Chapter 6
Eulerian and Hamiltonian Graphs

6.1 Introduction

The study of Eulerian graphs was initiated in the 18th century and that of Hamiltonian graphs in the 19th century. These graphs possess rich structures; hence, their study is a very fertile field of research for graph theorists. In this chapter, we present several structure theorems for these graphs.

6.2 Eulerian Graphs

Definition 6.2.1. An *Euler trail* in a graph G is a spanning trail in G that contains all the edges of G. An *Euler tour* of G is a closed Euler trail of G. G is called *Eulerian* (Fig. 6.1a) if G has an *Euler tour*. It was Euler who first considered these graphs, and hence their name.

It is clear that an Euler tour of G, if it exists, can be described from any vertex of G. Clearly, every Eulerian graph is connected.

Euler showed in 1736 that the celebrated *Königsberg bridge problem* has no solution. The city of Königsberg (now called Kaliningrad) has seven bridges linking two islands A and B and the banks C and D of the Pregel (now called Pregalya) River, as shown in Fig. 6.2.

The problem was to start from any one of the four land areas, take a stroll across the seven bridges, and get back to the starting point without crossing any bridge a second time. This problem can be converted into one concerning the graph obtained by representing each land area by a vertex and each bridge by an edge. The resulting graph H is the graph of Fig. 6.1b. The Königsberg bridge problem will have a solution provided that this graph H is Eulerian. But this is not the case since it has vertices of odd degrees (see Theorem 6.2.2).

Eulerian graphs admit, among others, the following two elegant characterizations, Theorems 6.2.2 and 6.2.3*.

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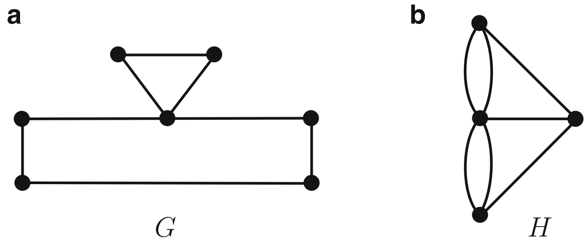


Fig. 6.1 (**a**) Eulerian graph G; (**b**) non-Eulerian graph H

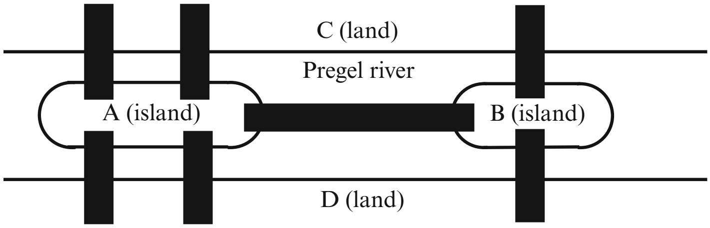


Fig. 6.2 Königsberg bridge problem

Theorem 6.2.2. *For a nontrivial connected graph G, the following statements are equivalent:*

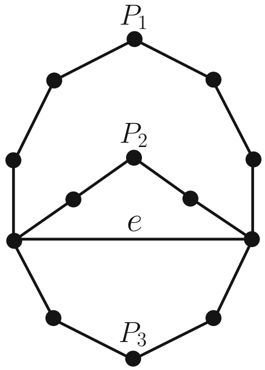
(i) G is Eulerian.
(ii) The degree of each vertex of G is an even positive integer.
(iii) G is an edge-disjoint union of cycles.

Proof. (i) $\Rightarrow$ (ii): Let T be an Euler tour of G described from some vertex $v_0 \in V(G)$. If $v \in V(G)$, and $v \neq v_0$, then every time T enters v, it must move out of v to get back to v_0. Hence two edges incident with v are used during a visit to v, and therefore, $d(v)$ is even. At v_0, every time T moves out of v_0, it must get back to v_0. Consequently, $d(v_0)$ is also even. Thus, the degree of each vertex of G is even.

(ii) $\Rightarrow$ (iii): As $\delta(G) \geq 2$, G contains a cycle C_1 (Exercise 11.11 of Chap. 1). In $G \backslash E(C_1)$, remove the isolated vertices if there are any. Let the resulting subgraph of G be G_1. If G_1 is nonempty, each vertex of G_1 is again of even positive degree. Hence $\delta(G_1) \geq 2$, and so G_1 contains a cycle C_2. It follows that after a finite number, say r, of steps, $G \backslash E(C_1 \cup \ldots \cup C_r)$ is totally disconnected. Then G is the edge-disjoint union of the cycles $C_1, C_2, \ldots, C_r$.

(iii) $\Rightarrow$ (i): Assume that G is an edge-disjoint union of cycles. Since any cycle is Eulerian, G certainly contains an Eulerian subgraph. Let G_1 be a longest closed trail in G. Then G_1 must be G. If not, let $G_2 = G \backslash E(G_1)$. Since G is an edge-disjoint union of cycles, every vertex of G is of even degree ≥ 2. Further, since G_1 is Eulerian, each vertex of G_1 is of even degree ≥ 2. Hence each vertex of G_2 is of even degree. Since G_2 is not totally disconnected and G is connected, G_2 contains a cycle C having a vertex v in common with G_1. Describe the Euler tour of G_1

Fig. 6.3 Eulerian graph with edge e belonging to three cycles



starting and ending at v and follow it by C. Then $G_1 \cup C$ is a closed trail in G longer than G_1. This contradicts the choice of G_1, and so G_1 must be G. Hence G is Eulerian. □

If $G_1, \ldots, G_r$ are subgraphs of a graph G that are pairwise edge-disjoint and their union is G, then this fact is denoted by writing $G = G_1 \oplus \ldots \oplus G_r$. In the above equation, if $G_i = C_i$, a cycle of G for each i, then $G = C_1 \oplus \ldots C_r$. The set of cycles $S = \{C_1, \ldots, C_r\}$ is then called a *cycle decomposition* of G. Thus, Theorem 6.2.2 implies that *a connected graph is Eulerian if and only if it admits a cycle decomposition.*

There is yet another characterization of Eulerian graphs due to McKee [138] and Toida [175]. Our proof is based on Fleischner [63, 64].

Theorem 6.2.3*. *A graph G is Eulerian if and only if each edge e of G belongs to an odd number of cycles of G.*

For instance, in Fig. 6.3, e belongs to the three cycles $P_1 \cup e$, $P_2 \cup e$, and $P_3 \cup e$.

Proof. Denote by γ_e the number of cycles of G containing e. Assume that γ_e is odd for each edge e of G. Since a loop at any vertex v of G is in exactly one cycle of G and contributes 2 to the degree of v in G, we may suppose that G is loopless.

Let $S = \{C_1, \ldots, C_p\}$ be the set of cycles of G. Replace each edge e of G by γ_e parallel edges and replace e in each of the γ_e cycles containing e by one of these parallel edges, making sure that none of the parallel edges is repeated. Let the resulting graph be G_0 and let the new set of cycles be $S_0 = \{C_1^0, \ldots, C_p^0\}$. Clearly, S_0 is a cycle decomposition of G_0. Hence, by Theorem 6.2.2, G_0 is Eulerian. But then $d_{G_0}(v) \equiv 0 \pmod 2$ for each $v \in V(G_0) = V(G)$. Moreover, $d_G(v) = d_{G_0}(v) - \sum_e (\gamma_e - 1)$, where e is incident at v in G and hence $d_G(v) \equiv 0 \pmod 2$, γ_e being odd for each $e \in E(G)$. Thus, G is Eulerian.

Conversely, assume that G is Eulerian. We proceed by induction on $n = |V(G)|$. If $n = 1$, each edge is a loop and hence belongs to exactly one cycle of G.

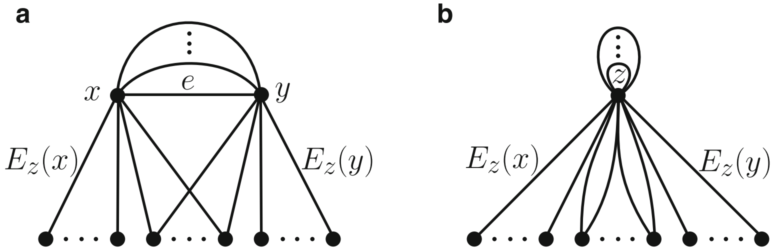


Fig. 6.4 Graph for proof of Theorem 6.2.3

Assume the result for graphs with fewer than n ($\geq$ 2) vertices. Let G be a graph with n vertices. Let $e = xy$ be an edge of G and let $\lambda(e)$ be the multiplicity of e in G.

