Chapter 3
Connectivity

3.1 Introduction

The connectivity of a graph is a "measure" of its connectedness. Some connected graphs are connected rather "loosely" in the sense that the deletion of a vertex or an edge from the graph destroys the connectedness of the graph. There are graphs at the other extreme as well, such as the complete graphs K_n, $n \geq 2$, which remain connected after the removal of any k vertices, $1 \leq k \leq n - 1$.

Consider a communication network. Any such network can be represented by a graph in which the vertices correspond to communication centers and the edges represent communication channels. In the communication network of Fig. 3.1a, any disruption in the communication center v will result in a communication breakdown, whereas in the network of Fig. 3.1b, at least two communication centers have to be disrupted to cause a breakdown. It is needless to stress the importance of maintaining reliable communication networks at all times, especially during times of war, and the reliability of a communication network has a direct bearing on its connectivity.

In this chapter, we study the two graph parameters, namely, vertex connectivity and edge connectivity. We also introduce the parameter cyclical edge connectivity. We prove Menger's theorem and several of its variations. In addition, the theorem of Ford and Fulkerson on flows in networks is established.

3.2 Vertex Cuts and Edges Cuts

We now introduce the notions of vertex cuts, edge cuts, vertex connectivity, and edge connectivity.

R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory,*
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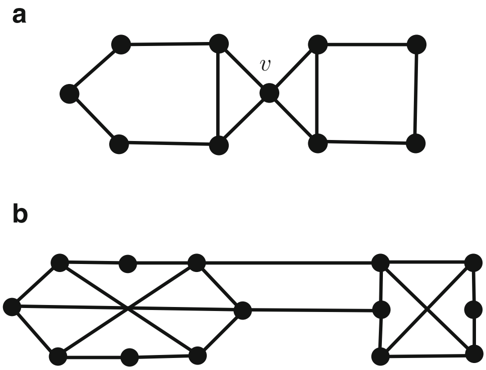


Fig. 3.1 Two types of communication networks

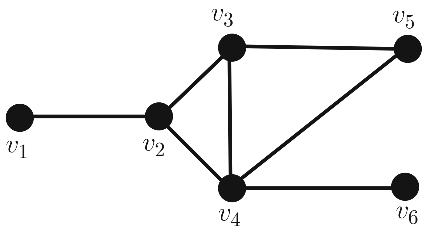


Fig. 3.2 Graph illustrating vertex cuts and edge cuts

Definitions 3.2.1. 1. A subset V' of the vertex set $V(G)$ of a connected graph G is a *vertex cut* of G if $G - V'$ is disconnected; it is a *k-vertex cut* if $|V'| = k$. V' is then called a *separating set of vertices* of G. A vertex v of G is a *cut vertex* of G if $\{v\}$ is a vertex cut of G.

2. Let G be a nontrivial connected graph with vertex set $V(G)$ and let S be a nonempty subset of $V(G)$. For $\bar{S} = V \backslash S \neq \emptyset$, let $[S, \bar{S}]$ denote the set of all edges of G that have one end vertex in S and the other in $\bar{S}$. A set of edges of G of the form $[S, \bar{S}]$ is called an *edge cut* of G. An edge e is a *cut edge* of G if $\{e\}$ is an edge cut of G. An edge cut of cardinality k is called a *k-edge cut* of G.

Example 3.2.2. For the graph of Fig. 3.2, $\{v_2\}$, and $\{v_3, v_4\}$ are vertex cuts. The edge subsets $\{v_3v_5, v_4v_5\}$, $\{v_1v_2\}$, and $\{v_4v_6\}$ are all edge cuts. Of these, v_2 is a cut vertex, and v_1v_2 and v_4v_6 are both cut edges. For the edge cut $\{v_3v_5, v_4v_5\}$, we may take $S = \{v_5\}$ so that $\bar{S} = \{v_1, v_2, v_3, v_4, v_6\}$.

Remarks 3.2.3. 1. If uv is an edge of an edge cut E', then all the edges having u and v as their ends also belong to E'.

2. No loop can belong to an edge cut.

Exercise 2.1. If $\{x, y\}$ is a 2-edge cut of a graph G, show that every cycle of G that contains x must also contain y.

Remarks 3.2.4. If G is connected and E' is a set of edges whose deletion results in a disconnected graph, then E' contains an edge cut of G. It is clear that if e is a cut edge of a connected graph G, then $G - e$ has exactly two components.

Remarks 3.2.5. Since the removal of a parallel edge of a connected graph does not result in a disconnected graph, such an edge cannot be a cut edge of the graph. A set of edges of a connected graph G whose deletion results in a disconnected graph is called a *separating set of edges.* In particular, any edge cut of a connected graph G is a separating set of edges of G.

We now characterize a cut vertex of G.

Theorem 3.2.6. *A vertex v of a connected graph G with at least three vertices is a cut vertex of G if and only if there exist vertices u and w of G distinct from v such that v is in every u-w path in G.*

Proof. If v is a cut vertex of G, then $G - v$ is disconnected and has at least two components, G_1 and G_2. Take $u \in V(G_1)$ and $w \in V(G_2)$. Then every u-w path in G must contain v, as otherwise u and w would belong to the same component of $G - v$.

Conversely, suppose that the condition of the theorem holds. Then the deletion of v destroys every u-w path in G, and hence u and w lie in distinct components of $G - v$. Therefore, $G - v$ is disconnected and v is a cut vertex of G. □

Theorems 3.2.7 and 3.2.8 characterize a cut edge of a graph.

Theorem 3.2.7. *An edge $e = xy$ of a connected graph G is a cut edge of G if and only if e belongs to no cycle of G.*

Proof. Let e be a cut edge of G and let $[S, \bar{S}] = \{e\}$ be the partition of V defined by $G - e$ so that one of x and y belongs to S, and the other to $\bar{S}$, say, $x \in S$ and $y \in \bar{S}$. If e belongs to a cycle of G, then $[S, \bar{S}]$ must contain at least one more edge, contradicting that $\{e\} = [S, \bar{S}]$. Hence, e cannot belong to a cycle.

Conversely, assume that e is not a cut edge of G. Then $G - e$ is connected, and hence there exists an x-y path P in $G - e$. Then $P \cup \{e\}$ is a cycle in G containing e. □

Theorem 3.2.8. *An edge $e = xy$ is a cut edge of a connected graph G if and only if there exist vertices u and v such that e belongs to every u-v path in G.*

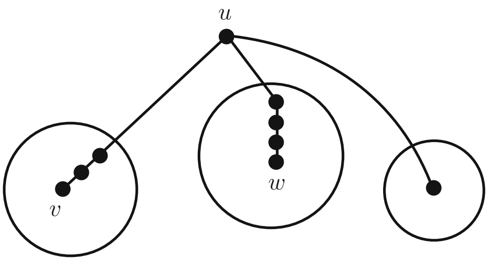
Proof. Let $e = xy$ be a cut edge of G. Then $G - e$ has two components, say, G_1 and G_2. Let $u \in V(G_1)$ and $v \in V(G_2)$. Then, clearly, every u-v path in G contains e.

Conversely, suppose that there exist vertices u and v satisfying the condition of the theorem. Then there exists no u-v path in $G - e$ so that $G - e$ is disconnected. Hence, e is a cut edge of G. □

Remark 3.2.9. There exist graphs in which every edge is a cut edge. It follows from Theorem 3.2.7 that if G is a simple connected graph with at least one edge and without cycles, then every edge of G is a cut edge of G. A similar result is not true for cut vertices. Our next result shows that not every vertex of a connected graph (with at least two vertices) can be a cut vertex of G.

