Chapter 2
Directed Graphs

2.1 Introduction

Directed graphs arise in a natural way in many applications of graph theory. The
street map of a city, an abstract representation of computer programs, and network
flows can be represented only by directed graphs rather than by graphs. Directed
graphs are also used in the study of sequential machines and system analysis in
control theory.

2.2 Basic Concepts

Definition 2.2.1. A *directed graph* D is an ordered triple $(V(D), A(D), I_D)$, where
$V(D)$ is a nonempty set called the set of *vertices* of D; $A(D)$ is a set disjoint from
$V(D)$, called the set of *arcs* of D; and I_D is an *incidence map* that associates with
each arc of D an ordered pair of vertices of D. If a is an arc of D, and $I_D(a) =$
(u, v), u is called the *tail* of a, and v is the *head* of a. The arc a is said to join v with
u. u and v are called the *ends* of a. A directed graph is also called a *digraph*.

With each digraph D, we can associate a graph G (written $G(D)$ when reference
to D is needed) on the same vertex set as follows: Corresponding to each arc
of D, there is an edge of G with the same ends. This graph G is called the
underlying graph of the digraph D. Thus, every digraph D defines a unique
(up to isomorphism) graph G. Conversely, given any graph G, we can obtain a
digraph from G by specifying for each edge of G an order of its ends. Such a
specification is called an *orientation* of G.

Just as with graphs, digraphs have a diagrammatic representation. A digraph is
represented by a diagram of its underlying graph together with arrows on its edges,
the arrow pointing toward the head of the corresponding arc. A digraph and its
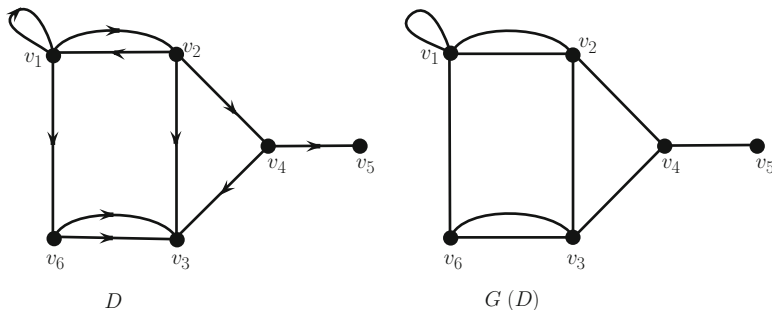underlying graph are shown in Fig. 2.1.

Fig. 2.1 Digraph D and its underlying graph $G(D)$

Many of the concepts and terminology for graphs are also valid for digraphs. However, there are many concepts of digraphs involving the notion of orientation that apply only to digraphs.

Definition 2.2.2. If $a = (u, v)$ is an arc of D, a is said to be *incident out of u* and *incident into v*. v is called an *outneighbor* of u, and u is called an *inneighbor* of v. $N_D^+(u)$ denotes the set of outneighbors of u in D. Similarly, $N_D^-(u)$ denotes the set of inneighbors of u in D. When no explicit reference to D is needed, we denote these sets by $N^+(u)$ and $N^-(u)$, respectively. An arc a is *incident with u* if it is either incident into or incident out of u. An arc having the same ends is called a *loop* of D. The number of arcs incident out of a vertex v is the *outdegree* of v and is denoted by $d_D^+(v)$ or $d^+(v)$. The number of arcs incident into v is its *indegree* and is denoted by $d_D^-(v)$ or $d^-(v)$.

For the digraph D of Fig. 2.2, we have $d^+(v_1) = 3$, $d^+(v_2) = 3$, $d^+(v_3) = 0$, $d^+(v_4) = 2$, $d^+(v_5) = 0$, $d^+(v_6) = 2$, $d^-(v_1) = 2$, $d^-(v_2) = 1$, $d^-(v_3) = 4$, $d^-(v_4) = 1$, $d^-(v_5) = 1$, and $d^-(v_6) = 1$. (The loop at v_1 contributes 1 each to $d^+(v_1)$ and $d^-(v_1)$.)

