Chapter 10
Domination in Graphs

10.1 Introduction

"Domination in graphs" is an area of graph theory that has received a lot of attention in recent years. It is reasonable to believe that "domination in graphs" has its origin in "chessboard domination." The "queen domination" problem asks: What is the minimum number of queens required to be placed on an 8×8 chessboard so that every square not occupied by any of these queens will be dominated (that is, can be attacked) by one of these queens? Recall that a queen can move horizontally, vertically, and diagonally on the chessboard. The answer to the above question is 5. Figure 10.1 gives one set of dominating locations for the five queens.

10.2 Domination in Graphs

The concept of chessboard domination can be extended to graphs in the following way:

Definition 10.2.1. Let G be a graph. A set $S \subseteq V$ is called a *dominating set* of G if every vertex $u \in V \backslash S$ has a neighbor $v \in S$. Equivalently, every vertex of G is either in S or in the neighbor set $N(S) = \bigcup\limits_{v \in S} N(v)$ of S in G. A vertex u is said to be *dominated by* a vertex $v \in G$ if either $u = v$ or $uv \in E(G)$.

Definition 10.2.2. A γ-*set* of G is a minimum dominating set of G, that is, a dominating set of G whose cardinality is minimum. A dominating set S of G is *minimal* if S properly contains no dominating set S' of G.

Definition 10.2.3. The *domination number* of G is the cardinality of a minimum dominating set (that is, γ-set) of G; it is denoted by $\gamma(G)$.

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Fig. 10.1 Queen domination

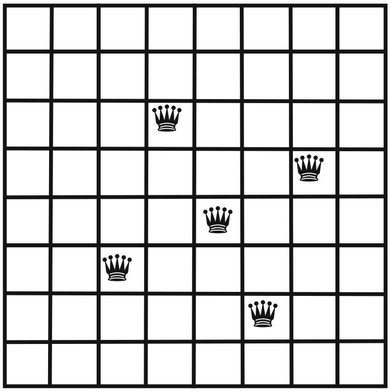
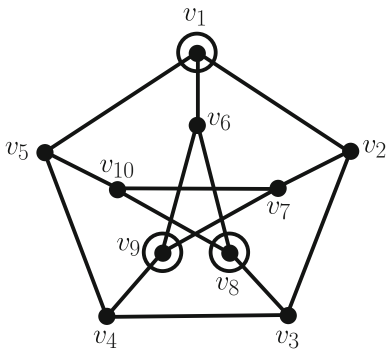


Fig. 10.2 Petersen graph P for which $\gamma(P) = 3$



Example 10.2.4. For the Petersen graph P, $\gamma(P) = 3$. In Fig. 10.2, $\{v_1, v_8, v_9\}$ is a γ-set of P while the set $\{v_1, v_2, v_3, v_4, v_5\}$ is a minimal dominating set of P.

The study of domination was formally initiated by Ore [151]. A comprehensive introduction to "domination in graphs" is given in the first volume of the two-volume book by Haynes, Hedetniemi, and Slater [100, 101]. The next three theorems are due to Ore [151]. Given a dominating set S of G, when is S a minimal dominating set? This question is answered in Theorem 10.2.5 below.

Theorem 10.2.5. *Let S be a dominating set of a graph G. Then S is a minimal dominating set of G if and only if for each vertex u of S, one of the following two conditions holds:*

(i) u is an isolated vertex of $G[S]$, the subgraph induced by S in G.
(ii) There exists a vertex $v \in V \backslash S$ such that u is the only neighbor of v in S.

Proof. Suppose that S is a minimal dominating set of G. Then for each vertex u of S, $S \backslash \{u\}$ is not a dominating set of G. Hence, there exists $v \in V \backslash (S \backslash \{u\})$ such that v is dominated by no vertex of $S \backslash \{u\}$. If $v = u$, then u is an isolated vertex of $G[S]$, and hence condition (i) holds. If $v \neq u$, as S is a dominating set of G, condition (ii) holds.

Conversely, assume that S is not a minimal dominating set of G. Then there exists a vertex $u \in S$, such that $S\backslash\{u\}$ is also a dominating set of G. Hence, u is dominated by some vertex of $S\backslash\{u\}$. This means that u is adjacent to some vertex of $S\backslash\{u\}$, and hence u is not an isolated vertex of $G[S]$. Moreover, if v is any vertex of $V\backslash S$, then v is adjacent to some vertex of $S\backslash\{u\}$. Hence, neither condition (i) nor condition (ii) holds. □

Notice that in Example 10.2.4, the set $S = \{v_1, v_2, v_3, v_4, v_5\}$ is a minimal dominating set of the Petersen graph P in which no vertex is an isolated vertex of $P[S]$ and for each $i = 1, 2, \ldots, 5$, v_i is the only vertex of S that is adjacent to v_{i+5}.

Theorem 10.2.5 suggests the following definition.

Definition 10.2.6. Let S be a dominating set of a graph G, and $u \in S$. The *private neighborhood of u relative to S in G* is the set of vertices which are in the closed neighborhood of u, but not in the closed neighborhood of any vertex in $S\backslash\{u\}$.

Thus, the private neighborhood $P_N(u, S)$ of u with respect to S is given by $P_N(u, S) = N[u]\backslash\Big(\bigcup\limits_{v\in S\backslash\{u\}} N[v]\Big)$.

Note that $u \in P_N(u, S)$ if and only if u is an isolated vertex of $G[S]$ in G.

Theorem 10.2.5 can now be restated as follows:

Theorem 10.2.5′. *A dominating set S of a graph G is a minimal dominating set of G if and only if $P_N(u, S) \neq \emptyset$ for every $u \in S$.*

Corollary 10.2.7. *Let G be a graph having no isolated vertices. If S is a minimal dominating set of G, then $V\backslash S$ is a dominating set of G.*

Proof. As S is a minimal dominating set, by Theorem 10.2.5′, $P_N(u, S) \neq \emptyset$ for every u ∈ S. This means that for every u ∈ S, there exists $v \in V\backslash S$ such that $uv \in E(G)$, and consequently, $V\backslash S$ is a dominating set of G. □

Corollary 10.2.8. *Let G be a graph of order $n \geq 2$. If $\delta(G) \geq 1$, then $\gamma(G) \leq \frac{n}{2}$.*

Proof. As $\delta(G) \geq 1$, G has no isolated vertices. If S is a minimal dominating set of G, by Corollary 10.2.7, both S and $V\backslash S$ are dominating sets of G. Certainly, at least one of them is of cardinality at most $\dfrac{n}{2}$. □

Corollary 10.2.9. *If G is a connected graph of order $n \geq 2$, $\gamma(G) \leq \frac{n}{2}$.*

Proof. As G is connected and $n \geq 2$, G has no isolated vertices. Now apply Corollary 10.2.8. □

We note that the conclusion in Corollary 10.2.9 would remain valid even if G is disconnected as long as no component of G is a K_1.