The graph $G \circ e$ obtained from G by contracting the edge e (cf. Sect. 4.4 of Chap. 4) is also Eulerian. Denote by z the new vertex of $G \circ e$ obtained by identifying the vertices x and y of G. The set of edges incident with z in $G \circ e$ is partitioned into three subsets (see Fig. 6.4):

1. $E_z(x)$ = set of edges arising out of edges of G incident with x but not with y
2. $E_z(y)$ = set of edges arising out of edges of G incident with y but not with x
3. $E_z(xy)$ = set of $\lambda(e) - 1$ loops of $G \circ e$ corresponding to the edges parallel to e in G

Let $k = |E_z(x)|$. Since G is Eulerian,

$$k + \lambda(e) = d_G(x) \equiv 0 (\mathrm{mod}\, 2). \tag{6.1}$$

Let Γ_f and $\Gamma(e_i, e_j)$ denote, respectively, the number of cycles in $G \circ e$ containing the edge f and the pair (e_i, e_j) of edges. Since $|V(G \circ e)| = n - 1$, and since $G \circ e$ is Eulerian by the induction assumption, Γ_f is odd for each edge f of $G \circ e$. Now, any cycle of G containing e either consists of e and an edge parallel to e in G (and there are $\lambda(e) - 1$ of them) or contains e, an edge e_i of $E_z(x)$, and an edge e'_j of $E_z(y)$. These correspond in $G \circ e$, respectively, to a loop at z and to a cycle containing the edges of $G \circ e$ that correspond to the edges e_i and e'_j of G. By abuse of notation, we denote these corresponding edges of $G \circ e$ also by e_i and e'_j, respectively. Moreover, any cycle of $G \circ e$ containing an edge e_i of $E_z(x)$ will also contain either an edge e_j of $E_z(x)$ or an edge e'_j of $E_z(y)$, but not both. A cycle of the former type is counted once in Γ_{e_i} and once in Γ_{e_j}, and these will not give rise to cycles in G containing e. Thus,

$$\gamma_e = (\lambda(e) - 1) + \sum_{e_i \in E_z(x)} \Gamma_{e_i} - \sum_{\substack{\{i,j\} \\ i \neq j \\ e_i, e_j \in E_z(x)}} \Gamma(e_i, e_j).$$

Now, by the induction hypothesis, $\Gamma_{e_i} \equiv 1 \pmod 2$ for each e_i, and $\Gamma(e_i, e_j) = \Gamma(e_j, e_i)$ in the last sum on the right, and hence this latter sum is even. Thus, $\gamma_e \equiv (\lambda(e) - 1) + k \pmod 2 \equiv 1 \pmod 2$ by relation (6.1). □

A consequence of Theorem 6.2.3 is a result of Bondy and Halberstam [26], which gives yet another characterization of Eulerian graphs.

Corollary 6.2.4*. *A graph is Eulerian if and only if it has an odd number of cycle decompositions.*

Proof. In one direction, the proof is trivial. If G has an odd number of cycle decompositions, then it has at least one, and hence G is Eulerian.

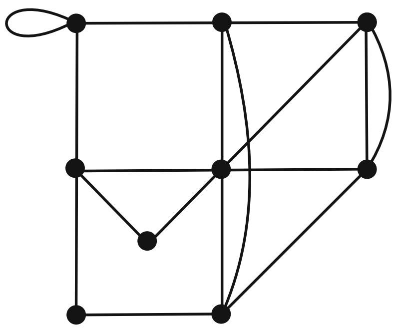
Conversely, assume that G is Eulerian. Let $e \in E(G)$ and let $C_1, \ldots, C_r$ be the cycles containing e. By Theorem 6.2.3, r is odd. We proceed by induction on $m = |E(G)|$ with G Eulerian.

If G is just a cycle, then the result is true. Assume then that G is not a cycle. This means that for each i, $1 \le i \le r$, by the induction assumption, $G_i = G - E(C_i)$ has an odd number, say s_i, of cycle decompositions. (If G_i is disconnected, apply the induction assumption to each of the nontrivial components of G_i.) The union of each of these cycle decompositions of G_i and C_i yields a cycle decomposition of G. Hence the number of cycle decompositions of G containing C_i is s_i, $1 \le i \le r$. Let $s(G)$ denote the number of cycle decompositions of G. Then

$$\begin{aligned} s(G) &= \sum_{i=1}^{r} s_i \equiv r \pmod 2 \ (\text{since } s_i \equiv 1 \pmod 2) \\ &\equiv 1 \pmod 2. \end{aligned}$$

□

Exercise 2.1. Find an Euler tour in the graph G below.



G

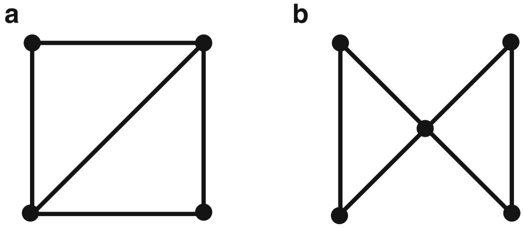


Fig. 6.5 (**a**) Hamiltonian graph; (**b**) non-Hamiltonian but traceable graph

Exercise 2.2. Does there exist an Eulerian graph with

(i) An even number of vertices and an odd number of edges?
(ii) An odd number of vertices and an even number of edges? Draw such a graph if it exists.

Exercise 2.3. Prove that a connected graph is Eulerian if and only if each of its blocks is Eulerian.

Exercise 2.4. If G is a connected graph with $2k(k > 0)$ vertices of odd degree, show that $E(G)$ can be partitioned into k open (i.e., not closed) trails.

Exercise 2.5. Prove that a connected graph is Eulerian if and only if each of its edge cuts has an even number of edges.

6.3 Hamiltonian Graphs

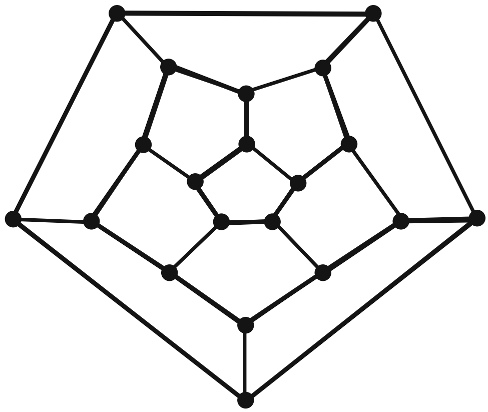
Definition 6.3.1. A graph is called *Hamiltonian* if it has a spanning cycle (see Fig. 6.5a). These graphs were first studied by Sir William Hamilton, a mathematician. A spanning cycle of a graph G, when it exists, is often called a *Hamilton cycle* (or *Hamiltonian cycle*) of G.

Definition 6.3.2. A graph G is called *traceable* if it has a spanning path of G (see Fig. 6.5b). A spanning path of G is also called a *Hamilton path* (or *Hamiltonian path*) of G.

6.3.1 Hamilton's "Around the World" Game

Hamilton introduced these graphs in 1859 through a game that used a solid dodecahedron (Fig. 6.6). A dodecahedron has 20 vertices and 12 pentagonal faces. At each vertex of the solid, a peg was attached. The vertices were marked Amsterdam, Ann Arbor, Berlin, Budapest, Dublin, Edinburgh, Jerusalem, London, Melbourne, Moscow, Novosibirsk, New York, Paris, Peking, Prague, Rio di Janeiro,

Fig. 6.6 Solid dodecahedron for Hamilton's "Around the World" problem



Rome, San Francisco, Tokyo, and Warsaw. Further, a string was also provided. The object of the game was to start from any one of the vertices and keep on attaching the string to the pegs as we move from one vertex to another along a particular edge with the condition that we have to get back to the starting city without visiting any intermediate city more than once. In other words, the problem asks one to find a Hamilton cycle in the graph of the dodecahedron (see Fig. 6.6). Hamilton solved this problem as follows: When a traveler arrives at a city, he has the choice of taking the edge to his right or left. Denote the choice of taking the edge to the right by R and that of taking the edge to the left by L. Let 1 denote the operation of staying where he is.

Define the product O_1O_2 of two operations O_1 and O_2 as O_1 followed by O_2. For example, LR denotes going left first and then going right. Two sequences of operations are *equal* if, after starting at a vertex, the two sequences lead to the same vertex. The product defined above is associative but not commutative. Further, it is clear (see Fig. 6.6) that

$$\begin{aligned} R^5 &= L^5 = 1 \\ RL^2R &= LRL, \\ LR^2L &= RLR, \\ RL^3R &= L^2, \text{ and} \\ LR^3L &= R^2. \end{aligned}$$

These relations give

$$\begin{aligned} 1 &= R^5 = R^2R^3 = (LR^3L)R^3 - (LR^3)(LR^3) = (LR^3)^2 = (LR^2R)^2 \\ &= (L(LR^3L)R)^2 = (L^2R^3LR)^2 = (L^2((LR^3L)R)LR)^2 = (L^3R^3LRLR)^2 \\ &= LLLRRRLRLRLLLRRRLRLR. \end{aligned} \tag{6.2}$$

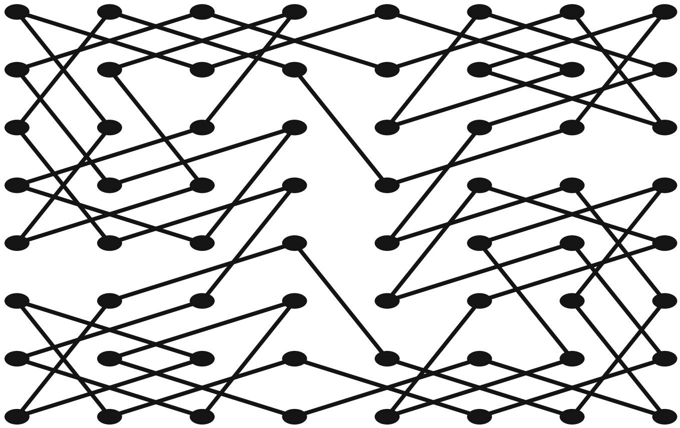


Fig. 6.7 A knight's tour in a chessboard

The last sequence of operations contains 20 operations and contains no partial sequence equal to 1. Hence, this sequence must represent a Hamilton cycle. Thus, starting from any vertex and following the sequence of operations (6.2), we do indeed get a Hamilton cycle of the graph of Fig. 6.6.

