Chapter 5
Independent Sets and Matchings

5.1 Introduction

Vertex-independent sets and vertex coverings as also edge-independent sets and edge coverings of graphs occur very naturally in many practical situations and hence have several potential applications. In this chapter, we study the properties of these sets. In addition, we discuss matchings in graphs and, in particular, in bipartite graphs. Matchings in bipartite graphs have varied applications in operations research. We also present two celebrated theorems of graph theory, namely, Tutte's 1-factor theorem and Hall's matching theorem. All graphs considered in this chapter are loopless.

5.2 Vertex-Independent Sets and Vertex Coverings

Definition 5.2.1. A subset S of the vertex set V of a graph G is called *independent* if no two vertices of S are adjacent in G. $S \subseteq V$ is a *maximum independent set* of G if G has no independent set S' with $|S'| > |S|$. A *maximal independent set* of G is an independent set that is not a proper subset of another independent set of G.

For example, in the graph of Fig. 5.1, $\{u, v, w\}$ is a maximum independent set and $\{x, y\}$ is a maximal independent set that is not maximum.

Definition 5.2.2. A subset K of V is called a *covering* of G if every edge of G is incident with at least one vertex of K. A covering K is *minimum* if there is no covering K' of G such that $|K'| < |K|$; it is *minimal* if there is no covering K_1 of G such that K_1 is a proper subset of K.

In the graph W_5 of Fig. 5.2, $\{v_1, v_2, v_3, v_4, v_5\}$ is a covering of W_5 and $\{v_1, v_3, v_4, v_6\}$ is a minimal covering. Also, the set $\{x, y\}$ is a minimum covering of the graph of Fig. 5.1.

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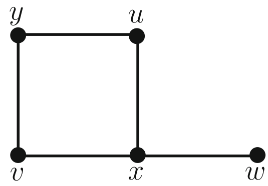


Fig. 5.1 Graph with maximum independent set $\{u, v, w\}$ and maximal independent set $\{x, y\}$

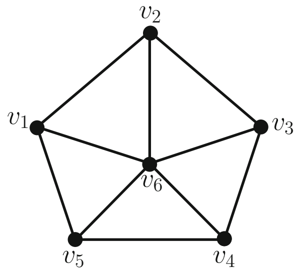


Fig. 5.2 Wheel W_5

The concepts of covering and independent sets of a graph arise very naturally in practical problems. Suppose we want to store a set of chemicals in different rooms. Naturally, we would like to store incompatible chemicals, that is, chemicals that are likely to react violently when brought together, in distinct rooms. Let G be a graph whose vertex set represents the set of chemicals and let two vertices be made adjacent in G if and only if the corresponding chemicals are incompatible. Then any set of vertices representing compatible chemicals forms an independent set of G.

Now consider the graph G whose vertices represent the various locations in a factory and whose edges represent the pathways between pairs of such locations. A light source placed at a location supplies light to all the pathways incident to that location. A set of light sources that supplies light to all the pathways in the factory forms a covering of G.

Theorem 5.2.3. *A subset S of V is independent if and only if $V \backslash S$ is a covering of G.*

Proof. S is independent if and only if no two vertices in S are adjacent in G. Hence, every edge of G must be incident to a vertex of $V \backslash S$. This is the case if and only if $V \backslash S$ is a covering of G. □

Definition 5.2.4. The number of vertices in a maximum independent set of G is called the *independence number* (or the *stability number*) of G and is denoted by $\alpha(G)$. The number of vertices in a minimum covering of G is the *covering number* of G and is denoted by $\beta(G)$. We denote these numbers simply by α and β when there is no confusion.

Corollary 5.2.5. *For any graph G, $\alpha + \beta = n$.*

Proof. Let S be a maximum independent set of G. By Theorem 5.2.3, $V \backslash S$ is a covering of G and therefore $|V \backslash S| = n - \alpha \geq \beta$. Similarly, let K be a minimum covering of G. Then $V \backslash K$ is independent and so $|V \backslash K| = n - \beta \leq \alpha$. These two inequalities together imply that $n = \alpha + \beta$. □

5.3 Edge-Independent Sets

Definitions 5.3.1. 1. A subset M of the edge set E of a loopless graph G is called *independent* if no two edges of M are adjacent in G.
2. A *matching* in G is a set of independent edges.
3. An *edge covering* of G is a subset L of E such that every vertex of G is incident to some edge of L. Hence, an edge covering of G exists if and only if $\delta > 0$.
4. A matching M of G is *maximum* if G has no matching M' with $|M'| > |M|$. M is *maximal* if G has no matching M' strictly containing M. $\alpha'(G)$ is the cardinality of a maximum matching and $\beta'(G)$ is the size of a minimum edge covering of G.
5. A set S of vertices of G is said to be *saturated* by a matching M of G or *M-saturated* if every vertex of S is incident to some edge of M. A vertex v of G is *M-saturated* if $\{v\}$ is M-saturated. v is *M-unsaturated* if it is not M-saturated.

For example, in the wheel W_5 (Fig. 5.2), $M = \{v_1v_2, v_4v_6\}$ is a maximal matching; $\{v_1v_5, v_2v_3, v_4v_6\}$ is a maximum matching and a minimum edge covering; the vertices v_1, v_2, v_4, and v_6 are M-saturated, whereas v_3 and v_5 are M-unsaturated.

Remark 5.3.2. The edge analog of Theorem 5.2.3 is not true, however. For instance, in the graph G of Fig. 5.3, the set $E' = \{e_3, e_4\}$ is independent, but $E \backslash E' = \{e_1, e_2, e_5\}$ is not an edge covering of G. Also, $E'' = \{e_1, e_3, e_4\}$ is an edge covering of G, but $E \backslash E''$ is not independent in G. Again, E' is a matching in G that saturates v_2, v_3, v_4 and v_5 but does not saturate v_1.

