timetable for the day that requires only the minimum number of periods. This problem is known as the "timetable problem."

To convert this problem into a graph-theoretic one, we form the bipartite graph $G = G(T, C)$ with bipartition (T, C), where T represents the set of teachers T_i and C represents the set of classes C_j. Further, T_i is made adjacent to C_j in G with p_{ij} parallel edges if and only if teacher T_i is to handle class C_j for p_{ij} periods. Now color the edges of G so that no two adjacent edges receive the same color. Then the edges in a particular color class, that is, the edges in that color, form a matching in G and correspond to a schedule of work for a particular period. Hence, the minimum number of periods required is the minimum number of colors in an edge coloring of G in which adjacent edges receive distinct colors; in other words, it is the edge-chromatic number of G. We now present these notions as formal definitions.

Definition 7.6.1. An *edge coloring* of a loopless graph G is a function $\pi : E(G) \to S$, where S is a set of distinct colors; it is *proper* if no two adjacent edges receive the same color. Thus, a proper edge coloring π of G is a function $\pi : E(G) \to S$ such that $\pi(e) \neq \pi(e')$ whenever edges e and e' are adjacent in G, and it is a proper k-edge coloring of G if $|S| = k$.

Definition 7.6.2. The minimum k for which a loopless graph G has a proper k-edge coloring is called the *edge-chromatic number* or *chromatic index* of G. It is denoted by $\chi'(G)$. G is *k-edge-chromatic* if $\chi'(G) = k$.

Further, if an edge uv is colored by color c, we say that c is represented at both u and v. If G has a proper k-edge coloring, $E(G)$ is partitioned into k edge-disjoint matchings.

It is clear that for any (loopless) graph G, $\chi'(G) \geq \Delta(G)$ since the $\Delta(G)$ edges incident at a vertex v of maximum degree $\Delta(G)$ must all receive distinct colors. For bipartite graphs, however, equality holds.

Theorem 7.6.3 (König). *If G is a bipartite graph, $\chi'(G) = \Delta(G)$.*

Proof. The proof is by induction on the size (i.e., number of edges) m of G. The result is true for $m = 1$. Assume the result for bipartite graphs of size at most $m-1$. Let G have m edges. Let $e = uv \in E(G)$. Then $G-e$ has [since $\Delta(G-e) \leq \Delta(G)$] a proper Δ-edge coloring, say c. Out of these Δ colors, suppose that one particular color is not represented at both u and v. Then in this coloring the edge uv can be colored with this color, and a proper Δ-edge coloring of G is obtained.

In the other case (that is, in the case in which each of the Δ colors is represented either at u or at v in $G-e$), since the degrees of u and v in $G-e$ are at most $\Delta - 1$, there exists a color out of the Δ colors that is not represented in $G-e$ at u, and similarly there exists a color not represented at v. Thus, if color j is not represented at u in c, then j is represented at v in c, and if color i is not represented at v in c, then i is represented at u in c. Since G is bipartite and u and v are not in the same parts of the bipartition, there can exist no u-v path in G in which the colors alternate between i and j.

Let P be a maximal path in $G-e$ starting from u in which the colors of the edges alternate between i and j. Interchange the colors i and j in P. This would still yield a proper edge coloring of $G-e$ using the Δ colors in which color i is not represented at both u and v. Now color the edge uv by the color i. This results in a proper Δ-edge coloring of G. □

Exercise 6.1. Disprove the converse of Theorem 7.6.3 by a counterexample.

Next, we determine the chromatic index of the complete graphs.

Theorem 7.6.4. $\chi'(K_n) = \begin{cases} n-1 & \textit{if } n \textit{ is even,} \\ n & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. (Berge) Since K_n is regular of degree $n-1$, $\chi'(K_n) \geq n-1$.

Case 1. n is even. We show that $\chi'(K_n) \leq n-1$ by exhibiting a proper $(n-1)$-edge coloring of K_n. Label the n vertices of K_n as $0, 1, \ldots, n-1$. Draw a circle with center at 0 and place the remaining $n-1$ numbers on the circumference of the circle so that they form a regular $(n-1)$-gon (Fig. 7.7). Then the $\frac{n}{2}$ edges $(0,1)$, $(2, n-1)$, $(3, n-2)$, $\ldots$, $(\frac{n}{2}, \frac{n}{2}+1)$ form a 1-factor of K_n. These $\frac{n}{2}$ edges are the thick edges of Fig.7.7. Rotation of these edges through the angle $\frac{2\pi}{n-1}$ in succession gives $(n-1)$ edge-disjoint 1-factors of K_n. This would account for $\frac{n}{2}(n-1)$ edges and hence all the edges of K_n. (Actually, the above construction displays a 1-factorization of K_n when n is even.) Each 1-factor can be assigned a distinct color. Thus, $\chi'(K_n) \leq n-1$. This proves the result in Case 1.