Theorem 3.2.10. *A connected graph G with at least two vertices contains at least two vertices that are not cut vertices.*

Fig. 3.3 Graph for proof of Theorem 3.2.10



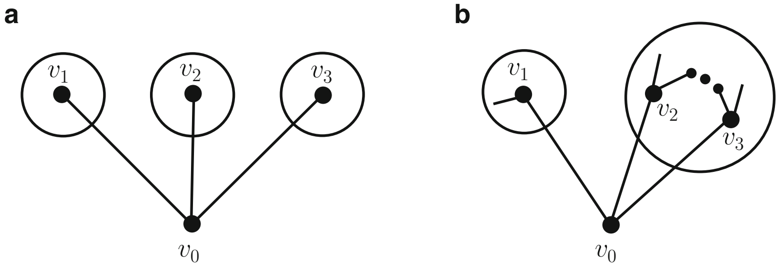


Fig. 3.4 Graph for proof of Proposition 3.2.11

Proof. First, suppose that $n(G) \geq 3$. Let u and v be vertices of G such that $d(u, v)$ is maximum. Then neither u nor v is a cut vertex of G. For if u were a cut vertex of G, $G - u$ would be disconnected, having at least two components. The vertex v belongs to one of these components. Let w be any vertex belonging to a component of $G - u$ not containing v. Then every v-w path in G must contain u (see Fig. 3.3). Consequently, $d(v, w) > d(v, u)$, contradicting the choice of u and v. Hence, u is not a cut vertex of G. Similarly, v is not a cut vertex of G.

If $n(G) = 2$, then K_2 is a spanning subgraph of G, and so no vertex of G is a cut vertex of G. This completes the proof of the theorem. □

Proposition 3.2.11. *A simple cubic (i.e., 3-regular) connected graph G has a cut vertex if and only if it has a cut edge.*

Proof. Let G have a cut vertex v_0. Let v_1, v_2, v_3 be the vertices of G that are adjacent to v_0 in G. Consider $G - v_0$, which has either two or three components. If $G - v_0$ has three components, no two of v_1, v_2, and v_3 can belong to the same component of $G - v_0$. In this case, each of v_0v_1, v_0v_2, and v_0v_3 is a cut edge of G. (See Fig. 3.4a.) In the case when $G - v_0$ has only two components, one of the vertices, say v_1, belongs to one component of $G - v_0$, and v_2 and v_3 belong to the other component. In this case, v_0v_1 is a cut edge. (See Fig. 3.4b.)

Conversely, suppose that $e = uv$ is a cut edge of G. Then the deletion of u results in the deletion of the edge uv. Since G is cubic, $G - u$ is disconnected. Accordingly, u is a cut vertex of G. □

Exercise 2.2. Find the vertex cuts and edge cuts of the graph of Fig. 3.2.

Exercise 2.3. Prove or disprove: Let G be a simple connected graph with $n(G) \geq 3$. Then G has a cut edge if and only if G has a cut vertex.

Exercise 2.4. Show that in a graph, the number of edges common to a cycle and an edge cut is even.

3.3 Connectivity and Edge Connectivity

We now introduce two parameters of a graph that in a way measure the connectedness of the graph.

Definition 3.3.1. For a nontrivial connected graph G having a pair of nonadjacent vertices, the minimum k for which there exists a k-vertex cut is called the *vertex connectivity* or simply the *connectivity* of G; it is denoted by $\kappa(G)$ or simply κ (kappa) when G is understood. If G is trivial or disconnected, $\kappa(G)$ is taken to be zero, whereas if G contains K_n as a spanning subgraph, $\kappa(G)$ is taken to be $n-1$.

A set of vertices and/or edges of a connected graph G is said to *disconnect* G if its deletion results in a disconnected graph.

When a connected graph G (on $n \geq 3$ vertices) does not contain K_n as a spanning subgraph, κ is the connectivity of G if there exists a set of κ vertices of G whose deletion results in a disconnected subgraph of G while no set of $\kappa - 1$ (or fewer) vertices has this property.

Exercise 3.1. Prove that a simple graph G with n vertices, $n \geq 2$, is complete if and only if $\kappa(G) = n - 1$.

Definition 3.3.2. The *edge connectivity* of a connected graph G is the smallest k for which there exists a k-edge cut (i.e., an edge cut having k edges). The edge connectivity of a trivial or disconnected graph is taken to be 0. The edge connectivity of G is denoted by $\lambda(G)$. If λ is the edge connectivity of a connected graph G, there exists a set of λ edges whose deletion results in a disconnected graph, and no subset of edges of G of size less than λ has this property.

Exercise 3.2. Prove that the deletion of edges of a minimum-edge cut of a connected graph G results in a disconnected graph with exactly two components. (Note that a similar result is not true for a minimum vertex cut.)

Definition 3.3.3. A graph G is *r-connected* if $\kappa(G) \geq r$. Also, G is *r-edge connected* if $\lambda(G) \geq r$.

An r-connected (respectively, r-edge-connected) graph is also ℓ-connected (respectively, ℓ-edge connected) for each ℓ, $0 \leq \ell \leq r - 1$.

For the graph G of Fig. 3.5, $\kappa(G) = 1$ and $\lambda(G) = 2$.

We now derive inequalities connecting $\kappa(G)$, $\lambda(G)$, and $\delta(G)$.

Theorem 3.3.4. *For any loopless connected graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

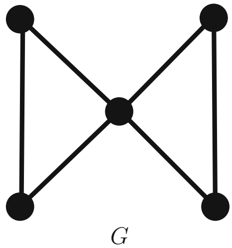


Fig. 3.5 A 1-connected graph

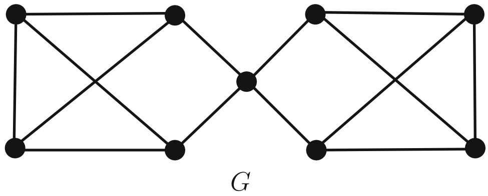


Fig. 3.6 Graph G with $\kappa = 1$, $\lambda = 2$ and $\delta = 3$

Proof. We observe that $\kappa = 0$ if and only if $\lambda = 0$. Also, $\delta = 0$ implies that $\kappa = 0$ and $\lambda = 0$. Hence we may assume that κ, λ, and δ are all at least 1. Let $\mathscr{E}$ be an edge cut of G with λ edges. Let u and v be the end vertices of an edge of $\mathscr{E}$. For each edge of $\mathscr{E}$ that does not have both u and v as end vertices, remove an end vertex that is different from u and v. If there are t such edges, at most t vertices have been removed. If the resulting graph, say H, is disconnected, then $\kappa \leq t < \lambda$. Otherwise, there will remain a subset of edges of E having u and v as end vertices, the removal of which from H would disconnect G. Hence, in addition to the already removed vertices, the removal of one of u and v will result in either a disconnected graph or a trivial graph. In the process, a set of at most $t + 1$ vertices has been removed and $\kappa \leq t + 1 \leq \lambda$.

Finally, it is clear that $\lambda \leq \delta$. In fact, if v is a vertex of G with $d_G(v) = \delta$, then the set $[\{v\}, V \setminus \{v\}]$ of δ edges of G incident at v forms an edge cut of G. Thus, $\lambda \leq \delta$. □

It is possible that the inequalities in Theorem 3.3.4 can be strict. See the graph G of Fig. 3.6, for which $\kappa = 1$, $\lambda = 2$, and $\delta = 3$.

Exercise 3.3. Prove or disprove: If H is a subgraph of G, then

(a) $\kappa(H) \leq \kappa(G)$ and
(b) $\lambda(H) \leq \lambda(G)$.

Exercise 3.4. Determine $\lambda(K_n)$.

Exercise 3.5. Determine the connectivity and edge connectivity of the Petersen graph P. (See graph P of Fig. 1.7. Note that P is a cubic graph.)

Theorem 3.3.5 gives a class of graphs for which $\kappa = \lambda$.