10.3 Bounds for the Domination Number

In this section, we present lower and upper bounds for the domination number $\gamma(G)$.

We first make two observations:

Observation 10.3.1. (i) A vertex v dominates $N(v)$ and $|N(v)| \leq \Delta(G)$.
(ii) Let v be any vertex of G. Then $V \backslash N(v)$ is a dominating set of G.

These two observations yield the following lower and upper bounds for $\gamma(G)$.

Theorem 10.3.2. *For any graph* G, $\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G)$.

Proof. By Observation 10.3.1 (i), for any vertex v of G, the vertices of $N[v]$ will be dominated by v. To cover all the vertices of G, at least $\left\lceil \frac{n}{1+\Delta(G)} \right\rceil$ closed neighborhoods are required. This gives the lower bound.

By Observation 10.3.1 (ii), $\gamma(G) \leq |V \backslash N(v)|$ for each $v \in V(G)$. The minimum is attained on the right when $|N(v)| = \Delta(G)$. Hence, $\gamma(G) \leq n - \Delta(G)$. □.

The lower bound in Theorem 10.3.2 is due to Walikar, Acharya, and Sampathkumar [187], while the upper bound is due to Berge [19].

10.4 Bound for the Size *m* in Terms of Order *n* and Domination Number $\gamma(G)$

In this section, we present a basic result of Vizing [184], which bounds m (the size of G) in terms of n (the order of G) and $\gamma = \gamma(G)$.

Theorem 10.4.1 (Vizing [184]). *Let G be a graph of order n, size m, and domination number γ. Then*

$$m \leq \left\lfloor \frac{1}{2}(n-\gamma)(n-\gamma+2) \right\rfloor. \tag{10.1}$$

Proof. If $\gamma = 1$, $\frac{1}{2}(n-\gamma)(n-\gamma+2) = \frac{1}{2}(n^2-1)$, while the maximum value for $m = \frac{1}{2}n(n-1)$ (when $G = K_n$), and the result is true. If $\gamma = 2$, $\frac{1}{2}(n-\gamma)(n-\gamma+2) = \frac{1}{2}n(n-2)$. Now when $\gamma = 2$ by Theorem 10.3.2, $\Delta \leq n-2$ and $m \leq \frac{1}{2}n(n-2)$ (by Euler's theorem), and the result is true. Thus, the result is true for $\gamma = 1$ and 2. We now assume that $\gamma \geq 3$. We apply induction on n. Let G be a graph of order n, size m, and $\gamma \geq 3$. If v is a vertex of maximum degree Δ of G, again by Theorem 10.3.2, $|N(v)| = \Delta \leq n - \gamma$, and hence $\Delta = n - \gamma - r$, where $0 \leq r$ (see Fig. 10.3).

Let $S = V \backslash N[v]$. Then

$$|S| = |V| - |N(v)| - 1 = n - (n - \gamma - r) - 1 = \gamma + r - 1. \tag{10.2}$$

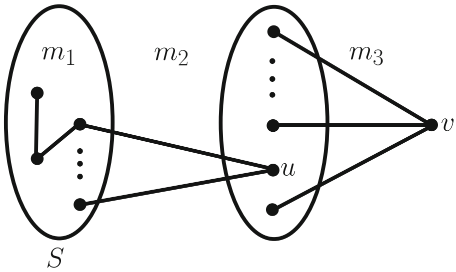


Fig. 10.3 The set S in the proof of Theorem 10.4.1

Let m_1 be the size of $G[S]$, m_2 be the number of edges between S and $N(v)$, and m_3 be the size of $G[N[v]]$. Clearly, $m = m_1 + m_2 + m_3$. If D is a γ-set of $G[S]$, then $D \cup \{v\}$ is a dominating set of G. Hence,

$$\gamma(G) = \gamma \leq |D| + 1. \tag{10.3}$$

By the induction hypothesis, this implies, by virtue of (10.2) and (10.3), that

$$\begin{aligned} m_1 &\leq \left\lfloor \frac{1}{2}(|S| - |D|)(|S| - |D| + 2) \right\rfloor \\ &\leq \left\lfloor \frac{1}{2}[(\gamma + r - 1) - (\gamma - 1)][(\gamma + r - 1) - (\gamma - 1) + 2] \right\rfloor \text{ (by (10.3))} \\ &= \frac{1}{2}r(r + 2). \end{aligned} \tag{10.4}$$

If $u \in N(v)$, then $(S \backslash N(u)) \cup \{u, v\}$ is a dominating set of G. Therefore,

$$\begin{aligned} \gamma &\leq |S \backslash N(u)| + 2 \\ &= |S| - |S \cap N(u)| + 2 \\ &\leq (\gamma + r - 1) - |S \cap N(u)| + 2 \text{ (by (10.2))}. \end{aligned}$$

This is turn implies that for each vertex $u \in N(v)$, $|S \cap N(u)| \leq r + 1$.
Consequently,

$$\begin{aligned} m_2 &= \text{ the number of edges between } N(v) \text{ and } S \\ &\leq |N(v)|(r + 1) \\ &= \Delta(r + 1). \end{aligned} \tag{10.5}$$

Now the sum of the degrees of the vertices of $N[v] \leq (\Delta + 1)\Delta$. As there are m_2 edges between $N(v)$ and S,

$$\begin{aligned}
&\text{the sum of the degrees of the vertices of } N[v] \text{ in } G[N[v]] \\
&\quad = (\text{the sum of the degrees of the vertices of } N[v] \text{ in } G) - m_2 \\
&\quad \leq \Delta(\Delta + 1) - m_2.
\end{aligned}$$