The *degree* $d_D(v)$ of a vertex v of a digraph D is the degree of v in $G(D)$. Thus, $d(v) = d^+(v) + d^-(v)$. As each arc of a digraph contributes 1 to the sum of the outdegrees and 1 to the sum of indegrees, we have

$$\sum_{v \in V(D)} d^+(v) = \sum_{v \in V(D)} d^-(v) = m(D),$$

where $m(D)$ is the number of arcs of D.

A vertex of D is *isolated* if its degree is 0; it is *pendant* if its degree is 1. Thus, for a pendant vertex v, either $d^+(v) = 1$ and $d^-(v) = 0$, or $d^+(v) = 0$ and $d^-(v) = 1$.

Definitions 2.2.3. 1. A digraph D' is a *subdigraph* of a digraph D if $V(D') \subseteq V(D)$, $A(D') \subseteq A(D)$, and $I_{D'}$ is the restriction of I_D to $A(D')$.
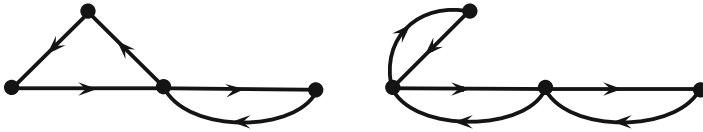
Fig. 2.2 A strong digraph (*left*) and a symmetric digraph (*right*)

2. A *directed walk* joining the vertex v_0 to the vertex v_k in D is an alternating sequence $W = v_0 a_1 v_1 a_2 v_2 \ldots a_k v_k$, $1 \leq i \leq k$, with a_i incident out of v_{i-1} and incident into v_i. *Directed trails, directed paths, directed cycles*, and *induced subdigraphs* are defined analogously as for graphs.
3. A vertex v is *reachable* from a vertex u of D if there is a directed path in D from u to v.
4. Two vertices of D are *diconnected* if each is reachable from the other in D. Clearly, diconnection is an equivalence relation on the vertex set of D, and if the equivalence classes are $V_1, V_2, \ldots, V_\omega$, the subdigraphs of D induced by $V_1, V_2, \ldots, V_\omega$ are called the *dicomponents* of D.
5. A digraph is *diconnected* (also called *strongly-connected*) if it has exactly one dicomponent. A diconnected digraph is also called a *strong digraph*.
6. A digraph is *strict* if its underlying graph is simple. A digraph D is *symmetric* if, whenever (u, v) is an arc of D, then (v, u) is also an arc of D (see Fig. 2.2).

Exercise 2.1. How many orientations does a simple graph of m edges have?

Exercise 2.2. Let D be a digraph with no directed cycle. Prove that there exists a vertex whose indegree is 0. Deduce that there is an ordering $v_1, v_2, \ldots, v_n$ of V such that, for $2 \leq i \leq n$, every arc of D with terminal vertex v_i has its initial vertex in $\{v_1, v_2, \ldots, v_{i-1}\}$.

2.3 Tournaments

A digraph D is a *tournament* if its underlying graph is a complete graph. Thus, in a tournament, for every pair of distinct vertices u and v, either (u, v) or (v, u), but not both, is an arc of D. Figures 2.3a, b display all tournaments on three and four vertices, respectively.

The word "tournament" derives its name from the usual round-robin tournament. Suppose there are n players in a tournament and that every player is to play against every other player. The results of such a tournament can be represented by a tournament on n vertices, where the vertices represent the n players and an arc (u, v) represents the victory of player u over player v.