Knight's Tour in a Chessboard 6.3.3. The knight's tour problem is the problem of determining a closed tour through all 64 squares of an 8×8 chessboard by a knight with the condition that the knight does not visit any intermediate square more than once. This is equivalent to finding a Hamilton cycle in the corresponding graph of 64 ($= 8 \times 8$) vertices in which two vertices are adjacent if and only if the knight can move from one vertex to the other following the rules of the chess game. Figure 6.7 displays a knight's tour.

Even though Eulerian graphs admit an elegant characterization, no decent characterization of Hamiltonian graphs is known as yet. In fact, it is one of the most difficult unsolved problems in graph theory. (Actually, it is an NP-complete problem; see reference [71].) Many sufficient conditions for a graph to be Hamiltonian are known; however, none of them happens to be an elegant necessary condition.

We begin with a necessary condition. Recall that $\omega(H)$ stands for the number of components of the graph H.

Theorem 6.3.4. *If G is Hamiltonian, then for every nonempty proper subset S of V, $\omega(G - S) \leq |S|$.*

Proof. Let C be a Hamilton cycle in G. Then, since C is a spanning subgraph of G, $\omega(G - S) \leq \omega(C - S)$. If $|S| = 1$, $C - S$ is a path, and therefore $\omega(C - S) = 1 = |S|$. The removal of a vertex from a path P results in one or two components, according to whether the removed vertex is an end vertex or an internal vertex of P.

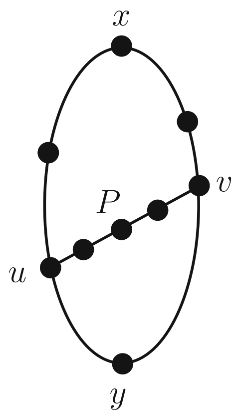


Fig. 6.8 Theta graph

Hence, by induction, the number of components in $C - S$ cannot exceed $|S|$. This proves that $\omega(G - S) \leq \omega(C - S) \leq |S|$. □

It follows directly from the definition of a Hamiltonian graph or from Theorem 6.3.4 that any Hamiltonian graph must be 2-connected. [If G has a cut vertex v, then taking $S = \{v\}$, we see that $\omega(G - S) > |S|$.] The converse, however, is not true. For example, the theta graph of Fig. 6.8 is 2-connected but not Hamiltonian. Here, P stands for a u-v path of any length ≥ 2 containing neither x nor y.

Exercise 3.1. Show by means of an example that the condition in Theorem 6.3.4 is not sufficient for G to be Hamiltonian.

Exercise 3.2. Use Theorem 6.3.4 to show that the Herschel graph (shown in Fig. 5.4) is non-Hamiltonian.

Exercise 3.3. Do Exercise 3.2 by using Theorem 1.5.10 (characterization theorem for bipartite graphs).

If a cubic graph G has a Hamilton cycle C, then $G \backslash E(C)$ is a 1-factor of G. Hence, for a cubic graph G to be Hamiltonian, G must have a 1-factor F such that $G \backslash E(F)$ is a Hamilton cycle of G. Now, the Petersen graph P (shown in Fig. 1.7) has two different types of 1-factors (see Fig. 6.9), and for any such 1-factor F of P, $P \backslash E(F)$ consists of two disjoint 5-cycles. Hence P is non-Hamiltonian.

Theorem 6.3.5 is a basic result due to Ore [150] which gives a sufficient condition for a graph to be Hamiltonian.

Theorem 6.3.5 (Ore [150]). *Let G be a simple graph with $n \geq 3$ vertices. If, for every pair of nonadjacent vertices u, v of G, $d(u) + d(v) \geq n$, then G is Hamiltonian.*

Proof. Suppose that G satisfies the condition of the theorem, but G is not Hamiltonian. Add edges to G (without adding vertices) and get a supergraph G^* of G such that G^* is a maximal simple graph that satisfies the condition of the

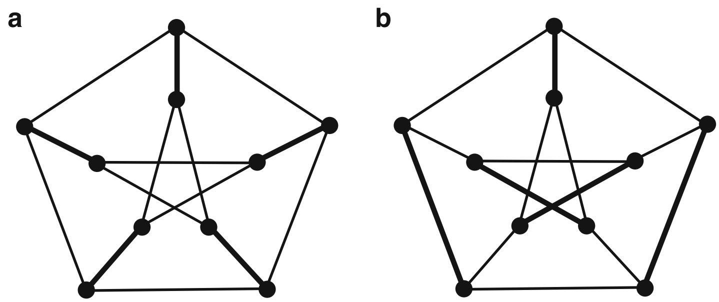


Fig. 6.9 Petersen graph. The solid edges form a 1-factor of P

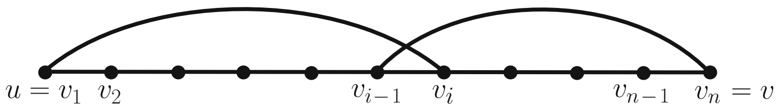


Fig. 6.10 Hamilton path for proof of Theorem 6.3.5

theorem, but G^* is non-Hamiltonian. Such a graph G^* must exist since G is non-Hamiltonian while the complete graph on $V(G)$ is Hamiltonian. Hence, for any pair u and v of nonadjacent vertices of G^*, $G^* + uv$ must contain a Hamilton cycle C. This cycle C would certainly contain the edge $e = uv$. Then $C - e$ is a Hamilton path $u = v_1 v_2 v_3 \ldots v_n = v$ of G^* (see Fig. 6.10).

Now, if $v_i \in N(u)$, $v_{i-1} \notin N(v)$; otherwise, $v_1\, v_2 \ldots v_{i-1}\, v_n\, v_{n-1}\, v_{n-2} \ldots v_{i+1}\, v_i\, v_1$ would be a Hamilton cycle in G^*. Hence, for each vertex v_i adjacent to u, the vertex v_{i-1} of $V - \{v\}$ is nonadjacent to v. But then

$$d_{G^*}(v) \leq (n-1) - d_{G^*}(u).$$

This gives that $d_{G^*}(u) + d_{G^*}(v) \leq n - 1$, and therefore $d_G(u) + d_G(v) \leq n - 1$, a contradiction. □

Corollary 6.3.6 (Dirac [54]). *If G is a simple graph with $n \geq 3$ and $\delta \geq \frac{n}{2}$, then G is Hamiltonian.* □

Corollary 6.3.7. *Let G be a simple graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n - 1$ for every pair of nonadjacent vertices u and v of G, then G is traceable.*

Proof. Choose a new vertex w and let G' be the graph $G \vee \{w\}$. Then each vertex of G has its degree increased by one, and therefore in G', $d(u) + d(v) \geq n + 1$ for every pair of nonadjacent vertices. Since $|V(G')| = n + 1$, by Theorem 6.3.5, G' is Hamiltonian. If C' is a Hamilton cycle of G', then $C' - w$ is a Hamilton path of G. Thus, G is traceable. □

Exercise 3.4. Show by means of an example that the conditions of Theorem 6.3.5 and its Corollary 6.3.6 are not necessary for a simple connected graph to be Hamiltonian.

Exercise 3.5. Show that if a cubic graph G has a spanning closed trail, then G is Hamiltonian.

Exercise 3.6. Prove that the n-cube Q_n is Hamiltonian for every $n \geq 2$.

Exercise 3.7. Prove that the wheel W_n is Hamiltonian for every $n \geq 4$.

Exercise 3.8. Prove that a simple k-regular graph on $2k-1$ vertices is Hamiltonian.

Exercise 3.9. For any vertex v of the Petersen graph P, show that $P - v$ is Hamiltonian. (A non-Hamiltonian graph G with this property, namely, for any vertex v of G the subgraph $G-v$ of G is Hamiltonian, is called a hypo-Hamiltonian graph. In fact, P is the lowest-order graph with this property.)

Exercise 3.10. For any vertex v of the Petersen graph P, show that a Hamilton path exists starting at v.

Exercise 3.11. If $G = G(X,Y)$ is a bipartite Hamiltonian graph, show that $|X| = |Y|$.

Exercise 3.12. Let G be a simple graph on $2k$ vertices with $\delta(G) \geq k$. Show that G has a perfect matching.

Exercise 3.13. Prove that a simple graph of order n with n even and $\delta \geq \frac{(n+2)}{2}$ has a 3-factor.