Thus,

$$m_3 \leq \frac{1}{2}[\Delta(\Delta + 1) - m_2]. \tag{10.6}$$

From (10.4), (10.5), and (10.6), we get

$$\begin{aligned}
m &= m_1 + m_2 + m_3 \\
&\leq \frac{1}{2}r(r+2) + m_2 + \frac{1}{2}[\Delta(\Delta+1) - m_2] \\
&= \frac{1}{2}r(r+2) + \frac{1}{2}[\Delta(\Delta+1) + m_2] \\
&\leq \frac{1}{2}r(r+2) + \frac{1}{2}[\Delta(\Delta+1) + \Delta(r+1)](by (10.5)) \\
&\leq \frac{1}{2}(n-\gamma-\Delta)(n-\gamma-\Delta+2) + \frac{1}{2}[\Delta(\Delta+1) + \Delta(r+1)](as \Delta = n-\gamma-r) \\
&= \frac{1}{2}(n-\gamma)(n-\gamma+2) - \frac{\Delta}{2}[(n-\gamma-\Delta+2) + (n-\gamma-\Delta) - \Delta - (\Delta+1)(r+1)] \\
&= \frac{1}{2}(n-\gamma)(n-\gamma+2) - \frac{\Delta}{2}[(n-\gamma+2) + (n-\gamma) - \Delta - (\Delta+1) - (r+1)] \\
&= \frac{1}{2}(n-\gamma)(n-\gamma+2) - \frac{\Delta}{2}[(\Delta+r+2) + (\Delta+r) - \Delta - (\Delta+1) - (r+1)] \\
&= \frac{1}{2}(n-\gamma)(n-\gamma+2) - \frac{\Delta}{2}r \\
&\leq \frac{1}{2}(n-\gamma)(n-\gamma+2)(\text{as } r \geq 0). \qquad \square
\end{aligned}$$

The bound given in Theorem 10.4.1 is sharp. In other words, there are graphs G for which

$$m = \frac{1}{2}(n-\gamma)(n-\gamma+2). \tag{10.7}$$

Example 10.4.2 (Vizing [184]). Let H_t be the graph obtained from K_t by removing the edges of a minimum edge cover (that is, the smallest number of edges containing all the vertices of K_t) $\mathscr{C}$. We construct for any positive integer $n \geq 2$, a graph G of order n with domination number γ satisfying (10.7).

Case (i). $\gamma = 2$. Take $t = n-2$ and $G = H_n$. Now $\mathscr{C}$ has $\lceil \frac{n-2}{2} \rceil$ edges. Hence, $m = m(G) = \{\frac{1}{2}(n-2)(n-3) - \lceil \frac{n-2}{2} \rceil\} + 2(n-2) = \lfloor \frac{1}{2}(n-2)n \rfloor = \lfloor \frac{1}{2}(n-\gamma)(n-\gamma+2) \rfloor$.

Case (ii.) $\gamma > 2$. Take $t = n - \gamma + 2$ and $G = H_t \cup K^c_{\gamma-2}$. Then $\gamma(G) = 2 + (\gamma - 2) = \gamma$, $|V(G)| = (n-\gamma) + 2 + (\gamma - 2) = n$, and

$$\begin{aligned} m &= \left\{ \frac{(n-\gamma)(n-\gamma-1)}{2} - \left\lceil \frac{n-\gamma}{2} \right\rceil \right\} + 2(n-\gamma) \\ &= \frac{(n-\gamma)(n-\gamma+3)}{2} - \left\lceil \frac{n-\gamma}{2} \right\rceil \\ &= \frac{1}{2}(n-\gamma)(n-\gamma+2). \end{aligned}$$

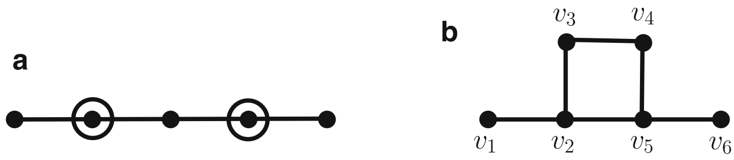


Fig. 10.4 (**a**) $\gamma(G) = 2 = i(G)$ (**b**) $\gamma(G) = 2$ while $i(G) = 3$

10.5 Independent Domination and Irredundance

Definition 10.5.1. A subset S of the vertex set of a graph G is an *independent dominating set* of G if S is both an independent and a dominating set. The *independent domination number* $i(G)$ of G is the minimum cardinality of an independent dominating set of G.

It is clear that $\gamma(G) \leq i(G)$ for any graph G. For the path P_5, $\gamma(P_5) = i(P_5) = 2$, (see Fig. 10.4(a)) while for the graph G of Fig. 10.4(b), $\gamma(G) = 2$ and $i(G) = 3$. In fact, $\{v_2, v_5\}$ is a γ-set for G, while $\{v_1, v_3, v_5\}$ is a minimum independent dominating set of G.

Theorem 10.5.2. *Every maximal independent set of a graph G is a minimal dominating set.*

Proof. Let S be a maximal independent set of G. Then S must be a dominating set of G. If not, there exists a vertex $v \in V \backslash S$ that is not dominated by S, and so $S \cup \{v\}$ is an independent set of G, violating the maximality of S. Further, S must be a minimal dominating set of G. If not, there exists a vertex u of S such that $T = S \backslash \{u\}$ is also a dominating set of G. This means, as $u \notin T$, u has a neighbor in T and hence S is not independent, a contradiction. $\square$

Definition 10.5.3. A set $S \subset V(G)$ is called *irredundant* if every vertex v of S has at least one private neighbor.

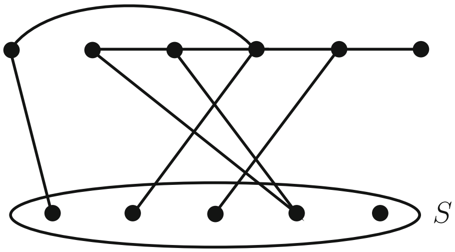


Fig. 10.5 S is an irredundant but not a dominating set

This definition means that either v is an isolated vertex of $G[S]$ or else v has a private neighbor in $V\backslash S$; that is, there exists at least one vertex $w \in V\backslash S$ that is adjacent only to v in S.

In Fig. 10.5, S is irredundant but not a dominating set. Hence an irredundant set S need not be dominating. If S is both irredundant and dominating, then it is minimal dominating, and vice versa.