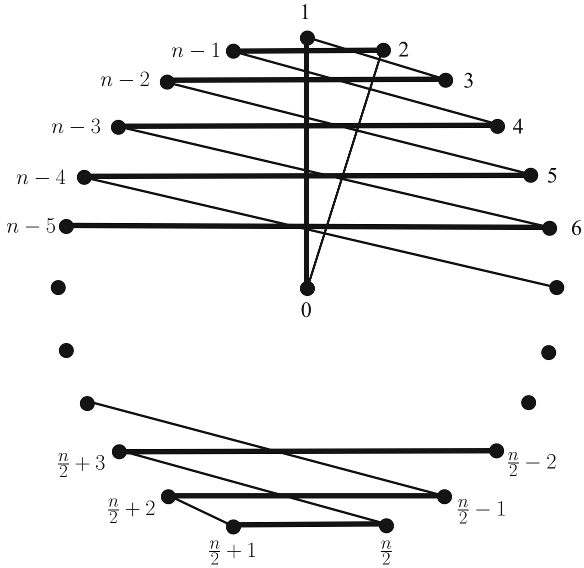


Fig. 7.7 Graph for proof of Theorem 7.6.4

Case 2. n is odd. Take a new vertex and make it adjacent to all the n vertices of K_n. This gives K_{n+1}. By Case 1, $\chi'(K_{n+1}) = n$. The restriction of this edge coloring to K_n yields a proper n-edge coloring of K_n. Hence, $\chi'(K_n) \leq n$. However, K_n cannot be edge colored properly with $n-1$ colors. This is because the size of any matching of K_n can contain no more than $\frac{n-1}{2}$ edges, and hence $n-1$ matchings of K_n can contain no more than $\frac{(n-1)^2}{2}$ edges. But K_n has $\frac{n(n-1)}{2}$ edges. Thus, $\chi'(K_n) \geq n$, and hence $\chi'(K_n) = n$. □

Exercise 6.2. Show that a Hamiltonian cubic graph is 3-edge-chromatic.

Exercise 6.3. Show that the Petersen graph is 4-edge-chromatic.

Exercise 6.4. Show that the Herschel graph (see Fig. 5.4) is 4-edge-chromatic.

Exercise 6.5. Determine the edge-chromatic number of the Grötzsch graph (Fig. 7.6).

Exercise 6.6. Show that a simple cubic graph with a cut edge is 4-edge-chromatic.

Exercise 6.7. Describe a proper k-edge coloring of a k-regular bipartite graph.

Exercise 6.8. Show that any bipartite graph G of maximum degree Δ is a subgraph of a Δ-regular bipartite graph. Hence, furnish an alternative proof of Theorem 7.6.3, using Exercise 6.7.

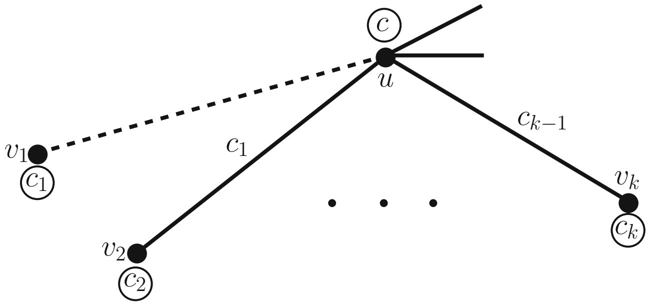


Fig. 7.8 Graph for proof of Theorem 7.6.5

7.6.2 Vizing's Theorem

Although it is true that for any loopless graph G, $\chi'(G) \geq \Delta(G)$, it turns out that for any simple graph G, $\chi'(G) \leq 1 + \Delta(G)$. This major result in edge coloring of graphs was established by Vizing [183] and independently by Gupta [81].

Theorem 7.6.5 (Vizing-Gupta). *For any simple graph* G, $\Delta(G) \leq \chi'(G) \leq 1 + \Delta(G)$.

Proof. In a proper edge coloring of G, $\Delta(G)$, colors are to be used for the edges incident at a vertex of maximum degree in G. Hence, $\chi'(G) \geq \Delta(G)$.

We now prove that $\chi'(G) \leq 1 + \Delta$, where $\Delta = \Delta(G)$.

If G is not $(1 + \Delta)$-edge-colorable, choose a subgraph H of G with a maximum possible number of edges such that H is $(1 + \Delta)$-edge-colorable. We derive a contradiction by showing that there exists a subgraph H_0 of G that is $(1 + \Delta)$-edge-colorable and has one edge more than H.

By our assumption, G has an edge $uv_1 \notin E(H)$. Since $d(u) \leq \Delta$, and $1 + \Delta$ colors are being used in H, there is a color c that is not represented at u (i.e., not used for any edge of H incident at u). For the same reason, there is a color c_1 not represented at v_1. (See Fig. 7.8, where the color not represented at a particular vertex is enclosed in a circle and marked near the vertex.)

There must be an edge, say uv_2 of H, colored c_1; otherwise, uv_1 can be assigned the color c_1, and $H \cup (uv_1)$, which has one edge more than H, would have a proper $(1 + \Delta)$-edge coloring. Again, there is a color, say c_2, not represented at v_2. Then as above, there is an edge uv_3 colored c_2 and there is a color, say c_3, not represented at v_3.

In this way, we construct a sequence of edges $\{uv_1, uv_2, \ldots, uv_k\}$ such that color c_i is not represented at vertex v_i, $1 \leq i \leq k$, and the edge uv_{j+1} receives the color c_j, $1 \leq j \leq k - 1$ (see Fig. 7.8).

Suppose at some stage, say the rth stage, where $1 \leq r \leq k$, c (the missing color at u) is not represented at v_r. We then "cascade" (i.e., shift in order) the colors $c_1, \ldots, c_{r-1}$ from $uv_2, uv_3, \ldots, uv_r$ to $uv_1, uv_2, \ldots, uv_{r-1}$. Under this new coloring,

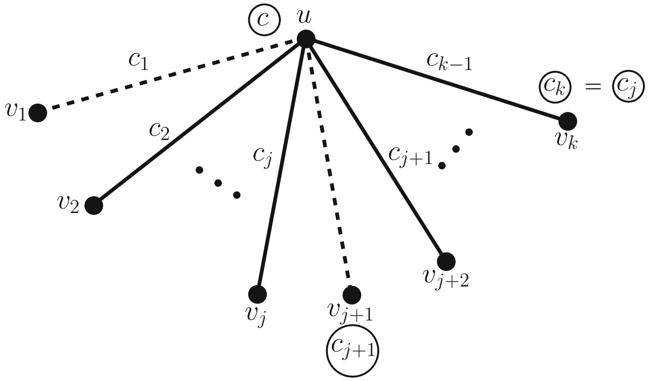


Fig. 7.9 Another graph for proof of Theorem 7.6.5

c is not represented both at u and at v_r, and therefore we can color uv_r with c. This yields a proper $(1 + \Delta)$-edge coloring to $H \cup (uv_1)$, contradicting the choice of H. Hence, we may assume that c is represented at each of the vertices $v_1, v_2, \ldots, v_k$.