Bondy and Chvátal [25] observed that the proof of Theorem 6.3.5 is essentially based on the following result.

Theorem 6.3.8. *Let G be a simple graph of order $n \geq 3$ vertices. Then G is Hamiltonian if and only if $G + uv$ is Hamiltonian for every pair of nonadjacent vertices u and v with $d(u) + d(v) \geq n$.*

The last result has been instrumental for Bondy and Chvátal to define the closure of a graph G.

Definition 6.3.9. The *closure* of a graph G, denoted $\mathrm{cl}(G)$, is defined to be that supergraph of G obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair exists.

This recursive definition does not stipulate the order in which the new edges are added. Hence, we must first show that the definition does not depend upon the order of the newly added edges. Figure 6.11 explains the construction of $\mathrm{cl}(G)$.

Theorem 6.3.10. *The closure $cl(G)$ of a graph G is well defined.*

Proof. Let G_1 and G_2 be two graphs obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair exists. We have to prove that $G_1 = G_2$.

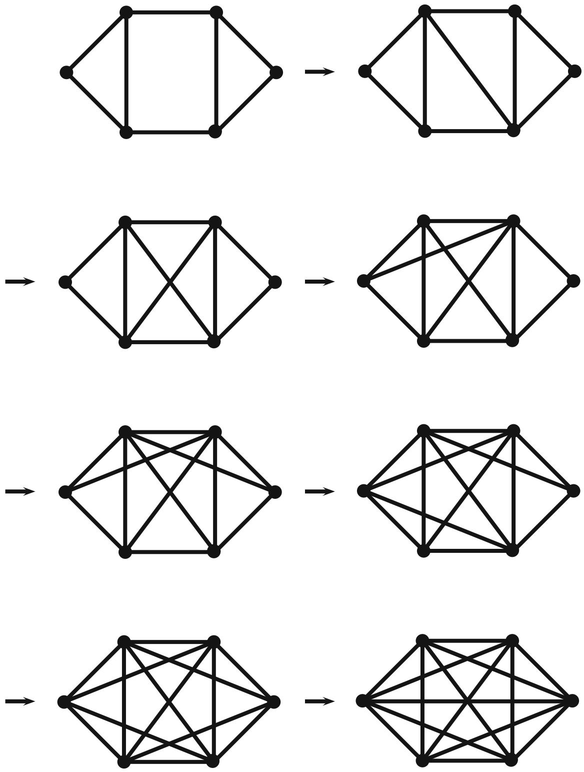


Fig. 6.11 Closure of a graph

Let $\{e_1, \ldots, e_p\}$ and $\{f_1, \ldots, f_q\}$ be the sets of new edges added to G in these sequential orderings to get G_1 and G_2, respectively. We want to show that each e_i is some f_j (and therefore belongs to G_2) and that each f_k is some e_l (and therefore belongs to G_1). Let e_i be the first edge in $\{e_1, \ldots, e_p\}$ not belonging to G_2. Then $\{e_1, \ldots, e_{i-1}\}$ are all in both G_1 and G_2, and $uv = e_i \notin E(G_2)$. Let $H = G + \{e_1, \ldots, e_{i-1}\}$. Then H is a subgraph of both G_1 and G_2. By the way cl(G) is defined,

$$d_H(u) + d_H(v) \geq n,$$

and hence,

$$d_{G_2}(u) + d_{G_2}(v) \geq n.$$

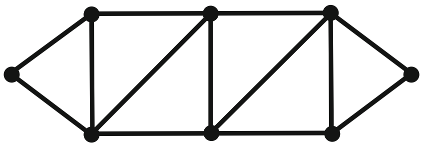
But this is a contradiction since u and v are nonadjacent vertices of G_2, and G_2 is a closure of G. Thus $e_i \in E(G_2)$ for each i and similarly, $f_k \in E(G_1)$ for each k. $\square$

An immediate consequence of Theorem 6.3.8 is the following.

Theorem 6.3.11. *If cl(G) is Hamiltonian, then G is Hamiltonian.*

Corollary 6.3.12. *If cl(G) is complete, then G is Hamiltonian.*

Exercise 3.14. Determine the closure of the following graph.



We conclude this section with a result of Chvátal and Erdös [39].

Theorem 6.3.13 (Chvátal and Erdös). *If, for a simple* 2*-connected graph* G*,* $\alpha \leq \kappa$*, then* G *is Hamiltonian. (*α *is the independence number of* G *and* κ *is the connectivity of* G*.)*

Proof. Suppose $\alpha \leq \kappa$ but G is not Hamiltonian. Let $C : v_0 v_1 \ldots v_{p-1}$ be a longest cycle of G. We fix this orientation on C. By Dirac's theorem (Exercise 6.4 of Chap. 3), $p \geq \kappa$. Let $v \in V(G) \setminus V(C)$. Then by Menger's theorem (see also Exercise 6.3 of Chap. 3), there exist κ internally disjoint paths $P_1, \ldots, P_\kappa$ from v to C. Let $v_{i_1}, v_{i_2}, \ldots, v_{i_\kappa}$ be the end vertices (with suffixes in the increasing order) of these paths on C. No two of the consecutive vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_\kappa}, v_{i_1}$ can be adjacent vertices of C, since otherwise we get a cycle of G longer than C. Hence, between any two consecutive vertices of $\{v_{i_1}, v_{i_2}, \ldots, v_{i_\kappa}, v_{i_1}\}$, there exists at least one vertex of G. Let u_{i_j} be the vertex next to v_{i_j} in the v_{i_j}-$v_{i_{j+1}}$ path along C (see Fig. 6.12a).

We claim that $\{u_{i_1}, \ldots, u_{i_\kappa}\}$ is an independent set of G. Suppose u_{i_j} is adjacent to u_{i_m}, $m > j$ (suffixes taken modulo κ); then

$$u_{i_j} \ldots v_{i_{j+1}} \ldots v_{i_m} P_m^{-1} v P_j v_{i_j} \ldots v_{i_{j-1}} \ldots u_{i_m} u_{i_j}$$

is a cycle of G longer than C, a contradiction.

Further, $\{v, u_{i_1} \ldots, u_{i_\kappa}\}$ is also an independent set of G. [Otherwise, $v u_{i_m} \in E(G)$ for some m. See Fig. 6.12b. Then

$$v u_{i_m} \ldots v_{i_{m+1}} \ldots v_{i_\kappa} \ldots v_{i_1} \ldots v_{i_m} P_m^{-1} v$$

is a cycle longer than C, a contradiction.] But this implies that $\alpha > \kappa$, a contradiction to our hypothesis. Thus G is Hamiltonian. □

This theorem, although interesting, is not powerful in that for the cycle C_n, $\kappa = 2$ while $\alpha = \left\lfloor \frac{n}{2} \right\rfloor$ and hence increases with n.

A graph G with at least three vertices is *Hamiltonian-connected* if any two vertices of G are connected by a Hamilton path in G. For example, for $n \geq 3$, K_n is Hamiltonian-connected, whereas for $n \geq 4$, C_n is not Hamiltonian-connected.

Theorem 6.3.14. *If* G *is a simple graph with* $n \geq 3$ *vertices such that* $d(u)+d(v) \geq n+1$ *for every pair of nonadjacent vertices of* G*, then* G *is Hamiltonian-connected.*

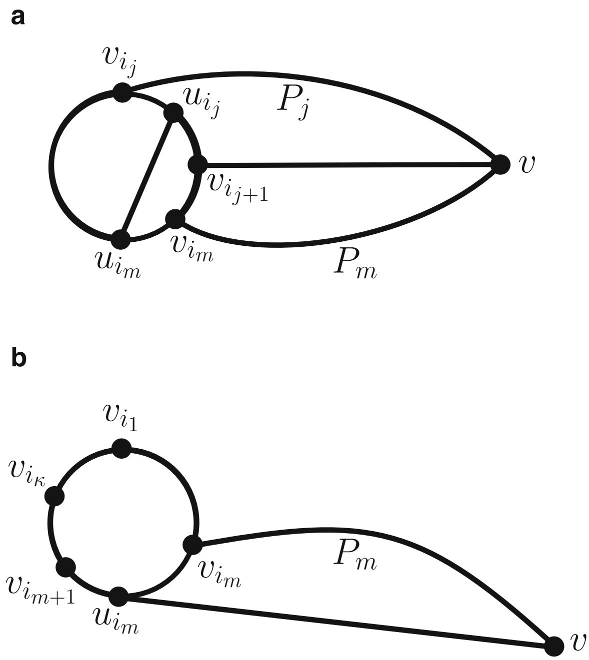


Fig. 6.12 Graphs for proof of Theorem 6.3.13

Proof. Let u and v be any two vertices of G. Our aim is to show that a Hamilton path exists from u to v in G.