Theorem 3.3.5. *The connectivity and edge connectivity of a simple cubic graph G are equal.*

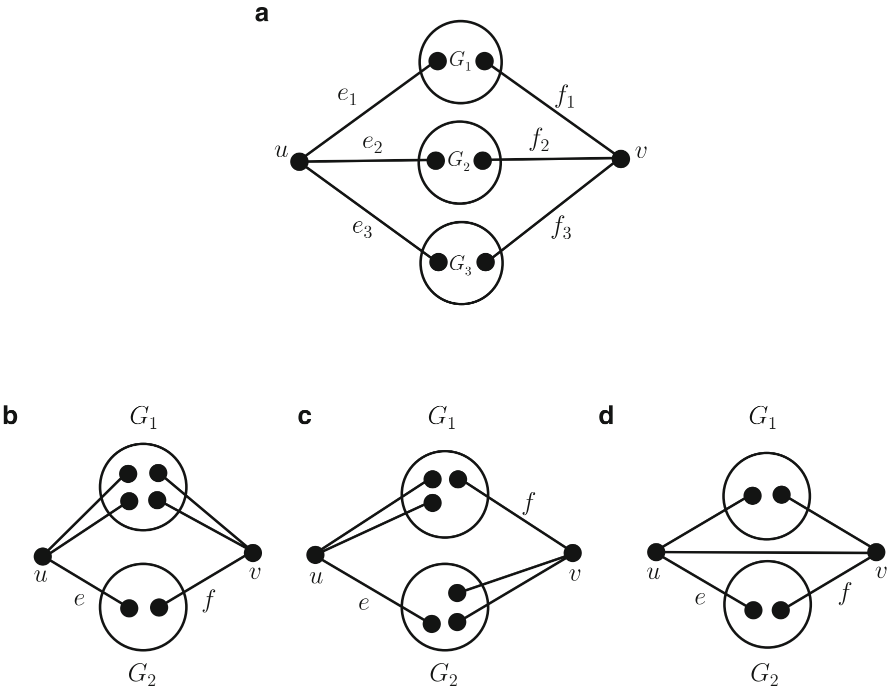


Fig. 3.7 Connected cubic graphs for proof of Theorem 3.3.5

Proof. We need only consider the case of a connected cubic graph. Again, since $\kappa \leq \lambda \leq \delta = 3$, we have only to consider the cases when $\kappa = 1, 2$, or 3. Now, Proposition 3.2.11 implies that for a simple cubic graph G, $\kappa = 1$ if and only if $\lambda = 1$.

If $\kappa = 3$, then by Theorem 3.3.4, $3 = \kappa \leq \lambda \leq \delta = 3$, and hence $\lambda = 3$.

We shall now prove that $\kappa = 2$ implies that $\lambda = 2$.

Suppose $\kappa = 2$ and $\{u, v\}$ is a 2-vertex cut of G. The deletion of $\{u, v\}$ results in a disconnected subgraph G' of G. Since each of u and v must be joined to each component of G', and since G is cubic, G' can have at most three components. If G' has three components, G_1, G_2, and G_3, and if e_i and f_i, $i = 1, 2, 3$, join, respectively, u and v with G_i, then each pair $\{e_i, f_i\}$ is an edge cut of G (see Fig. 3.7a).

If G' has only two components, G_1 and G_2, then each of u and v is joined to one of G_1 and G_2 by a single edge, say, e and f, respectively, so that $\{e, f\}$ is an edge cut of G (see Fig. 3.7b–d).

Hence, in either case there exists an edge cut consisting of two edges. As such, $\lambda \leq 2$. But by Theorem 3.3.4, $\lambda \geq \kappa = 2$. Hence $\lambda = 2$. Finally, the above arguments show that if $\lambda = 3$, then $\kappa = 3$, and if $\lambda = 2$, then $\kappa = 2$. □

Exercise 3.6. Give examples of cubic graphs G_1, G_2, and G_3 with $\kappa(G_1) = 1$, $\kappa(G_2) = 2$, and $\kappa(G_3) = 3$.

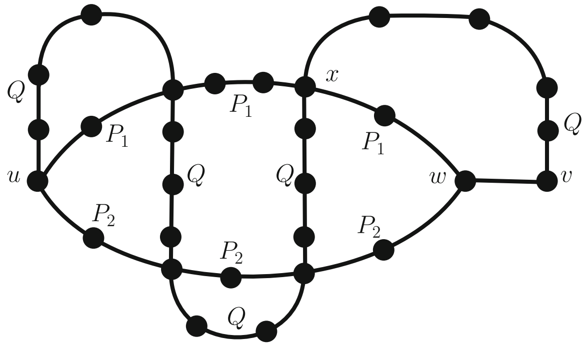


Fig. 3.8 Graph for proof of Theorem 3.3.7

Definition 3.3.6. A family of two or more paths in a graph G is said to be *internally disjoint* if no vertex of G is an internal vertex of more than one path in the family.

We now state and prove *Whitney's characterization theorem* of 2-connected graphs.

Theorem 3.3.7 (Whitney [193]). *A graph G with at least three vertices is 2-connected if and only if any two vertices of G are connected by at least two internally disjoint paths.*

Proof. Let G be 2-connected. Then G contains no cut vertex. Let u and v be two distinct vertices of G. We now use induction on $d(u,v)$ to prove that u and v are joined by two internally disjoint paths.

If $d(u,v) = 1$, let $e = uv$. As G is 2-connected and $n(G) \geq 3$, e cannot be a cut edge of G, since if e were a cut edge, at least one of u and v must be a cut vertex. By Theorem 3.2.7, e belongs to a cycle C in G. Then $C - e$ is a u-v path in G, internally disjoint from the path uv.

Now assume that any two vertices x and y of G with $d(x,y) = k-1$, $k \geq 2$, are joined by two internally disjoint x-y paths in G. Let $d(u,v) = k$. Let P be a u-v path of length k and w be the vertex of G just preceding v on P. Then $d(u,w) = k-1$. By an induction hypothesis, there are two internally disjoint u-w paths, say P_1 and P_2, in G. As G has no cut vertex, $G - w$ is connected and hence there exists a u-v path Q in $G - w$. Q is clearly a u-v path in G not containing w. Let x be the vertex of Q such that the x-v section of Q contains only the vertex x in common with $P_1 \cup P_2$ (see Fig. 3.8).

We may suppose, without loss of generality, that x belongs to P_1. Then the union of the u-x section of P_1 and x-v section of Q and $P_2 \cup (wv)$ are two internally disjoint u-v paths in G. This gives the proof in one direction.

In the other direction, assume that any two distinct vertices of G are connected by at least two internally disjoint paths. Then G is connected. Further, G cannot contain a cut vertex, since if v were a cut vertex of G, there must exist vertices u and w such that every u-w path contains v (compare with Theorem 3.2.6), contradicting the hypothesis. Hence, G is 2-connected. □

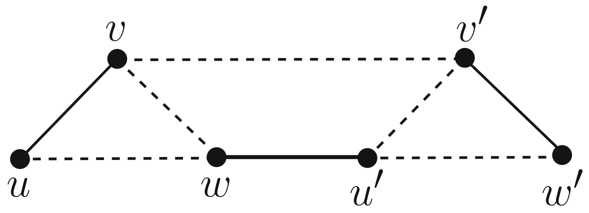


Fig. 3.9 Graph for Remark 3.3.9

Theorem 3.3.8. *A graph G with at least three vertices is 2-connected if and only if any two vertices of G lie on a common cycle.*

Proof. Let u and v be any two vertices of a 2-connected graph G. By Theorem 3.3.7, there exist two internally disjoint paths in G joining u and v. The union of these two paths is a cycle containing u and v.

Conversely, if any two vertices u and v lie on a cycle C, then C is the union of two internally disjoint u-v paths. Again, by Theorem 3.3.7, G is 2-connected. □

Remark 3.3.9. If G is 2-connected, if u and v are distinct vertices of G, and if P is a u-v path in G, it is not in general true that there exists another u-v path Q in G such that P and Q are internally disjoint. For example, in the 2-connected graph of Fig. 3.9, if P is the u-w' path $uwvv'u'w'$, there exists no u-w' path Q in G that is internally disjoint from P. However, there do exist two internally disjoint u-w' paths in G.