Now we need to know why the sequence of edges uv_i, $1 \le i \le k$, had stopped. There are two possible reasons. Either there is no edge incident to u that is colored c_k, or the color $c_k = c_j$ for some $j < k-1$ and so has already been represented at u. Note that the sequence must stop at some finite stage since $d(u)$ is finite; however, it may as well stop before all the edges incident to u are exhausted.

If c_k is not represented at u in H, then we can cascade as before so that uv_i gets color c_i, $1 \le i \le k-1$, and then color uv_k with color c_k. Once again, we have a contradiction to our assumption on H.

Thus, we must have $c_k = c_j$ for come $j < k-1$. In this case, cascade the colors $c_1, c_2, \ldots, c_j$ so that uv_i has color c_i, $1 \le i \le j$, and leave uv_{j+1} uncolored (Fig. 7.9). Let $S = (H \cup (uv_1)) - uv_{j+1}$. Then S and H have the same number of edges.

Now consider S_{cc_j}, the subgraph of S defined by the edges of S with colors c and c_j. Clearly, each component of S_{cc_j} is either an even cycle or a path in which the adjacent edges alternate with colors c and c_j.

Now, c is represented at each of the vertices $v_1, v_2, \ldots, v_k$, and in particular at v_{j+1} and v_k. But c_j is not represented at v_{j+1} and v_k, since we have just moved c_j to uv_j, and $c_j = c_k$ is not represented at v_k. Hence in S_{cc_j}, the degrees of v_{j+1} and v_k are both equal to 1. Moreover, c_j is represented at u, but c is not. Therefore, u also has degree 1 in S_{cc_j}. As each component of S_{cc_j} is either a path or an even cycle, not all of u, v_{j+1}, and v_k can be in the same component of S_{cc_j} (since a nontrivial path has only two vertices of degree 1).

If u and v_{j+1} are in different components of S_{cc_j}, interchange the colors c and c_{j+1} in the component containing v_{j+1}. Then c is not represented at both u and v_{j+1}, and so we can color the edge uv_{j+1} with c. This gives a $(1+\Delta)$-edge coloring to the graph $S \cup (uv_{j+1})$.

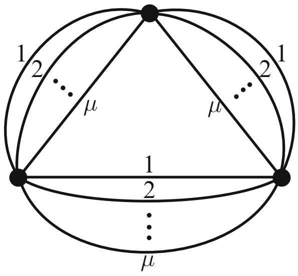


Fig. 7.10 Graph illustrating the generalized Vizing's theorem

Suppose then u and v_{j+1} are in the same components of S_{cc_j}. Then, necessarily, v_k is not in this component. Interchange c and c_j in the component containing v_k. In this case, *further* cascade the colors so that uv_i has color c_i, $1 \leq i \leq k-1$. Now color uv_k with color c.

Thus, we have extended our edge coloring of S with $1 + \Delta$ colors to one more edge of G. This contradiction proves that $H = G$, and thus $\chi'(G) \leq 1 + \Delta$. □

Actually, Vizing proved a more general result than the one given above. Let G be any loopless graph and let μ denote the maximum number of edges joining two vertices in G. Then the generalized Vizing's theorem states that $\Delta \leq \chi' \leq \Delta + \mu$. This theorem is the best possible in that there are graphs with $\chi' = \Delta + \mu$. For example, let G be the graph of Fig. 7.10. Since any two edges of G are adjacent, $\chi' = m(G) = 3\mu = \Delta + \mu$. For a proof of the generalized Vizing's theorem, see Yap [194].

Definition 7.6.6. Graphs for which $\chi' = \Delta$ are called *Class* 1 graphs and those for which $\chi' = 1 + \Delta$ are called *Class* 2 graphs.

Example 7.6.7. Bipartite graphs are of class 1 (see Theorem 7.6.3), whereas the Petersen graph (see Exercise 6.3) and any simple cubic graph with a cut edge (see Exercise 6.6) are of class 2.

For details relating to graphs of class 1 and class 2, see [62, 194].

Exercise 6.9. Let G be a simple Δ-edge-chromatic critical graph [i.e., G is of class 1 and for every edge e of G, $\chi'(G - e) < \chi'(G)$]. Prove that if $uv \in E(G)$, then $d(u) + d(v) \geq \Delta + 2$.

We now return to the timetable problem. Following are some examples of such a problem.

Problem 1. In a social health checkup scheme, specialist physicians are to visit various health centers. Given the places each physician has to visit and also the time interval of his or her visit, how can we fit in an itinerary? The assumption is that each health center can accommodate only one doctor at a time.

Problem 2. Mobile laboratories are to visit various schools in a city. Given the places each lab has to visit and also the time interval (period) of visits in a day, how can we fit in a timetable for the laboratories?

Problem 3. In an educational institution, as is well known, teachers have to instruct various classes. Given the various classes each teacher has to instruct in a day, how can we fit in a timetable? It is presumed that a teacher can teach only one class at a time and that each class could be taught by only one teacher at a time!

