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G. George Yin
Qing Zhang

Continuous-Time Markov Chains and Applications

A Two-Time-Scale Approach

Second Edition

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G. George Yin
Department of Mathematics
Wayne State University
Detroit, Michigan
USA

Qing Zhang
Department of Mathematics
University of Georgia
Athens, Georgia
USA

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To Our Mentors

Harold J. Kushner and Wendell H. Fleming

Contents

Preface

It has been fourteen years since the first edition of this book was published. Two-time-scale Markovian systems have drawn continuing and resurgent attention, and have been used in many existing and emerging applications. To bring some of the most recent progress up to date, we decided to put together this second, expanded, edition. The main theme remains the same, but the contents have been revised and updated. The main changes include the addition of two chapters and the reorganization of two chapters. One of the new chapters is on asymptotic expansions of solutions of backward equations, which serves as a complement to the chapter on asymptotic expansions of solution of forward equations. The other new chapter presents near-optimal controls of linear quadratic Gaussian systems with random switching. Apart from the addition of the new chapters, we have combined Chapters 4 and 6 in the first edition to form the current Chapter 4, and combined Chapters 5 and 7 in the first edition to form Chapter 5 in this edition. Moreover, we have made an effort to simplify the notation throughout so as to make the book more reader-friendly.

Detroit, Michigan — G. George Yin
Athens, Georgia — Qing Zhang
August — 2012

Preface to the First Edition

This book is concerned with continuous-time Markov chains. It develops an integrated approach to singularly perturbed Markovian systems, and reveals interrelations of stochastic processes and singular perturbations. In recent years, Markovian formulations have been used routinely for numerous real-world systems under uncertainties. Quite often, the underlying Markov chain is subject to rather frequent fluctuations and the corresponding states are naturally divisible to a number of groups such that the chain fluctuates very rapidly among different states within a group, but jumps less frequently from one group to another. Various applications in engineering, economics, and biological and physical sciences have posed increasing demands on an in-depth study of such systems. A basic issue common to many different fields is the understanding of the distribution and the structure of the underlying uncertainty. Such needs become even more pressing when we deal with complex and/or large-scale Markovian models, whose closed-form solutions are usually very difficult to obtain.

Markov chain, a well-known subject, has been studied by a host of researchers for many years. While nonstationary cases have been treated in the literature, much emphasis has been on stationary Markov chains and their basic properties such as ergodicity, recurrence, and stability. In contrast, this book focuses on singularly perturbed nonstationary Markov chains and their asymptotic properties.

Singular perturbation theory has a long history and is a powerful tool for a wide variety of applications. Complementing to the ever growing literature in singular perturbations, by using the basic properties of Markov chains, this book aims to provide a systematic treatment for singularly per-

turbed Markovian models. It collects a number of ideas on Markov chains and singular perturbations scattered through the literature.

This book reports our recent research findings on singularly perturbed Markov chains. We obtain asymptotic expansions of the probability distributions, validate the asymptotic series, deduce the error estimates, establish asymptotic normality, derive exponential type of bounds, and investigate the structure of the weak and strong interactions. To demonstrate the applicability of the asymptotic theory, we focus on hierarchical production planning of manufacturing systems, Markov decision processes, and control and optimization of stochastic dynamic systems. Since numerical methods are viable and indispensable alternatives to many applications, we also consider numerical solutions of control and optimization problems involving Markov chains and provide computationally feasible algorithms.

Originating from a diverse range of applications in production planning, queueing network, communication theory, system reliability, and control and optimization of uncertain systems, this book is application oriented. It is written for applied mathematicians, operations researchers, physical scientists, and engineers. Selected material from the book can also be used for a one semester course for advanced graduate students in applied probability and stochastic processes.

We take great pleasure to acknowledge those who have made it possible for us to bring the book into being. We express our profound gratitude to Wendell Fleming and Harold Kushner, who introduced the intellectual horizon–stochastics to us and whose mathematical inspiration and constant encouragement have facilitated our progress. We have had the privilege to work with Rafail Khasminskii and have greatly benefited from his expertise in probability and singular perturbations. We are very much indebted to Alain Bensoussan, Wendell Fleming, Ruihua Liu, Zigang Pan, Zeev Schuss and the four reviewers for their reviews of earlier versions of the manuscript, and for their comments, criticisms and suggestions. Our thanks also go to Petar Kokotovic for providing us with references that led to further study, investigation, and discovery. We have benefited from discussions with Thomas Kurtz, whose suggestions are very much appreciated. We are very grateful to the series editor Ioannis Karatzas for his encouragement, and to the Springer-Verlag senior editor of statistics John Kimmel and the Springer-Verlag professionals for their help in finalizing the book. This research has been supported in part by the National Science Foundation and the Office of Naval Research, to whom we extend our hearty thanks.

Detroit, Michigan — G. George Yin
Athens, Georgia — Qing Zhang
March — 1998

Convention

Here we clarify the numbering system and cross-reference conventions used in the book. Within a chapter, equations are numbered consecutively, e.g., (3.10) indicates the tenth equation in Chapter 3. Corollaries, definitions, examples, lemmas, propositions, remarks, and theorems are treated as one entity and numbered sequentially throughout the chapter, e.g., Definition 4.1, Theorem 4.2, Corollary 4.3, etc. Likewise, assumptions are also marked consecutively within a chapter, e.g., (A6.1) stands for Assumption 1 of Chapter 6. For cross reference either within a chapter or to another chapter, equations are identified by the chapter number and the equation number, e.g., (5.2) refers to Equation 2 of Chapter 5. Similar methods apply to theorems, remarks, assumptions, etc.

Throughout the book, all deterministic processes are assumed to be Borel measurable and all stochastic processes are assumed to be measurable with respect to a given filtration. The notation $|\cdot|$ denotes either an Euclidean norm or a norm on the appropriate function spaces, which will be clear from the context. The ith component of a vector $z \in \mathbb{R}^r$ is denoted by z_i; the ijth entry of a matrix A is denoted by a_{ij}. In the asymptotic expansions, to simplify the notation, we use $\varphi_n(t)$ and $\psi_n(t)$ etc. to denote sequences, whereas $\varphi^i(t)$ denotes the ith partitioned vector of dimension $\mathbb{R}^{1\times m_i}$ or the ith component of $\varphi(t)$ when $m_i = 1$.

Glossary of Symbol and Notation

A'	transpose of a matrix (or a vector) A
B^c	complement of a set B
$\mathrm{Cov}(\zeta)$	covariance of a random variable ζ
$C([0,T];S)$	space of continuous functions on $[0,T]$ taking values in S
$C(\mathcal{O};S)$	space of continuous functions on $\mathcal{O}$ taking values in S
C^k	space of functions with continuous derivatives up to the kth order (space of C^k functions)
C^2_L	space of C^2 functions with bounded derivatives up to the second order and with Lipschitz second derivatives
$D([0,T];S)$	space of right continuous defined on $[0,T]$ taking values is S functions with left-hand limits
$E\xi$	expectation of a random variable ξ
$\mathcal{F}$	σ-algebra
$\{\mathcal{F}_t\}$	filtration $\{\mathcal{F}_t, t \geq 0\}$
I_A	indicator function of a set A
I_n	$n \times n$ identity matrix
K	generic positive constant with convention $K + K = K$ and $KK = K$
$L^2([0,T];S)$	space of square integrable functions on $[0,T]$ with values in S
$\mathcal{M}$	state space of the Markov chain