Choose a new vertex w, and let $G^* = G \cup \{wu, wv\}$. We claim that $\text{cl}(G^*) = K_{n+1}$. First, the recursive addition of the pairs of nonadjacent vertices u and v of G with $d(u) + d(v) \geq n + 1$ gives K_n. Further, each vertex of K_n is of degree $n - 1$ in K_n and $d_{G^*}(w) = 2$. Hence, $\text{cl}(G^*) = K_{n+1}$. So by Corollary 6.3.12, G^* is Hamiltonian. Let C be a Hamilton cycle in G^*. Then $C - w$ is a Hamilton path in G from u to v. □

6.4* Pancyclic Graphs

Definition 6.4.1. A graph G of order n (≥ 3) is *pancyclic* if G contains cycles of all lengths from 3 to n. G is called *vertex-pancyclic* if each vertex v of G belongs to a cycle of every length l, $3 \leq l \leq n$.

Example 6.4.2. Clearly, a *vertex-pancyclic graph is pancyclic.* However, the converse is not true. Figure 6.13 displays a pancyclic graph that is not vertex-pancyclic.

The study of pancyclic graphs was initiated by Bondy [24], who showed that Ore's sufficient condition for a graph G to be Hamiltonian (Theorem 6.3.5) actually implies much more. Note that if $\delta \geq \frac{n}{2}$, then $m \geq \frac{n^2}{4}$.

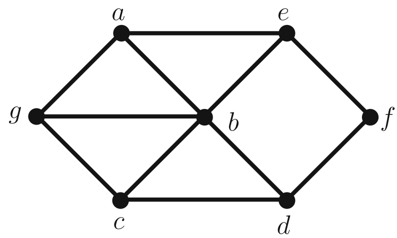


Fig. 6.13 Pancyclic graph that is not vertex-pancyclic

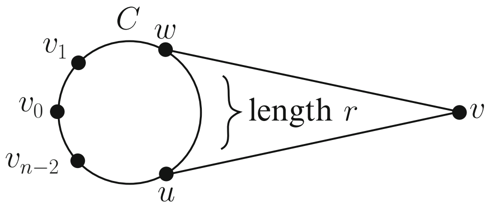


Fig. 6.14 Graph for proof of Theorem 6.4.3

Theorem 6.4.3. *Let G be a simple Hamiltonian graph on n vertices with at least $\lceil \frac{n^2}{4} \rceil$ edges. Then G either is pancyclic or else is the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$. In particular, if G is Hamiltonian and $m > \frac{n^2}{4}$, then G is pancyclic.*

Proof. The result can directly be verified for $n = 3$. We may therefore assume that $n \geq 4$. We apply induction on n. Suppose the result is true for all graphs of order at most $n-1 (n \geq 4)$, and let G be a graph of order n.

First, assume that G has a cycle $C = v_0 v_1 \ldots v_{n-2} v_0$ of length $n-1$. Let v be the (unique) vertex of G not belonging to C. If $d(v) \geq \frac{n}{2}$, v is adjacent to two consecutive vertices on C, and hence G has a cycle of length 3. Suppose for some r, $2 \leq r \leq \frac{n-1}{2}$, C has no pair of vertices u and w on C adjacent to v in G with $d_C(u,w) = r$. Then, if $v_{i_1}, v_{i_2}, \ldots v_{i_{d(v)}}$ are the vertices of C that are adjacent to v in G (recall that C contains all the vertices of G except v,) then $v_{i_1+r}, v_{i_2+r}, \ldots, v_{i_{d(v)}+r}$ are nonadjacent to v in G, where the suffixes are taken modulo $(n-1)$. Hence, $2d(v) \leq n-1$, a contradiction. Thus, for each r, $2 \leq r \leq \frac{n-1}{2}$, C has a pair of vertices u and w on C adjacent to v in G with $d_C(u,w) = r$. So for each r, $2 \leq r \leq \frac{n-1}{2}$, G has a cycle of length $r+2$ as well as a cycle of length $n-1-r+2 = n-r+1$ (see Fig. 6.14). Consequently, G is pancyclic. (Recall that G is already Hamiltonian.)

If $d(v) \leq \frac{n-1}{2}$, then $G[V(C)]$, the subgraph of G induced by $V(C)$ has at least $\frac{n^2}{4} - d(v) \geq \frac{n^2}{4} - \frac{n-1}{2} > \frac{(n-1)^2}{4}$ edges. So by the induction assumption, $G[V(C)]$ is pancyclic and hence G is pancyclic. (By hypothesis, G is Hamiltonian.)

Next, assume that G has no cycle of length $n-1$. Then G is not pancyclic. In this case, we show that G is $K_{\frac{n}{2},\frac{n}{2}}$.

Let $C = v_0 v_1 v_2 \ldots v_{n-1} v_0$ be a Hamiltonian cycle of G. We claim that of the two pairs $v_i v_k$ and $v_{i+1} v_{k+2}$ (where suffixes are taken modulo n), at most one of them can be an edge of G. Otherwise, $v_k v_{k-1} v_{k-2} \ldots v_{i+1} v_{k+2} v_{k+3} v_{k+4} \ldots v_i v_k$ is an $(n-1)$-cycle in G (as it misses only the vertex v_{k+1} of G), a contradiction.

Hence, if $d(v_i) = r$, then there are r vertices adjacent to v_i in G and hence at least r vertices that are nonadjacent to v_{i+1}. Thus, $d(v_i + 1) \leq n - r$, and $d(v_i) + d(v_{i+1}) \leq n$.

Summing the last inequality over i from 0 to $n-1$, we get $4m \leq n^2$. But by hypothesis, $4m \geq n^2$. Hence, $m = \frac{n^2}{4}$ and so n must be even. Again, this yields $d(v_i) + d(v_{i+1}) = n$ for each i, and therefore for each i and k,

$$\text{exactly one of } v_i\, v_k \text{ and } v_{i+1}\, v_{k+2} \text{ is an edge of } G. \tag{*}$$

Thus, if $G \neq K_{\frac{n}{2},\frac{n}{2}}$, then certainly there exist i and j such that $v_i v_j \in E$ and $i \equiv j \pmod 2$. Hence, for some j, there exists an *even* positive integer s such that $v_{j+1}\, v_{j+1+s} \in E$. Choose s to be the least *even* positive integer with the above property. Then $v_j\, v_{j+s-1} \notin E$. Hence, $s \geq 4$ (as $s = 2$ would mean that $v_j\, v_{j+1} \notin E$). Again by (*), $v_{j-1}\, v_{j+s-3} = v_{j-1}\, v_{j-1+(s-2)} \in E(G)$ contradicting the choice of s. Thus, $G = K_{\frac{n}{2},\frac{n}{2}}$. The last part follows from the fact that $|E(K_{\frac{n}{2},\frac{n}{2}})| = \frac{n^4}{4}$. □

Corollary 6.4.4. *Let $G \neq K_{\frac{n}{2},\frac{n}{2}}$, be a simple graph with $n \geq 3$ vertices, and let $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices of G. Then G is pancyclic.*

Proof. By Ore's theorem (Theorem 6.3.5), G is Hamiltonian. We show that G is pancyclic by first proving that $m \geq \frac{n^2}{2}$ and then invoking Theorem 6.4.3. This is true if $\delta \geq \frac{n}{2}$ (as $2m = \sum_{i=1}^{n} d_i \geq \delta\, n \geq n^2/2$). So assume that $\delta < \frac{n}{2}$.

Let S be the set of vertices of degree δ in G. For every pair (u, v) of vertices of degree δ, $d(u) + d(v) < \frac{n}{2} + \frac{n}{2} = n$. Hence, by hypothesis, S induces a clique of G and $|S| \leq \delta + 1$. If $|S| = \delta + 1$, then G is disconnected with $G[S]$ as a component, which is impossible (as G is Hamiltonian). Thus $|S| \leq \delta$. Further if $v \in S$, v is nonadjacent to $n - 1 - \delta$ vertices of G. If u is such a vertex, $d(v) + d(u) \geq n$ implies that $d(u) \geq n - \delta$. Further, v is adjacent to at least one vertex $w \notin S$ and $d(w) \geq \delta + 1$ by the choice of S. These facts give that

$$2m = \sum_{i=1}^{n} d_i \geq (n - \delta - 1)(n - \delta) + \delta^2 + (\delta + 1),$$

where the last $(\delta + 1)$ comes out of the degree of w. Thus,

$$2m \geq n^2 - n(2\delta + 1) + 2\delta^2 + 2\delta + 1$$

which implies that

$$\begin{aligned} 4m &\geq 2n^2 - 2n(2\delta + 1) + 4\delta^2 + 4\delta + 2 \\ &= (n - (2\delta + 1))^2 + n^2 + 1 \\ &\geq n^2 + 1, \quad \text{since } n > 2\delta. \end{aligned}$$

Consequently, $m > \frac{n^2}{4}$, and by Theorem 6.4.3, G is pancyclic. □

6.5 Hamilton Cycles in Line Graphs

We now turn our attention to the existence of Hamilton cycles in line graphs.