We shall now discuss Problem 3. Let $x_1, x_2, \ldots, x_n$ denote the teachers and $y_1, y_2, \ldots, y_m$ the classes. Let t_{ij} denote the number of periods for which teacher x_i has to meet class y_j. How can we draw up a timetable? If there are constraints on the availability of classrooms, what is the minimum number of periods required to implement a timetable? If the number of periods in a day is specified, what is the minimum number of rooms required to implement the timetable? All these problems could be analyzed by using a suitable graph.

Let $G(T, C)$ be a bipartite graph formed with $T = \{x_1, x_2, \ldots, x_p\}$ and $C = \{y_1, y_2, \ldots, y_q\}$ as the bipartition and in which there are t_{ij} parallel edges with x_i and y_j as their common ends. If T denotes the set of teachers and C the set of classrooms, a teaching assignment for a period determines a matching in the bipartite graph G. Conversely, any matching in G corresponds to a teaching assignment for one period. The edges of G could be partitioned into Δ edge-disjoint matchings (see Theorem 7.6.3). Corresponding to the Δ matchings, a Δ-period timetable can be drawn up.

Let N be the total number of periods to be taught by all teachers put together. Then, on average, N/Δ classes are to be taught per period. Hence, at least $\lceil N/\Delta \rceil$ rooms are necessary to implement a Δ-period timetable. We present below a method for drawing up such a timetable. For this, we need Lemma 7.6.8.

Lemma 7.6.8. *Let M and N be disjoint matchings of a graph G with $|M| > |N|$. Then there are disjoint matchings M' and N' of G with $|M'| = |M| - 1$ and $|N'| = |N| + 1$ and with $M' \cup N' = M \cup N$.*

Proof. Consider the subgraph $H = G[M \cup N]$. Each component of H is either an even cycle or a path with edges alternating between M and N. Since $|M| > |N|$, some path component P of H must have its initial and terminal edges in M. Let $P = v_0 e_1 v_1 e_2 v_2 \ldots e_{2r+1} v_{2r+1}$.

Now set

$$M' = (M \backslash \{e_1, e_3, \ldots, e_{2r+1}\}) \cup \{e_2, e_4, \ldots, e_{2r}\}$$

and

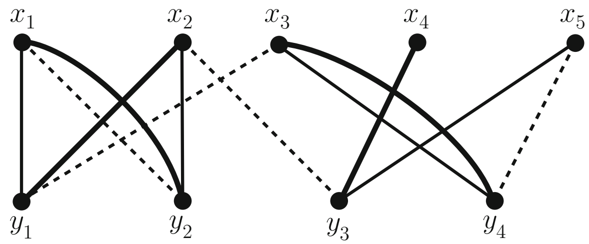
$$N' = (N \backslash \{e_2, e_4, \ldots, e_{2r}\}) \cup \{e_1, e_3, \ldots, e_{2r+1}\}.$$

Then M' and N' are disjoint matchings of G satisfying the conditions of the lemma. □

Theorem 7.6.9. *If G is a bipartite graph (with m edges), and if $m \geq t \geq \Delta$, then there exist t disjoint matchings $M_1, M_2, \ldots, M_t$ of G such that*

$$E = M_1 \cup M_2 \cup \ldots \cup M_t$$

Fig. 7.11 Bipartite graph corresponding to Problem 1



and, for $1 \leq i \leq t$,

$$\lfloor m/t \rfloor \leq |M_i| \leq \lceil m/t \rceil .$$

(In other words, any connected bipartite graph G is equitably t-edge-colorable, where $m \geq t \geq \Delta$.)

Proof. By Theorem 7.6.3, $\chi' = \Delta$. Hence, $E(G)$ can be partitioned into Δ matchings $M'_1, M'_2, \ldots, M'_\Delta$. So for $t \geq \Delta$, there exist disjoint matchings $M'_1, M'_2, \ldots, M'_t$, where $M'_i = \emptyset$ for $\Delta + 1 \leq i \leq t$, and

$$E = M'_1 \cup M'_2 \cup \ldots M'_t.$$

Now repeatedly apply Lemma 7.6.8 to pairs of matchings that differ by more than one in size. This would eventually result in matchings $M_1, M_2, \ldots, M_t$ of G satisfying the condition stated in the theorem. □

Coming back to our timetable problem, if the number of rooms available, say r, is less than N/Δ (so that $N/r > \Delta$), then the number of periods is to be correspondingly increased. Hence, starting with an edge partition of $E(G)$ into matchings $M'_1, M'_2, \ldots, M'_\Delta$, we apply Lemma 7.6.8 repeatedly to get an edge partition of $E(G)$ into disjoint matchings $M_1, M_2, \ldots, M_{\lceil N/r \rceil}$. This partition gives a $\lceil N/r \rceil$-period timetable that uses r rooms.

Illustration. The teaching assignments of five professors, x_1, x_2, x_3, x_4, x_5, in the mathematics department of a particular university are given by the following array:

	I Year	II Year	III Year	IV Year
	y_1	y_2	y_3	y_4
x_1	1	2	—	—
x_2	1	1	1	—
x_3	1	—	—	2
x_4	—	—	1	—
x_5	—	—	1	1

The bipartite graph G corresponding to the above problem is shown in Fig. 7.11. Each of the sets of edges drawn by the ordinary lines, dashed lines, and thick lines

Table 7.1 Timetable

		Period		
		I	II	III
Professor:	x_1	y_1	y_2	y_2
	x_2	y_2	y_3	y_1
	x_3	y_4	y_1	y_4
	x_4	—	—	y_3
	x_5	y_3	y_4	—

gives a matching in G. The three matchings cover the edges of G. Hence, they can be the basis of a three-period timetable. The corresponding timetable is given in Table 7.1.