Theorem 5.3.3. *For any graph G for which $\delta > 0$, $\alpha' + \beta' = n$.*

Proof. Let M be a maximum matching in G so that $|M| = \alpha'$. Let U be the set of M-unsaturated vertices in G. Since M is maximum, U is an independent set of vertices with $|U| = n - 2\alpha'$. Since $\delta > 0$, we can pick one edge for each vertex in

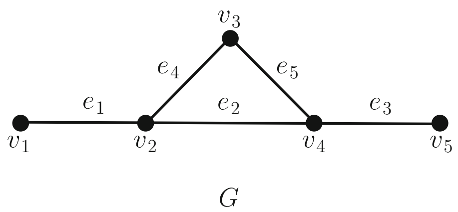


Fig. 5.3 Graph illustrating edge relationships

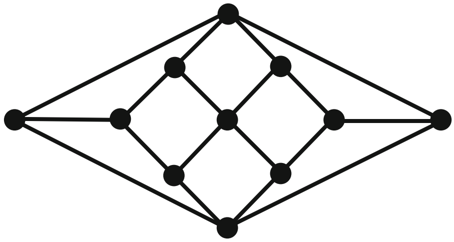


Fig. 5.4 Herschel graph

U incident with it. Let F be the set of edges thus chosen. Then $M \cup F$ is an edge covering of G. Hence, $|M \cup F| = |M| + |F| = \alpha' + n - 2\alpha' \geq \beta'$, and therefore

$$n \geq \alpha' + \beta'. \tag{5.1}$$

Now let L be a minimum edge covering of G so that $|L| = \beta'$. Let $H = G[L]$ be the edge subgraph of G defined by L, and let M_H be a maximum matching in H. Denote the set of M_H-unsaturated vertices in H by U. As L is an edge covering of G, H is a spanning subgraph of G. Consequently, $|L| - |M_H| = |L \backslash M_H| \geq |U| = n - 2|M_H|$ and so $|L| + |M_H| \geq n$. But since M_H is a matching in G, $|M_H| \leq \alpha'$. Thus,

$$n \leq |L| + |M_H| \leq \beta' + \alpha'. \tag{5.2}$$

Inequalities (5.1) and (5.2) imply that $\alpha' + \beta' = n$. □

Exercise 3.1. Determine the values of the parameters α, α', β, and β' for

1. K_n,
2. The Petersen graph P,
3. The Herschel graph (see Fig. 5.4).

Exercise 3.2. For any graph G with $\delta > 0$, prove that $\alpha \leq \beta'$ and $\alpha' \geq \beta$.

Exercise 3.3. Show that for a bipartite graph G, $\alpha\,\beta \geq m$ and that equality holds if and only if G is complete.

5.4 Matchings and Factors

Definition 5.4.1. A *matching* of a graph G is (as given in Definition 5.3.1) a set of independent edges of G. If $e = uv$ is an edge of a matching M of G, the end vertices u and v of e are said to be *matched* by M.

If M_1 and M_2 are matchings of G, the edge subgraph defined by $M_1 \Delta M_2$, the symmetric difference of M_1 and M_2, is a subgraph H of G whose components are paths or even cycles of G in which the edges alternate between M_1 and M_2.

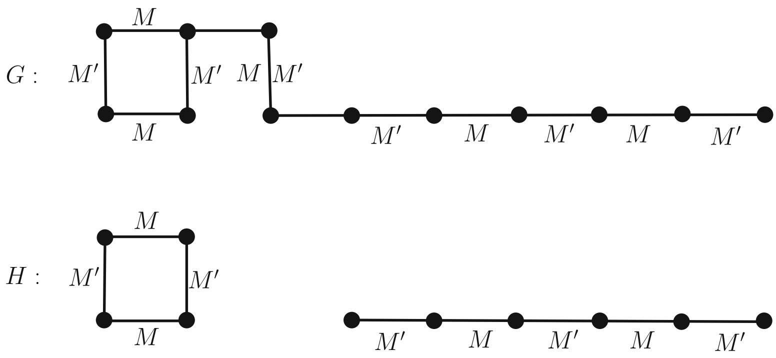


Fig. 5.5 Graphs for proof of Theorem 5.4.4

Definition 5.4.2. An *M-augmenting path* in G is a path in which the edges alternate between $E\backslash M$ and M and its end vertices are M-unsaturated. An *M-alternating path* in G is a path whose edges alternate between $E\backslash M$ and M.

Example 5.4.3. In the graph G of Fig. 5.2, $M_1 = \{v_1v_2, v_3v_4, v_5v_6\}$, $M_2 = \{v_1v_2, v_3v_6, v_4v_5\}$, and $M_3 = \{v_3v_4, v_5v_6\}$ are matchings of G. Moreover, $G[M_1 \Delta M_2]$ is the even cycle $(v_3v_4v_5v_6v_3)$. The path $v_2v_3v_4v_6v_5v_1$ is an M_3-augmenting path in G.

Maximum matchings have been characterized by Berge [19].

Theorem 5.4.4. *A matching M of a graph G is maximum if and only if G has no M-augmenting path.*

Proof. Assume first that M is maximum. If G has an M-augmenting path P : $v_0v_1v_2 \ldots v_{2t+1}$ in which the edges alternate between $E\backslash M$ and M, then P has one edge of $E\backslash M$ more than that of M. Define

$$M' = (M \cup \{v_0v_1, v_2v_3, \ldots, v_{2t}v_{2t+1}\})\backslash\{v_1v_2, v_3v_4, \ldots, v_{2t-1}v_{2t}\}.$$

Clearly, M' is a matching of G with $|M'| = |M| + 1$, which is a contradiction since M is a maximum matching of G.

Conversely, assume that G has no M-augmenting path. Then M must be maximum. If not, there exists a matching M' of G with $|M'| > |M|$. Let H be the edge subgraph $G[M \Delta M']$ defined by the symmetric difference of M and M'. Then the components of H are paths or even cycles in which the edges alternate between M and M'. Since $|M'| > |M|$, at least one of the components of H must be a path starting and ending with edges of M'. But then such a path is an M-augmenting path of G, contradicting the assumption (see Fig. 5.5). □

Definition 5.4.5. A *factor* of a graph G is a spanning subgraph of G. A *k-factor* of G is a factor of G that is k-regular. Thus, a 1-*factor* of G is a matching that saturates

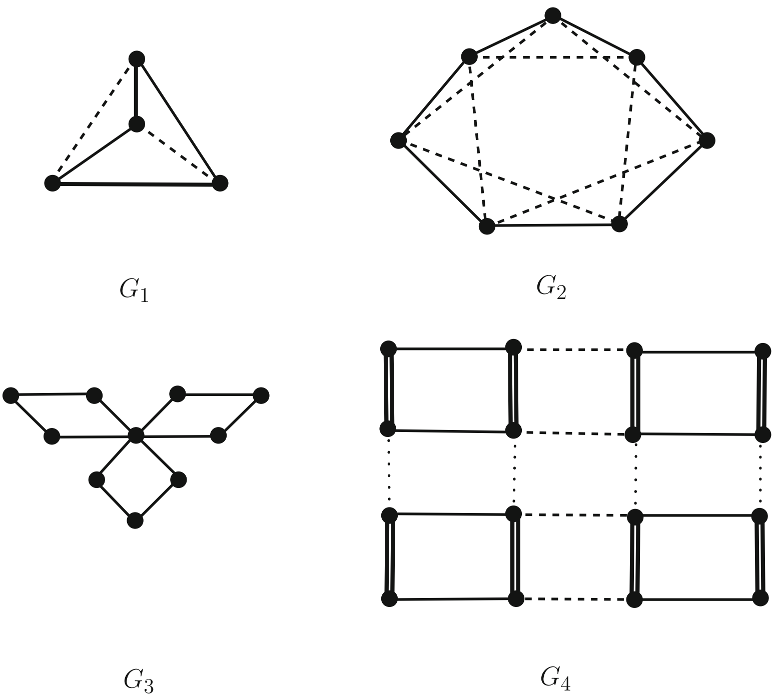


Fig. 5.6 Graphs illustrating factorability

all the vertices of G. For this reason, a 1-factor of G is called a *perfect matching* of G. A 2-factor of G is a factor of G that is a disjoint union of cycles of G. A graph G is k-*factorable* if G is an edge-disjoint union of k-factors of G.