$N(x)$	neighborhood of x
$O(y)$	function of y such that $\sup_y \lvert O(y)\rvert/\lvert y\rvert < \infty$
$O_1(y)$	function of y such that $\sup_y \lvert O(y)\rvert/\lvert y\rvert \le 1$
$P(\xi \in \cdot)$	probability distribution of a random variable ξ
Q or Q^ε	generator of a Markov chain
$Q(t)$ or $Q^\varepsilon(t)$	generator of a Markov chain
$Qf(\cdot)(i)$	$= \sum_{j\neq i} q_{ij}(f(j) - f(i))$ where $Q = (q_{ij})$
$\mathbb{R}^r$	r-dimensional Euclidean space
a^+	$= \max\{a, 0\}$ for a real number a
a^-	$= \max\{-a, 0\}$ for a real number a
$a_1 \wedge \cdots \wedge a_l$	$= \min\{a_1, \ldots, a_l\}$ for $a_i \in \mathbb{R}$, $i = 1, \ldots, l$
$a_1 \vee \cdots \vee a_l$	$= \max\{a_1, \ldots, a_l\}$ for $a_i \in \mathbb{R}$, $i = 1, \ldots, l$
a.e.	almost everywhere
a.s.	almost surely
$\mathrm{diag}(A_1, \ldots, A_l)$	diagonal matrix of blocks $A_1, \ldots, A_l$
$\exp(Q)$	e^Q for any argument Q
$\log x$	natural logarithm of x
$o(y)$	a function of y such that $\lim_{y\to 0} o(y)/\lvert y\rvert = 0$
$p^\varepsilon(t)$	$P(\alpha^\varepsilon(t) = 1, \ldots, \alpha^\varepsilon(t) = m)$ or $P(\alpha^\varepsilon(t) = 1, \alpha^\varepsilon(t) = 2, \ldots)$
w.p.1	with probability one
$p^{\varepsilon,i}(t)$	ith partitioned vector of $p^\varepsilon(t)$ $(\in \mathbb{R}^{1\times m_i})$
p^0	$= p^\varepsilon(0)$, initial data
$(\Omega, \mathcal{F}, P)$	probability space
$\alpha(t)$ or $\alpha^\varepsilon(t)$	Markov chain with finite or countable state space
δ_{ij}	equals 1 if $i = j$ and 0 otherwise
ε	positive small parameter
ι	pure imaginary number with $\iota^2 = -1$
$\nu(t)$	quasi-stationary distribution
$\nu^i_j(t)$	the jth component of $\nu^i(t) \in \mathbb{R}^{1\times m_i}$
$\varphi_n(t)$, $\psi_n(t)$	sequences of functions
$\varphi^i_n(t)$	ith partitioned vector or ith component of $\varphi_n(t)$
$\varphi^i(t)$	ith partitioned vector or ith component of $\varphi(t)$
$\sigma\{\alpha(s) : s \le t\}$	σ-algebra generated by the process $\alpha(\cdot)$ up to t
$\vartheta_k(t)$	$= (\vartheta^1_k(t), \ldots, \vartheta^l_k) \in \mathbb{R}^{1\times l}$
$\mathbb{1}$	column vector with all components equal to one
$:=$	defined to be equal to
$\doteq$	approximately equal to
∇f	gradient of a function f
$\square$	end of a proof

$(a_1, \ldots, a_l) > 0$	$a_1 > 0, \ldots, a_l > 0$
$(a_1, \ldots, a_l) \geq 0$	$a_1 \geq 0, \ldots, a_l \geq 0$
$\lvert(a_1, \ldots, a_l)\rvert$	$= \sqrt{a_1^2 + \cdots + a_l^2}$
$\lvert y\rvert_T$	$= \max\limits_{i,j} \sup\limits_{0 \leq t \leq T} \lvert y_{ij}(t)\rvert$, where $y = (y_{ij}) \in \mathbb{R}^{r_1 \times r_2}$
$\langle a, b \rangle$	scalar product of vectors a and b
$\xi_n \Rightarrow \xi$	ξ_n converges to ξ weakly

Part I

Prologue and Preliminaries

1
Introduction and Overview

1.1 Introduction

This book presents asymptotic analysis and applications for two-time-scale Markov chains and reveals the interrelations of Markov chains and singular perturbations. It is the second edition of our book Yin and Zhang [237]. Treating a wide variety of real-world systems under uncertainties, one frequently uses Markovian models. Quite often, the formulations lead to two-time-scale or singularly perturbed Markov chains. In many applications, various factors change at different rates: Some evolve slowly, whereas others vary rapidly. As a result, the separation of fast and slow time scales arises. The phenomena are often described by introducing a small parameter $\varepsilon > 0$, which leads to a singularly perturbed system involving two-time scales, namely, the actual time t and the stretched time t/ε. To analyze such systems, one seeks to "average out" the fast variables and to consider only certain averaged characteristics via asymptotic methods.

Our study originates from a large class of problems in engineering, operations research, management, and biological and physical sciences. To proceed, we present a couple of problems in what follows to further elaborate the motivation of our investigation. The system in the first problem involves a rapidly fluctuating Markov chain, whereas the second one entails modeling of large-scale systems via decomposition and aggregation. More examples, applications, and models will be given in Chapter 3 and in Chapters 7–10.

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Consider a model for production planning of a failure-prone manufacturing system. The system consists of a single machine whose production capacity is modeled by a finite-state Markov chain $\alpha^\varepsilon(\cdot) = \{\alpha^\varepsilon(t) : t \geq 0\}$ taking values in $\{\alpha_1, \ldots, \alpha_m\}$. Let $x(t)$, $u(t)$, and $z(t)$ denote the surplus, the rate of production, and the rate of demand, respectively. The system is given by

$$\frac{dx(t)}{dt} = u(t) - z(t), \quad x(0) = x^0.$$

Assume that the Markov chain $\alpha^\varepsilon(\cdot)$ is generated by $Q(u(t))/\varepsilon$ (see Sethi and Zhang [192, Chapter 5]). Our objective is to choose the production rate $u(t)$, $t \geq 0$, that is subject to the production constraint

$$0 \leq u(t) \leq \alpha^\varepsilon(t),$$

and that minimizes the discounted cost function

$$E \int_0^\infty e^{-\rho t} G(x(t), u(t)) dt,$$

where $\rho > 0$ is a discount factor and $G(\cdot)$ is a running cost function.

It is difficult to obtain the closed-form solution of the optimal control. Nevertheless, as was shown in [192], for sufficiently small $\varepsilon > 0$, the given problem can be approximated by a limit control problem

$$\frac{d\overline{x}(t)}{dt} = \sum_{i=1}^m \nu_i(t) u_i(t) - z(t), \quad \overline{x}(0) = x^0,$$

where $\nu(t) = (\nu_1(t), \ldots, \nu_m(t)) \in \mathbb{R}^{1\times m}$ is the "average distribution" (a precise definition is given in Definition 2.8) of the Markov chain generated by $Q(u(t))$. The corresponding cost function becomes

$$\int_0^\infty e^{-\rho t} \sum_{i=1}^m \nu_i(t) G(x(t), u_i(t)) dt.$$

To solve the limit problem, we choose $(u_1(t), \ldots, u_m(t))$ over time to minimize the corresponding cost function. The optimal solution of the limit system can be used to construct a control for the given problem and to show that it is asymptotically optimal as $\varepsilon \to 0$.

The second problem is concerned with a large-scale controlled dynamic system. It is necessary to develop adequate models for such large-scale systems with complex structures that are otherwise difficult to handle, even with powerful computing devices. As an effective way, using the so-called weak and strong interaction models will result in the formulation of singularly perturbed systems. By taking appropriate decomposition and aggregation, we will be able to divide a formidable, large-dimensional system into a number of simpler subsystems with lower dimensions thereby making the underlying problem solvable.

To be more specific, for a given $T > 0$, suppose $t \in [0, T]$, and $x(t) \in \mathbb{R}^n$ represents the state of the dynamic system given by

$$\frac{dx(t)}{dt} = b(x(t), u(t), \alpha(t)), \quad x(0) = x^0,$$

where $b(\cdot)$ is an appropriate function, $\alpha(\cdot)$ is a finite-state Markov chain having a large state space $\mathcal{M} = \{1, \ldots, m\}$ with $m \gg 1$, and $u(t) \in \Gamma \in \mathbb{R}^{r_1}$ is the control or input of the system. The random process may be regarded as a driving noise. One aims to select $u(\cdot)$ within the class of "admissible controls" to minimize the expected cost

$$J(u(\cdot)) = E \int_0^T G(x(t), u(t), \alpha(t))dt.$$

The analysis is quite involved and the computation is likely to be costly due to the high dimensionality. Consequently, a direct application of the dynamic programming (or maximum principle) approach may not be feasible and alternative methods are therefore desirable.