In each period, four classes are to be met. Hence, at least four rooms are needed to implement this timetable. Here $\Delta = 3$ and $N = 12$. Consequently, G could be covered by three matchings each containing $\lfloor 12/3 \rceil$ or $\lceil 12/3 \rceil$ edges, that is, exactly four edges. This gives the edge partition

$$M' = \{M_1', M_2', M_3'\},$$

where

$$M_1' = \{x_1y_1, x_2y_2, x_3y_4, x_5y_3\},$$
$$M_2' = \{x_1y_2, x_2y_3, x_3y_1, x_5y_4\},$$

and

$$M_3' = \{x_1y_2, x_2y_1, x_3y_4.x_4y_3\}.$$

Now, take $M'' = \{M_1', M_2', M_3', M_4' = \emptyset\}$, and apply Lemma 7.6.8. This gives an edge partition $M = \{M_1, M_2, M_3, M_4\}$, where $M_1 = \{x_1y_1, x_2y_2, x_3y_4\}$, $M_2 = \{x_1y_2, x_2y_3, x_5y_4\}$, $M_3 = \{x_2y_1, x_3y_4, x_4y_3\}$, and $M_4 = \{x_5y_3, x_3y_1, x_1y_2\}$. The above partition yields a four-period timetable using three rooms.

7.7 Snarks

A consequence of the Vizing–Gupta theorem is that *if G is a simple cubic graph,* $\chi'(G) = 3$ *or* 4. By Exercise 6.6, *if G is a simple cubic graph with a cut edge,* $\chi'(G) = 4$. So the natural question is: Are there 2-edge-connected, simple cubic graphs that are 4-edge-chromatic? Such graphs are important in their own right, since their existence is related to the four-color problem (see Chap. 8). The search for such graphs has led to the study of snarks.

Definition 7.7.1. A *snark* is a cyclically 4-edge-connected cubic graph of girth at least 5 that has chromatic index 4.

Exercise 7.1. Prove that no snark can be Hamiltonian.

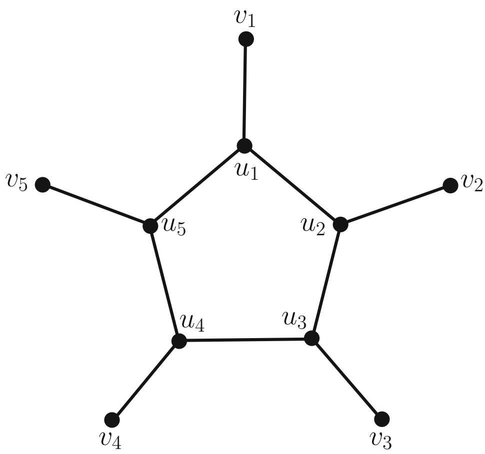


Fig. 7.12 Graph for proof of Theorem 7.7.2

Clearly, the Petersen graph is a snark. In fact, Theorem 7.7.2 is an interesting result.

Theorem 7.7.2. *The Petersen graph P is the smallest snark and it is the unique snark on* 10 *vertices.*

Proof. Let G be a snark and A a cyclical edge cut of G. Then $G - A$ has two components, each having a cycle of length at least 5 (since G is of girth at least 5). Hence, $|V(G)| \geq 10$. Thus, P is a smallest snark since $|V(P)| = 10$.

We now show that any snark G on 10 vertices must be isomorphic to P. Let A be a cyclical edge cut of G. If $|A| = 4$, then each component of $G - A$ is a 5-cycle. But this will not account for all the 15 edges of G. If $|A| > 5$, then $|E(G)| > 5 + 5 + 5 = 15$, a contradiction. Hence, $|A| = 5$, and let $A = \{u_i v_i : 1 \leq i \leq 5\}$. Then $G - A$ consists of two 5-cycles. Let one of these cycles be $\{u_1, u_2, u_3, u_4, u_5\}$. Let v_i be the third neighbor of u_i not belonging to the set $\{u_1, u_2, u_3, u_4, u_5\}$ for each i. If $v_1 v_2$ or $v_1 v_5$ is an edge of G, then G contains a 4-cycle (see Fig. 7.12).

Since G is cubic, $v_1 v_3 \in E(G)$ and $v_1 v_4 \in E(G)$. Similarly, $v_2 v_4$, $v_2 v_5$, and $v_3 v_5$ are edges of G and hence $G \simeq P$. □

The construction of snarks is not easy. In 1975, Isaacs constructed two infinite classes of snarks. Prior to that, only four kinds of snarks were known: (1) the Petersen graph on 10 vertices, (2) Blanusǎ's graphs on 18 vertices, (3) Szekeres' graph on 50 vertices, and (4) Blanche Descartes' graph on 210 vertices.

7.8 Kirkman's Schoolgirl Problem

Kirkman's schoolgirl problem was introduced in 1850 by Reverend Thomas J. Kirkman as "query 6" in page 48 of the *Ladies and Gentlemen's Diary*. The problem is the following: A teacher would like to take 15 schoolgirls out for a walk,