Exercise 3.7. (a) Show that a graph G with at least three vertices is 2-connected if and only if any vertex and any edge of G lie on a common cycle of G.
(b) Show that a graph G with at least three vertices is 2-connected if and only if any two edges of G lie on a common cycle.

Exercise 3.8. Prove that a graph is 2-connected if and only if for every pair of disjoint connected subgraphs G_1 and G_2, there exist two internally disjoint paths P_1 and P_2 of G between G_1 and G_2.

Exercise 3.9. *Edge form of Whitney's theorem:* Prove that a graph G with $n \geq 3$ is 2-edge connected if and only if any two distinct vertices of G are connected by at least two edge-disjoint paths in G. [Hint: Imitate the proof of Theorem 3.3.7, or pass on to $L(G)$.]

Exercise 3.10. (a) Disprove by a counterexample: If $\kappa(G) = k$, then $\kappa(L(G))=k$.
(b) Prove: $\lambda(G) \leq \kappa(L(G))$. Give an example of a graph G for which $\lambda(G) < \kappa(L(G))$.

Theorem 3.3.10. *In a 2-connected graph G, any two longest cycles have at least two vertices in common.*

Proof. Let $C_1 = u_1u_2 \ldots u_ku_1$ and $C_2 = v_1v_2 \ldots v_kv_1$ be two longest cycles in G. If C_1 and C_2 are disjoint, there exist (since G is 2-connected) two disjoint paths,

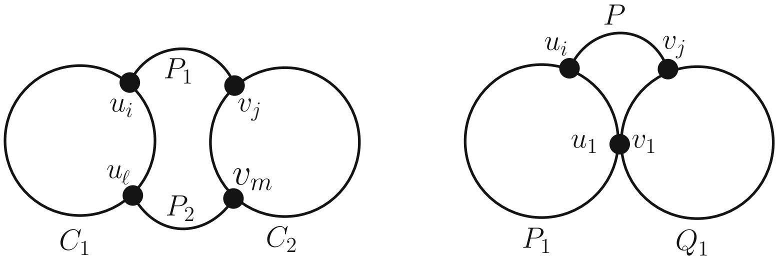
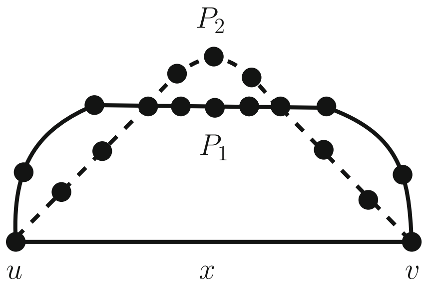


Fig. 3.10 Graphs for proof of Theorem 3.3.10

Fig. 3.11 Graph for proof of Theorem 3.3.11



say, P_1 joining u_i and v_j and P_2 joining u_ℓ and v_p, connecting C_1 and C_2 such that $u_i \neq u_\ell$ and $v_j \neq v_p$ (see Exercise 3.8). u_i and u_ℓ divide C_1 into two subpaths. Let L_1 be the longer of these subpaths. (If both subpaths are of equal length, we take either one of them to be L_1.) Let L_2 be defined in a similar manner in C_2. Then $L_1 \cup P_1 \cup L_2 \cup P_2$ is a cycle of length greater than that of C_1 (or C_2). Hence, C_1 and C_2 cannot be disjoint. (See Fig. 3.10.)

Suppose that C_1 and C_2 have exactly one vertex, say $u_1 = v_1$, in common. Since G is 2-connected, u_1 is not a cut vertex of G, and so there exists a path P with one end vertex u_i in $C_1 - u_1$ and the other end vertex v_j in $C_2 - v_1$, which is internally disjoint from $C_1 \cup C_2$. Let P_1 denote the longer of the two u_1-u_i sections of C_1, and Q_1 denote the longer of the two v_1-v_j sections of C_2. If the two sections of C_1 or of C_2 are of equal length, take any one of them. Then $P_1 \cup P \cup Q_1$ is a cycle longer than C_1 (or C_2). But this is impossible. Thus, C_1 and C_2 must have at least two vertices in common. □

Theorem 3.3.11 gives a simple characterization of 3-edge-connected graphs.

Theorem 3.3.11. *A connected simple graph G is 3-edge connected if and only if every edge of G is the (exact) intersection of the edge sets of two cycles of G.*

Proof. Let G be 3-edge connected and let $x = uv$ be an edge of G. Since $G - x$ is 2-edge connected, there exist two edge-disjoint u-v paths P_1 and P_2 in $G - x$ (see Exercise 3.9). Now, $P_1 \cup \{x\}$ and $P_2 \cup \{x\}$ are two cycles of G, the intersection of whose edge sets is precisely $\{x\}$ (see Fig. 3.11).

Conversely, suppose that for each edge $x = uv$ there exist two cycles C and C' such that $\{x\} = E(C) \cap E(C')$. G cannot have a cut edge since, by hypothesis, each edge belongs to two cycles and no cut edge can belong to a cycle; nor can G contain an edge cut consisting of two edges x and y, by Exercise 2.1. (Since any cycle that contains x also contains y, the intersection of any two such cycles must contain both x and y, a contradiction.) Hence, $\lambda(G) \geq 3$, and G is 3-edge connected. □

3.4 Blocks

In this section, we focus on connected graphs without cut vertices.

Definition 3.4.1. A graph G is *nonseparable* if it is nontrivial and connected and has no cut vertices. A *block of a graph* is a maximal nonseparable subgraph of G. If G has no cut vertex, G itself is a block.

In Fig. 3.12, a graph G and its blocks B_1, B_2, B_3, and B_4 are indicated. B_1, B_3, and B_4 are the *end blocks* of G (i.e., blocks having exactly one cut vertex of G). The following facts are worthy of observation.

Remarks 3.4.2. Let G be a connected graph with $n \geq 3$.

1. Each block of G with at least three vertices is a 2-connected subgraph of G.
2. Each edge of G belongs to one and only one of its blocks. Hence G is an edge-disjoint union of its blocks.
3. Any two blocks of G have at most one vertex in common. (Such a common vertex is a cut vertex of G.)
4. A vertex of G that is not a cut vertex belongs to exactly one of its blocks.
5. A vertex of G is a cut vertex of G if and only if it belongs to at least two blocks of G.

Whitney's theorem (Theorem 3.3.7) implies that a graph with at least three vertices is a block if and only if any two vertices of the graph are connected by at least two internally disjoint paths. Again by Theorem 3.3.8, we see that any two vertices of a block with at least three vertices belong to a common cycle. Thus, a block with at least three vertices contains a cycle.

Theorem 3.4.3 (Ear decomposition of a block). *If C is any cycle of a simple block G, then there exists a sequence of nonseparable subgraphs $C = B_0, B_1, \ldots, B_r = G$ such that B_{i+1} is an edge-disjoint union of B_i and a path P_i, where the only vertices common to B_i and P_i are the end vertices of P_i, $0 \leq i \leq r-1$.*

Proof. Assume that we have already determined B_i (see Fig. 3.13). If $B_i \neq G$, there exists (as G is connected) an edge $e = uv$ not belonging to B_i but with u in B_i. If v also belongs to B_i, take $P_i = uv$ and $B_{i+1} = B_i \cup P_i$. Otherwise, $e = uv$ is an edge of G having only one of its ends, namely u, in B_i. Let u' be any other