Theorem 6.5.1. *If G is Eulerian, then $L(G)$, the line graph of G is both Hamiltonian and Eulerian.*

Proof. As G is Eulerian, it is connected and hence $L(G)$ is also connected. If $e_1 e_2 \dots e_m$ is the edge sequence of an Euler tour in G, and if vertex u_i in $L(G)$ represents the edge e_i, $1 \leq i \leq m$, then $u_1 u_2 \dots u_m u_1$ is a Hamilton cycle of $L(G)$. Further, if $e = v_1 v_2 \in E(G)$ and the vertex u in $L(G)$ represents the edge e, then $d_{L(G)}(u) = d_G(v_1) + d_G(v_2) - 2$, which is even (and ≥ 2) since both $d_G(v_1)$ and $d_G(v_2)$ are even (and ≥ 2). Hence in $L(G)$ every vertex is of even degree (≥ 2). So $L(G)$ is also Eulerian. □

Exercise 5.1. Disprove the converse of Theorem 6.5.1 by a counterexample.

Definition 6.5.2. A *dominating trail* of a graph G is a closed trail T in G (which may be just a single vertex) such that every edge of G not in T is incident with T.

Example 6.5.3. For instance, in the graph of Fig. 6.13, the trail *abcdbea* is a dominating trail.

Harary and Nash–Williams [94] characterized graphs that have Hamiltonian line graphs.

Theorem 6.5.4 (Harary and Nash–Williams). *The line graph of a graph G with at least three edges is Hamiltonian if and only if G has a dominating trail.*

Proof. Let T be a dominating trail of G and let $\{e_1, e_2, \dots, e_s\}$ be the edge sequence representing T. Then every edge of G not in T is incident to some vertex of T. Assume that e_1 and e_2 are incident at v_1. Replace the subsequence $\{e_1, e_2\}$ of $\{e_1, e_2, \dots, e_s\}$ by the sequence $\{e_1, e_{11}, e_{12}, \dots, e_{1r_1}, e_2\}$, where $e_{11}, e_{12}, \dots e_{1r_1}$ are the edges of $E(G)\backslash E(T)$ incident at v_1 other than e_1 and e_2. Assume that we have already replaced the subsequence $\{e_i, e_{i+1}\}$ by $\{e_i, e_{i1}, \dots, e_{ir_i}, e_{i+1}\}$. Then replace $\{e_{i+1}, e_{i+2}\}$ by the sequence $\{e_{i+1}, e_{(i+1)1}, \dots, e_{(i+1)r_{(i+1)}}, e_{i+2}\}$ in $E(G)\backslash E(T)$, where the new edges $e_{(i+1)1} \dots, e_{(i+1)r_{(i+1)}}$ have not appeared in the previous i subsequences. (Here we take $e_{s+1} = e_1$.) The resulting edge sequence is $e_1 e_{11} e_{12} \dots e_{1r_1}$ $e_2 e_{21} e_{22} \dots e_{2r_2} e_3 \dots e_s e_{s1} e_{s2} \dots e_{sr_s} e_1$ and this gives the Hamilton cycle $u_1 u_{11} u_{12} \dots u_{1r_1} u_2 u_{21} u_{22} \dots u_{2r_2} u_3 \dots u_s u_{s1} u_{s2} \dots u_{sr_s} u_1$ in $L(G)$. [Here u_1 is the vertex of $L(G)$ that corresponds to the edge e_1 of G, and so on.]

Conversely, assume that $L(G)$ has a Hamilton cycle C. Let $C = u_1 u_2 \dots u_m u_1$ and let e_i be the edge of G corresponding to the vertex u_i of $L(G)$. Let T_0 be the edge sequence $e_1 e_2 \dots e_m e_1$. We now delete edges from T_0 one after another as follows: Let $e_i\, e_j\, e_k$ be the first three distinct consecutive edges of T_0 that have a common vertex; then delete e_j from the sequence. Let $T_0' = T_0 - e_j$ $= \{e_1 e_2 \dots e_i e_k \dots e_m e_1\}$.

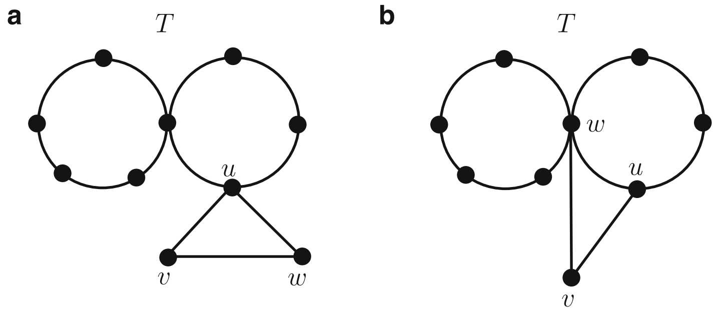


Fig. 6.15 Graphs for proof of Theorem 6.5.6. (**a**) $T \cup \{uv, vw, wu\}$ is longer than T; (**b**) $(T \backslash \{uw\}) \cup \{uv, vw\}$ is longer than T

Now proceed with T_0' as we did with T_0. Continue this process until no such triad of edges exists. Then the resulting subsequence of T_0 must be a dominating trail or a pair of adjacent edges incident at a vertex, say, v_0. In the latter case, all the edges of G are incident at v_0, and hence we take $\{v_0\}$ as the dominating trail of G. □

Corollary 6.5.5. *The line graph of a Hamiltonian graph is Hamiltonian.*

Proof. Let G be a Hamiltonian graph with Hamilton cycle C. Then C is a dominating trail of G. Hence, $L(G)$ is Hamiltonian. □

Exercise 5.2. Show that the line graph of a graph G has a Hamilton path if and only if G has a trail T such that every edge of G not in T is incident with T.

Exercise 5.3. Draw the line graph of the graph of Fig. 6.13 and display a Hamilton cycle in it.

Theorem 6.5.6 ([12]). *Let G be any connected graph. If each edge of G belongs to a triangle in G, then G has a spanning, Eulerian subgraph.*

Proof. Since G has a triangle, G has a closed trail. Let T be a longest closed trail in G. Then T must be a spanning Eulerian subgraph of G. If not, there exists a vertex v of G with $v \notin T$ and v is adjacent to a vertex u of T.

By hypothesis, uv belongs to a triangle, say uvw. If none of the edges of this triangle is in T, then $T \cup \{uv, vw, wu\}$ yields a closed trail longer than T (see Fig. 6.15). If $uw \in T$, then $(T - uw) \cup \{uv, vw\}$ would be a closed trail longer than T. These contradictions prove that T is a spanning closed trail of G. □

Corollary 6.5.7. *Let G be any connected graph. If each edge of G belongs to a triangle, then $L(G)$ is Hamiltonian.*

Proof. The proof is an immediate consequence of Theorems 6.5.4 and 6.5.6. □

Corollary 6.5.8 (Chartrand and Wall [35]). *If G is connected and $\delta(G) \geq 3$, then $L^2(G)$ is Hamiltonian.*

(Note: For $n > 1$, $L^n(G) = L(L^{n-1}(G))$, and $L^0(G) = G$.)

Proof. Since $\delta(G) \geq 3$, each vertex of $L(G)$ belongs to a clique of size at least three, and hence each edge of $L(G)$ belongs to a triangle. Now apply Corollary 6.5.7. $\square$

Corollary 6.5.9 (Nebesky [146]). *If G is a connected graph with at least three vertices, then $L(G^2)$ is Hamiltonian.*

Proof. Since G is a connected graph with at least three vertices, every edge of G^2 belongs to a triangle. Hence, $L(G^2)$ is Hamiltonian by Corollary 6.5.7. $\square$

Theorem 6.5.10. *Let G be a connected graph in which every edge belongs to a triangle. If e_1 and e_2 are edges of G such that $G\backslash\{e_1, e_2\}$ is connected, then there exists a spanning trail of G with e_1 and e_2 as its initial and terminal edges.*

Proof. The proof is essentially the same as for Theorem 6.5.6 and is based on considering the longest trail T in G with e_1 and e_2 as its initial and terminal edges, respectively. $\square$

Corollary 6.5.11 ([12]). *Let G be any connected graph with $\delta(G) \geq 4$. Then $L^2(G)$ is Hamiltonian-connected.*

Proof. The edges incident to a vertex v of G will yield a clique of size $d(v)$ in $L(G)$. Since $\delta(G) \geq 4$, each vertex of $L(G)$ belongs to a clique of order at least 4, and hence $L(G)$ is 3-edge connected. Therefore, for any pair of distinct edges e_1 and e_2 of $L(G)$, $L(G)\backslash\{e_1, e_2\}$ is connected. Further, each edge of $L(G)$ belongs to a triangle. Hence, by Theorem 6.5.10, a spanning trail T in $L(G)$ exists having e_1 and e_2 as the initial and terminal edges, respectively. Thus, if there are any edges of $L(G)$ not belonging to T, they can only be "chords" of T. It follows (see Exercise 5.2) that in $L^2(G)$ there exists a Hamilton path starting and ending at the vertices corresponding to e_1 and e_2, respectively. Since e_1 and e_2 are arbitrary, $L^2(G)$ is Hamiltonian-connected. $\square$

Corollary 6.5.12 (Jaeger [113]). *The line graph of a 4-edge-connected graph is Hamiltonian.*

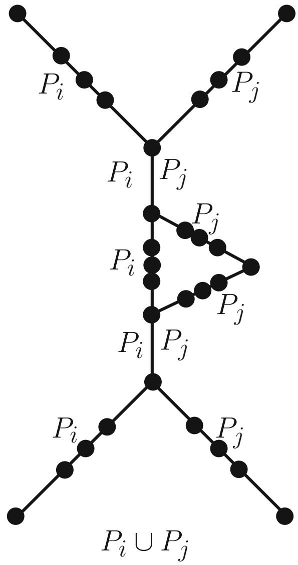
To prove Corollary 6.5.12, we need the following lemma.