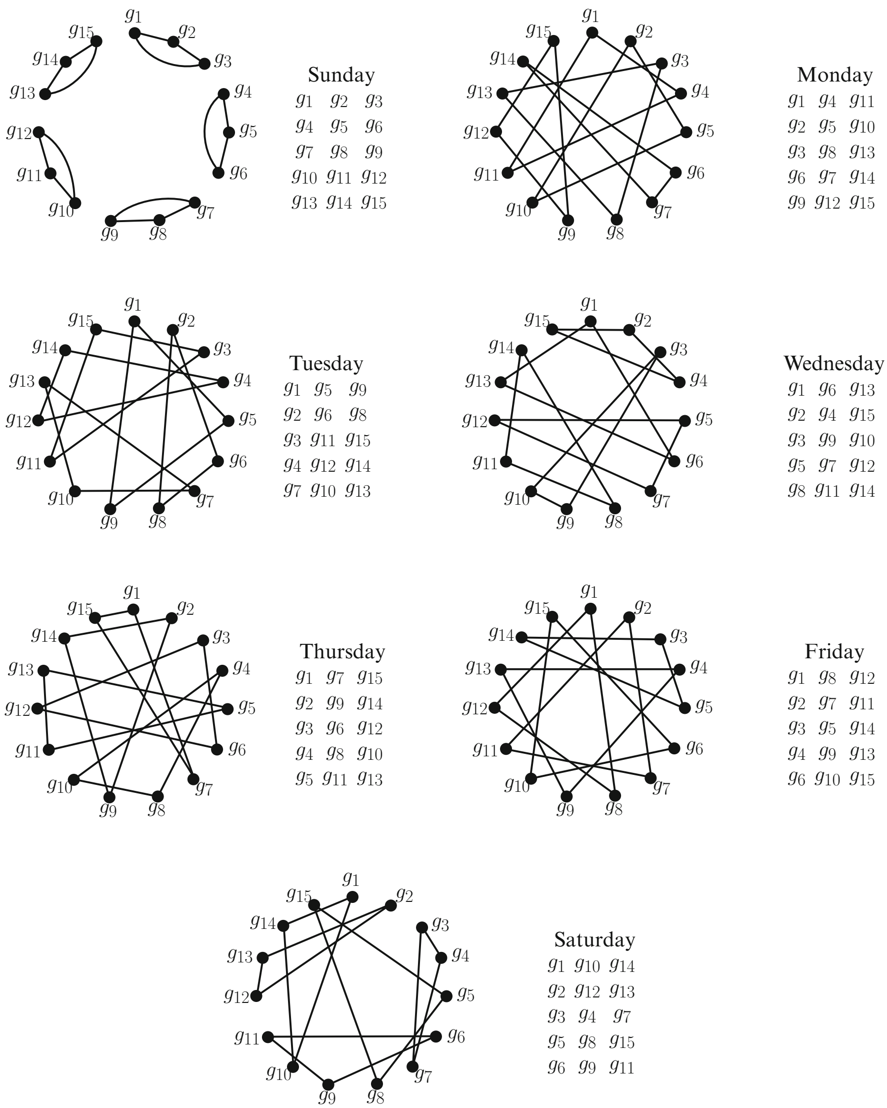


Fig. 7.13 Kirkman's schoolgirl problem

the girls being arranged in five rows of three. The teacher would like to ensure equal chances of friendship between any two girls. Hence, it is desirable to find different row arrangements for the seven days of the week such that any pair of girls walk in the same row on exactly one day of the week.

Kirkman's 15-schoolgirl problem has a solution. In fact, one of the possible schedules is given in Fig. 7.13.

In the general case, one wants to arrange $6n + 3$ girls in $2n + 1$ rows of three. The problem is to find different row arrangements for $3n + 1$ different days in such a way that any pair of girls walks in the same row on exactly one day out of the $3n + 1$ days. The existence of such an arrangement was proved by Ray-Chaudhuri and R. M. Wilson [164]. In graph-theoretic terminology, Kirkman's schoolgirl problem corresponds to an edge coloring $\mathscr{C} : E(K_{6n+3}) \to \{c_1, \ldots, c_{3n+1}\}$ of the complete graph $G = K_{6n+3}$ with $3n + 1$ colors such that if E_i denotes the set of all edges receiving the color c_i and $G_i = G[E_i]$, then G_i has $2n + 1$ components, each component being a triangle.

The general problem can be tackled as follows: Consider the *triangle graph* T of K_{6n+3} defined as follows: The vertex set of T is the set of all triads of $V(K_{6n+3})$, and two distinct vertices of T are joined by an edge in T if and only if the corresponding triads have two elements in common. Let S be any independent set of T. Each vertex of S gives rise to three pairs of vertices of K_{6n+3}, and each such pair belongs to at most one vertex of S. Hence, we have $3\,|S| \le \binom{6n+3}{2}$, that is, $|S| \le (2n + 1)(3n + 1)$. We must then find an independent set S' of cardinality $|S'| = (2n + 1)(3n + 1)$. Such a set exists since every solution of the Kirkman's schoolgirl problem yields an independent set of T with $(2n + 1)(3n + 1)$ vertices. We observe that S' covers each pair of $V(K_{6n+3})$ exactly once. Having found a maximum independent set S' in T, we form a new graph T' as follows: We take S' as its vertex set and join two vertices of T' by an edge if and only if the corresponding triads have exactly one vertex in common. We note that each independent set of T' is a partition of a subset of $V(K_{6n+3})$ into subsets of cardinality 3, and hence each independent set of T' has at most $(2n + 1)$ vertices. If the chromatic number of T' is $3n + 1$, then there is a partition $(V_1, V_2, \ldots, V_{3n+1})$ of $V(T')$ into parts of size $2n + 1$ each. This partition is a solution to the Kirkman's schoolgirl problem, and conversely, each solution to the Kirkman's schoolgirl problem yields such a partition.

Exercise 8.1. Let $m \ge n + 2$ and let there exist edge partitions $\mathscr{F}$ and $\mathscr{G}$ of K_n and K_m, respectively, into triangles with $\mathscr{F} \subset \mathscr{G}$. Prove that $m \ge 2n + 1$.