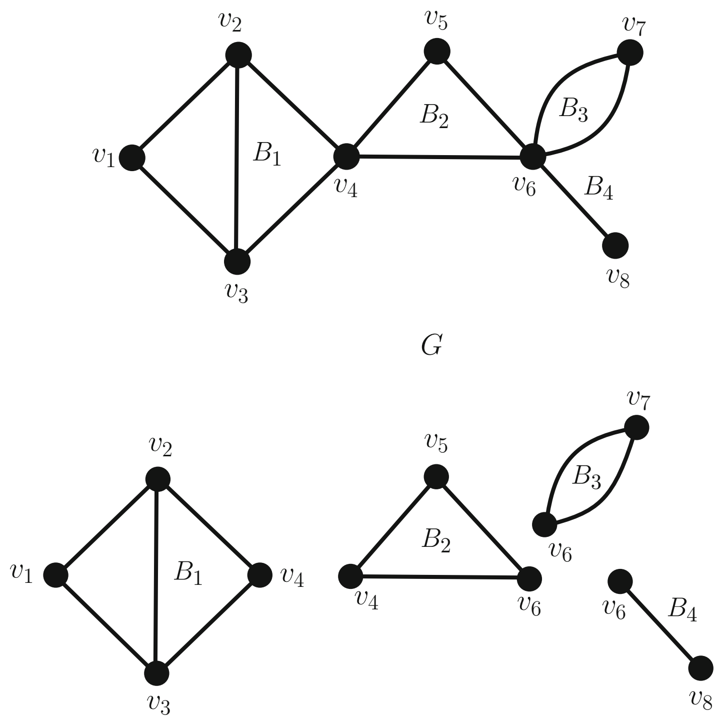
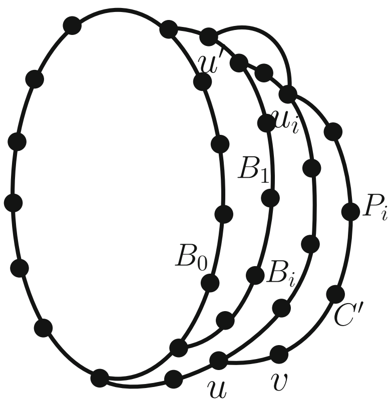


Fig. 3.12 A graph G and its blocks

Fig. 3.13 Graph for proof of Theorem 3.4.3



vertex of B_i. Then since G is 2-connected, e and u' belong to a common cycle C_i (see Exercise 3.7). Let u_i be the first vertex of B_i after u in the u-u' section C' of C_i containing v, and let P_i be the u-u_i section of C'. Define $B_{i+1} = B_i \cup P_i$. Then B_{i+1} is nonseparable and the proof follows by induction on i. □

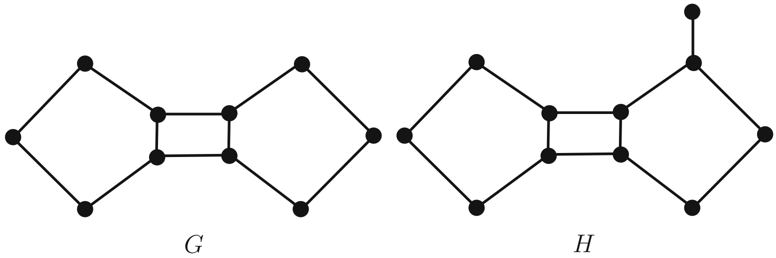


Fig. 3.14 Graphs $G(\lambda(G) = \lambda_c(G) = 2)$ and $H(\lambda(H) = 1, \lambda_c(H) = 2)$

3.5 Cyclical Edge Connectivity of a Graph

In this section we introduce the parameter "cyclical edge connectivity of a graph." Unlike connectivity and edge connectivity, cyclical edge connectivity is not defined for all graphs.

Definition 3.5.1. Let G be a simple connected graph containing at least two disjoint cycles. Then the *cyclical edge connectivity* of G is defined to be the minimum number of edges of G whose deletion results in a graph having two components, each containing a cycle. It is denoted by $\lambda_c(G)$.

It is clear that $\lambda \leq \lambda_c$. The graphs G and H of Fig. 3.14 show that both $\lambda = \lambda_c$ and $\lambda < \lambda_c$ can happen.

Exercise 5.1. Show that the cyclical edge connectivity of the Petersen graph P is 5.

3.6 Menger's Theorem

In this section we prove different versions of the celebrated Menger's theorem, which generalizes Whitney's theorem (Theorem 3.3.7). Menger's theorem [140] relates the connectivity of a graph G to the number of internally disjoint paths between pairs of vertices of G. The proofs given here make use of network analysis. Hence we begin with the definition of a network.

Definition 3.6.1. A *network* N is a digraph D with two distinguished vertices s and t, $(s \neq t)$, and a nonnegative integer-valued function c defined on its arc set A. s is called the *source* and t is called the *sink* of N. The source corresponds to the supply center and the sink corresponds to a market. Vertices of N, other than s and t, are called the *intermediate vertices* of N. The digraph D is called the underlying digraph of N. The function c is called the *capacity function* of N and $c(a)$, for an arc a, denotes the *capacity* of a.

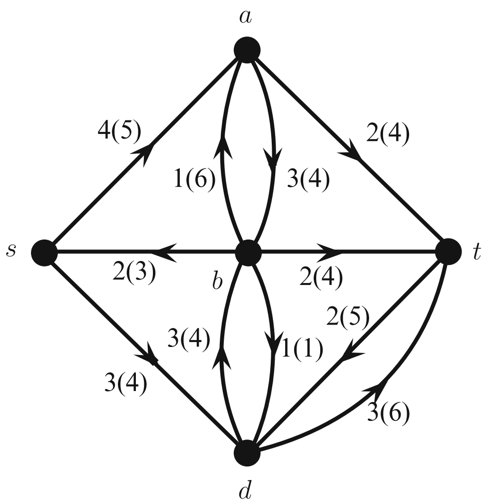


Fig. 3.15 Network with source (s), sink (t), and three intermediate vertices

Example 3.6.2. A network N is diagrammatically represented by the underlying digraph D, labeling each arc with its capacity. Figure 3.15 is a network with source s and sink t, and three intermediate vertices. The numbers inside the brackets denote the capacities of the respective arcs.

For a real-valued function f defined on A, and $K \subseteq A$, $\sum_{a \in K} f(a)$ will be denoted by $f(K)$. If K is a set of arcs of D of the form $[S, \bar{S}]$, that is, the set of arcs with heads in S and tails in $\bar{S}$, where $S \subseteq V(D)$, $\bar{S} = V(D) \setminus S$, then $f^+(S)$ and $f^-(S)$ denote $f([S, \bar{S}])$ and $f([\bar{S}, S])$, respectively. If $S = \{v\}$, then $f^+(S)$ and $f^-(S)$ are denoted by $f^+(v)$ and $f^-(v)$, respectively.

Definition 3.6.3. A *flow* in a network N is an integer-valued function f defined on $A = A(N)$ such that $0 \le f(a) \le c(a)$ for all $a \in A$ and $f^+(v) = f^-(v)$ for all the intermediate vertices v of N.

Remarks 3.6.4. 1. $f^+(v)$ is the flow out of v and $f^-(v)$ is the flow into v. *The condition $f^+(v) = f^-(v)$ for each intermediate vertex v then signifies that there is conservation of flow at every such vertex.*

2. If $a = (u, v)$, we denote $f(a)$ by f_{uv}. Every network N has at least one flow since the function f defined by $f(a) = 0$ for all $a \in A$ is a flow. It is called the *zero flow* in N.
3. A less trivial example of a flow in the network of Fig. 3.15 is given by f, where $f_{sa} = 4$, $f_{sd} = 3$, $f_{bs} = 2$, $f_{at} = 2$, $f_{ab} = 3$, $f_{ba} = 1$, $f_{bd} = 1$, $f_{db} = 3$, $f_{bt} = 2$, $f_{dt} = 3$, and $f_{td} = 2$.
4. If S is a subset of vertices in a network N and f is a flow in N, $f^+(S) - f^-(S)$, is called the *resultant flow out of* S and $f^-(S) - f^+(S)$, the *resultant flow into* S, relative to f.