Theorem 10.5.4. *A set $S \subset V$ is a minimal dominating set of G if and only if S is both dominating and irredundant.*

Proof. Assume that S is both a dominating and an irredundant set of G. If S were not a minimal dominating set, there exists $v' \in S$ such that $S \setminus \{v'\}$ is also a dominating set. But as S is irredundant, v' has a private neighbor w' (may be equal to v'). Since w' has no neighbor in $S \setminus v'$, $S \setminus \{v'\}$ is not a dominating set of G. Thus, S is a minimal dominating set of G.

The proof of the converse is similar. □

We define below a few more well-known graph parameters:

(i) The minimum cardinality of a maximal irredundant set of a graph G is known as the *irredundance number* and is denoted by $ir(G)$.
(ii) The maximum cardinality of an irredundant set is known as the *upper irredundance number* and is denoted by $IR(G)$.
(iii) The maximum cardinality of a minimal dominating set is known as the *upper domination number* and is denoted by $\Gamma(G)$.

From our earlier results and definitions, one can prove the following result of Cockeyne, Hedetniemi and Miller [45].

Theorem 10.5.5 ([45]). *For any graph G, the following inequality chain holds:*

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G).$$

Proof. Exercise. □

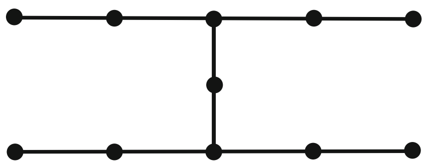
10.6 Exercises

6.1. If G is a graph of diameter 2, show that $\gamma(G) \le \delta(G)$.
6.2. If D is a dominating set of a graph G, show that D meets every closed neighborhood of G.
6.3. Show that for any edge e of a graph G, $\gamma(G) \le \gamma(G-e) \le \gamma(G)+1$, and that for any vertex v of G, $\gamma(G)-1 \le \gamma(G-v)$.
6.4. If $d_1 \ge d_2 \ge \ldots \ge d_n$ is the degree sequence of a graph G, prove that

$$\gamma(G) \ge \min\{k : k + (d_1 + \cdots + d_k) \ge n\}.$$

6.5. For any graph G, prove that $\gamma(G) \le \chi(G^c)$.
6.6. Show that every minimal dominating set in a graph G is a maximal irredundant set of G.
6.7. Prove that an independent set is maximal independent if and only if it is dominating and independent.
6.8. Prove: If $\gamma(G^c) \ge 3$, then $\mathrm{diam}(G) \le 2$.
6.9. Prove: If G is connected, then $\left\lceil \frac{\mathrm{diam}(G)+1}{3} \right\rceil \le \gamma(G)$.
6.10. For any graph G, $n - m \le \gamma(G) \le n + 1 - \sqrt{1+2m}$. Prove further that $\gamma(G) = n - m$ if and only if G is a forest in which each component is a star. [Hint: To establish the upper bound, use Vizing's theorem (Theorem 10.4.1).]
6.11. Give the proof of Theorem 10.5.5.
6.12. For the graph G of Fig. 10.6, determine the six parameters given in Theorem 10.5.5.

Fig. 10.6 Refer Question 6.12



6.13. Exhibit a graph (different from the graph of Fig. 10.4b) for which no minimum dominating set is independent.

10.7 Vizing's Conjecture

All graphs considered in this section are simple. In this section we present Vizing's conjecture on the domination number of the Cartesian product of two graphs. In 1963, Vizing [182] proposed the problem of determining a lower bound for the domination number of the Cartesian product of two graphs. Five years later, in 1968, he presented it as a conjecture [185]. In the same year, E. Cockayne included it in his survey article [44]. This conjecture is one of the major unsolved problems in graph theory.

Conjecture 10.7.1 (Vizing [185]). For any two graphs G and H, $\gamma(G \square H) \geq \gamma(G)\gamma(H)$.

In what follows, we dwell upon some partial results toward this conjecture as well as some of the techniques that have been adopted in attempts to tackle this conjecture.

We write $G \leq H$ to denote the fact that G is a spanning subgraph of H. By definition (see Chap. 1),

$G \boxtimes H \leq G[H]$,

$G \square H \leq G \boxtimes H$, and

$G \times H \leq G \boxtimes H$.

It is clear that if $G \leq H$, then $\gamma(G) \geq \gamma(H)$. Consequently, we have the following result.

Theorem 10.7.2. *For any two graphs G and H,*

$\gamma(G[H]) \leq \gamma(G \boxtimes H) \leq min\{\gamma(G \square H), \gamma(G \times H)\}$.

In the absence of a proof of Vizing's conjecture, what is normally done is to fix one of the two graphs, say G, and allow the other graph H to vary and see if the conjecture 10.7.1 holds for all graphs H. Since the Cartesian product is commutative, it is immaterial as to which of the two graphs is fixed and which is varied. With this in view, we make the following definition:

Definition 10.7.3. A graph G is said to *satisfy Vizing's conjecture if and only if* $\gamma(G \square H) \geq \gamma(G)\gamma(H)$ *for every graph H.*

Definition 10.7.4. A graph G is *edge-maximal with respect to domination if* $\gamma(G + uv) < \gamma(G)$ *for every pair of nonadjacent vertices u, v of G.*

For example, C_4 is edge-maximal since $\gamma(C_4 + e) = 1 < 2 = \gamma(C_4)$.

Definition 10.7.5. A 2-*packing* of a graph G is a set P of vertices of G such that $N[x] \cap N[y] = \emptyset$ for every pair of (distinct) vertices x, y of P. The 2-*packing number* $\rho(G)$ of a graph G is the largest cardinality of a 2-packing of G. In other words, $\rho(G)$ is the maximum number of pairwise disjoint closed neighborhoods of G.

Before we set out to prove some theorems relating to Vizing's conjecture, we point out that in the relation $\gamma(G\square H) \geq \gamma(G)\,\gamma(H)$, both equality and strict inequality are possible.

For instance, $\gamma(C_4 \square P_3) = 3 > 2 \times 1 = \gamma(C_4)\,\gamma(P_3)$ (see Fig. 1.28), while $\gamma(C_4 \square P_2) = 2 = \gamma(C_4)\gamma(P_2)$.