Example 5.4.6. In Fig. 5.6, G_1 is 1-factorable and G_2 is 2-factorable, whereas G_3 has neither a 1-factor nor a 2-factor. The dotted, solid, and ordinary lines of G_1 give the three distinct 1-factors, and the dotted and ordinary lines of G_2 give its two distinct 2-factors.

Exercise 4.1. Give an example of a cubic graph having no 1-factor.

Exercise 4.2. Show that $K_{n,n}$ and K_{2n} are 1-factorable.

Exercise 4.3. Show that the number of 1-factors of

(i) $K_{n,n}$ is $n!$,
(ii) K_{2n} is $\frac{(2n)!}{2^n\, n!}$.

Exercise 4.4. The n-*cube* Q_n is the graph whose vertices are binary n-tuples. Two vertices of Q_n are adjacent if and only if they differ in exactly one place. Show that Q_n ($n \geq 2$) has a perfect matching. (The 3-cube Q_3 and the 4-cube Q_4 are displayed in Fig. 5.7.) It is easy to see that $Q_n \simeq K_2 \square K_2 \square \ldots \square K_2$ (n times).

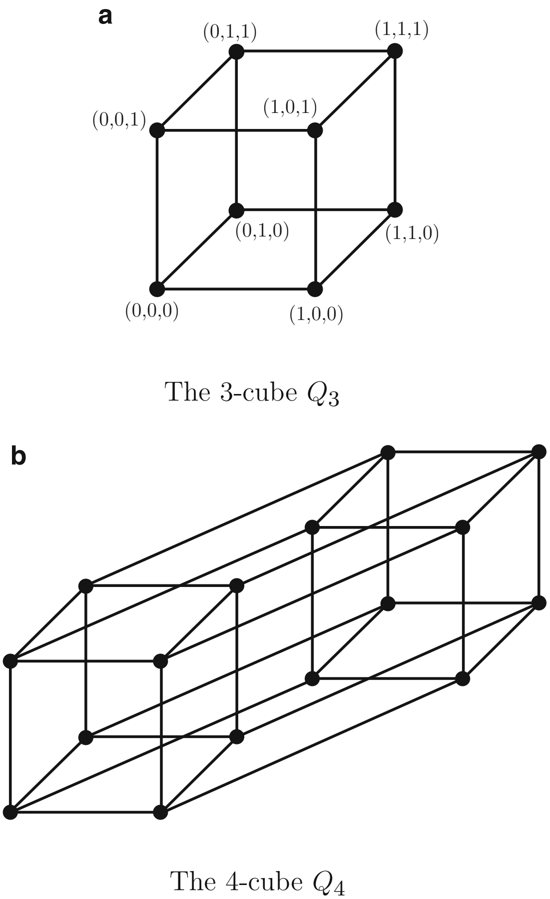


Fig. 5.7 (**a**) 3-cube Q_3 and (**b**) 4-cube Q_4

Exercise 4.5. Show that the Petersen graph P is not 1-factorable. (Hint: Look at the possible types of 1-factors of P.)

Exercise 4.6. Show that every tree has at most one perfect matching.

Exercise 4.7*. Show that if a 2-edge-connected graph has a 1-factor, then it has at least two distinct 1-factors.

Exercise 4.8. Show that the graph G_4 of Fig. 5.6 is not 1-factorable.

An Application to Physics 5.4.7. In crystal physics, a crystal is represented by a three-dimensional lattice in which each face corresponds to a two-dimensional lattice. Each vertex of the lattice represents an atom of the crystal, and an edge between two vertices represents the bond between the two corresponding atoms.

In crystallography, one is interested in obtaining an analytical expression for certain surface properties of crystals consisting of diatomic molecules (also called dimers). For this, one must find the number of ways in which all the atoms of the crystal can be paired off as molecules consisting of two atoms each. The problem is clearly equivalent to that of finding the number of perfect matchings of the corresponding two-dimensional lattice.

Two different dimer coverings (perfect matchings) of the lattice defined by the graph G_4 are exhibited in Fig. 5.6—one in solid lines and the other in parallel lines.

5.5 Matchings in Bipartite Graphs

Assignment Problem 5.5.1. Suppose in a factory there are n jobs $j_1, j_2, \ldots, j_n$ and s workers $w_1, w_2, \ldots, w_s$. Also suppose that each job j_i can be performed by a certain number of workers and that each worker w_j has been trained to do a certain number of jobs. Is it possible to assign each of the n jobs to a worker who can do that job so that no two jobs are assigned to the same worker?

We convert this job assignment problem into a problem in graphs as follows: Form a bipartite graph G with bipartition (J, W), where $J = \{j_1, j_2, \ldots, j_n\}$ and $W = \{w_1, w_2, \ldots, w_s\}$, and make j_i adjacent to w_j if and only if worker w_j can do the job j_i. Then our assignment problem translates into the following graph problem: Is it possible to find a matching in G that saturates all the vertices of J?

A solution to the above matching problem in bipartite graphs has been given by Hall [90] (see also Hall, Jr. [91]).

For a subset $S \subseteq V$ in a graph G, $N(S)$ denotes the neighbor set of S, that is, the set of all vertices each of which is adjacent to at least one vertex in S.

Theorem 5.5.2 (Hall). *Let G be a bipartite graph with bipartition (X, Y). Then G has a matching that saturates all the vertices of X if and only if*

$$|N(S)| \geq |S| \tag{5.3}$$

for every subset S of X.

Proof. If G has a matching that saturates all the vertices of X, then distinct vertices of X are matched to distinct vertices of Y. Hence, trivially, $|N(S)| \geq |S|$ for every subset $S \subseteq X$.