5. The flow along any arc (u, v) is both the outflow at u along (u, v) and the inflow at v along (u, v). Hence, $\sum_{v \in V(N)} f^+(v) - \sum_{v \in V(N)} f^-(v) = 0$. This gives us

$$[f^+(s) - f^-(s)] + \sum_{\substack{v \in V(N) \\ v \neq s,t}} (f^+(v) - f^-(v)) + [f^+(t) - f^-(t)] = 0.$$

But $f^+(v) = f^-(v)$ for each $v \in V(N)$, $v \neq s, t$. Hence,

$$f^+(s) - f^-(s) = f^-(t) - f^+(t).$$

Thus, relative to any flow f, the resultant flow out of s is equal to the resultant flow into t. For a similar reason, if S is any subset of $V(N)$ containing s but not t,

$$\sum_{v \in S} f^+(v) - \sum_{v \in S} f^-(v) = f^+(s) - f^-(s). \tag{3.1}$$

This common quantity is called the *value* of f and is denoted by val f. Thus,

$$\text{val } f = f^+(s) - f^-(s) = f^-(t) - f^+(t).$$

The value of the flow f of the network of Fig. 3.15 is 5.

Definition 3.6.5. 1. A flow f in N is a *maximum flow* if there is no flow f' in N such that val $f' >$ val f.

2. A cut K in N is a set of arcs of the form $[S, \bar{S},]$ where $s \in S$ and $t \in \bar{S}$. Such a cut is said to *separate* s and t. For example, $K = \{(a, t), (b, t), (d, t)\}$ is a cut in the network of Fig. 3.15, where $S = \{s, a, b, d\}$.
3. The *capacity* of a cut K is the sum of the capacities of its arcs. We denote the capacity of K by cap K. Thus, cap $K = \sum_{a \in K} c(a)$. For the network of Fig. 3.15, cap $K = 2 + 2 + 3 = 7$.

Theorem 3.6.6 gives the relation between the value of a flow and the capacity of a cut in a network.

Theorem 3.6.6. *In any network N, the value of any flow f is less than or equal to the capacity of any cut K.*

Proof. Let $[S, \bar{S}]$ be any cut with $s \in S$ and $t \in T$. We have, by (3.1),

$$\begin{aligned}
\text{val } f &= f^+(s) - f^-(s) \\
&= \sum_{v \in S} f^+(v) - \sum_{v \in S} f^-(v) \\
&= \sum_{\substack{v \in S,\ u \in S, \\ (v,u) \in A(D)}} f_{vu} + \sum_{\substack{v \in S,\ u \in \bar{S}, \\ (v,u) \in A(D)}} f_{vu} - \sum_{\substack{u \in S,\ v \in S, \\ (u,v) \in A(D)}} f_{uv} - \sum_{\substack{u \in \bar{S},\ v \in S, \\ (u,v) \in A(D)}} f_{uv}.
\end{aligned}$$

But

$$\sum_{\substack{v\in S,\ u\in S,\\ (v,u)\in A(D)}} f_{vu} - \sum_{\substack{u\in S,\ v\in S,\\ (u,v)\in A(D)}} f_{uv} = 0.$$

Hence

$$\text{val } f = \sum_{\substack{v\in S,\ u\in \bar{S},\\ (v,u)\in A(D)}} f_{vu} - \sum_{\substack{u\in \bar{S},\ v\in S,\\ (u,v)\in A(D)}} f_{uv}. \tag{3.2}$$

Since

$$\sum_{\substack{u\in \bar{S},\ v\in S,\\ (u,v)\in A(D)}} f_{uv} \geq 0$$

(recall that f is a nonnegative integer-valued function), we get
$\text{val } f \leq \sum_{\substack{v\in S,\ u\in \bar{S},\\ (v,u)\in A(D)}} f_{vu} \leq \sum_{\substack{v\in S,\ u\in \bar{S},\\ (v,u)\in A(D)}} c(v,u) = c([S,\bar{S}]).$ □

Note 3.6.7. Note that we have shown in (3.2) that val f is the flow out of S minus the flow into S for any $S \subset V$ with $s \in S$ and $t \in \bar{S}$.

By Theorem 3.6.6, in any network N, the value of any flow f does not exceed the capacity of any cut K. In particular, if f^* is a maximum flow in N and K^* is a minimum cut, that is, a cut with minimum capacity, then val $f^* \leq$ cap K^*.

Lemma 3.6.8. *Let f be a flow and K a cut in a network N such that val f = cap K. Then f is a maximum flow and K is a minimum cut.*

Proof. Let f^* be a maximum flow and K^* be a minimum cut in N. Then we have, by Theorem 3.6.6, val $f \leq$ val $f^* \leq$ cap $K^* \leq$ cap K. But by hypothesis, val f = cap K. Hence, val f = val f^* = cap K^* = cap K. Thus, f is a maximum flow and K is a minimum cut. □

Theorem 3.6.9 is the celebrated *max-flow min-cut theorem* due to Ford and Fulkerson [65], which establishes the equality of the value of a maximum flow and the minimum capacity of a cut separating s and t.

Theorem 3.6.9 (Ford and Fulkerson). *In a given network N (with source s and sink t), the maximum value of a flow is equal to the minimum value of the capacities of all the cuts in N.*

Proof. In view of Lemma 3.6.8, we need only prove that there exists a flow in N whose value is equal to $c([S,\bar{S}])$ for some cut $[S,\bar{S}]$ separating s and t in N. Let f be a maximum flow in N with val $f = w_0$. Define $S \subset N$ recursively as follows:

(a) $s \in S$, and
(b) If a vertex $u \in S$ and either $f_{uv} < c(u,v)$ or $f_{vu} > 0$, then include v in S.

Any vertex not belonging to S belongs to $\bar{S}$. We claim that t cannot belong to S; indeed, if we suppose that $t \in S$, then there exists a path P from s to t, say

$P : sv_1v_2 \dots v_jv_{j+1} \dots v_kt$, with its vertices in S such that for any arc of P, either $f_{v_jv_{j+1}} < c(v_j, v_{j+1})$ or $f_{v_{j+1}v_j} > 0$. Call an arc joining v_j and v_{j+1} of P a *forward arc* if it is directed from v_j to v_{j+1}; otherwise, it is a *backward arc*.

Let δ_1 be the minimum of all differences $(c(v_j, v_{j+1}) - f_{v_jv_{j+1}})$ for forward arcs, and let δ_2 be the minimum of all flows in backward arcs of P. Both δ_1 and δ_2 are positive, by the definition of S. Let $\delta = \min\{\delta_1, \delta_2\}$. Increase the flow in each forward arc of P by δ and also decrease the flow in each backward arc of P by δ. Keep the flows along the other arcs of N unaltered. Then there results a new flow whose value is $w_0 + \delta > w_0$, leading to a contradiction. [This is because among all arcs incident at s, only in the initial arc of P, the flow value is increased by δ if it is a forward arc or decreased by δ if it is a backward arc; see (5) of Remarks 3.6.4.] This contradiction shows that $t \notin S$, and therefore $t \in \bar{S}$. In other words, $[S, \bar{S}]$ is a cut separating s and t. If $v \in S$ and $u \in \bar{S}$, we have, by the definition of S, $f_{vu} = c(v, u)$ if (v, u) is an arc of N, and $f_{uv} = 0$ if (u, v) is an arc of N. Hence, as in the proof of Theorem 3.6.6,

$$w_0 = \sum_{\substack{u \in S \\ v \in \bar{S}}} f_{uv} - \sum_{\substack{v \in \bar{S} \\ u \in S}} f_{vu} = \sum_{\substack{u \in S \\ v \in \bar{S}}} f_{uv} - 0 = \sum_{\substack{u \in S \\ v \in \bar{S}}} c(u, v) = c([S, \bar{S}]). \qquad \square$$

We now use the max-flow min-cut theorem to prove a number of results due to Menger. We shall first prove a result for a network in which each arc has unit capacity.