Most of the results supporting Vizing's conjecture are of the following two types:

(i) If H is a graph related to G in some way, and if G satisfies Vizing's conjecture, then H also does.
(ii) Let $\mathscr{P}$ be a graph property. If G satisfies $\mathscr{P}$, then G satisfies Vizing's conjecture.

First, we present two results (Lemmas 10.7.6 and 10.7.7) that come under the first category.

Lemma 10.7.6. *Let $K \leq G$ such that $\gamma(K) = \gamma(G)$. If G satisfies Vizing's conjecture, then K also does.*

Proof. The graph K is obtained from G by removing edges of G (if $K = G$, there is nothing to prove). Let $e \in E(G)\backslash E(K)$. Then $K \leq G - e \leq G$. Hence, $\gamma(K) \geq \gamma(G-e) \geq \gamma(G)$. By hypothesis, $\gamma(K) = \gamma(G)$. Hence $\gamma(G-e) = \gamma(G)$, and since $(G-e)\Box H \leq G\Box H$, we have

$$\begin{aligned}\gamma((G-e)\Box H) &\geq \gamma(G\Box H)\\ &\geq \gamma(G)\,\gamma(H)\text{(by hypothesis)}\\ &= \gamma(G-e)\,\gamma(H).\end{aligned}$$

Hence, $G - e$ also satisfies Vizing's conjecture. Now start from $G - e$ and delete edges in succession until the resulting graph is K. Thus, K also satisfies Vizing's conjecture. □

Lemma 10.7.6 is about edge deletion. We now consider vertex deletion.

Lemma 10.7.7. *Let $v \in V(G)$ such that $\gamma(G - v) < \gamma(G)$. If G satisfies Vizing's conjecture, then so does $G - v$.*

Proof. The inequality $\gamma(G - v) < \gamma(G)$ means that $\gamma(G - v) = \gamma(G) - 1$. Set $K = G - v$ so that $\gamma(K) = \gamma(G) - 1$. Suppose the result is false. Then there exists a graph H such that

$$\gamma(K\Box H) < \gamma(K)\,\gamma(H).$$

Let A be a γ-set of $K\Box H$ and B a γ-set of H. (Recall that a γ-set stands for a minimum dominating set.) Set $D = A \cup \{(v,b) : b \in B\} = A \cup (\{v\} \times B)$. Then D is a dominating set of $G\Box H$. But then, as the sets A and $\{v\} \times B$ are disjoint,

$$\begin{aligned}\gamma(G\Box H) \leq |D| = |A| + |(\{v\} \times B)| &= |A| + |B|\\ &= \gamma(K\Box H) + \gamma(H)\\ &< \gamma(K)\,\gamma(H) + \gamma(H)\\ &= \gamma(H)\,(\gamma(K) + 1)\\ &= \gamma(H)\,\gamma(G),\end{aligned}$$

and this contradicts the hypothesis that G satisfies Vizing's conjecture. □

We next present a lower bound (Theorem 10.7.8) and an upper bound (Theorem 10.7.10) for $\gamma(G\Box H)$.

Theorem 10.7.8 (El-Zahar and Pareek [59]). $\gamma(G\Box H) \geq \min\{|V(G)|, |V(H)|\}$.

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_p\}$ and $V(H) = \{v_1, v_2, \ldots, v_q\}$. We have to prove that $\gamma(G\Box H) \geq \min\{p, q\}$. Suppose D is a dominating set of $G\Box H$ with

$$|D| < \min\{p, q\}. \tag{10.8}$$

Then $|D| < p$ and $|D| < q$.

Recall that the G-fibers of $G\Box H$ are pairwise disjoint. A similar statement applies for the H-fibers of $G\Box H$ as well. In view of (10.8), D does not meet all the G-fibers nor does it meet all the H-fibers. Hence, there exist a G-fiber, say G_y, with $y \in V(H)$, and a H-fiber, say H_x, with $x \in V(G)$, which are both disjoint from D. Now any vertex that dominates (x, y) must belong either to G_y or to H_x. But this is not the case as D is disjoint from both G_y and H_x. This contradicts the fact that D is a dominating set of $G\Box H$. Thus, $|D| \geq \min\{p, q\}$. □

Corollary 10.7.9 (Rall [99]). *Let H be an arbitrary graph. Then there exists a positive integer $r = r(H)$ such that if G is any graph with $\gamma(G) \leq r$ and $|V(G)| \geq |V(H)|$, then $\gamma(G\Box H) \geq \gamma(G)\,\gamma(H)$.*

Proof. Recall that $\gamma(H) \leq \frac{1}{2}|V(H)|$ (Corollary 10.2.7). Let $c = \frac{\gamma(H)|}{|V(H)|}$, and $r = \left\lfloor \frac{1}{c} \right\rfloor$. Then

$$\begin{aligned}
\gamma(G\Box H) &\geq \min\{|V(G)|, |V(H)|\} \text{(by Theorem 10.7.8)}\\
&= |V(H)| \text{(since by hypothesis } |V(G)| \geq |V(H)|)\\
&= \frac{\gamma(H)|}{c}\\
&\geq r\,\gamma(H)\\
&\geq \gamma(G)\,\gamma(H) \text{ (since } \gamma(G) \leq r).
\end{aligned}$$
□

Theorem 10.7.10 (Vizing [182]). *For any two graphs G and H,*

$$\gamma(G\Box H) \leq \min\{\gamma(G)\,|V(H)|,\ \gamma(H)\,|V(G)|\}.$$

Proof. Let D be a γ-set for G. Then $\bigcup\limits_{v\in D} H_v$ is a dominating set for $G\Box H$. To see this, consider any vertex (x, y) of $G\Box H$. As $x \in V(G)$, and D is a dominating set of G, either $x \in D$ or there exists $v \in D$ with $vx \in E(G)$. Hence, either $(x, y) \in D \times V(H)$ or (v, y) dominates (x, y) in $G\Box H$. Thus, $\bigcup\limits_{v\in D} H_v$ is a dominating set of $G\Box H$. Further, $|D| = \gamma(G)$, and so

$$\gamma(G\Box H) \le |\bigcup_{v\in D} H_v| = |D|\,|V(H)| = \gamma(G)\,|V(H)|.$$

Similarly, $\gamma(G\Box H) \le \gamma(H)\,|V(G)|$. □

Lemma 10.7.11. *If D is any dominating set of $G\Box H$ and x is any vertex of G, then $|D \cap (N[x] \times V(H))| \ge \gamma(H)$.*