Hierarchical structure, a feature common to most systems of practical concerns (see Simon [195] and Simon and Ando [196]), can help us to surmount these obstacles. In a large-scale system, various "components" may change at different rates. By taking advantage of these, the system may be decomposed, and the states of the Markov chain may be aggregated. The introduction of a small parameter $\varepsilon > 0$ makes the system belong to the category of two-time-scale systems; see Section 3.6 for a simple illustration. For the dynamic system given above, assume (for simplicity) that the generator of the Markov chain is time-invariant and has the form

$$Q^\varepsilon = \frac{1}{\varepsilon}\widetilde{Q} + \widehat{Q},$$

where $\widetilde{Q}$ and $\widehat{Q}$ are constant matrices and are themselves generators of suitable Markov chains, in which $\widetilde{Q}/\varepsilon$ represents the fast motion part and $\widehat{Q}$ models the slow motion part. Suppose

$$\widetilde{Q} = \text{diag}\left(\widetilde{Q}^1, \widetilde{Q}^2, \ldots, \widetilde{Q}^l\right) = \begin{pmatrix} \widetilde{Q}^1 & & & \\ & \widetilde{Q}^2 & & \\ & & \ddots & \\ & & & \widetilde{Q}^l \end{pmatrix},$$

where all $\widetilde{Q}^k \in \mathbb{R}^{m_k \times m_k}$ are irreducible generators (see Definition 2.7) for $k = 1, 2, \ldots, l$, and $\sum_{k=1}^l m_k = m$. Denote

$$\mathcal{M} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l,$$

where $\mathcal{M}_1 \ldots, \mathcal{M}_l$ is a partition of $\mathcal{M}$. Define the aggregation of the corresponding singularly perturbed chain $\alpha^\varepsilon(t)$ by

$$\overline{\alpha}^\varepsilon(t) = \sum_{i=1}^{l} i I_{\{\alpha^\varepsilon(t) \in \mathcal{M}_i\}},$$

where I_A denotes the indicator function of a set A. We may also write similar expressions of other types of generators for $\widetilde{Q}$ (see Chapter 4) such as Markov chains including absorbing states and Markov chains including transient states.

In this book, we show that $\overline{\alpha}^\varepsilon(\cdot)$ converges to a stochastic process $\overline{\alpha}(\cdot)$ in an appropriate sense, and the limit $\overline{\alpha}(\cdot)$ is a Markov chain generated by

$$\begin{aligned}\overline{Q} &= \operatorname{diag}(\nu^1, \ldots, \nu^l)\, \widehat{Q}\, \operatorname{diag}(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_l}) \\ &= \begin{pmatrix} \nu^1 & & & \\ & \nu^2 & & \\ & & \ddots & \\ & & & \nu^l \end{pmatrix} \widehat{Q} \begin{pmatrix} \mathbb{1}_{m_1} & & & \\ & \mathbb{1}_{m_2} & & \\ & & \ddots & \\ & & & \mathbb{1}_{m_l} \end{pmatrix},\end{aligned}$$

where ν^k denotes the stationary distribution corresponding to the Markov chain generated by $\widetilde{Q}^k$, $\mathbb{1}_{m_k} = (1, \ldots, 1)' \in \mathbb{R}^{m_k \times 1}$, and a' denotes the transpose of a. Note that $\overline{Q}$ is an $l \times l$ matrix. If $l \ll m$, the large-dimensional problem originally encountered is replaced by an averaged problem having a generator with substantially reduced dimension. The essence is to examine a limit problem with an "averaged" Markov chain in lieu of solving the original problem. Applying the optimal control of the limit problem to the original problem leads to the construction of asymptotic optimal controls.

Next, we consider a related problem in networked control systems. Very often one needs to consider stability problems. With the presence of both continuous dynamics and discrete events, hybrid systems are capable of describing complex systems and their inherent uncertainty and randomness in the environment. The hybrid formulation provides more opportunity for realistic models, but adds more difficulties in analyzing the underlying systems. One class of such systems is hybrid systems with Markovian switching. Such systems have been found in emerging applications of financial engineering, wireless communications, manufacturing systems, and other related fields; see for example, Barone-Adesi and Whaley [6], Mariton [155], Yin, Krishnamurthy, and Ion [225], Yin and Zhou [243], Zhang and Yin [255], Zhang, Yin, and Liu [257], Zhou and Yin [259], and many references therein. Much of the contemporary study of stochastic stability of dynamic systems can be traced back to the original work of Kac and Krasovskii [101], in which a systematic approach was developed for stability

of systems with Markovian switching using Liapunov function methods. For nonlinear differential equations, the well-known Hartman-Grobman theorem (see Perko [173, Section 2.8]) provides an important result concerning the local qualitative behavior. It says that near x_0, a hyperbolic equilibrium point, the nonlinear system $\dot{x} = f(x)$ has the same qualitative structure as that of the linear system $\dot{x} = \nabla f(x_0)x$. While the topological equivalence may not hold for a non-hyperbolic equilibrium point (e.g., a center). Treating hybrid systems, consider the differential equations $\dot{x} = f(x, \alpha(t))$ and $\dot{x} = \nabla f(x_0, \alpha(t))x$ for $\alpha(t)$ being in a finite set. We showed in a recent work of Zhu, Yin, and Song [260], although some of the linear equations have centers, as long as the spectrum of the coefficients of the differential equation corresponding to the stable node dominates that of the centers, we may still use linearization in the analysis.

In fact, it is known that the switching systems are notoriously more difficult to treat. Sometimes rather unexpected events happen. Treating purely deterministic systems, Wang, Khargonecker, and Beydoun [212] dealt with an interesting system. Consider a linear in x system given by

$$\dot{x} = [A(\alpha^\varepsilon(t)) - B(\alpha^\varepsilon(t))K(\alpha^\varepsilon(t))]x, \tag{1.1}$$

where $\alpha^\varepsilon(t)$ is a purely deterministic discrete event process that is running in the continuous time and that takes values in $\{1.2\}$. Using a linear in x feedback control $u(t) = K(i)x$ for $i = 1, 2$ and let

$$G(1) = A(1) - B(1)K(1) = \begin{pmatrix} -100 & 20 \\ 200 & -100 \end{pmatrix},$$

$$G(2) = A(2) - B(2)K(2) = \begin{pmatrix} -100 & 200 \\ 20 & -100 \end{pmatrix}.$$

Then the above two closed-loop systems are stable individually. It is demonstrated in [212], when the switching takes place at $k\varepsilon$ for $\varepsilon = 0.01$ and $k = 1, 2, 3, \ldots$, then the resulting switched system is unstable. Intuitively, one might expect that when one puts two stable systems together, the combined system should also be stable. Nevertheless, the aforementioned reference provides a counterintuitive example to the common belief. Naturally, we would like to ask: What is the reason behind this example? Turning the question into stochastic setup, we assume that the switching process $\alpha^\varepsilon(\cdot)$ is a continuous-time Markov chain with state space $\{1, 2\}$. Using the analytic techniques provided in this book, we can show that associated with the original switching system, there is a limit system. Even if the two individual systems are stable, the limit one is not. Using large deviations type estimates, we can conclude that in any finite time interval, with large probability, the combined system will reside in the unstable mode. Furthermore,

using a perturbed Liapunov function method, the reason that the switched system being unstable can be revealed. Such a question cannot be precisely addressed without using properties obtained in the current book. Further discussions on this example will be provided in Section 5.6.

To handle singularly perturbed systems driven by Markov chains, it is essential to have a thorough understanding of the underlying probabilistic structure, to which a large portion of the book is devoted. Our main interests are the asymptotic properties of singularly perturbed Markov chains with nonstationarity. In the subsequent chapters, the asymptotic properties of such systems will be closely examined through their probability distributions.