Suppose the players of a tournament have to be ranked. The corresponding digraph T, a tournament, could be used for such a ranking. The ranking of the vertices of T is as follows: One way of doing it is by looking at the sequence of outdegrees of T. This is because $d_T^+(v)$ stands for the number of players defeated by
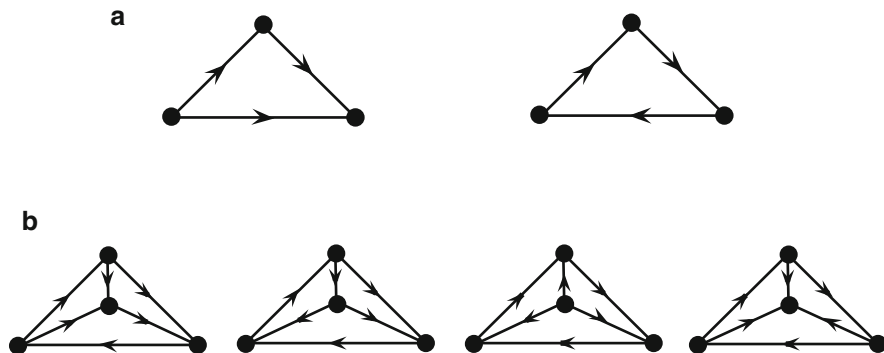
Fig. 2.3 Tournaments on (**a**) three and (**b**) four vertices

the player v. Another way of doing it is by finding a *directed Hamilton path,* that is, a spanning directed path in T. One could rank the players as per the sequence of this path so that each player defeats his or her successor. We now prove the existence of a directed Hamilton path in any tournament.

Theorem 2.3.1 (Rédei [165]). *Every tournament contains a directed Hamilton path.*

Proof. (By induction on the number of vertices n of the tournament.) The result can be directly verified for all tournaments having two or three vertices. Hence, suppose that the result is true for all tournaments on $n \geq 3$ vertices. Let T be a tournament on $n + 1$ vertices $v_1, v_2, \ldots, v_{n+1}$. Now, delete v_{n+1} from T. The resulting subdigraph T' of T is a tournament on n vertices and hence by the induction hypothesis contains a directed Hamilton path. Assume that the Hamilton path is $v_1 v_2 \ldots v_n$, relabeling the vertices, if necessary.

If the arc joining v_1 and v_{n+1} has v_{n+1} as its tail, then $v_{n+1} v_1 v_2 \ldots v_n$ is a directed Hamilton path in T and the result stands proved (see Fig. 2.4a).

If the arc joining v_n and v_{n+1} is directed from v_n to v_{n+1}, then $v_1 v_2 \ldots v_n v_{n+1}$ is a directed Hamilton path in T (see Fig. 2.4b).

Now suppose that none of (v_{n+1}, v_1) and (v_n, v_{n+1}) is an arc of T. Hence, (v_1, v_{n+1}) and (v_{n+1}, v_n) are arcs of T—the first arc incident into v_{n+1} and the second arc incident out of v_{n+1}. Thus, as we pass on from v_1 to v_n, we encounter a reversal of the orientation of edges incident with v_{n+1}. Let v_i, $2 \leq i \leq n$, be the first vertex where this reversal takes place, so that (v_{i-1}, v_{n+1}) and (v_{n+1}, v_i) are arcs of T. Then $v_1 v_2 \ldots v_{i-1} v_{n+1} v_i v_{i+1} \ldots v_n$ is a directed Hamilton path of T (see Fig. 2.4c).   □

Theorem 2.3.2 (Moon [141, 143]). *Every vertex of a diconnected tournament T on n vertices with $n \geq 3$ is contained in a directed k-cycle, $3 \leq k \leq n$. (T is then said to be vertex-pancyclic.)*