Lemma 6.5.13. *Let S be a set of vertices of a nontrivial tree T, and let $|S| = 2k$, $k \geq 1$. Then there exists a set of k pairwise edge-disjoint paths in T whose end vertices are all the vertices of S.*

Proof. Certainly there exists a set of k paths in T whose end vertices are all the vertices of S. (This is because between any two vertices of T, there is a unique path in T.) Choose such a set of k paths, say $\mathscr{P} = \{P_1, P_2, \ldots, P_k\}$ with the additional condition that the sum of their lengths is minimum.

We claim that the paths of $\mathscr{P}$ are pairwise edge-disjoint. If not, there exists a pair $\{P_i, P_j\}$, $i \neq j$, with P_i and P_j having an edge in common. In this case, P_i and P_j have one or more disjoint paths of length at least 1 in common. Then $P_i \,\Delta\, P_j$, the symmetric difference of P_i and P_j, properly contains a disjoint union of two paths,

Fig. 6.16 Graph for proof of Lemma 6.5.13



say Q_i and Q_j, with their end vertices being disjoint pairs of vertices belonging to S (Fig. 6.16).

If we replace P_i and P_j by Q_i and Q_j in $\mathscr{P}$, then the resulting set of paths has the property that their end vertices belong to S and that the sum of the lengths of Q_i and Q_j is less than that of the sum of the lengths of the paths P_i and P_j in $\mathscr{P}$. This contradicts the choice of $\mathscr{P}$, and hence $\mathscr{P}$ has the stated property. $\square$

Proof of Corollary 6.5.12. Let G be a 4-edge-connected graph. In view of Theorem 6.5.4, it suffices to show that G contains a dominating trail.

By Corollary 4.4.6, G contains two edge-disjoint spanning trees T_1 and T_2. Let S be the set of vertices of odd degree in T_1. Then $|S|$ is even. Let $|S| = 2k$, $k \geq 1$. By Lemma 6.5.13, there exists a set of k pairwise edge-disjoint paths $\{P_l, P_2, \ldots, P_k\}$ in T_2 with the property stated in Lemma 6.5.13. Then $G_0 = T_1 \cup (P_1 \cup P_2 \cup \ldots P_k)$ is a connected spanning subgraph of G in which each vertex is of even degree. Thus, G_0 is a dominating trail of G. $\square$

We conclude this section with a theorem on locally connected graphs (see Definition 1.5.9 of Chap. 1).

Theorem 6.5.14* (Oberly and Sumner [149]). *A connected, locally connected, nontrivial $K_{1,3}$-free graph is Hamiltonian.*

Proof. Let G be a connected, locally connected, nontrivial $K_{1,3}$-free graph. We may assume that G has at least four vertices. Since G is locally connected, G certainly has a cycle. Let C be a longest cycle of G. If C is not a Hamilton cycle, there exists a vertex $v \in V(G) \backslash V(C)$ that is adjacent to a vertex u of C. Let u_1 and u_2 be the neighbors of u on C. Then, as the edges uv, uu_1 and uu_2 do not induce a $K_{1,3}$ in G,

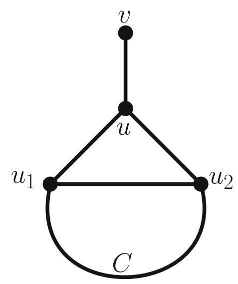


Fig. 6.17 Graph for proof of Theorem 6.5.14

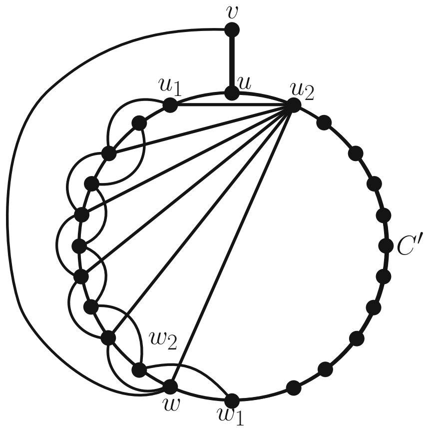


Fig. 6.18 Graph for case 1 of proof of Theorem 6.5.14

$u_1u_2 \in E(G)$, since otherwise v is adjacent either to u_1 or u_2 and we get a cycle longer than C, a contradiction (see Fig. 6.17).

For each $x \in V(G)$, denote by $G_0(x)$ the subgraph $G[N_G(x)]$ of G. By hypothesis, $G_0(u)$ is connected, and hence there exists either a v-u_1 path P in $G_0(u)$ not containing u_2 or a v-u_2 path Q in $G_0(u)$ not containing u_1. Let us say it is the former. For the purpose of the proof of this theorem, we call a vertex $w \in V_0 = (V(C) \cap V(P))\backslash\{u_1\}$ *singular* if neither of the two neighbors of w on C is in $N_G(u)$.

Case 1. Each vertex of V_0 is singular. Then for any $w \in V_0$, w is adjacent to u (since $w \in V(P) \subseteq V(G_0(u))$) but since w is singular, neither of the neighbors w_1 and w_2 of w on C is adjacent to u in G. Then considering the $K_{1,3}$ subgraph $\{ww_1, ww_2, wu\}$, we see that $w_1 w_2 \in E(G)$. Now, describe the cycle C' as follows: Start from u_2, move away from u along C, and whenever we encounter a singular vertex w, bypass it by going through the edge w_1w_2. After reaching u_1, retrace the u_1-v path P^{-1} and follow it up by the path vuu_2. Then C' traverses through each vertex of $C \cup P$ exactly once. Thus, C' is a cycle longer than C, a contradiction to the choice of C (see Fig. 6.18).

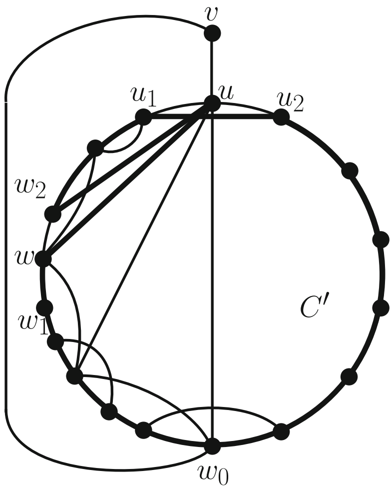


Fig. 6.19 Cycle C' for case 2 of proof of Theorem 6.5.14

Case 2. V_0 has a nonsingular vertex. Let w be the first nonsingular vertex as P is traversed from v to u_1. As before, let w_1 and w_2 be the neighbors of w along C. Then at least one of w_1 and w_2 is adjacent to u. Without loss of generality, assume that w_2 is adjacent to u. Let

$$C' = (C \cup \{w_2u, uw, u_1u_2\})\backslash\{w_2w, uu_1, uu_2\}.$$

(See Fig. 6.19.)

Clearly, C and C' are of the same length, and therefore C' is also a longest cycle of G. Then, by the choice w, the v-w section of P contains the only nonsingular vertex w. Let w_0 be the first singular vertex on this section. Consider the cycle C'' described as follows: Start from w_2 and move along C' away from u until we reach the vertex preceding w_0. Bypass w_0 by moving through the neighbors of w_0 along C' (as in case 1), and repeat it for each nonsingular vertex after w_0. After reaching w, move along the w-v section of P^{-1} and follow it by the path vuw_2 (see Fig. 6.20). Then C'' is a cycle longer than C' (as in case 1), a contradiction.

Hence, in any case, C cannot be a longest cycle of G. Thus, G is Hamiltonian. $\square$

6.6 2-Factorable Graphs

It is clear that if a graph G is r-factorable with k r-factors, then the degree of each vertex of G is rk. In particular, if G is 2-factorable, then G is regular of even degree, say, $2k$. That the converse is also true is a result due to Petersen [158].