Consider the following illustrative model. Let $Q = (q_{ij}) \in \mathbb{R}^{m\times m}$ be a matrix with

$$Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1m} \\ q_{21} & q_{22} & \cdots & q_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ q_{m1} & q_{m2} & \cdots & q_{mm} \end{pmatrix}, \tag{1.2}$$

satisfying

$$q_{ij} \geq 0, \text{ for } i \neq j, \text{ and } q_{ii} = -\sum_{j\neq i} q_{ij}.$$

Let $\alpha^\varepsilon(t) \in \mathcal{M}$, for $t \geq 0$, denote a finite-state Markov chain generated by $Q^\varepsilon := Q/\varepsilon$, whose state space is $\mathcal{M} = \{1, \ldots, m\}$. Then, the probability distribution of $\alpha^\varepsilon(t)$ denoted by $p^\varepsilon(t) = (P(\alpha^\varepsilon(t) = 1), \ldots, P(\alpha^\varepsilon(t) = m))$, which lies in $\mathbb{R}^{1\times m}$, satisfies the differential equation

$$\begin{aligned} &\frac{dp^\varepsilon(t)}{dt} = \frac{1}{\varepsilon} p^\varepsilon(t)Q, \ 0 \leq t \leq T, \\ &p^\varepsilon(0) = p^0 = (p_1^0, \ldots, p_m^0) \ \text{ such that} \\ &p_i^0 \geq 0 \ \text{ for each } i, \text{ and } \sum_{i=1}^m p_i^0 = 1, \end{aligned} \tag{1.3}$$

where $\varepsilon > 0$ is a small parameter and $T < \infty$. The unique solution of (1.3) can be written explicitly as

$$p^\varepsilon(t) = p^0 \exp\left(\frac{Qt}{\varepsilon}\right).$$

Assume that the Markov chain is irreducible, that is, the system of equations

$$\nu Q = 0 \text{ and } \sum_{i=1}^m \nu_i = 1$$

has a unique positive solution $\nu = (\nu_1, \ldots, \nu_m)$, which is the stationary distribution of the Markov chain generated by Q. In addition,

$$p^0 \exp(Qt) \to \nu \text{ as } t \to \infty.$$

Therefore, for each $t > 0$, $p^\varepsilon(t)$ converges to ν, as $\varepsilon \to 0$ because $t/\varepsilon \to \infty$. Moreover, it is not difficult to prove that the convergence rate is exponential, i.e.,

$$p^\varepsilon(t) - \nu = O\left(\exp\left(-\frac{\kappa_0 t}{\varepsilon}\right)\right) \text{ for some } \kappa_0 > 0.$$

Much of our effort in this book concerns obtaining asymptotic properties of $p^\varepsilon(\cdot)$, when the generator Q is a function of time. With time-dependent generator $Q(t)$, $t \geq 0$, we will address the following issues: (1) When $\varepsilon \to 0$, does the limit of $p^\varepsilon(t)$ exist? (2) If $p^\varepsilon(t)$ converges, how can one determine the limit? (3) What is the convergence rate? (4) Suppose $p^\varepsilon(t) \to \nu(t) = (\nu_1(t), \ldots, \nu_m(t))$, a probability distribution as $\varepsilon \to 0$. Define

$$\chi^\varepsilon(t) = (I_{\{\alpha^\varepsilon(t)=1\}}, \ldots, I_{\{\alpha^\varepsilon(t)=m\}}).$$

Consider the centered and scaled occupation measure

$$n^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t (\chi^\varepsilon(s) - \nu(s)) ds.$$

As $\varepsilon \to 0$, what is the limit distribution of the random process $n^\varepsilon(\cdot)$? (5) Will the results carry over to singularly perturbed Markov chains with weak and strong interactions (when the states of the Markov chain belong to multiple irreducible classes)? (6) Is there anything that can be said about merely measurable generators? The subsequent chapters provide detailed answers to these questions and related topics.

This book concentrates on continuous-time singularly perturbed Markov chains. The phrase singular perturbation used herein is defined in a broad sense that includes both deterministic and probabilistic methods. One of the principal component of the work is to develop various approximation methods of Markov chains. In many applications, an approximate solution provides results that are nearly as good as the analytical one. An approximation is often more desirable since an exact solution is not obtainable or it requires too much effort to obtain, especially if on-line computations are involved.

To summarize, various real-world systems under uncertainties can be modeled as singularly perturbed Markov chains. The singular perturbation approach is applicable to problems involving processes with rapid fluctuations; it can also serve as a convenient machinery for handling large-dimensional systems not manageable otherwise. The formulation of such systems are achieved by introducing a small parameter $\varepsilon > 0$, which indicates the relative order of magnitude; only the relative order is important in

applications. For instance, $\varepsilon = 0.1$ might be considered as a small quantity from a practical point of view. The mathematical results to be presented can serve as a guide for various approximations and for estimating error bounds; the asymptotic results of the underlying systems (as $\varepsilon \to 0$) provide insights into the structure of the system and heuristics in applications. A thorough understanding of the intrinsic behavior of the systems will be instructive and beneficial for in-depth studies of applications in hierarchical production planning, Markov decision processes, random evolution, and control and optimization of stochastic dynamic systems involving singularly perturbed Markov chains.

1.2 A Brief Survey

The review of the literature is composed of two parts, namely, Markov chains and singular perturbations; additional review and references are given at the end of each chapter. The references provided are mostly related to the materials treated in this book and are by no means exhaustive.

1.2.1 Markov Chains

The theory of Markov chains belongs to that of Markov processes, which is named after A. A. Markov who introduced the concept in 1907 for discrete-time processes with finite-state spaces. Perhaps, his original intention was to generalize classical theory for sums of independent random variables to that of dependent random variables. Rapid and continued progress has been made for several decades. The development of the theory began with the systematic treatment of A. N. Kolmogorov in the early 1930s, and was followed by Doeblin's important contribution. Fundamental work on continuous-time chains was done by J. L. Doob in the 1940s and P. Lévy in the 1950s. To consolidate and to continue the pioneering work, D. G. Kendall, G. E. H. Reuter, and K. L. Chung among others launched comprehensive studies. Extensive surveys of the development of Markov chains can be found in, for example, Anderson [3], Chung [31], Davis [41], Doob [49], Dynkin and Yushkevich [53], Feller [60], Hou and Guo [89], Kannan [103], Revuz [180], Rosenblatt [183], and Wang and Yang [213]. Classical work on Markov processes with continuous-state space is contained in Dynkin [51]. An exposition of elementary theory of Markov chains is in Karlin and Taylor [105]. A detailed study of discrete-time Markov chains and their stability is in Meyn and Tweedie [159]; a modern treatment of the theory of Markov processes is in Ethier and Kurtz [59]. Effort has also been made to treat evolution of systems in random media in Korolyuk and Swishchuk [130], and semi-Markov processes in Korolyuk and Limnios [131, 132] and references therein. The nonstationary cases were first treated by Markov

himself; subsequently, a number of people made important contributions, to name just a few, S. N. Bernstein, W. Doeblin, Yu. V. Linnik, and R. L. Dobrushin among others; see Iosifescu [95] for discussion on this aspect and the corresponding historical notes. In depth studies of such topics as probability metrics, coupling methods, and spectral gaps are covered in the book by Chen [25]. The recent work of Davis [41] on piecewise-deterministic processes sets up a framework for the treatment of nonstationary Markov processes. For the subsequent study herein, the main results of Markov chains are taken from Chung [31] and Davis [41] among others.

1.2.2 Singular Perturbations

The topics of this book are at the verge of singular perturbation theory, which also has a long history. At the beginning of this century, L. Prandtl published his seminal paper "On fluid motion with small friction," which established the foundation of the boundary layer theory. The origin of the nowadays well-known WKB method initiated independently by three physicists, G. Wentzell, H. A. Kramers, and L. Brillouin in 1926, can be traced back to the work of Liouville and Green in 1837; see Wasow [216] for a historical remark.