Proof. Let T be a diconnected tournament with $n \geq 3$ and u, a vertex of T. Let $S = N^+(u)$, the set of all outneighbors of u in T, and $S' = N^-(u)$, the set of all
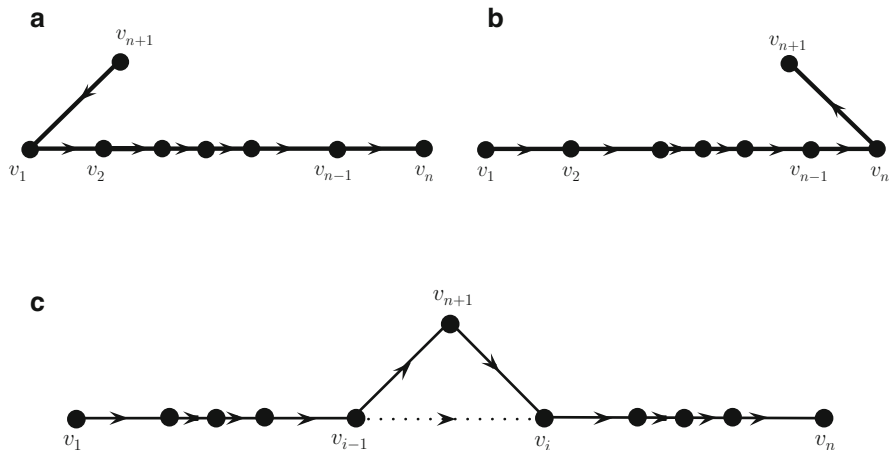
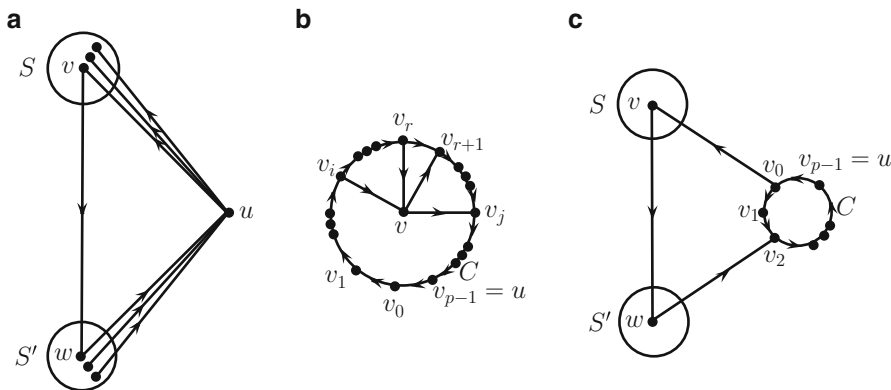**Fig. 2.4** Digraphs for proof of Theorem 2.3.1

Fig. 2.5 Digraphs for proof of Theorem 2.3.2

inneighbors of u in T. As T is diconnected, none of S and S' is empty. If $[S, S']$ denotes the set of all arcs of T having their tails in S and heads in S', then $[S, S']$ is also nonempty for the same reason. If (v, w) is an arc of $[S, S']$, then (u, v, w, u) is a directed 3-cycle in T containing u. (see Fig. 2.5a.)

Suppose that u belongs to directed cycles of T of all lengths k, $3 \leq k \leq p$, where $p < n$. We shall prove that there is a directed $(p + 1)$-cycle of T containing u.

Let C : $(v_0, v_1, \ldots, v_{p-1}, v_0)$ be a directed p-cycle containing u, where $v_{p-1} = u$. Suppose that v is a vertex of T not belonging to C such that for some i and j, $0 \leq i$, $j \leq p-1$, $i \neq j$, there exist arcs (v_i, v) and (v, v_j) of T (see Fig. 2.5b). Then there must exist arcs (v_r, v) and (v, v_{r+1}) of $A(T)$, $i \leq r \leq j - 1$ (suffixes taken modulo p), and hence $(v_0, v_1, \ldots, v_r, v, v_{r+1}, \ldots, v_{p-1}, v_0)$ is a directed $(p + 1)$-cycle containing u (see Fig. 2.5b).