Proof. Let h be any vertex of H so that (x,h) is an arbitrary vertex of the fiber $H_x = \{x\}\Box H$. Let $N_G[x] = \{x, u_1, \ldots, u_k\} \subseteq V(G)$, and let $p : (N[x] \times V(H)) \to H_x$ be the projection map defined by $p((x,h)) = (x,h)$ and $p((u_i,h)) = (x,h)$ for $i = 1,\ldots,k$. Since D is a dominating set of $G\Box H$, D must meet each closed neighborhood in $G\Box H$ (see Exercise 6.2.). Now the closed neighborhood of (x,h) in $G\Box H$ is $N[(x,h)] = \{(x,h)\} \cup (\{x\} \times N(h)) \cup (N(x) \times \{h\})$, and therefore D must contain (x,h) or a vertex either of the form (x,h'), where $h' \in N(h) \subset V(H)$ or of the form (u_i,h), $i = 1,\ldots,k$. Now $p((x,h)) = (x,h)$ and $p((x,h')) = (x,h')$, while $p((u_i,h)) = (x,h)$. Thus, $p(D \cap (N[x] \times V(H)))$ dominates H_x and so $|D \cap (N[x] \times V(H))| \ge |p(D \cap (N[x] \times V(H)))| \ge \gamma(H_x) = \gamma(H)$ (as $H_x \simeq H$). □

Next consider a set $\mathscr{S}$ of pairwise disjoint closed neighborhoods of G. Let H be any graph, and D, a dominating set of $G\Box H$. Then D must meet every star in $\mathscr{S}$. Hence if $N[x] \in \mathscr{S}$, then by Lemma 10.7.11,

$$|D \cap (N[x] \times V(H))| \ge \gamma(H).$$

As this is true for each of the closed neighborhoods in $\mathscr{S}$, we have

$$|D| \ge |\mathscr{S}|\gamma(H). \tag{10.9}$$

Recall that $\rho(G)$ denotes the maximum number of pairwise disjoint closed neighborhoods in the graph G. Replacing $|\mathscr{S}|$ by $\rho(G)$ in (10.9), we get the following result of Jacobson and Kinch.

Theorem 10.7.12 (Jacobson and Kinch [112]). *For any two graphs G and H, $\gamma(G\Box H) \ge \max\{\rho(G)\,\gamma(H),\ \rho(H)\,\gamma(G)\}$.*

Proof. Replacing $|\mathscr{S}|$ by $\rho(G)$ in (10.9), we get

$$\gamma(G\Box H) \ge \rho(G)\,\gamma(H).$$

In a similar manner,

$$\gamma(H\Box G) \ge \rho(H)\,\gamma(G).$$

The result now follows from the fact that $G\Box H \simeq H\Box G$. □

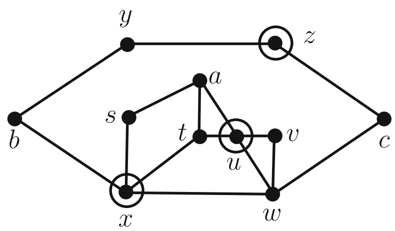


Fig. 10.7 A graph G satisfying Vizing's conjecture

If G is a graph for which $\rho(G) = \gamma(G)$, then Theorem 10.7.12 implies that $\gamma(G\Box H) \geq \gamma(G)\,\gamma(H)$. In other words, Vizing's conjecture is true for such graphs G. Now Meir and Moon [139] have shown that for any tree T, $\rho(T) = \gamma(T)$. Hence, Vizing's conjecture is true for all trees. This fact was first proved by Barcalkin and German [15]. We state this result as a corollary.

Corollary 10.7.13 ([15]). *If T is any tree, then T satisfies Vizing's conjecture.* $\square$

Example 10.7.14. Let G be the graph of Fig. 10.7 The set $D = \{x, u, z\}$ is a γ-set for G so that $\gamma(G) = 3$. Moreover, the closed neighborhoods $N[a]$, $N[b]$, and $N[c]$ are pairwise disjoint in G. Hence, by (10.9) or by Theorem 10.7.12, $|D| \geq 3\,\gamma(H) = \gamma(G)\,\gamma(H)$, and hence G satisfies Vizing's conjecture.

10.8 Decomposable Graphs

In 1979, Barcalkin and German [15] showed that Vizing's conjecture is true for a very large class of graphs. Their result was published in Russian and remained unnoticed until 1991. The result of Barcalkin–German is on the validity of Vizing's conjecture for any decomposable graph. So we now give the definition of a decomposable graph.

Definition 10.8.1. A graph G is called *decomposable* if its vertex set can be partitioned into $\gamma(G)$ subsets with each part inducing a clique (that is a complete subgraph) of G.

Figure 10.8 displays a decomposable graph with $\gamma = 2$.

Theorem 10.8.2 (Barcalkin and German [15]). *If a graph G is decomposable, then G satisfies Vizing's conjecture.*

However, the converse of Theorem 10.8.2 is false. For instance, the graph of Fig. 10.9 is not decomposable, but it satisfies Vizing's conjecture (as $\rho = \gamma = 2$).

Theorem 10.8.2, when taken in conjunction with Lemma 10.7.6, yields the following result.

Theorem 10.8.3. *If $H \leq G$, $\gamma(H) = \gamma(G)$ and G is decomposable, then H satisfies Vizing's conjecture.*

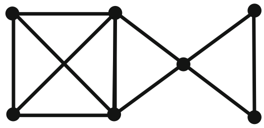


Fig. 10.8 A decomposable graph with $\gamma = 2$

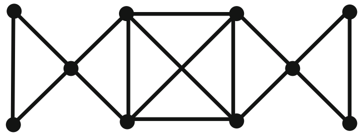


Fig. 10.9 Graph not decomposable but satisfies Vizing's conjecture

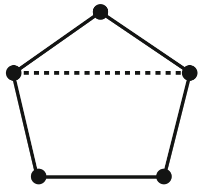


Fig. 10.10 Graph G of Example 10.8.4

Example 10.8.4. C_5 satisfies Vizing's conjecture. This is because if G stands for the graph of Fig. 10.10, then $C_5 \leq G$, $\gamma(C_5) = \gamma(G) = 2$ and G is decomposable as its vertex set can be partitioned into the cliques K_3 and K_2.