7.9 Chromatic Polynomials

In 1946, Birkhoff and Lewis [23] introduced the chromatic polynomial of a graph in their attempt to tackle the four-color problem (see Chap. 8) through algebraic techniques.

For a graph G and a given set of λ colors, the function $f(G;\lambda)$ is defined to be the number of ways of (vertex) coloring G properly using the λ colors. Hence, $f(G;\lambda) = 0$ when G has no proper λ-coloring. Clearly, the minimum λ for which $f(G;\lambda) > 0$ is the chromatic number $\chi(G)$ of G.

It is easy to see that $f(K_n;\lambda) = \lambda(\lambda - 1)\ldots(\lambda - n + 1)$ for $\lambda \ge n$. This is because any vertex of K_n can be colored by any one of the given λ colors. After coloring a vertex of K_n, a second vertex of K_n can be colored by any one of the remaining $(\lambda - 1)$ colors, and so on. In particular, $f(K_3; \lambda) = \lambda(\lambda - 1)(\lambda - 2)$. Also, $f(K_n^c; \lambda) = \lambda^n$.

Let $e = uv$ be any edge of G. Recall (see Sect. 4.3, Chap. 4) that the graph $G \circ e$ is obtained from G by contracting the edge e. Theorem 7.9.1 presents a simple reduction formula to compute $f(G; \lambda)$.

Theorem 7.9.1. *Let G be any graph. Then* $f(G; \lambda) = f(G - e; \lambda) - f(G \circ e; \lambda)$ *for any edge e of G.*

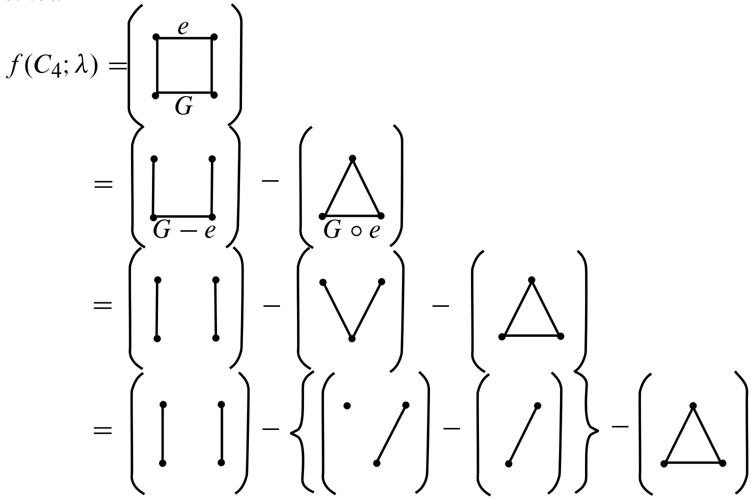
Proof. $f(G - e; \lambda)$ denotes the number of proper colorings of $G - e$ using λ colors. Hence, it is the sum of the number of proper colorings of $G - e$ in which u and v receive the same color and the number of proper colorings of $G - e$ in which u and v receive distinct colors. The former number is $f(G \circ e; \lambda)$, and the latter number is $f(G; \lambda)$. □

Exercise 9.1. If G and H are disjoint graphs, show that

$$f(G \cup H; \lambda) = f(G; \lambda) f(H; \lambda).$$

Theorem 7.9.1 could be used recursively to determine $f(G; \lambda)$ for graphs of small size by taking the given graph on n vertices as G and successively deleting edges until we end up with the totally disconnected graph K_n^c. It can also be determined by taking the given graph as $G - e$ and recursively adding a new edge e until we end up with the complete graph K_n. For a fixed n, when $m(G)$, the number of edges of G is small, the first method is preferable, and when it is large, the second method is preferable. These two methods are illustrated for the graph C_4. [Here the function $f(G; \lambda)$ is represented by the graph itself.]

Method 1

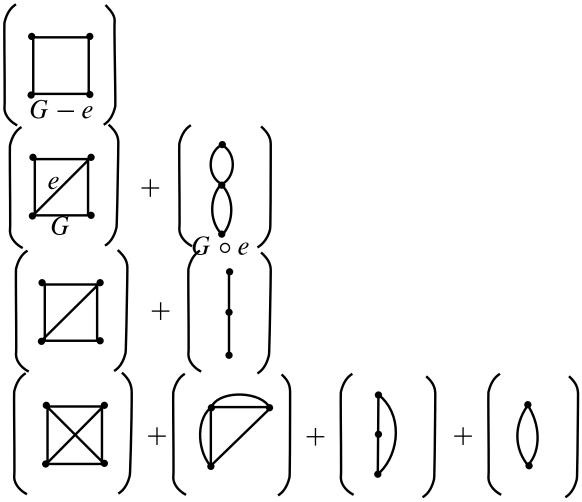


$$= (\lambda(\lambda - 1)^2 - \{\lambda^2(\lambda - 1) - \lambda(\lambda - 1)\} - \lambda(\lambda - 1)(\lambda - 2)$$
$$= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda.$$

Method 2

$f(C_4;\lambda) =$

4



$$
\begin{aligned}
&= f(K_4;\lambda) + f(K_3;\lambda) + f(K_3;\lambda) + f(K_2;\lambda)\\
&= f(K_4;\lambda) + 2f(K_3;\lambda) + f(K_2;\lambda)\\
&= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + 2\lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1)\\
&= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda.
\end{aligned}
$$

The function $f(C_4;\lambda)$ computed above is a monic polynomial with integer coefficients of degree $n = 4$ in which the coefficient of $\lambda^3 = -4 = -m$, the constant term is zero, and the coefficients alternate in sign. That this is the case with all such functions $f(G;\lambda)$ is the content of Theorem 7.9.2. For this reason, the function $f(G;\lambda)$ is called the *chromatic polynomial* of the graph G.