If no such v exists, then for every vertex v of T not belonging to $V(C)$, either $(v_i, v) \in A(T)$ for every i, $0 \le i \le p - 1$, or $(v, v_i) \in A(T)$ for every i, $0 \le i \le p - 1$. Let $S = \{v \in V(T) \setminus V(C) : (v_i, v) \in A(T)$ for each i, $0 \le i \le p - 1\}$ and $S' = \{w \in V(T) \setminus V(C) : (w, v_i) \in A(T)$ for each i, $0 \le i \le p - 1\}$. The diconnectedness of T implies that none of S, S', and $[S, S']$ is empty. Let (v, w) be an arc of $[S, S']$. Then $(v_0, v, w, v_2, \ldots, v_{p-1}, v_0)$ is a directed $(p + 1)$-cycle of T containing $v_{p-1} = u$ (see Fig. 2.5c). □

Remark 2.3.3. Theorem 2.3.2 shows, in particular, that every diconnected tournament is Hamiltonian; that is, it contains a directed spanning cycle.

Exercise 3.1. Show that every tournament T is diconnected or can be made into one by the reorientation of just one arc of T.

Exercise 3.2. Show that a tournament is diconnected if and only if it has a spanning directed cycle.

Exercise 3.3. Show that every tournament of order n has at most one vertex v with $d^+(v) = n - 1$.

Exercise 3.4. Show that for each positive integer $n \ge 3$, there exists a non-Hamiltonian tournament of order n (that is, a tournament not containing a spanning directed cycle).

Exercise 3.5. Show that if a tournament contains a directed cycle, then it contains a directed cycle of length 3.

Exercise 3.6. Show that every tournament T contains a vertex v such that every other vertex of T is reachable from v by a directed path of length at most 2.

2.4 k-Partite Tournaments

Definition 2.4.1. A *k-partite graph*, $k \ge 2$, is a graph G in which $V(G)$ is partitioned into k nonempty subsets $V_1, V_2, \ldots, V_k$, such that the induced subgraphs $G[V_1], G[V_2], \ldots, G[V_k]$ are all totally disconnected. It is said to be *complete* if, for $i \ne j$, each vertex of V_i is adjacent to every vertex of V_j, $1 \le i, j \le k$. A *k-partite tournament* is an oriented complete k-partite graph (see Fig. 2.6). The subsets $V_1, V_2, \ldots, V_k$ are often referred to as the *partite sets* of G.

The next three theorems are based on Goddard et al. [74]. We now give a characterization of a k-partite tournament containing a 3-cycle.
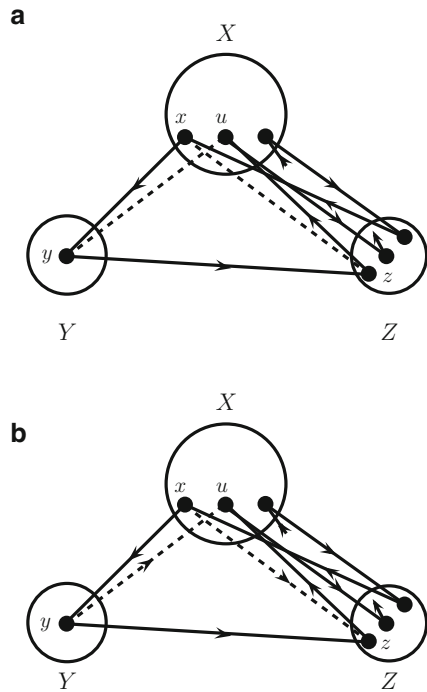
Theorem 2.4.2. *Let T be a k-partite tournament, $k \ge 3$. Then T contains a directed 3-cycle if and only if there exists a directed cycle in T that contains vertices from at least three partite sets.*

Proof. Suppose that T contains a directed 3-cycle C. Then the three vertices of C must belong to three distinct partite sets of T.

Fig. 2.6 A 3-partite
tournament
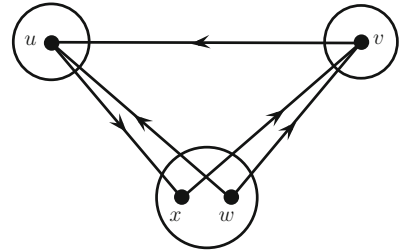
Fig. 2.7 Digraphs for proof
of Theorem 2.4.2

Conversely, suppose that T contains a directed cycle C that in turn contains
vertices from at least three partite sets. Assume that C has the least possible length.
Then there exist three consecutive vertices x, y, z on C that belong to distinct partite
sets of T, say X, Y, Z, respectively. We claim that C is a directed 3-cycle.