Owing to its wide spectrum of applications, singular perturbation theory has witnessed tremendous progress for decades. The Kiev school headed by N. M. Krylov and N. N. Bogoliubov developed the so- called averaging methods to treat oscillations. Their work was further continued by K. Friedrichs, N. Levinson, and Y. A. Mitropolskii among others. For detailed survey and historical development, consult the work of Bogoliubov and Mitropolskii [18], Eckhaus [54], Erdélyi [58], Il'in [92], Kevorkian and Cole [108, 109], Krylov and Bogoliubov [133], Lochak and Meunier [149], Nayfeh [161], O'Malley [163], Smith [199], Vasil'eava and Butuzov [210], Wasow [215, 216], and the references therein.

Singular perturbation methods have been extensively used in various branches of physics including statistical mechanics, solid state physics, chemical physics, molecular biophysics (see Gardiner [68], Risken [182], van Kampen [208], Schuss [188], Hänggi, Talkner, and Borkovec [80] and the over hundreds of references cited there). For the related applications in control theory and optimization, we refer the reader to Bensoussan [8], Kokotovic [126], Kokotovic, Bensoussan, and Blankenship [127], Kokotovic and Khalil [128], Kokotovic, Khalil, and O'Reilly [129], Pervozvanskii and Gaitsgori [174], Phillips and Kokotovic [175], and the large reference citations contained therein. The idea of two-time-scale expansion has also found emerging applications in communication theory (see Tse, Gallager, and Tsitsiklis [206] among others).

Parallel to the advances in the deterministic theory, there is a stochastic version of the averaging methods. It began with the work of Khasminskii [112], continued by the large deviations approach of Friedlin and Wentzell [67], and the martingale averaging methods of Kushner [139].

In summary, the theories of Markov chains and singular perturbation have flourished. They have now become important techniques in various branches of applied mathematics and have a diverse range of applications.

1.3 Outline of the Book

This book consists of three parts including ten chapters and an appendix. Part I and Part II each consists three chapters, whereas Part III comprises four chapters.

Including an introduction, mathematical preliminaries, and a number of Markovian models, Part I provides background material. It begins with Chapter 1 that contains an overview, a brief review of the literature, and the plan of the book.

Chapter 2 is concerned with mathematical preliminaries and technical aspects such as Chapman–Kolmogorov equations, forward and backward differential equations, irreducibilities, quasi-stationary distributions, and piecewise-deterministic processes. The definition of Markov chains is given through the formulation of martingales, which appears to be natural for nonstationary processes. This chapter also collects certain properties of martingales and Gaussian processes. It serves as a quick reference of results to be used later in the book. A list of related textbooks is given at the end of the chapter for further consultation.

To demonstrate the versatility of Markov chains, we present a number of models and examples in Chapter 3. They include birth and death processes, queueing systems with finite capacity, competing risk theory, singularly perturbed Cox processes, seasonal variation, simulated annealing and stochastic optimization algorithms, system reliability, and optimal control of jump linear systems. These models are used to exhibit the scope of the diversity of applications rather than to provide complete solutions of the underlying problems.

Consisting of Chapters 4–6, Part II is devoted to asymptotic properties of singularly perturbed Markov chains. To allow a better understanding, this part begins with a thorough treatment of singularly perturbed chains under weak irreducibility. Then more complex cases are treated following the logical path of development from simpler to more complex problems.

In Chapter 4, we begin the study with the case that the Markov chain is generated by an irreducible generator $Q^\varepsilon(t) = Q(t)/\varepsilon$, where $\varepsilon > 0$ is a small parameter. The smaller the ε is, the more rapidly the underlying Markov chain fluctuates. To analyze the asymptotic properties of the Markov chains for small ε, we make use of the forward differential equations and an asymptotic expansion of the probability distribution. The asymptotic expansion is composed of a regular part (outer expansion) and initial layer corrections. The regular part and the initial layer corrections are matched through appropriate choices of the initial conditions. We then

give rigorous justification on the validity of the asymptotic series and obtain the upper bounds on the approximation errors. Then much more complex models of Markov chains generated by $Q^\varepsilon(t) = \widetilde{Q}(t)/\varepsilon + \widehat{Q}(t)$ are considered. Since the fast varying part of the generator $\widetilde{Q}(t)/\varepsilon$ is the dominant factor, we treat recurrent chains, chains having absorbing states, and chains containing transient states in accordance with the classification of states for the underlying chains. We first detail the asymptotic development of the model with recurrent states. As will be seen in this chapter, the analysis becomes much more involved due to the complexity of the model. Similar techniques are then applied to treat chains with absorbing states and chains with transient states. The choice of the initial conditions is a delicate issue in dealing with models with fast and slow motions and/or weak and strong interactions. It is interesting to note that the proper choices of initial conditions are so critical that without them an *ad hoc* formal expansion will not yield the desired asymptotic estimates.

Also considered here is another generalization involves the study of countable state space cases. We first consider the case where $\widetilde{Q}(t)$ and $\widehat{Q}(t)$ are themselves generators of appropriate Markov chains, in which $\widetilde{Q}(t)$ is a block-diagonal matrix with infinitely many blocks each of which is a generator of a Markov chain with a finite-state space. Then we study the case $Q^\varepsilon(t) = Q(t)/\varepsilon$ with $Q(t)$ being an infinite-dimensional generator, and provide sufficient conditions ensuring the validity of the asymptotic expansion. Finally, remarks on the corresponding results of singularly perturbed diffusion processes are discussed briefly.

As another main core of the book, Chapter 5 concerns the asymptotic distribution and exponential error estimates of unscaled and scaled occupation measures. Chapter 4 focuses on purely analytic properties and uses mainly analysis tool to treat the underlying problems, whereas this chapter takes up the related probabilistic issues. Starting with the case that the Markov chain is weakly irreducible, by use of the asymptotic expansions developed in Chapter 4, it is shown that a sequence of unscaled occupation measures converges to that of an "integrated" quasi-stationary distribution, and a scaled sequence of the occupation measures verifies a mixing condition and is tight in an appropriate function space. We establish the weak convergence of the sequence and derive the explicit representation of the covariance of the Gaussian process. One of the distinct features of the central limit result is that its asymptotic covariance involves explicit expression of the initial layers, a property not shared by many of the existing results of asymptotic normality. In various applications, it is often needed to make use of an exponential error estimate to obtain the corresponding large deviations results. Deriving such upper bounds is another main task of this chapter.

Next, we ask the question: If the irreducible Markovian models are replaced by Markovian models with weak and strong interactions, can we still

get desired asymptotic distribution results? A natural question is whether the asymptotic normality holds for the scaled occupation measures. The situation becomes more complex. To begin, for the weakly irreducible case, the scaled occupation measures are centered about a non-random quantity, namely, the quasi-stationary distribution. When weak and strong iteration models are treated, the centering term can no longer be a constant, but a random process that stems from the idea of aggregations and that is in a form of conditional mean. In fact, the limit of the scaled sequence of occupation measures does not possess independent increments. To characterize the limit process, one has to examine a combined process of the Markov chain and the scaled occupation measure. Owing to the interactions among different blocks, although there is an analog to the central limit result, the limit distribution is no longer Gaussian but Gaussian mixture (or a switching diffusion characterized by solutions of martingale problems with appropriate operators). Thus, strictly speaking, we no longer have the asymptotic normality. Nevertheless, a limit in distribution result still holds. The limit process displays a certain mixture property; it resembles both diffusion processes and jump processes. Note that for small ε, the Markov chain $\alpha^\varepsilon(\cdot)$ jumps more frequently within each block and less frequently from one block to another. To further the understanding of the underlying process, we study the structural properties of the Markov chain by aggregating the states in the kth block by a single state k and approximating $\alpha^\varepsilon(\cdot)$ by the aggregated process. In addition to analyzing the aggregated chain, we also take up the issue of generators being merely measurable and obtain a number of results concerning the probability distribution under the weak topology in $L^2[0,T]$, the space of square integrable functions.

With emphases on the "deterministic" and the "probabilistic" aspects of the distributions of the Markov chains, both Chapters 4 and 5 are concerned with forward equations. Chapter 6 serves as the "adjoint" of Chapter 4; it treats the backward equations, also known as Kolmogorov backward equations. There are plenty cases that we need to deal with backward equations. We ask the following questions. Do there exist asymptotic expansions for the backward equations? We provide an affirmative answer to this question in Chapter 6. In contrast to the Kolmogorov forward equations, instead of having initial conditions, we now have terminal conditions. Similar to what have been done in Chapter 4, we construct outer expansions and terminal layer corrections, and obtain the desired error bounds. One of the crucial results is Lemma 6.1.