Conversely, assume that the condition (5.3) above holds but that G has no matching that saturates all the vertices of X. Let M be a maximum matching of G. As M does not saturate all the vertices of X, there exists a vertex $x_0 \in X$ that is M-unsaturated. Let Z denote the set of all vertices of G connected to x_0 by M-alternating paths. Since M is a maximum matching, by Theorem 5.4.4, G has no M-augmenting path. As x_0 is M-unsaturated, x_0 is the only vertex of Z that is M-unsaturated. Let $A = Z \cap X$ and $B = Z \cap Y$. Then the vertices of $A \backslash \{x_0\}$ get matched under M to the vertices of B, and $N(A) = B$. Thus, since $|B| = |A| - 1$, $|N(A)| = |B| = |A| - 1 < |A|$, and this contradicts the assumption (5.3) (see Fig. 5.8). □

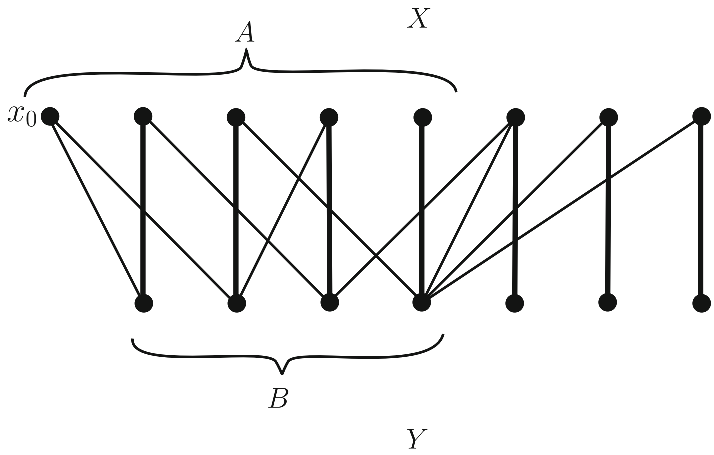


Fig. 5.8 Figure for proof of Theorem 5.5.2 (matching edges are *boldfaced*)

We now give some important consequences of Hall's theorem.

Theorem 5.5.3. *A k ($\geq$ 1)-regular bipartite graph is 1-factorable.*

Proof. Let G be k-regular with bipartition (X, Y). Then $E(G) =$ the set of edges incident to the vertices of $X =$ the set of edges incident to the vertices of Y. Hence, $k|X| = |E(G)| = k|Y|$, and therefore $|X| = |Y|$. If $S \subseteq X$, then $N(S) \subseteq Y$, and $N(N(S))$ contains S. Let E_1 and E_2 be the sets of edges of G incident to S and $N(S)$, respectively. Then $E_1 \subseteq E_2$, $|E_1| = k|S|$, and $|E_2| = k|N(S)|$. Hence, as $|E_2| \geq |E_1|$, $|N(S)| \geq |S|$. So by Hall's theorem (Theorem 5.5.2), G has a matching that saturates all the vertices of X; that is, G has a perfect matching M. Deletion of the edges of M from G results in a $(k-1)$-regular bipartite graph. Repeated application of the above argument shows that G is 1-factorable. □

König's theorem: Consider any matching M of a graph G. If K is any (vertex) covering for the graph, then it is clear that to cover each edge of M, we have to choose at least one vertex of K. Thus, $|M| \leq |K|$. In particular, if M^* is a maximum matching and K^* is a minimum covering of G, then

$$|M^*| \leq |K^*|. \tag{5.4}$$

König's theorem asserts that for bipartite graphs, equality holds in relation (5.4). Before we establish this theorem, we present a lemma that is interesting in its own right and is similar to Lemma 3.6.8.

Lemma 5.5.4. *Let K be any covering and M any matching of a graph G with $|K| = |M|$. Then K is a minimum covering and M is a maximum matching.*

Proof. Let M^* be a maximum matching and K^* a minimum covering of G. Then $|M| \leq |M^*|$ and $|K| \geq |K^*|$. Hence, by (5.4) we have $|M| \leq |M^*| \leq |K^*| \leq |K|$. Since $|M| = |K|$, we must have $|M| = |M^*| = |K^*| = |K|$, proving the lemma. □

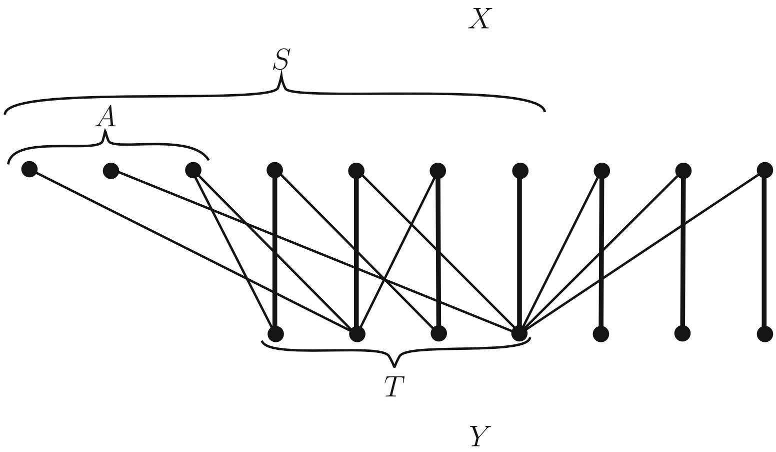


Fig. 5.9 Graph for proof of Theorem 5.5.5

Theorem 5.5.5 (König). *In a bipartite graph the minimum number of vertices that cover all the edges of G is equal to the maximum number of independent edges; that is,* $\alpha'(G) = \beta(G)$.

Proof. Let G be a bipartite graph with bipartition (X, Y). Let M be a maximum matching in G. Denote by A the set of vertices of X unsaturated by M (see Fig. 5.9). As in the proof of Theorem 5.5.2, let Z stand for the set of vertices connected to A by M-alternating paths starting in A. Let $S = X \cap Z$ and $T = Y \cap Z$. Then clearly, $T = N(S)$ and $K = T \cup (X \backslash S)$ is a covering of G, because if there is an edge e not incident to any vertex in K, then one of the end vertices of e must be in S and the other in $Y \backslash T$, contradicting the fact that $N(S) = T$. Clearly, $|K| = |M|$, and so by Lemma 5.5.4, M is a maximum matching and K a minimum covering of G. □

Let A be a binary matrix (so that each entry of A is 0 or 1). A *line* of A is a row or column of A. A line covers all of its entries. Two 1's of A are called *independent* if they do not lie in the same line of A. The matrix version of König's theorem is given in Theorem 5.5.6.