As x and z are in different partite sets of the k-partite tournament T, either
$(z, x) \in A(T)$, the arc set of T, or $(x, z) \in A(T)$. If $(z, x) \in A(T)$, then (x, y, z, x)
is a directed 3-cycle containing vertices from three partite sets of T. If $(x, z) \in A(T)$,
then consider $C' = (C - y) + (x, z)$. C' is a directed cycle of length one less than
that of C. So by assumption on C, C' contains vertices from only two partite sets,
namely, X and Z. Let u be the vertex of C immediately following z on C'. Then
$u \in X$. If $(u, y) \in A(T)$, then $C_1'' = (y, z, u, y)$ is a directed 3-cycle containing
vertices from three partite sets of T. Hence, assume that $(u, y) \notin A(T)$, and so
$(y, u) \in A(T)$ (see Fig. 2.7a, b). Now consider $C_2'' = (C - z) + (y, u)$. C_2'' is a
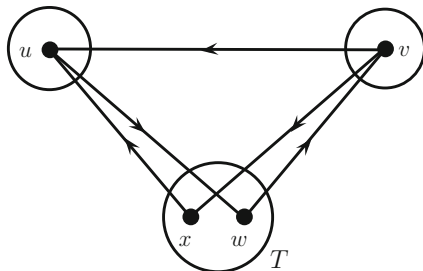
directed cycle that is shorter than C and contains at least one vertex other than x, y, and u. The successor of u in C_2'' belongs to Z, and thus C_2'' contains vertices from three partite sets of T. This is a contradiction to the choice of C. Thus, (y, u) does not belong to $A(T)$, a contradiction. This proves the result. □

Theorem 2.4.3. *Let T be a k-partite tournament, $k \geq 3$. Then every vertex u belonging to a directed cycle in T must belong to either a directed 3-cycle or a directed 4-cycle.*

Proof. Let C be a shortest directed cycle in T that contains u. Suppose that C is not a directed 3-cycle. We shall prove that u is a vertex of a directed 4-cycle. Let u, x, y, and z be four consecutive vertices of C. If $(u, y) \in A(T)$, then $C' = (C - x) + (u, y)$ is a directed cycle in T containing u and having a length shorter than C. This contradicts the choice of C. Hence $(u, y) \notin A(T)$. Also, if $(y, u) \in A(T)$, then (u, x, y, u) is a directed 3-cycle containing u. This contradicts our assumption on C. Hence, $(y, u) \notin A(T)$. Consequently, y and u belong to the same partite set of T. This means that u and z must belong to distinct partite sets of T. If $(u, z) \in A(T)$, then $C'' = (C - \{x, y\}) + (u, z)$ is a directed cycle containing u and having length shorter than that of C. Hence $(u, z) \notin A(T)$. Therefore, $(z, u) \in A(T)$ and (u, x, y, z, u) is a directed 4-cycle containing u. □

Remark 2.4.4. Theorem 2.3.2 states that every vertex of a diconnected tournament lies on a k-cycle for every k, $3 \leq k \leq n$. However, this property is not true for a diconnected k-partite tournament. The tournament T of Fig. 2.8 is a counterexample. T is a 3-partite tournament with $\{x, w\}$, $\{u\}$, and $\{v\}$ as partite sets, $(uwvxu)$ is a spanning directed cycle in T, and hence T is strongly connected, but x is not a vertex of any directed 3-cycle.

Definition 2.4.5. The *score* of a vertex v in a tournament T is its outdegree. (This corresponds to the number of players who are beaten by player v.) If $v_1, v_2, \ldots, v_n$ are the vertices of T and $S(v_i)$ is the score of v_i in T, then $(S(v_1), S(v_2), \ldots, S(v_n))$ is the *score vector* of T. An ordered triple (u, v, w) of vertices of T is a *transitive triple* of T if $(u, v) \in A(T)$ and $(v, w) \in A(T)$, then $(u, w) \in A(T)$.