Part III deals with several applications including Markov decision processes, nearly optimal controls of stochastic dynamic systems, numerical solutions of control and optimization of Markov chains, and hybrid two-time-scale LQG problems. The materials are divided into Chapters 7–10.

The studies of many operations research and operations management problems can be boiled down to analysis of Markov decision processes. One of the main advantages of Markov decision processes is that the dynamics

of the model are governed purely by the generator $Q^{\varepsilon}(u)$, as a function of control u, without involving differential equations. Chapter 7 focuses on optimal decisions of these problems involving weak and strong interactions. Specifically, the generator of the underlying Markov chain is assumed to be a function of the control variable, i.e., the generator is given by $Q^{\varepsilon}(t) = \widetilde{Q}(u(t))/\varepsilon + \widehat{Q}(u(t))$ such that $\widetilde{Q}(u(t))$ is a block-diagonal matrix with each block being irreducible. This yields a limit control problem obtained by replacing the states in each group with a single state and by using the corresponding average distributions. A nearly optimal solution for the original problem is constructed by using an optimal solution to the limit problem. Both discounted cost and long-run average cost criteria are considered. Error bounds of the constructed controls are obtained; related computational methods are also discussed.

Stemming from hierarchical decomposition of large and complex systems, Chapter 8 is about asymptotic optimal controls of singularly perturbed dynamic systems under Markovian disturbance. Assuming that the Markov chain is under weak and strong interactions, we obtain the asymptotic optimal control and derive the corresponding error bounds. As specific examples, hierarchical control of manufacturing systems are examined. Moreover, obtaining near optimality via weak convergence methods is also demonstrated. The main idea is that in lieu of dealing with the more difficult singularly perturbed problems, one considers the limit problems and uses them as a bridge to establish nearly optimal controls of the actual systems. In this process, the asymptotic properties of the singularly perturbed Markov chains play an essential role.

To implement a control policy in practice, numerical methods are often necessary and indispensable. In Chapter 9, we develop numerical algorithms for approximating control and optimization problems of finite-state Markov chains. It encompasses two parts. The first part concentrates on approximation of the controlled dynamic systems by an appropriate controlled Markov chain on a finite-state space using discretization; the suggested algorithm leads to the desired optimal control. The second part converts a class of control problems to an optimization procedure; in lieu of approximating the optimal controls, we focus our attention on the threshold control policies, and obtain the optimal threshold values by stochastic approximation methods.

A basic model used extensively in the control and systems literature is the LQP (linear quadratic Gaussian) model, which has enjoyed a wide variety of applications range from traditional setup to manufacturing systems, telecommunication, financial engineering, and networked systems. Due to the uncertain world, to reflect the random environment reality and/or to aim at more "robust" control designs, one often has to allow the system parameters to change within a set, which leads to the regime-switching Markov models. Chapter 10 takes up the issues of Markovian switching

diffusion models. It illustrates how the two-time-scale methods of the book can be used to design nearly optimal strategies.

In the entire book, each chapter begins with an introduction that gives the outline of the chapter, and ends with a section "notes" that provides further remarks, literature citations, and other related matters. The appendix contains brief discussions, basic notion, and results on the topics of viscosity solutions, piecewise-deterministic processes, weak convergence, relaxed controls, and a number of technical complements. The flow chart in Figure 1.1 should help to visualize the logical dependence, relationship, and connection among various chapters.

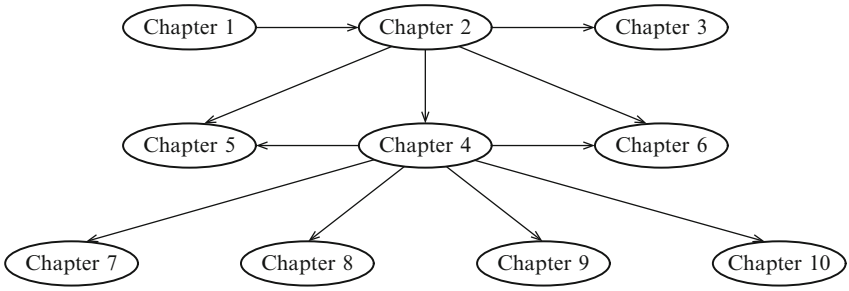

FIGURE 1.1. Relationship and Dependence Among Chapters

2

Mathematical Preliminaries

2.1 Introduction

To prepare us for the subsequent study, this chapter summarizes certain background materials used in the rest of the book. Section 2.2 begins with the definitions of stochastic processes and filtrations, which lead to a very important concept in stochastic processes, namely, the notion of martingales. In Section 2.3, we recall the definition of Markov chains. Rather than working exclusively with their transition probabilities, this book concentrates on their generators. In view of various applications, it is practical and natural to characterize a Markov chain by using its generator. Given a generator, the construction of the associated Markov chain is described in Section 2.4 by means of the piecewise-deterministic process approach. Since one of the central themes of the book encompasses quasi-stationary distributions of singularly perturbed chains, we introduce this notion together with the weak and strong irreducibilities in Section 2.5, which are used extensively in the chapters to follow. Section 2.6 reviews Gaussian and diffusion processes. Section 2.7 discusses switching diffusion processes. Finally, Section 2.8 closes the chapter with some postscript notes.

2.2 Martingales

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Denote the space of $\mathbb{R}^r$-valued continuous functions defined on $[0, T]$ by $C([0, T]; \mathbb{R}^r)$, and the space of functions

G.G. Yin and Q. Zhang, *Continuous-Time Markov Chains and Applications*, Stochastic Modelling 37, DOI 10.1007/978-1-4614-4346-9_2,

that are right continuous with left-hand limits by $D([0,T];\mathbb{R}^r)$. Consider $x(\cdot) = \{x(t) \in \mathbb{R}^r : \ t \geq 0\}$. If for each $t \geq 0$, $x(t)$ is a random vector (or an $\mathbb{R}^r$-valued random variable), we call $x(\cdot)$ a *stochastic process* and write it as $x(t)$, $t \geq 0$, or simply $x(t)$ if there is no confusion.

- A collection of σ-algebras $\{\mathcal{F}_t, \ t \geq 0\}$, or simply $\{\mathcal{F}_t\}$, is called a *filtration* if $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$. It is understood that $\mathcal{F}_t$ is complete in the sense that it contains all null sets. A probability space $(\Omega, \mathcal{F}, P)$ together with a filtration $\{\mathcal{F}_t\}$ is termed a *filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$.

- A process $x(\cdot)$ is *adapted* to a filtration $\{\mathcal{F}_t\}$, if for each $t \geq 0$, $x(t)$ is an $\mathcal{F}_t$-measurable random variable; $x(\cdot)$ is *progressively measurable* if for each $t \geq 0$, the process restricted to $[0,t]$ is measurable with respect to the σ-algebra $\mathcal{B}[0,t] \times \mathcal{F}_t$ in $[0,t] \times \Omega$, where $\mathcal{B}[0,t]$ denotes the Borel sets of $[0,t]$. A progressively measurable process is measurable and adapted, whereas the converse is not generally true. However, any measurable and adapted process with right-continuous sample paths is progressively measurable (see Davis [41, p. 19]).

- A *stopping time* τ on $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}$ is a nonnegative random variable such that $\{\tau \leq t\} \in \mathcal{F}_t$, for all $t \geq 0$.

- A stochastic process $\{x(t) : \ t \geq 0\}$ (real or vector valued) is said to be a *martingale* on $(\Omega, \mathcal{F}, P)$ with respect to $\{\mathcal{F}_t\}$ if

  (a) For each $t \geq 0$, $x(t)$ is $\mathcal{F}_t$-measurable,

  (b) $E|x(t)| < \infty$, and

  (c) $E[x(t)|\mathcal{F}_s] = x(s)$ w.p.1 for all $t \geq s$.