Theorem 3.6.10. *Let N be a network with source s and sink t. Let each arc of N have unit capacity. Then,*

(a) The value of a maximum flow in N is equal to the maximum number k of arc-disjoint directed (s, t)-paths in N, and
(b) The capacity of a minimum cut in N is equal to the minimum number ℓ of arcs whose deletion destroys all (s, t)-paths in N.

Proof. Let f^* be a maximum flow in N, and let D^* denote the digraph obtained from D, the underlying digraph of N, by deleting all arcs whose flow is zero in f^*. Now, note that $0 < f^*(a) \le c(a) = 1$ for all $a \in A(D^*)$, and therefore, $f^*(a) = 1$ for all $a \in A(D^*)$. Hence,

(i) $d^+_{D^*}(s) - d^-_{D^*}(s) = {f^*}^+(s) - {f^*}^-(s) = \text{val } f^* = {f^*}^-(t) - {f^*}^+(t) = d^-_{D^*}(t) - d^+_{D^*}(t)$ and
(ii) $d^+_{D^*}(v) = d^-_{D^*}(v)$ for $v \in V(N) \setminus \{s, t\}$.

(i) and (ii) imply that there are val f^* arc-disjoint directed (s, t)-paths in D^* and hence also in D. Thus, val $f^* \le k$. Now let $P_1, P_2, \dots, P_k$ be any system of k arc-disjoint directed (s, t)-paths in N. Define a function f on $A(N)$ by

$$f(a) = \begin{cases} 1 & \text{if } a \text{ is an arc of } \bigcup_{i=1}^{k} P_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is a flow in N with value k. Since f^* is a maximum flow, we have val $f^* \geq k$. Consequently, val $f^* = k$, proving (a).

Let $K^* = [S, \bar{S}]$ be a minimum cut in N so that $|K^*| \geq \ell$ by the definition of ℓ. Then, cap $K^* = |K^*| \geq \ell$.

Now let Z be a set of ℓ arcs whose deletion destroys all directed (s, t)-paths, and let T denote the set of all vertices including s joined to s by a directed path in $N - Z$. Then since $s \in T$, and $t \in \bar{T}$, $K = [T, \bar{T}]$ is a cut in N. By the definition of T, $N - Z$ can contain no arc of $[T, \bar{T}]$, and hence $K \subseteq Z$. Since K^* is a minimum cut, we conclude that cap $K^* \leq$ cap $K = |K| \leq |Z| = \ell$. Thus, cap $K^* = \ell$. □

We now state and prove the *edge version of Menger's theorem for directed graphs.*

Theorem 3.6.11. *Let x and y be two vertices of a digraph D. Then the maximum number of arc-disjoint directed (x, y)-paths in D is equal to the minimum number of arcs whose deletion destroys all directed (x, y)-paths in D.*

Proof. Apply Theorem 3.6.9 to the two results of Theorem 3.6.10. □

Theorem 3.6.12 is the *edge version of Menger's theorem for undirected graphs.*

Theorem 3.6.12. *Let x and y be two vertices of a graph G. Then the maximum number of edge-disjoint (x, y)-paths in G is equal to the minimum number of edges of G whose deletion destroys all (x, y)-paths in G.*

Proof. Construct a digraph $D(G)$ from G as follows: $V(G)$ is also the vertex set of $D(G)$ and if $u, v \in V(G)$, then $(u, v) \in A(D(G))$ if and only if u and v are adjacent in G; that is, $D(G)$ is obtained from G by replacing each edge uv of G by a symmetric pair of arcs (u, v) and (v, u). By Theorem 3.6.11, the maximum number of arc-disjoint directed (x, y)-paths in $D(G)$ is equal to the minimum number of arcs whose deletion destroys all directed (x, y)-paths in $D(G)$. But each directed (x, y)-path in $D(G)$ gives rise to a unique (x, y)-path in G, and conversely an (x, y)-path in G yields a unique directed (x, y)-path in $D(G)$. Hence, the deletion of a set of λ edges in G destroys all (x, y)-paths in G if and only if the deletion of the corresponding set of λ arcs in $D(G)$ destroys all directed (x, y)-paths in $D(G)$. □

Theorem 3.6.13 is the *vertex version of Menger's theorem for digraphs.*

Theorem 3.6.13. *Let x and y be two vertices of a digraph D such that $(x, y) \notin A(D)$. Then the maximum number of internally disjoint directed (x, y)-paths in D is equal to the minimum number of vertices whose deletion destroys all directed (x, y)-paths in D.*

Proof. Construct a new digraph D' from D as follows:

(a) Split each vertex $v \in V \setminus \{x, y\}$ into two new vertices, v' and v'', and join them by an arc (v', v''), and
(b) Replace

  (i) Each arc (u, v) of D where $u \notin \{x, y\}$ and $v \notin \{x, y\}$ by the arc (u'', v'),

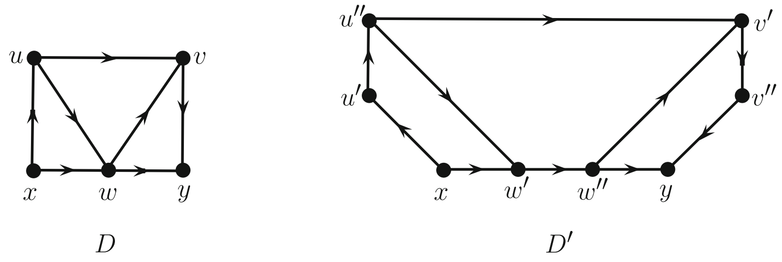


Fig. 3.16 Digraphs D and D' for proof of Theorem 3.6.13

(ii) Each arc (x, v) of D by (x, v') and (v, x) by (v'', x), and
(iii) Each arc (v, y) of D by (v'', y) and (y, v) by (y, v').
(See Fig. 3.16.)

Now, to each directed (x, y)-path in D', there corresponds a directed (x, y)-path in D obtained by contracting all arcs of the type (v', v'') [that is, delete the arc (v', v'') and identify the vertices v' and v''], and, conversely, to each directed (x, y)-path in D, there corresponds a directed (x, y)-path in D' obtained by splitting each intermediate vertex of the path. Furthermore, two directed (x, y)-paths in D' are arc-disjoint if and only if the corresponding directed paths in D are internally disjoint. Hence, the maximum number of arc-disjoint directed (x, y)-paths in D' is equal to the maximum number of internally disjoint directed (x, y)-paths in D.

Similarly, the minimum number of arcs in D' whose deletion destroys all directed (x, y)-paths in D' is equal to the minimum number of vertices in D whose deletion destroys all directed (x, y)-paths in D. To see this, let A' be a minimum set of p arcs of D' whose deletion destroys all directed (x, y)-paths in D', and let B' be a minimum set of q vertices of D whose deletion destroys all directed (x, y)-paths in D. We have to show that $p = q$. Any arc of A' must contain either v' or v'' corresponding to the vertex v of D. Then the deletion of all vertices v corresponding to such arcs of D' separates x and y in D and hence $q \leq p$. Conversely, if $v \in B'$, delete the corresponding arc (v', v'') in D'. Then the deletion of the q arcs (v', v'') (which correspond to the q vertices of B') from D' destroys all directed (x, y)-paths in D', and therefore $p \leq q$. Thus, $p = q$. The result now follows from Theorem 3.6.11. □

Theorem 3.6.14 is the *vertex version of Menger's theorem for undirected graphs.*

Theorem 3.6.14. *Let x and y be two nonadjacent vertices of a graph G. Then the maximum number of internally disjoint (x, y)-paths in G is equal to the minimum number of vertices whose deletion destroys all (x, y)-paths.*

Proof. Define $D(G)$ as in Theorem 3.6.12 and apply Theorem 3.6.13. □

Let G be an undirected graph with $n \geq k + 1$ vertices. Suppose G satisfies the condition $(*)$:

$(*)$ Any two distinct vertices of G are connected by k internally disjoint paths in G.