Theorem 5.5.6 (Matrix version of König's theorem). *In a binary matrix, the minimum number of lines that cover all the* 1*'s is equal to the maximum number of independent* 1*'s.*

Proof. Let $A = (a_{ij})$ be a binary matrix of size p by q. Form a bipartite graph G with bipartition (X, Y), where X and Y are sets of cardinality p and q, respectively, say, $X = \{v_1, v_2, \ldots, v_p\}$ and $Y = \{w_1, w_2, \ldots, w_q\}$. Make v_i adjacent to w_j in G if and only if $a_{ij} = 1$. Then an entry 1 in A corresponds to an edge of G, and two independent 1's in A correspond to two independent edges of G. Further, each vertex of G corresponds to a line of A. Thus, the matrix version of König's theorem is actually a restatement of König's theorem. □

A consequence of Theorem 5.5.2 is the theorem on the existence of a *system of distinct representatives* (SDR) for a family of subsets of a given finite set.

Definition 5.5.7. Let $\mathscr{F} = \{A_\alpha : \alpha \in J\}$ be a family of sets. An *SDR* for the family $\mathscr{F}$ is a family of elements $\{x_\alpha : \alpha \in J\}$ such that $x_\alpha \in A_\alpha$ for every $\alpha \in J$ and $x_\alpha \neq x_\beta$ whenever $\alpha \neq \beta$.

Example 5.5.8. For instance, if $A_1 = \{1\}$, $A_2 = \{2,3\}$, $A_3 = \{3,4\}$, $A_4 = \{1,2,3,4\}$, and $A_5 = \{2,3,4\}$, then the family $\{A_1, A_2, A_3, A_4\}$ has $\{1,2,3,4\}$ as an SDR, whereas the family $\{A_1, A_2, A_3, A_4, A_5\}$ has no SDR. It is clear that for $\mathscr{F}$ to have an SDR, it is necessary that for any positive integer k, the union of any k sets of $\mathscr{F}$ must contain at least k elements. That this condition is also sufficient when $\mathscr{F}$ is a finite family of finite sets is the assertion of Hall's theorem on the existence of an SDR.

Theorem 5.5.9 (Hall's theorem on the existence of an SDR [90]). *Let $\mathscr{F} = \{A_i : 1 \leq i \leq r\}$ be a family of finite sets. Then $\mathscr{F}$ has an SDR if and only if the union of any k members of $\mathscr{F}$, $1 \leq k \leq r$, contains at least k elements.*

Proof. We need only prove the sufficiency part. Let $\bigcup_{i=1}^{r} A_i = \{y_1, y_2, \ldots, y_n\}$. Form a bipartite graph $G = G(X, Y)$ with $X = \{x_1, x_2, \ldots, x_r\}$, where x_i corresponds to the set A_i, $1 \leq i \leq r$, and $Y = \{y_1, y_2, \ldots, y_n\}$. Make x_i adjacent to y_j in G if and only if $y_j \in A_i$. Then it is clear that $\mathscr{F}$ has an SDR if and only if G has a matching that saturates all the vertices of X. But this is the case, by Theorem 5.5.2, if for each $S \subseteq X$, $|N(S)| \geq |S|$, that is, if and only if $|\bigcup_{x_i \in S} A_i| \geq |S|$, which is precisely the condition stated in the theorem. □

Exercise 5.1. Prove Theorem 5.5.5 (König's theorem) assuming Theorem 5.5.9.

Exercise 5.2. Show that a bipartite graph has a 1-factor if and only if $|N(S)| \geq |S|$ for every subset S of V. Does this hold for any graph G?

When does a graph have a 1-factor? Tutte's celebrated 1-*factor theorem* answers this question. The proof given here is due to Lovász [135]. A component of a graph is *odd* or *even* according to whether it has an odd or even number of vertices. Let $O(G)$ denote the number of odd components of G.

Theorem 5.5.10 (Tutte's 1-factor theorem [179]). *A graph G has a 1-factor if and only if*

$$O(G - S) \leq |S|, \tag{5.5}$$

for all $S \subseteq V$.

Proof. While considering matchings in graphs, we are interested only in the adjacency of pairs of vertices. Hence, we may assume without loss of generality that G is simple. If G has a 1-factor M, each of the odd components of $G - S$ must have at least one vertex, which is to be matched only to a vertex of S under M. Hence, for each odd component of $G - S$, there exists an edge of the matching with one end in S. Hence, the number of vertices in S should be at least as large as the number of odd components in $G - S$; that is, $O(G - S) \leq |S|$.

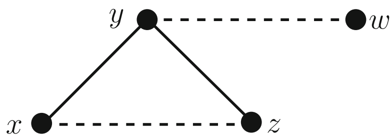


Fig. 5.10 Supergraph G^* for proof of Theorem 5.5.10. *Unbroken lines* correspond to edges of G^* and *broken lines* correspond to edges not belonging to G^*

Conversely, assume that condition (5.5) holds. If G has no 1-factor, we join pairs of nonadjacent vertices of G until we get a maximal supergraph G^* of G with G^* having no 1-factor. Condition (5.5) holds clearly for G^* as

$$O(G^* - S) \le O(G - S). \tag{5.6}$$

(When two odd components are joined by an edge, the result is an even component.)

Taking $S = \phi$ in (5.5), we see that $O(G) = 0$, and so $n(G^*)\ (= n(G)) = n$ is even. Further, for every pair of nonadjacent vertices u and v of G^*, $G^* + uv$ has a 1-factor, and any such 1-factor must necessarily contain the edge uv.

Let K be the set of vertices of G^* of degree $(n - 1)$. $K \neq V$, since otherwise $G^* = K_n$ has a perfect matching. We claim that each component of $G^* - K$ is complete. Suppose to the contrary that some component G_1 of $G^* - K$ is not complete. Then in G_1 there are vertices $x,\ y$ and z such that $xy \in E(G^*)$, $yz \in E(G^*)$, but xz does not belong to $E(G^*)$ (Exercise 5.11 of Chap. 1). Moreover, since $y \in V(G_1)$, $d_{G^*}(y) < n - 1$ and hence there exists a vertex w of G^* with $yw \notin E(G^*)$. Necessarily, w does not belong to K. (See Fig. 5.10.)

By the choice of G^*, each of $G^* + xz$ and $G^* + yw$ has a 1-factor, say M_1 and $M2$, respectively. Necessarily, $xz \in M_1$ and $yw \in M_2$. Let H be the subgraph of $G^* + \{xz, yw\}$ induced by the edges in the symmetric difference $M_1 \Delta M_2$ of M_1 and M_2. Since M_1 and M_2 are 1-factors, each vertex of G^* is saturated by both M_1 and M_2, and H is a disjoint union of even cycles in which the edges alternate between M_1 and M_2. There arise two cases:

Case 1. xz and yw belong to different components of H (Fig. 5.11a). If yw belongs to the even cycle C, then the edges of M_1 in C together with the edges of M_2 not belonging to C form a 1-factor in G^*, contradicting the choice of G^*.