  If we only say that $x(\cdot)$ is a martingale (without specifying the filtration $\mathcal{F}_t$), $\mathcal{F}_t$ is taken to be $\sigma\{x(s) : \ s \leq t\}$.

- If there exists a sequence of stopping times $\{\tau_n\}$ such that $0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq \tau_{n+1} \leq \cdots$, $\tau_n \to \infty$ w.p.1 as $n \to \infty$, and the process $x^{(n)}(t) := x(t \wedge \tau_n)$ is a martingale, then $x(\cdot)$ is a *local martingale*.

2.3 Markov Chains

A *jump process* is a right-continuous stochastic process with piecewise-constant sample paths. Let $\alpha(\cdot) = \{\alpha(t) : t \geq 0\}$ denote a jump process defined on $(\Omega, \mathcal{F}, P)$ taking values in either $\mathcal{M} = \{1, 2, \ldots, m\}$ or $\mathcal{M} = \{1, 2, \ldots\}$. Then $\{\alpha(t) : t \geq 0\}$ is a *Markov chain* with state space $\mathcal{M}$ if

$$P(\alpha(t) = i | \alpha(r) : r \leq s) = P(\alpha(t) = i | \alpha(s)), \tag{2.1}$$

for all $0 \leq s \leq t$ and $i \in \mathcal{M}$. Note that (2.1) is known as Markov property and that the state space is either finite or countable.

For any $i, j \in \mathcal{M}$ and $t \geq s \geq 0$, let $p_{ij}(t,s)$ denote the transition probability $P(\alpha(t) = j | \alpha(s) = i)$, and $P(t,s)$ the matrix $(p_{ij}(t,s))$. We name $P(t,s)$ the transition matrix of the Markov chain $\alpha(\cdot)$, and postulate that

$$\lim_{t \to s^+} p_{ij}(t,s) = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. It follows that, for $0 \leq s \leq \varsigma \leq t$,

$$\begin{cases} p_{ij}(t,s) \geq 0, \ i,j \in \mathcal{M}, \\ \sum_{j \in \mathcal{M}} p_{ij}(t,s) = 1, \ i \in \mathcal{M}, \\ p_{ij}(t,s) = \sum_{k \in \mathcal{M}} p_{ik}(\varsigma,s) p_{kj}(t,\varsigma), \ i,j \in \mathcal{M}. \end{cases}$$

The last identity is usually referred to as the Chapman–Kolmogorov equation. If the transition probability $P(\alpha(t) = j | \alpha(s) = i)$ depends only on $(t-s)$, then $\alpha(\cdot)$ is *stationary*. In this case, we define $p_{ij}(h) := p_{ij}(s+h, s)$ for any $h \geq 0$. The process is nonstationary otherwise.

Definition 2.1 (q-Property). Denote $Q(t) = (q_{ij}(t))$, for $t \geq 0$. It satisfies the q-Property, if

(a) $q_{ij}(t)$ is Borel measurable for all $i, j \in \mathcal{M}$ and $t \geq 0$;

(b) $q_{ij}(t)$ is uniformly bounded, that is, there exists a constant K such that $|q_{ij}(t)| \leq K$, for all $i, j \in \mathcal{M}$ and $t \geq 0$;

(c) $q_{ij}(t) \geq 0$ for $j \neq i$ and $q_{ii}(t) = -\sum_{j \neq i} q_{ij}(t)$, $t \geq 0$.

For any real-valued function f on $\mathcal{M}$ and $i \in \mathcal{M}$, write

$$Q(t) f(\cdot)(i) = \sum_{j \in \mathcal{M}} q_{ij}(t) f(j) = \sum_{j \neq i} q_{ij}(t)(f(j) - f(i)).$$

We are now in a position to define the generator of a Markov chain.

Definition 2.2 (Generator). A matrix $Q(t)$, $t \geq 0$, is an *infinitesimal generator* (or simply a *generator*) of $\alpha(\cdot)$ if it satisfies the q-Property, and for all bounded real-valued functions f defined on $\mathcal{M}$

$$f(\alpha(t)) - \int_0^t Q(\varsigma) f(\cdot)(\alpha(\varsigma)) d\varsigma \tag{2.2}$$

is a martingale.

Remark 2.3. Motivated by the many applications we are interested in, a generator is defined for a matrix satisfying the q-Property. Different definitions, including other classes of matrices, may be devised as in Chung [31]. To proceed, we give an equivalent condition for a finite-state Markov chain generated by $Q(t)$.

Lemma 2.4. *Let $\mathcal{M} = \{1, \ldots, m\}$. Then $\alpha(t) \in \mathcal{M}$, $t \geq 0$, is a Markov chain generated by $Q(t)$ iff*

$$\left(I_{\{\alpha(t)=1\}}, \ldots, I_{\{\alpha(t)=m\}}\right) - \int_0^t \left(I_{\{\alpha(\varsigma)=1\}}, \ldots, I_{\{\alpha(\varsigma)=m\}}\right) Q(\varsigma) d\varsigma \quad (2.3)$$

is a martingale.

Proof: If $Q(t)$ is a generator of $\alpha(\cdot)$, for any $f(\cdot)$, (2.2) defines a martingale. For any $k \in \mathcal{M}$, choose $f_k(\alpha) = I_{\{\alpha=k\}}$. Then

$$f_k(\alpha(t)) = I_{\{\alpha(t)=k\}}$$

and

$$\begin{aligned} Q(\varsigma) f_k(\cdot)(\alpha(\varsigma)) &= \sum_{i=1}^{m} I_{\{\alpha(\varsigma)=i\}} [Q(\varsigma) f_k(\cdot)(i)] \\ &= \sum_{i=1}^{m} \sum_{j=1}^{m} I_{\{\alpha(\varsigma)=i\}} q_{ij}(\varsigma) f_k(j) \\ &= \sum_{i=1}^{m} I_{\{\alpha(\varsigma)=i\}} q_{ik}(\varsigma). \end{aligned}$$

Thus, (2.3) defines a martingale.

Conversely, note that

$$\begin{aligned} f(\alpha(\varsigma)) &= \sum_{i=1}^{m} I_{\{\alpha(\varsigma)=i\}} f(i) \\ &= \left(I_{\{\alpha(\varsigma)=1\}}, \ldots, I_{\{\alpha(\varsigma)=m\}}\right) (f(1), \ldots, f(m))' \end{aligned}$$

and

$$\begin{aligned} Q(\varsigma) f(\cdot)(\alpha(\varsigma)) &= \sum_{i=1}^{m} I_{\{\alpha(\varsigma)=i\}} [Q(\varsigma) f(\cdot)(i)] \\ &= \left(I_{\{\alpha(\varsigma)=1\}}, \ldots, I_{\{\alpha(\varsigma)=m\}}\right) Q(\varsigma) \left(f(1), \ldots, f(m)\right)'. \end{aligned}$$

Multiplying (2.3) by $(f(1), \ldots, f(m))'$ and using the equations above confirms that (2.2) defines a martingale. □

We will show in the next section that for any given $Q(t)$ satisfying the q-Property, there exists a Markov chain $\alpha(\cdot)$ generated by $Q(t)$. For convenience, call any matrix $Q(t)$ that possesses q-Property a generator.

2.4 Piecewise-Deterministic Processes

This section gives an account of the construction of nonstationary Markov chains generated by $Q(t)$ for $t \geq 0$. If $Q(t) = Q$, a constant matrix, the idea of Ethier and Kurtz [59] can be utilized for the construction. For time-varying generator $Q(t)$, we need to use the piecewise-deterministic process approach, described in Davis [41], to define the Markov chain $\alpha(\cdot)$.