Theorem 6.6.1 (Petersen). *Every $2k$-regular graph, $k \geq 1$, is 2-factorable.*

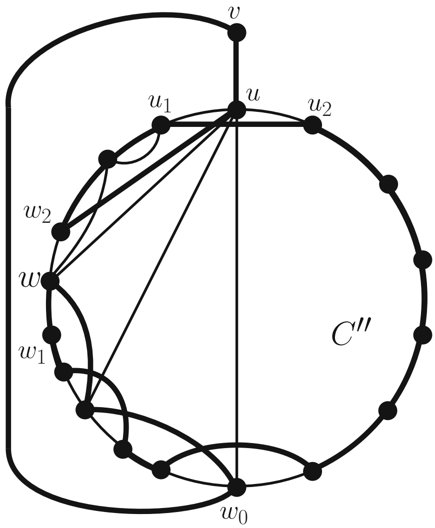


Fig. 6.20 Cycle C'' for case 2 of proof of Theorem 6.5.14

Proof. Let G be a $2k$-regular graph with $V = \{v_1, v_2, \ldots, v_n\}$. We may assume without loss of generality that G is connected. (Otherwise, we can consider the components of G separately.) Since each vertex of G is of even degree, by Theorem 6.2.2, G is Eulerian. Let T be an Euler tour of G. Form a bipartite graph H with bipartition (V, W), where $V = \{v_1, v_2, \ldots, v_n\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ and in which v_i is made adjacent to w_j if and only if v_j follows v_i immediately in T. Since at every vertex of G there are k incoming edges and k outgoing edges along T, H is k-regular. Hence, by Theorem 5.5.3, H is 1-factorable. Let the k 1-factors be $M_1, \ldots, M_k$. Label the edges of M_i with the label i, $1 \le i \le k$. Then the k edges incident at each v_i of H receive the k labels $1, 2, \ldots, k$, and hence if the edges $v_i\, w_j$ and $v_j\, w_r$ are in M_p, $1 \le p \le k$, identifying the vertex w_j with the vertex v_j for each j in M_p gives an edge labeling to G in which the edges $v_i\, v_j$ and $v_j\, v_r$ receive the label p. It is then clear that the edges of M_p yield a 2-factor of G with label p. Note that v_i is nonadjacent to w_i in H, $1 \le i \le k$. Since this is true for each of the 1-factors M_p, $1 \le p \le k$, we get a 2-factorization of G into k 2-factors. □

A special case of Theorem 6.6.1 is the 2-factorization of K_{2p+1}, which is $2p$-regular. Actually, K_{2p+1} has a 2-factorization into Hamilton cycles.

Theorem 6.6.2. *K_{2p+1} is 2-factorable into p Hamilton cycles.*

Proof. Label the vertices of K_{2p+1} as $v_0, v_1, \ldots, v_{2p}$. For $i = 0, 1, \ldots, p$, let P_i be the path $v_i\, v_{i-1}\, v_{i+1}\, v_{i-2}\, v_{i+2} \ldots v_{i+p-1}\, v_{i-(p-1)}$ (suffixes taken modulo $2p$), and let C_i be the Hamilton cycle obtained from P_i by joining v_{2p} to the end vertices of P_i. The cycles C_i are edge-disjoint. This may be seen by placing the $2p$ vertices $v_0, v_1, \ldots, v_{2p-1}$ symmetrically on a circle and placing v_{2p} at the center of the

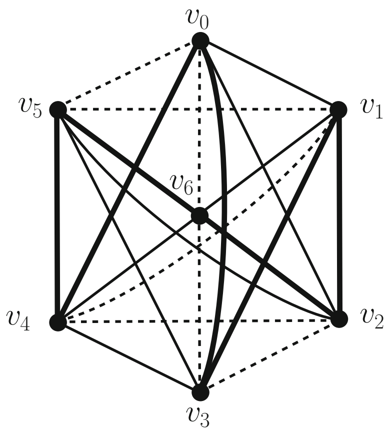


Fig. 6.21 Parallel chords and edge-disjoint Hamilton cycles in K_7

circle and noting that the edges $v_i\, v_{i-1},\ v_{i+1}\, v_{i-2},\ \ldots,\ v_{i+p-1}\, v_{i-p}$ form a set of p parallel chords of this circle. □

Figure 6.21 displays the three sets of parallel chords and three edge-disjoint Hamilton cycles in K_7. The 2-factors are

$$F_1 : v_6\, v_0\, v_5\, v_1\, v_4\, v_2\, v_3\, v_6,$$

$$F_2 : v_6\, v_1\, v_0\, v_2\, v_5\, v_3\, v_4\, v_6,$$

$$F_3 : v_6\, v_2\, v_1\, v_3\, v_0\, v_4\, v_5\, v_6.$$

6.7 Exercises

7.1. Prove: A Hamiltonian-connected graph is Hamiltonian. (Note: The converse is not true. See the next exercise.)

7.2. Show that a Hamiltonian-connected graph is 3-connected. Display a Hamiltonian graph of connectivity 3 that is not Hamiltonian-connected.

7.3. If G is traceable, show that for every proper subset S of $V(G)$, $\omega(G - S) \leq |S| + 1$. Disprove the converse by a counterexample.

7.4. If G is simple and $\delta \geq \frac{n-1}{2}$ show that G is traceable. Disprove the converse.

7.5. If G is simple and $\delta \geq \frac{n+1}{2}$ show that G is Hamiltonian-connected. Is the converse true?

7.6. Give an example of a non-Hamiltonian simple graph G of order n ($n \geq 3$) such that for every pair of nonadjacent vertices u and v, $d(u) + d(v) \geq n - 1$. [This shows that the condition in Ore's theorem (Theorem 6.3.5) cannot be weakened further.]

7.7. Show that if a cubic graph is Hamiltonian, then it has three disjoint 1-factors.

7.8.* Show that if a cubic graph has a 1-factor, then it has at least three distinct 1-factors.

7.9. Show that a complete k-partite graph G is Hamiltonian if and only if $|V(G)\backslash N| \geq |N|$, where N is the size of a maximum part of G. (See Aravamudhan and Rajendran [9].)

7.10. A graph is called *locally Hamiltonian* if $G[N(v)]$ is Hamiltonian for each vertex v of G. Show that a locally Hamiltonian graph is 3-connected.

7.11. If $|V(G)| \geq 5$, prove that $L(G)$ is locally Hamiltonian if and only if $G \cong K_{1,n}$.

7.12. If G is a 2-connected graph that is both $K_{1,3}$-free and $(K_{1,3} + e)$-free, prove that G is Hamiltonian. (Recall that a graph G is H*-free* if G does not contain an isomorphic copy of H as an induced subgraph.)

7.13. Let G be a simple graph of order $2n$ ($n \geq 2$). If for every pair of nonadjacent vertices u and v, $d(u) + d(v) > 2n + 2$, show that G contains a spanning cubic graph.

7.14. Show by means of an example that the square of a 1-connected (i.e., connected) graph need not be Hamiltonian. (A celebrated result of H. Fleischner states that the square of any 2-connected graph is Hamiltonian—a result that was originally conjectured by M. D. Plummer.)

7.15.* Let G be a simple graph with degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \leq d_2 \leq \ldots \leq d_n$ and $n \geq 3$. Suppose that there is no value of r less than $\frac{n}{2}$ for which $d_r \leq r$ and $d_{n-r} < n - r$. Show that G is Hamiltonian. (See Chvátal [38] or reference [27].)

7.16. Does there exist a simple non-Hamiltonian graph with degree sequence $(2, 3, 5, 5, 5, 6, 6, 6, 6, 6)$?

7.17. Draw a non-Hamiltonian simple graph with degree sequence $(3, 3, 3, 6, 6, 6, 9, 9, 9)$.

7.18. Let G be a $(2k + 1)$-regular graph with the property that $|[S, \bar{S}]| \geq 2k$ for every proper nonempty set S of V. Prove that G has k edge-disjoint 2-factors. (Note that when $k = 1$, this is just Petersen's result: Corollary 5.5.11. Hint: Use Tutte's 1-factor Theorem 5.6.5 to show that G has a 1-factor. Then apply Petersen's Theorem 6.6.1.)

Notes

Königsberg was part of East Prussia before Germany's defeat in World War II. It has been renamed Kaliningrad, and perhaps before long it will get back its original name. It is also the birthplace of the German mathematician David Hilbert as well as the German philosopher Immanuel Kant. It is interesting to note that even though the Königsberg bridge problem did give birth to Eulerian graphs, Euler himself did not use the concept of Eulerian graphs to solve this problem; instead, he relied on an exhaustive case-by-case verification (see reference [24]).

Ore's theorem (Theorem 6.3.5) can be restated as follows: *If G is a simple graph with $n \geq 3$ vertices and $|N(u)| + |N(v)| \geq n$, for every pair of nonadjacent vertices of G, then G is Hamiltonian.* This statement replaces $d(u)$ in Theorem 6.3.5 by $|N(u)|$. There are several sufficient conditions for a graph to be Hamiltonian using the neighborhood conditions. A nice survey of these results is given in Lesniak [131]. To give a flavor of these results, we give three results here of Faudree, Gould, Jacobson, and Lesniak:

Theorem 1. *If G is a 2-connected graph of order n such that $|N(u) \cap N(u)| \geq \frac{2n-1}{3}$ for each pair u, v of nonadjacent vertices of G, then G is Hamiltonian.*

Theorem 2. *If G is a connected graph of order n such that $|N(u) \cap N(v)| \geq \frac{2n-2}{3}$ for each pair u, v of nonadjacent vertices of G, then G is traceable.*

Theorem 3. *If G is a 3-connected graph of order n such that $|N(u) \cap N(v)| > \frac{2n}{3}$ for each pair u, v of nonadjacent vertices of G, then G is Hamiltonian-connected.*