Case 2. xz and yw belong to the same component C of H. Since each component of H is a cycle, C is a cycle (Fig. 5.11b). By the symmetry of x and z, we may suppose that the vertices $x,\ y,\ w$, and z occur in that order on C. Then the edges of M_1 belonging to the $yw \ldots z$ section of C together with the edge yz and the edges of M_2 not in the $yw \ldots z$ section of C form a 1-factor of G^*, again contradicting the choice of G^*. Thus, each component of $G^* - K$ is complete.

By condition (5.6), $O(G^* - K) \le |K|$. Hence, a vertex of each of the odd components of $G^* - K$ is matched to a vertex of K. (This is possible since each

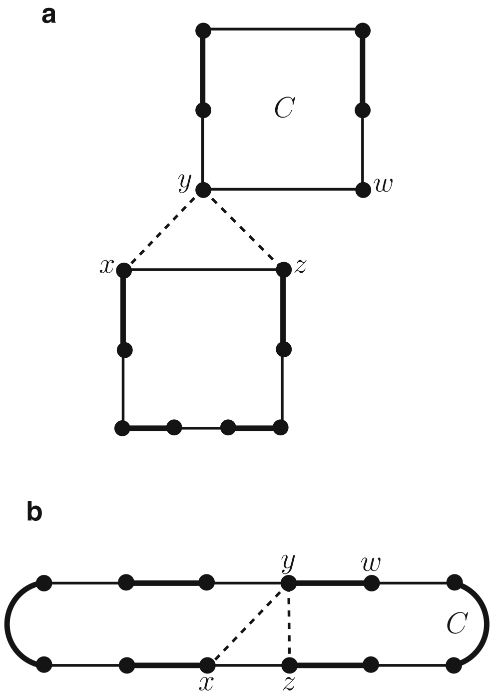


Fig. 5.11 1-factors M_1 and M_2 for (**a**) case 1 and (**b**) case 2 in proof of Theorem 5.5.10. *Ordinary lines* correspond to edges of M_1 and *bold lines* correspond to edges of M_2

vertex of K is adjacent to every other vertex of G^*.) Also, the remaining vertices in each of the odd and even components of $G^* - K$ can be matched among themselves (see Fig. 5.12). The total number of vertices thus matched is even. Since $|V(G^*)|$ is even, the remaining vertices, if any, of K can be matched among themselves. This gives a 1-factor of G^*. Note that if $K = \emptyset$, $O(G^*) = 0$, and the existence of a 1-factor in G^* is trivially true. But by choice, G^* has no 1-factor. This contradiction proves that G has a 1-factor. □

Corollary 5.5.11 (Petersen [158]). *Every connected 3-regular graph having no cut edges has a 1-factor.*

Proof. Let G be a connected 3-regular graph without cut edges. Let $S \subseteq V$. Denote by $G_1, G_2, \ldots, G_k$ the odd components of $G - S$. Let m_i be the number of edges of G having one end in $V(G_i)$ and the other end in S. Since G is a cubic graph,

$$\sum_{v \in V(G_i)} d(v) = 3n(G_i), \tag{5.7}$$

and

$$\sum_{v \in S} d(v) = 3|S|. \tag{5.8}$$

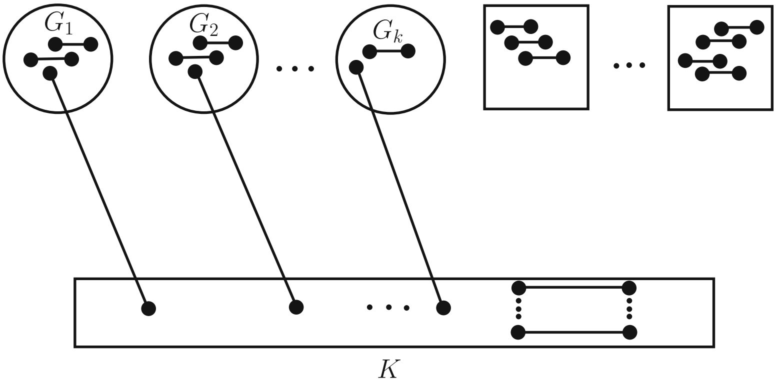


Fig. 5.12 Components of $G^* - K$ for proof of Theorem 5.5.10

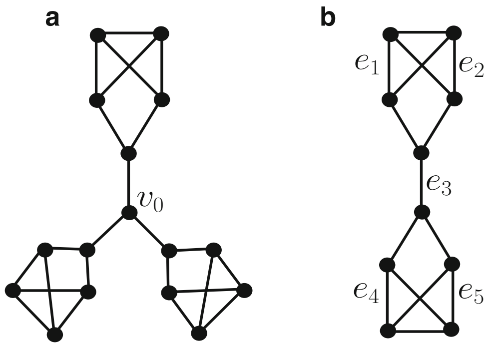


Fig. 5.13 (**a**) 3-regular graph with cut edges having no 1-factor; (**b**) cubic graph with a 1-factor having a cut edge

Now $E(G_i) = [V(G_i), V(G_i) \cup S] \backslash [V(G_i), S]$, where $[A, B]$ denotes the set of edges having one end in A and the other end in B, $A \subseteq V$, $B \subseteq V$. Hence, $m_i = |[V(G_i), S]| = \sum_{v \in V(G_i)} d(v) - 2m(G_i)$, and since $d(v)$ is 3 for each v and $V(G_i)$ is an odd component, m_i is odd for each i. Further, as G has no cut edges, $m_i \geq 3$. Thus, $O(G - S) = k \leq \frac{1}{3} \sum_{i=1}^{k} m_i \leq \frac{1}{3} \sum_{v \in S} d(v) = \frac{1}{3} 3|S| = |S|$. Therefore, by Tutte's theorem (Theorem 5.5.10), G has a 1-factor. □

Example 5.5.12. A 3-regular graph with cut edges may not have a 1-factor (see Fig. 5.13a). Again, a cubic graph with a 1-factor may have cut edges (see Fig. 5.13b).

In Fig. 5.13a, if $S = \{v_0\}$, $O(G - S) = 3 > 1 = |S|$, and so G has no 1-factor. In Fig. 5.13b, $\{e_1, e_2, e_3, e_4, e_5\}$ is a 1-factor, and e_3 is a cut edge of G.

If G has no 1-factor, by Theorem 5.5.10 there exists $S \subset V(G)$ with $O(G - S) > |S|$. Such a set S is called an *antifactor* set of G; clearly, S is a proper subset of $V(G)$.

Let G be a graph of even order n and let S be an antifactor set of G. Then $|S|$ and $O(G - S)$ have the same parity, and therefore $O(G - S) \equiv |S| \pmod 2$. Thus, we make the following observation.