Remarks 2.4.6. 1. If v is any vertex of a tournament T and u, w are two outneighbors of v, then $\{u, v, w\}$ determines a unique transitive triple in T. Such a transitive

Fig. 2.9 Three-partite
tournaments with eight
directed 3-cycles (T_1) and
five directed 3-cycles (T_2).
Both T_1 and T_2 have the same
score vector

T_1

T_2
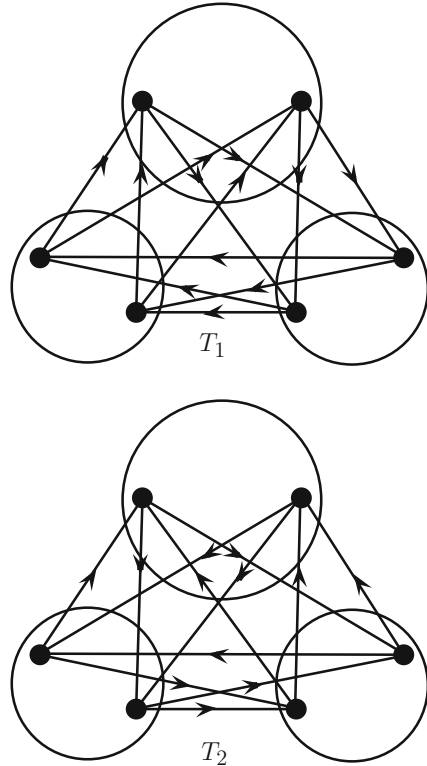
triple is said to be defined by the vertex v. Clearly, any transitive triple of T is
defined by some vertex of T. Further, the number of transitive triples defined by
v is $\binom{S(v)}{2}$.

2. The number of directed 3-cycles in a tournament T of order n is obtained by
subtracting the total number of transitive triples of vertices of T from the total
number of triples of vertices of T. Thus, the total number of directed 3-cycles in
T is equal to $\binom{n}{3} - \sum_{v \in V(T)} \binom{S(v)}{2} = \frac{n(n-1)(n-2)}{6} - \frac{1}{2} \sum_{v \in V(T)} S(v)(S(v)-1)$.

Thus, the score vector of a tournament T determines the number of directed
3-cycles in T. But in a general k-partite tournament, the score vector need not
determine the number of directed 3-cycles. Consider the two 3-partite tournaments
T_1 and T_2 of Fig. 2.9. Both have the same score vector $(2, 2, 2, 2, 2, 2)$. But T_1 has
eight directed 3-cycles, whereas T_2 has only five directed 3-cycles.

Theorem 2.4.7 gives a formula for the number of directed 3-cycles in a k-partite
tournament.

Theorem 2.4.7. *Let T be a k-partite tournament, $k \geq 3$, having partite sets
$V_0, V_1, \ldots, V_{k-1}$. Then the number of directed 3-cycles in T is given by*

$$\sum_{0 \le i < j < \ell \le k-1} |V_i| \, |V_j| \, |V_\ell| - \sum_{v \in V(T)} \sum_{i<j} O_i(v) \, O_j(v),$$

where $O_i(v)$ denotes the number of outneighbors of v in V_i.

Proof. Let S denote the set of triples of vertices of T such that the three vertices of the triple belong to three different partite sets, and let $N = |S|$. Then

$$N = \sum_{0 \le i < j < \ell \le k-1} |V_i| \, |V_j| \, |V_\ell|.$$

Any orientation of a triangle gives a directed 3-cycle or a transitive triple. Hence the number of directed 3-cycles in $T = N - N_1$, where N_1 is the number of transitive triples in T. Also, a triple of vertices of T is transitive if and only if there exists a vertex of the triple having the other two vertices as outneighbors. The number of such triples of T to which a vertex v can belong and for which the other two vertices are outneighbors of v is $\sum_{i<j} O_i(v) \, O_j(v)$. Hence $N_1 = \sum_{v \in V(T)} \sum_{i<j} O_i(v) \, O_j(v)$. Thus the number of directed 3-cycles in T is given by

$$N - N_1 = \sum_{0 \le i < j < \ell \le k-1} |V_i| \, |V_j| \, |V_\ell| - \sum_{v \in V(T)} \sum_{i<j} O_i(v) \, O_j(v). \qquad \square$$