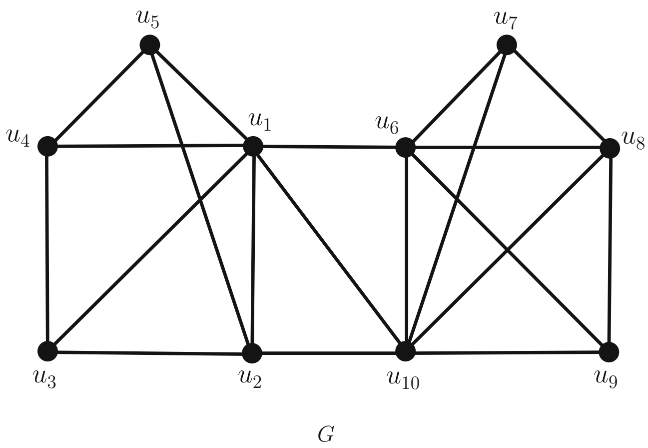


Fig. 3.17 A 2-connected and 3-edge connected graph

Then Theorem 3.6.14 implies that to separate two nonadjacent vertices x and y of G, at least k vertices are to be removed. Hence if $(*)$ holds, G is k-connected.

Conversely, if G is k-connected, to separate any pair of nonadjacent vertices x and y of G, at least k vertices are to be removed, and by Theorem 3.6.14, there are at least k internally disjoint (x, y)-paths in G. However, if x and y are adjacent, then since $G-xy$ is $(k-1)$-connected, there are at least k internally disjoint (x, y)-paths, including the edge xy. Thus, we have the following result of Whitney's generalizing Theorem 3.3.7.

Theorem 3.6.15 (Whitney). *A graph G with $n \geq k+1$ vertices is k-connected if and only if any two vertices of G are connected by at least k internally disjoint paths.*

Example 3.6.16. The graph G of Fig. 3.17 is 2-connected and 3-edge connected. The pair of vertices u_5 and u_{10} are connected by the following two internally disjoint paths:

$$u_5u_1u_{10} \text{ and } u_5u_4u_3u_2u_{10}$$

Moreover, they are connected by the following 3-edge-disjoint paths:

$$u_5u_1u_{10};\ u_5u_2u_{10};\ \text{and } u_5u_4u_1u_6u_{10}.$$

Exercise 6.1. If u and v are vertices of a graph G such that any two u-v paths in G have an internal common vertex, show that all the u-v paths in G have an internal common vertex.

Exercise 6.2. Show that if G is k-connected, then $G \vee K_1$ is $(k+1)$-connected.

Exercise 6.3. Let S be a subset of the vertex set of a k-connected graph G with $|S| = k$. If $v \in V \setminus S$, show that there exist k internally disjoint paths from v to

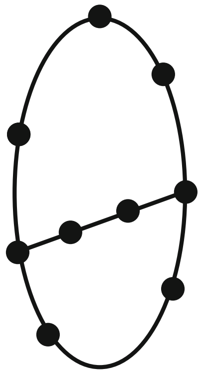


Fig. 3.18 A θ-graph

the k vertices of S. [Remark: In particular, if C is a cycle of length at least k in a k-connected graph G, and $v \in V(G) \setminus V(C)$, then there are k internally disjoint paths from v to C.]

The remark in Exercise 6.3 yields the following theorem of Dirac [55], which generalizes Theorem 3.3.8.

Exercise 6.4. *Dirac's theorem* [55]: If a graph is k-connected ($k \geq 2$), then any set of k vertices of G lie on a cycle of G. (Note: The cycle may contain additional vertices besides these k vertices.) Hint: Use induction on k. If G is $(k+1)$-connected, and $\{v_1, v_2, \ldots, v_k, v_{k+1}\}$ is any set of $k+1$ vertices of G, by the induction assumption, $v_1, v_2, \ldots, v_k$ all lie on a cycle C of G. If $V(C) = \{v_1, v_2, \ldots, v_k\}$, then the k disjoint paths from v_{k+1} to C must end in $v_1, v_2, \ldots, v_k$. Otherwise, $V(C) \supsetneqq \{v_1, v_2, \ldots, v_k\}$. Since G is $(k+1)$-connected, by the pigeonhole principle, the end vertices of two of the $(k+1)$ disjoint paths from v_{k+1} to C must belong to one of the k closed paths $[v_i, v_{i+1}]$, $1 \leq i \leq k-1$ and $[v_k, v_1]$ on C. (Here $[v_i, v_{i+1}]$ and $[v_k, v_1]$ are those paths on C that contain no other v_j.)

Exercise 6.5. Show by means of an example that the converse of Dirac's theorem (Exercise 6.4) is false.

Exercise 6.6. Show that a k-connected simple graph on $(k+1)$ vertices is K_{k+1}.

Exercise 6.7. *Dirac's theorem* [55]; *see also* [93]: Show that a graph G with at least $2k$ vertices is k-connected if and only if for any two disjoint sets V_1 and V_2 of k vertices each, there exist k disjoint paths from V_1 to V_2 in G.

Exercise 6.8. Show that a 2-connected non-Hamiltonian graph contains a θ-subgraph. (A θ-graph is a graph of the form $C \cup P$, where C is a cycle of length at least 4 and P is a path of length at least 2 that joins two nonadjacent vertices of C and is internally disjoint from C.) (See Fig. 3.18.)

3.7 Exercises

7.1. Prove that there exists no simple connected cubic graph with fewer than 10 vertices containing a cut edge. (For a simple connected cubic graph having exactly 10 vertices and having a cut edge, see Exercise 7.13)

7.2. Show that no vertex v of a simple graph can be a cut vertex of both G and G^c.

7.3. Show that a simple connected graph that is not a block contains at least two end blocks.

7.4. Show that a connected k-regular bipartite graph is 2-connected.

7.5. Let $b(v)$ denote the number of blocks of a simple connected graph G to which a vertex v belongs. Then prove that the number of blocks $b(G)$ of G is given by $b(G) = 1 + \sum_{v \in V(G)} (b(v) - 1)$.
(Hint: Use induction on the number of blocks of G.)

7.6. If $c(B)$ denotes the number of cut vertices of a simple connected graph G belonging to the block B, prove that the number of cut vertices $c(G)$ of G is given by $c(G) = 1 + \sum (c(B) - 1)$, the summation being over the blocks of G.

7.7. Show that a simple connected graph with at least three vertices is a path if and only if it has exactly two vertices that are not cut vertices.

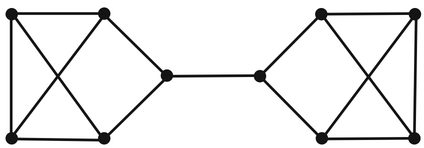
7.8. Prove that if a graph G is k-connected or k-edge connected, then $m \geq \frac{nk}{2}$.

7.9. Construct a graph with $\kappa = 3$, $\lambda = 4$, and $\delta = 5$.

7.10. For any three positive integers a, b, c, with $a \leq b \leq c$, construct a simple graph with $\kappa = a$, $\lambda = b$, and $\delta = c$.

7.11. Let G be a cubic graph with a 1-*factor* (i.e., a 1-regular spanning subgraph) F of G. Prove that any cut edge of G belongs to F.

7.12. Let G be a k-connected graph and let S be a separating set of G^2 such that $G^2 - S$ has q components. Show that $|S| \geq qk$.



7.13. Find all the edge cuts of the above graph.

7.14. Let G be a 2-connected graph and let $v_1, v_2 \in V(G)$. Let n_1 and n_2 be positive integers with $n = n_1 + n_2$. Show that there exists a partition of V into $V_1 \cup V_2$ with $|V_i| = n_i$, $G[V_i]$ connected, and $v_i \in V_i$ for each $i = 1, 2$. (Remark: The generalization of this result to k-connected graphs is also true [55].)

Notes

Chronologically, Menger's theorem appeared first [140]. Then followed Whitney's generalizations [193] of Menger's theorem. Our proof of Menger's theorem is based on the max-flow min-cut theorem of Ford and Fulkerson [65, 66].