Proof of Barcalkin–German theorem. Our proof is based on [30].

Let G be a decomposable graph with $\gamma(G) = k$, and let $\mathscr{C} = (C_1, \ldots, C_k)$ be a partition of $V(G)$ into cliques. Let $\{C_{i_1}, \ldots, C_{i_p}\}$ be a set of $p < k$ cliques belonging to $\mathscr{C}$. Suppose that S is a smallest set of vertices in $G \setminus (C_{i_1} \cup \ldots \cup C_{i_p})$ which dominates (all the vertices of) $C_{i_1} \cup \ldots \cup C_{i_p}$. In other words, S is a set of vertices of G outside $C_{i_1} \cup \ldots \cup C_{i_p}$ dominating the latter. Suppose further that $C_{j_1}, \ldots, C_{j_q}$ are those cliques of $\mathscr{C}$ that have a nonempty intersection with S so that $(\bigcup_{t=1}^{q} C_{j_t}) \cap S = S$. We then claim the following:

$$Claim: \quad \sum_{t=1}^{q}(|(C_{j_t} \cap S)| - 1) \geq p. \tag{10.10}$$

Since any vertex of a clique will dominate that clique, the vertices in S will dominate the $p + q$ cliques $C_{i_1}, \ldots, C_{i_p}; C_{j_1}, \ldots, C_{j_q}$. Hence, if $|S| < p + q$, then all the cliques of $\mathscr{C}$ will be dominated by $|S| + (k - p - q) < k$ vertices; equivalently, $\gamma(G) < k$, a contradiction. This contradiction proves our claim.

We now complete the proof of the theorem. Let D be a minimum dominating set of $G \square H$. The main idea of the proof is that each vertex from D will get a label from 1 to k, and for each label i, the projections to H of the vertices from D that are labeled i form a dominating set of H. This means that there are at least $\gamma(H)$ vertices in D that are labeled i, $1 \leq i \leq k$, and this implies that $|D| \geq k\,\gamma(H) = \gamma(G)\gamma(H)$.

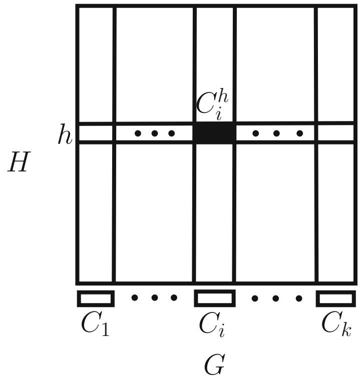


Fig. 10.11 The partition of $G \square H$ into G-cells

For each $h \in V(H)$ and i, $1 \leq i \leq k$, we call $C_i^h = V(C_i) \times \{h\}$ a *G-cell* (see Fig. 10.11, where the cell C_i^h is shaded).

We adopt the following labeling procedure: If a G-cell C_i^h contains a vertex from D, then one of the vertices from $D \cap C_i^h$ is given the label i. Hence, in the projection to H, h will also get the label i. Note that we have not yet determined the labels of the remaining vertices in $D \cap C_i^h$, if any.

Fix an arbitrary vertex $h \in V(H)$. We need to prove that for an arbitrary i, $1 \leq i \leq k$, there exists a vertex from D, labeled by i, that is projected to the neighborhood of h.

There are two cases. First, if there exists a vertex of D in $V(C_i) \times N[h]$, then by our labeling procedure, there will be a vertex in $N[h]$ to which the label i is projected, and so this case is settled.

The second case is that there is no vertex of D in $V(C_i) \times N[h]$, and we call such C_i^h *a missing G-cell for h*. Let $C_{i_1}^h, \ldots, C_{i_p}^h$ be the missing G-cells for h. Now by the definition of the Cartesian product, the missing G-cells for h must be dominated within the G-fiber G^h. Here there must be vertices in $D \cap G^h$ that dominate $C_{i_1}^h \cup \ldots \cup C_{i_p}^h$. Let $C_{j_1}^h, \ldots, C_{j_q}^h$ be the G-cells in G^h that intersect D. Since G^h is isomorphic to G, by inequality (10.10) we have

$$\sum_{t=1}^{q}(|C_{j_t}^h \cap D| - 1) \geq p.$$

Thus, there are enough additional vertices in $D \cap G^h$ (that have not been already labeled) so that for each missing G-cell C_i^h, the label i can be given to one of the vertices in $C_{j_t}^h \cap D$, where $|C_{j_t}^h \cap D| \geq 2$ (so that C_{j_t} has at least one unlabeled vertex of D at this stage). Hence, in this case, the label i will be projected to h. This concludes the proof. $\square$

We now present two applications of Barcalkin–German theorem.

Corollary 10.8.5. *If $\gamma(G) = 1$, then G satisfies Vizing's conjecture.*

Proof. If $\gamma(G) = 1$, G is a spanning subgraph of the complete graph K_n. As $\gamma(K_n) = 1$, the corollary follows. [Any complete graph satisfies Vizing's conjecture as $\gamma(K_n \Box H) = \gamma(H)$.]

Corollary 10.8.6. *If $\gamma(G) = 2$, then G satisfies Vizing's conjecture.*

Proof. Let G' be the graph obtained from G by adding edges so that G' is edge-maximal and $\gamma(G') = 2$. We prove that G' is decomposable. This would mean, by virtue of Theorem 10.8.3, that G satisfies Vizing's conjecture.

Let Q_1 and Q_2 be disjoint cliques of G' such that $|V(Q_1)| + |V(Q_2)|$ is maximum. We claim that $|V(Q_1)| + |V(Q_2)| = |V(G')|\ (= |V(G)|)$.