Remark 2.4.8. For $k = 3$, the results of Theorem 2.4.7 simplify as follows:

(i) The number of transitive triples in a 3-partite tournament equals

$$\sum_{i=0}^{2} \sum_{v \in V_i} O_{i+1}(v) \, O_{i+2}(v),$$

where the suffixes are taken modulo 3.

(ii) The number of directed 3-cycles in a 3-partite tournament is given by

$$|V_0| \, |V_1| \, |V_2| - \sum_{i=0}^{2} \sum_{v \in V_i} O_{i+1}(v) \, O_{i+2}(v),$$

where the suffixes are taken modulo 3.

Remark 2.4.9. Consider the two 3-partite tournaments of Fig. 2.10. T_1 has $|V_0| |V_1| |V_2|$ directed 3-cycles and has no transitive triples, whereas T_2 contains no directed 3-cycles but contains $|V_0| |V_1| |V_2|$ transitive triples.
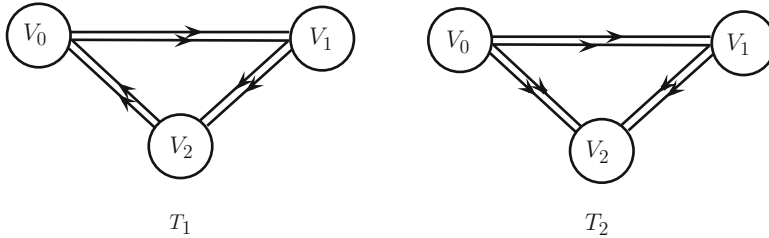
Fig. 2.10 Three-partite tournaments T_1 (with directed 3-cycles and no transitive triples) and T_2 (with transitive triples and no directed 3-cycles). *Double arrows* indicate that all arcs joining corresponding partite sets have the same orientation

2.5 Exercises

5.1. If $|V_i| = n_i$, $1 \leq i \leq k$, find the number of edges in the complete multipartite graph $G(V_1, V_2, \ldots, V_k)$. (See [27], p. 6.)

5.2. Show that if T is a strongly connected 3-partite tournament with partite sets V_0, V_1, V_2, then the maximum number of transitive triples in T is $|V_0||V_1||V_2| - 1$ unless $|V_0| = |V_1| = |V_2| = 2$, in which case T has at most $|V_0||V_1||V_2| - 2 = 6$ transitive triples.

5.3. Construct a strongly connected 3-partite tournament containing exactly six transitive triples.

5.4. Give a definition of digraph isomorphism similar to that of graph isomorphism.

5.5. Give an example of two nonisomorphic tournaments on five vertices. Justify your answer.

5.6. If u and v are distinct vertices of a tournament T such that both $d(u, v)$ and $d(v, u)$ are defined [where $d(u, v)$ denotes the length of a shortest directed (u, v)-path in T], show that $d(u, v) \neq d(v, u)$.

5.7. (A tournament T is called *transitive* if (a, b) and (b, c) are arcs of T, then (a, c) is also an arc of T.) Prove that a transitive tournament contains a Hamilton path with any preassigned orientation. [Hint: Use the fact that T has a vertex of outdegree $(n - 1)$ and a vertex of outdegree zero.]

Notes

The earliest of the books on directed graphs is by Harary, Norman, and Cartwright [97]. *Topics on Tournaments* by Moon [143] deals exclusively with tournaments. Theorems 2.4.2, 2.4.3 and 2.4.7 are based on [74].