Theorem 7.9.2. *For a simple graph G of order n and size m, $f(G;\lambda)$ is a monic polynomial of degree n in λ with integer coefficients and constant term zero. In addition, its coefficients alternate in sign and the coefficient of λ^{n-1} is $-m$.*

Proof. The proof is by induction on m. If $m = 0$, G is K_n^c and $f(K_n^c;\lambda) = \lambda^n$, and if $m = 1$, G is K_2 and $f(K_2;\lambda) = \lambda^2 - \lambda$, and the statement of the theorem is trivially true in these cases. Suppose now that the theorem holds for all graphs with fewer than m edges, where $m \geq 2$. Let G be any simple graph of order n and size m, and let e be any edge of G. Both $G - e$ and $G \circ e$ (after removal of multiple edges, if necessary) are simple graphs with at most $m - 1$ edges, and hence, by the induction hypothesis,

$$f(G-e;\lambda) = \lambda^n - a_0\lambda^{n-1} + a_1\lambda^{n-2} - \ldots + (-1)^{n-1}a_{n-2}\lambda,$$

and

$$f(G \circ e;\lambda) = \lambda^{n-1} - b_1\lambda^{n-2} + \ldots + (-1)^{n-2}b_{n-2}\lambda,$$

where $a_0, \ldots, a_{n-2}$; $b_1, \ldots, b_{n-2}$ are nonnegative integers (so that the coefficients alternate in sign), and a_0 is the number of edges in $G - e$, which is $m - 1$. By Theorem 7.9.1, $f(G;\lambda) = f(G-e;\lambda) - f(G \circ e;\lambda)$, and hence

$$f(G;\lambda) = \lambda^n - (a_0 + 1)\lambda^{n-1} + (a_1 + b_1)\lambda^{n-2} - \ldots + (-1)^{n-1}(a_{n-2} + b_{n-2})\lambda.$$

Since $a_0 + 1 = m$, $f(G;\lambda)$ has all the stated properties. □

Theorem 7.9.3. *A simple graph G on n vertices is a tree if and only if* $f(G;\lambda) = \lambda(\lambda-1)^{n-1}$.

Proof. Let G be a tree. We prove that $f(G;\lambda) = \lambda(\lambda-1)^{n-1}$ by induction on n. If $n = 1$, the result is trivial. So assume the result for trees with at most $n-1$ vertices, $n \geq 2$. Let G be a tree with n vertices, and e be a pendent edge of G. By Theorem 7.9.1, $f(G;\lambda) = f(G-e;\lambda) - f(G \circ e;\lambda)$. Now, $G-e$ is a forest with two component trees of orders $n-1$ and 1, and hence $f(G-e;\lambda) = (\lambda(\lambda-1)^{n-2})\lambda$ (see Exercise 9.1). Since $G \circ e$ is a tree with $n-1$ vertices, $f(G \circ e;\lambda) = \lambda(\lambda-1)^{n-2}$. Thus, $f(G;\lambda) = (\lambda(\lambda-1)^{n-2})\lambda - \lambda(\lambda-1)^{n-2} = \lambda(\lambda-1)^{n-1}$.

Conversely, assume that G is a simple graph with $f(G;\lambda) = \lambda(\lambda-1)^{n-1} = \lambda^n - (n-1)\lambda^{n-1} + \ldots + (-1)^{n-1}\lambda$. Hence, by Theorem 7.9.2, G has n vertices and $n-1$ edges. Further, the last term, $(-1)^{n-1}\lambda$, ensures that G is connected (see Exercise 9.2). Hence, G is a tree (see Theorem 4.2.4). □

Remark 7.9.4. Theorem 7.9.3 shows that the chromatic polynomial of a graph G does not fix the graph uniquely up to isomorphism. For example, even though the graphs $K_{1,3}$ and P_4 are not isomorphic, they have the same chromatic polynomial, namely, $\lambda(\lambda-1)^3$.

Exercise 9.2. If G has ω components, show that λ^{ω} is a factor of $f(G;\lambda)$.

Exercise 9.3. Show that there exists no graph with the following polynomials as chromatic polynomial (i) $\lambda^5 - 4\lambda^4 + 8\lambda^3 - 4\lambda^2 + \lambda$; (ii) $\lambda^4 - 3\lambda^3 + \lambda^2$; (iii) $\lambda^7 - \lambda^6 + 1$.

Exercise 9.4. Find a graph G whose chromatic polynomial is $\lambda^5 - 6\lambda^4 + 11\lambda^3 - 6\lambda^2$.

Exercise 9.5. Show that for the cycle C_n of length n, $f(C_n;\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1)$, $n \geq 3$.

Exercise 9.6. Show that for any graph G, $f(G \vee K_1;\lambda) = \lambda f(G;\lambda-1)$, and hence prove that $f(W_n;\lambda) = \lambda(\lambda-2)^n + (-1)^n\lambda(\lambda-2)$.

Notes

A good reference for graph colorings is the book by Jensen and Toft [116]. The book by Fiorini and Wilson [62] concentrates on edge colorings. Theorem 7.5.7 (Mycielski's theorem) has also been proved independently by Blanche Descartes [50] as well as by Zykov [195]. For a complete description of graph homomorphisms, see [105].

The proof of Brooks' theorem given in this chapter is based on the proof given by Fournier [67] (see also references [27] and [106]).

The term "snark" was given to the snark graph by Martin Gardner after the unusual creature that is described in Lewis Carroll's poem, T*he Hunting of the Snark*. A detailed account of the snarks, including their constructions, can be found in the interesting book by Holton and Sheehan [106].