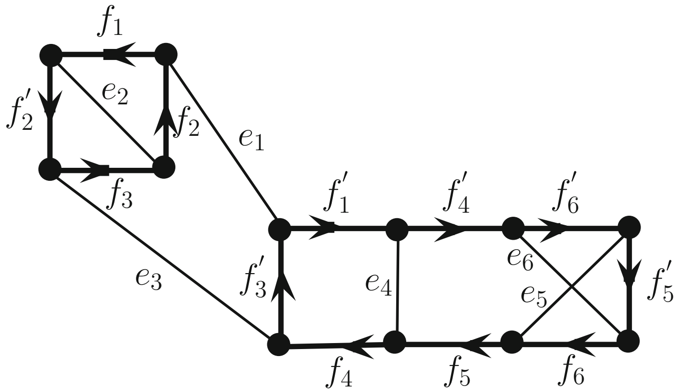


Fig. 5.14 Figure for the proof of Corollary 5.5.14

Observation 5.5.13. If S is an antifactor set of a graph G of even order, then $O(G - S) \geq |S| + 2$.

Corollary 5.5.14 (W. H. Cunnigham; see [119]). *The edge set of a simple 2-edge-connected cubic graph G can be partitioned into paths of length 3.*

Proof. By Corollary 5.5.11, G is a union of a 1-factor and a 2-factor. Orient the edges of each cycle of the above 2-factor in any manner so that each cycle becomes a directed cycle. Then if e_1 is any edge of the 1-factor, and f_1, f_1' are the two arcs of G having their tails at the end vertices of e_1, then $\{e_1, f_1, f_1'\}$ forms a typical 3-path of the edge partition of G (see Fig. 5.14). □

Corollary 5.5.15. *A $(p-1)$-regular connected simple graph on $2p$ vertices has a 1-factor.*

Proof. Proof is by contradiction. Let G be a $(p-1)$-regular connected simple graph on $2p$ vertices having no 1-factor. Then G has an antifactor set S. By Observation 5.5.13, $O(G - S) \geq |S| + 2$. Hence, $|S| + (|S| + 2) \leq 2p$, and therefore $|S| \leq p - 1$. Let $|S| = p - r$. Then $r \neq 1$ since if $r = 1$, $|S| = p - 1$, and therefore $O(G - S) = p + 1$. (Recall that G has $2p$ vertices.) Hence, each odd component of $G - S$ is a singleton, and therefore each such vertex must be adjacent to all the $p-1$ vertices of S [as G is $(p-1)$-regular]. But this means that every vertex of S is of degree at least $p + 1$, a contradiction. Thus, $|S| = p - r$, $2 \leq r \leq p - 1$. If G' is any component of $G - S$ and $v \in V(G')$, then v can be adjacent to at most $|S|$ vertices of S. Therefore, as G is $(p-1)$-regular, v must be adjacent to at least $(p-1)-(p-r) = r-1$ vertices of G'. Thus, $|V(G')| \geq r$. Counting the vertices of all the odd components of $G-S$ and the vertices of S, we get $(|S|+2)r + |S| \leq 2p$, or $(p - r + 2)r + (p - r) \leq 2p$. This gives $(r - 1)(r - p) \geq 0$, violating the condition on r. □

Our next result shows that there is another special family of graphs for which we can immediately conclude that all the graphs of the family have a 1-factor.

Theorem 5.5.16* (D. P. Sumner [174]). *Let G be a connected graph of even order n. If G is claw-free (i.e., contains no $K_{1,3}$ as an induced subgraph), then G has a 1-factor.*

Proof. If G has no 1-factor, G contains a minimal antifactor set S of G. There must be an edge between S and each odd component of $G - S$. Since $O(G - S) > |S|$ and G is of even order, by Observation 5.5.13, $O(G - S) \geq |S| + 2$. Hence, there are two possibilities: (i) There exists $v \in S$, and vx, vy, vz are edges of G with x, y and z belonging to distinct odd components of $G - S$. This cannot occur since by hypothesis G is $K_{1,3}$-free. (ii) There exist a vertex v of S, and edges vu and vw of G with u and w in distinct odd components of $G - S$. Suppose G_u and G_w are the odd components containing u and w, respectively. Then $< G_u \cup G_w \cup \{v\} >$ is an odd component of $G - S_1$, where $S_1 = S - \{v\}$. Further, $O(G - S_1) = O(G - S) - 1 > |S| - 1 = |S_1|$, and hence S_1 is an antifactor set of G with $|S_1| = |S| - 1$, a contradiction to the choice of S. Thus, G must have a 1-factor. [Note that by Observation 5.5.13, the case $|S| = 1$ and $O(G-S) = 2$ cannot arise.] □

Exercise 5.3. Find a 1-factorization of (i) Q_3, (ii) Q_4.

Exercise 5.4. Prove that Q_n, $n \geq 2$, is 1-factorable.

Exercise 5.5. Display a 2-factorization of K_9.

Exercise 5.6. Show that a k-regular $(k-1)$-edge-connected graph of even order has a 1-factor. (This result of F. Babler generalizes Petersen's result (Corollary 5.5.11) and can be shown by imitating the proof of Corollary 5.5.11).

Exercise 5.7. If G is a k-connected graph of even order having no $K_{1,k+1}$ as an induced subgraph, show that G has a 1-factor.

Exercise 5.8. Show that if G is a connected graph of even order, then G^2 has a 1-factor.

Exercise 5.9. (A square matrix $A = (a_{ij})$ is called *doubly stochastic* if $a_{ij} \geq 0$ for each i and j, and the sum of the entries in each row and column of A is 1.) Let $A = (a_{ij})$ be a doubly stochastic matrix of order n. Let $G = G(X, Y)$ be the bipartite graph with $|X| = |Y| = n$ obtained by setting $x_i x_j \in E(G)$ if and only if $a_{ij} = 0$. Prove that G has a perfect matching. (Hint: Apply Hall's theorem.)

5.6* Perfect Matchings and the Tutte Matrix

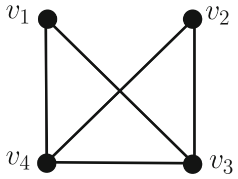
It has been established by Tutte that the existence of a perfect matching in a simple graph is related to the nonsingularity of a certain square matrix. This matrix is called the "Tutte matrix" of the graph. We now define the Tutte matrix.

Definition 5.6.1. Let $G = (V, E)$ be a simple graph of order n and let $V = \{v_1, v_2, \ldots, v_n\}$. Let $\{x_{ij} : 1 \leq i < j \leq n\}$ be a set of indeterminates. Then the *Tutte matrix* of G is defined to be the n by n matrix $T = (t_{ij})$, where

$$t_{ij} = \begin{cases} x_{ij} & \text{if } v_i v_j \in E(G) \text{ and } i < j \\ -x_{ji} & \text{if } v_i v_j \in E(G) \text{ and } i > j \\ 0 & \text{otherwise} \end{cases}$$

Thus, T is a skew-symmetric matrix of order n.