If not, there exists a vertex v of G, with $v \notin V(Q_1) \cup V(Q_2)$. v cannot be adjacent to all the vertices of Q_1 [else the subgraph induced by $V(Q_1) \cup \{v\}$ is a clique Q_1' of G with $|V(Q_1')| + |V(Q_2)| = |V(Q_1)| + |V(Q_2)| + 1$, a contradiction]. Similarly, v cannot be adjacent to all the vertices of Q_2. Let w be a vertex of Q_1 which is nonadjacent to v in G'. As G' is edge-maximal with respect to γ, $\gamma(G' + vw) = 1$. Hence, $G' + vw$ has a single vertex that forms a minimum dominating set of $G' + vw$. Such a vertex cannot be v (as v is not adjacent to all the vertices of Q_2). Hence, it can only be w (as only w dominates v in $G' + vw$). This means that w is adjacent to all the other vertices in $V(G') \cup \{v\}$, and in particular to all vertices of Q_2. Let A be the set of vertices in Q_1 not adjacent to v in G'. Then each vertex of A is adjacent to all the vertices of Q_2. Then $(Q_1 \setminus A) \cup \{v\}$ and $Q_2 \cup A$ span cliques in G' whose union contains one vertex more than $Q_1 \cup Q_2$. This contradiction proves that $|V(Q_1)| + |V(Q_2)| = |V(G')|$, and so G' is decomposable. $\square$

The preceding theorems gave graphs G for which Vizing's conjecture holds. However, a result providing a general lower bound for $\gamma(G \Box H)$ for all graphs G and H was given by Clark and Suen [42], stating that $\gamma(G \Box H) \geq \frac{1}{2}\gamma(G)\gamma(H)$ for all graphs G and H. Another interesting result concerning Vizing's conjecture is the following result of Clark, Ismail, and Suen [43]. If G and H are both δ-regular graphs, then with only a few possible exceptions, Vizing's conjecture holds for the graph $G \Box H$.

While so much is known about the domination number of the Cartesian product of two graphs, not much is known with regard to other products. We now present two easy results on the direct product.

10.9 Domination in Direct Products

Theorem 10.9.1. *Let G_1 and G_2 be graphs without isolated vertices. Then*

$$\gamma(G_1 \times G_2) \leq 4\,\gamma(G_1)\gamma(G_2).$$

The proof of Theorem 10.9.1 uses Lemma 10.9.2, which is an immediate consequence of Theorem 10.2.5′.

Lemma 10.9.2 ([160]). *Let D be a γ-set of a graph G, (that is, a minimum dominating set) without isolated vertices. Then there exists a matching in $E(D, V \setminus D)$ that saturates all the vertices of D.*

Proof of theorem 10.9.1. Let D_1 and D_2 be γ-sets for G_1 and G_2, respectively. Let D_1' and D_2' be the matching vertex sets of D_1 and D_2, respectively, as given by Lemma 10.9.2. Then $|D_1| = |D_1'| = \gamma(G_1)$, and $|D_2| = |D_2'| = \gamma(G_2)$. Clearly,
$(D_1 \times D_2) \cup (D_1 \times D_2') \cup (D_1' \times D_2) \cup (D_1' \times D_2')$ is a dominating set of $G_1 \times G_2$, of cardinality $4\gamma(G_1)\gamma(G_2)$. □

Definition 10.9.3. A graph G is a *split graph* if $V(G)$ can be partitioned into two subsets K and I such that the subgraph, $G[K]$, induced by K in G is a clique in G, and I is an independent subset of G.

Definition 10.9.4. A subset D of the vertex set of a graph is called a *total dominating set* of G if any vertex v of G has a neighbor in D. (In other words, D dominates not only vertices outside D but also vertices in D.) The *total dominating number*, $\gamma_t(G)$, of G is the minimum cardinality of a total dominating set of G.

Lemma 10.9.5. *For any split graph G, $\gamma_t(G) = \gamma(G)$.*

Proof. Let $G = (K|I)$ be a split graph with K, a clique of G, and I, an independent set of G. Let D be any minimum dominating set of G, and let $D \cap K = K^*$ and $D \cap I = I^*$. By Lemma 10.9.2, G contains a matching in $H = E[D, V \setminus D]$ that saturates all the vertices of D. Let $I^* = \{u_1, \dots, u_k\}$. Then G contains matching edges $u_1v_1, \dots, u_kv_k$, where $\{v_1, \dots, v_k\} \subset K \setminus K^*$. Clearly, $K^* \cup \{v_1, \dots, v_k\}$ is a minimum dominating set D_1 of G, and hence $|D_1| = \gamma(G)$. Now since K is a clique, D_1 is a total dominating set. Thus, $\gamma_t(G) \leq |D_1| = \gamma(G)$. Since $\gamma(G) \leq \gamma_t(G)$ always (as any total dominating set of G is a dominating set of G), we have $\gamma(G) = \gamma_t(G)$. □

Lemma 10.9.6. *For any two graphs G_1 and G_2 with no isolates, $\gamma(G_1 \times G_2) \leq \gamma_t(G_1)\gamma_t(G_2)$.*

Proof. Let A and B be minimum total dominating sets of G_1 and G_2, respectively. Then for any $(x, y) \in V(G_1) \times V(G_2)$, there exist $a \in A$ and $b \in B$ with $(x, a) \in E(G_1)$ and $(y, b) \in E(G_2)$. Hence, (x, y) is adjacent to (a, b) in $G_1 \times G_2$. This means that $A \times B$ is a dominating set for $G_1 \times G_2$, and hence $\gamma(G_1 \times G_2) \leq |A \times B| = |A||B| = \gamma_t(G_1)\gamma_t(G_2)$. □

Theorem 10.9.7. *If G_1 and G_2 are split graphs, then $\gamma(G_1 \times G_2) \leq \gamma(G_1)\gamma(G_2)$.*

Proof. The proof is an immediate consequence of Lemmas 10.9.5 and 10.9.6. □

Notes

"Domination in graphs" is one of the major areas of current research in graph theory. The two-volume book by Haynes, Hedetniemi, and Slater [100, 101] is a comprehensive reference work on graph domination. Several special types of domination in graphs have been studied by researchers—strong domination, weak domination, global domination, connected domination, independent domination, and so on.

As regards Vizing's conjecture, the technique of partitioning the vertex set of a graph, adopted by Barcalkin and German [15], has been exploited in two different ways to expand the classes of graphs satisfying Vizing's conjecture. In [98], Hartnell and Rall introduce the Type χ partition, which includes the Barcalkin–German class of graphs. The second has been proposed by Brešar and Rall [29]. Chordal graphs form yet another family that satisfies Vizing's conjecture. This was first established by Aharoni and Szabó [2] by using the approach of Clark and Suen [42], who showed that $\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)\gamma(H)$ for *all* graphs G and H. For further details on Vizing's conjecture, the interested reader can refer to the article by Brešar et al. [30].

Domination in graph products, other than the Cartesian product, remains an area that has still not been fully explored.