Example 5.6.2. For example, if G is the graph



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then

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} = \begin{bmatrix} 0 & 0 & x_{13} & x_{14} \\ 0 & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{bmatrix}. \tag{5.9}$$

Now, by the definition of a determinant of a square matrix, the determinant of $T(=\det T)$ is given by $\det T = \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\, t_{1\pi(1)}\, t_{2\pi(2)} \ldots t_{n\pi(n)}$, where $\pi \in S_n$ (i.e., π is a permutation on $\{1, 2, \ldots, n\}$), and $\operatorname{sgn}(\pi) = 1$ or -1, according to whether π is an even or odd permutation. We denote the expression $t_{1\pi(1)} t_{2\pi(2)} \ldots t_{n\pi(n)}$ by t_π. Hence, $\det T = \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\, t_\pi$. Further, if n is odd, say, $\pi = (123)$, then $t_\pi = t_{12} t_{23} t_{31} = x_{12} x_{23} (-x_{13})$ [Note: We take $x_{ij} = 0$ if $v_i v_j \notin E(G)$). Also, $\pi^{-1} = (321)$ and $t_{\pi^{-1}} = t_{13} t_{21} t_{32} = x_{13}(-x_{12})(-x_{23})$, so that $t_\pi + t_{\pi^{-1}} = 0$. It is clear that the same relation is true for any odd $n \geq 3$.]

Now, for the Tutte matrix of relation (5.9), we have

$$\det T = x_{13}^2 x_{24}^2 + x_{14}^2 x_{23}^2 - 2 x_{13} x_{24} x_{14} x_{23}.$$

In this expression, the term $x_{13}^2 x_{24}^2$ is obtained by choosing the entries $x_{13}, x_{24}, -x_{13} = x_{31}$, and $-x_{24} = x_{42}$ of T, and hence it corresponds to the 1-factor $\{v_1 v_3, v_2 v_4\}$. Similarly, the term $x_{14}^2 x_{23}^2$ corresponds to the 1-factor $\{v_1 v_4, v_2 v_3\}$, and the term $x_{13} x_{24} x_{14} x_{23}$ corresponds to the cycle $(v_1 v_3 v_2 v_4)$ consisting of the edges $v_1 v_3, v_3 v_2, v_2 v_4$, and $v_4 v_1$.

We are now ready to prove Tutte's theorem, but before doing so, we make two useful observations.

Observation 5.6.3. If $\pi \in S_n$ is a product of disjoint even cycles, then $\text{sgn}(\pi)t_\pi =$ is a product of squares of the form x_{ij}^2.

Indeed, in this case, n is even, and the edges of G corresponding to the alternate transpositions in all of the even cycles of π form a 1-factor of G. [For example, for the even cycle (1234), we take the alternate transpositions (12) and (34).] Further, if v_i, v_j $(i < j)$ is an edge of this 1-factor, the partial product $t_{ij}t_{ji} = -x_{ij}^2$ occurs in t_π. The number of such products is $\frac{n}{2}$, and therefore

$$\text{sgn}(\pi)t_\pi = (-1)^{\frac{n}{2}}(-1)^{\frac{n}{2}} \prod x_{ij}^2 = \prod x_{ij}^2,$$

where the product runs over all pairs (i, j) with $i < j$ such that v_i, v_j is an edge of the 1-factor corresponding to π.

Observation 5.6.4. If $\pi \in S_n$ has an odd cycle α in its decomposition into the product of disjoint cycles, consider $\pi_1 \in S_n$, where π_1 is obtained from π by replacing α by α^{-1} and retaining the remaining cycles in π. Then, from our earlier remarks, it is clear that $\text{sgn}(\pi)t_\pi + \text{sgn}(\pi_1)t_{\pi_1} = 0$.

Theorem 5.6.5 (W. T. Tutte). *A simple graph G has a 1-factor if and only if its Tutte matrix is invertible.*

Proof. Let G be a simple graph having T as its Tutte matrix. Suppose that $\det T \neq 0$. Then by Observation 5.6.4 and the fact that $\det T = \sum_{\pi \in S_n} \text{sgn}(\pi)\, t_{1\pi(1)}\, t_{2\pi(2)} \ldots t_{n\pi(n)}$, there exists a $\pi \in S_n$ containing no odd cycle in its cycle decomposition. Then π is a product of even cycles and, by Observation 5.6.3, $\text{sgn}(\pi)t_\pi = \prod x_{ij}^2$. The alternate transpositions of the even cycles of π then yield a 1-factor of G.

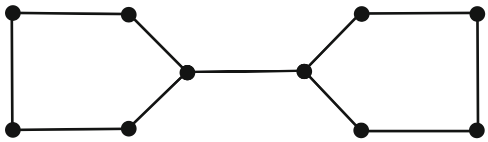
Conversely, assume that G has a 1-factor. Let $\pi \in S_n$ be the product of those transpositions corresponding to the 1-factor of G. [If $v_i v_j$ is an edge of the 1-factor, the corresponding transposition is (ij).] Then by Observation 5.6.3, $\text{sgn}(\pi)t_\pi = \prod x_{ij}^2$. Now set

$$x_{ij} = \begin{cases} 1 \text{ if } x_{ij}^2 \text{ appears in the product for } \text{sgn}(\pi)t_\pi \\ 0 \text{ otherwise.} \end{cases}$$

Then $\text{sgn}(\pi)t_\pi = 1$, and for these values of $x_{ij}, \text{sgn}(\sigma)t_\sigma = 0$ for any $\sigma \in S_n$, $\sigma \neq \pi$. This means that the polynomial $\det T$ is not the zero polynomial. □

Remark 5.6.6. Actually, our definition of the Tutte matrix of G depends on the order of the vertices of G. That is to say, the definition of T is based on regarding G as a labeled graph. However, if T is nonsingular with regard to one labeling of G, then the Tutte matrix of G will remain nonsingular with regard to any other labeling of G. This is because if T and T' are the Tutte matrices of G with regard to two labelings of G, $T' = PTP^{-1}$, where P is a permutation matrix of order n. Hence, T is nonsingular if and only if T' is nonsingular.

Exercise 6.1. By evaluating the Tutte matrix of the following graph G, show that G has a 1-factor.



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Notes

Readers who are more interested in matching theory can consult [91, 136], and [155]. Our proof of Tutte's 1-factor theorem is due to Lovász [135] (see also [27]).