2.4.1 Construction of Markov Chains

Let $0 = \tau_0 < \tau_1 < \cdots < \tau_l < \cdots$ denote a sequence of jump times of $\alpha(\cdot)$ such that the random variables $\tau_1, \tau_2 - \tau_1, \ldots, \tau_{k+1} - \tau_k, \ldots$ are independent. Let $\alpha(0) = i \in \mathcal{M}$. Then $\alpha(t) = i$ on the interval $[\tau_0, \tau_1)$. The first jump time τ_1 has the probability distribution

$$P(\tau_1 \in B) = \int_B \exp\left\{\int_0^t q_{ii}(s)ds\right\}(-q_{ii}(t))\,dt,$$

where $B \subset [0, \infty)$ is a Borel set. The post-jump location of $\alpha(t) = j$, $j \neq i$, is given by

$$P(\alpha(\tau_1) = j|\tau_1) = \frac{q_{ij}(\tau_1)}{-q_{ii}(\tau_1)}.$$

Note that $q_{ii}(\tau_1)$ may equal 0. In this case, define $P(\alpha(\tau_1) = j|\tau_1) = 0$, $j \neq i$. We claim $P(q_{ii}(\tau_1) = 0) = 0$. In fact, if we let $B_i = \{t : q_{ii}(t) = 0\}$, then

$$\begin{aligned} P(q_{ii}(\tau_1) = 0) &= P(\tau_1 \in B_i) \\ &= \int_{B_i} \exp\left\{\int_0^t q_{ii}(s)ds\right\}(-q_{ii}(t))\,dt = 0. \end{aligned}$$

In general, $\alpha(t) = \alpha(\tau_l)$ on the interval $[\tau_l, \tau_{l+1})$. The jump time τ_{l+1} has the conditional probability distribution

$$\begin{aligned} &P(\tau_{l+1} - \tau_l \in B_l|\tau_1, \ldots, \tau_l, \alpha(\tau_1), \ldots, \alpha(\tau_l)) \\ &= \int_{B_l} \exp\left\{\int_{\tau_l}^{t+\tau_l} q_{\alpha(\tau_l)\alpha(\tau_l)}(s)ds\right\}\left(-q_{\alpha(\tau_l)\alpha(\tau_l)}(t+\tau_l)\right)dt. \end{aligned}$$

The post-jump location of $\alpha(t) = j$, $j \neq \alpha(\tau_l)$ is given by

$$P(\alpha(\tau_{l+1}) = j | \tau_1, \ldots, \tau_l, \tau_{l+1}, \alpha(\tau_1), \ldots, \alpha(\tau_l)) = \frac{q_{\alpha(\tau_l)j}(\tau_{l+1})}{-q_{\alpha(\tau_l)\alpha(\tau_l)}(\tau_{l+1})}.$$

Theorem 2.5. *Suppose that the matrix $Q(t)$ satisfies the q-Property for $t \geq 0$. Then*

(a) *The process $\alpha(\cdot)$ constructed above is a Markov chain.*

(b) *The process*

$$f(\alpha(t)) - \int_0^t Q(\varsigma) f(\cdot)(\alpha(\varsigma)) d\varsigma \tag{2.4}$$

is a martingale for any uniformly bounded function $f(\cdot)$ on $\mathcal{M}$. Thus $Q(t)$ is indeed the generator of $\alpha(\cdot)$.

(c) *The transition matrix $P(t,s)$ satisfies the forward differential equation*

$$\begin{aligned} \frac{dP(t,s)}{dt} &= P(t,s)Q(t), \ t \geq s, \\ P(s,s) &= I, \end{aligned} \tag{2.5}$$

where I is the identity matrix.

(d) *Assume further that $Q(t)$ is continuous in t. Then $P(t,s)$ also satisfies the backward differential equation*

$$\begin{aligned} \frac{dP(t,s)}{ds} &= -Q(s)P(t,s), \ t \geq s, \\ P(t,t) &= I. \end{aligned} \tag{2.6}$$

Remark 2.6. In (c) and (d) above, the derivatives can also be written as partial derivatives, $(\partial/\partial t)P(t,s)$ and $(\partial/\partial s)P(t,s)$, respectively. Nevertheless, we note that the s in (2.5) and the t in (2.6) only appear in the formulas as parameters. That is, they main represents the initial and terminal conditions. For notational simplicity, we write them as in (c) and (d), and keep this convention throughout. It should be clear from the context.

Proof of Theorem 2.5: Parts (a) and (b) are in Davis [41, pp. 62-69]. To prove part (c), take $f(\alpha) = I_{\{\alpha=j\}}$ for $j \in \mathcal{M}$. Owing to the Markov property and the definition of martingales,

$$\begin{aligned} &P(\alpha(t) = j | \alpha(s) = i) \\ &\quad = \delta_{ij} + \int_s^t \sum_{k \in \mathcal{M}} P(\alpha(\varsigma) = k | \alpha(s) = i) q_{kj}(\varsigma) d\varsigma. \end{aligned} \tag{2.7}$$

To prove (d), note that the continuity of $Q(t)$ imply that

$$\frac{p_{ij}(s, s-\Delta)}{-\Delta} \to -q_{ij}(s) \quad \text{for } i \neq j \quad \text{and}$$
$$\frac{p_{ii}(s, s-\Delta) - 1}{-\Delta} \to -q_{ii}(s),$$

as $\Delta \to 0$. The rest of the proof follows from the standard line of argument using the Chapman–Kolmogorov equation (see [27, pp. 398-403].). □

2.5 Irreducibility and Quasi-Stationary Distributions

Let $\mathcal{M} = \{1, 2, \ldots, m\}$ for some integer $m \geq 2$. Suppose that $\alpha(t)$, $t \geq 0$, is a Markov chain generated by an $m \times m$ matrix $Q(t)$. This section concentrates on the irreducibility and quasi-stationary distribution, which are key points for the rest of the book.

Definition 2.7 (Irreducibility).

(a) A generator $Q(t)$ is said to be *weakly irreducible* if, for each fixed $t \geq 0$, the system of equations

$$\begin{aligned} \nu(t)Q(t) &= 0, \\ \sum_{i=1}^{m} \nu_i(t) &= 1 \end{aligned} \tag{2.8}$$

has a unique solution $\nu(t) = (\nu_1(t), \ldots, \nu_m(t))$ and $\nu(t) \geq 0$.

(b) A generator $Q(t)$ is said to be *strongly irreducible*, or simply *irreducible*, if for each fixed $t \geq 0$ the systems of equations (2.8) has a unique solution $\nu(t)$ and $\nu(t) > 0$.

The expression $\nu(t) \geq 0$ means that for each $i \in \mathcal{M}$, $\nu_i(t) \geq 0$. Similar interpretation holds for $\nu(t) > 0$. It follows from the definitions above that irreducibility implies weak irreducibility. However, the converse does not hold. For example, the generator

$$Q = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$

is weakly irreducible, but it is not irreducible because it contains an absorbing state corresponding to the row $(0, 0)$. A moment of reflection reveals that for a two-state Markov chain with generator

$$Q = \begin{pmatrix} -\lambda(t) & \lambda(t) \\ \mu(t) & -\mu(t) \end{pmatrix},$$

the weak irreducibility requires only $\lambda(t) + \mu(t) > 0$ whereas the irreducibility requires that both $\lambda(t)$ and $\mu(t)$ be positive. If a weakly irreducible Markov chain contains only one communicating class of recurrent states, and if there are no transient states, then the Markov chain is irreducible.

Definition 2.8 (Quasi-Stationary Distribution). For $t \geq 0$, $\nu(t)$ is termed a *quasi-stationary distribution* if it is the solution of (2.8) satisfying $\nu(t) \geq 0$.

Remark 2.9. While studying problems of stationary Markov chains, the stationary distributions play an important role. In the context of nonstationary Markov chains, they are replaced by the quasi-stationary distributions, which will be used extensively in this book.

If $\nu(t) = \nu > 0$, it is termed a stationary distribution. In view of Definitions 2.7 and 2.8, if $Q(t)$ is weakly irreducible, then there is a quasi-stationary distribution. Note that the rank of a weakly irreducible $m \times m$ matrix $Q(t)$ is $m - 1$, for each $t \geq 0$. The definition given above emphasizes the probabilistic interpretation. An equivalent definition for the weak irreducibility that pinpoints the algebraic properties of $Q(t)$ is given below. One can verify their equivalence using the Fredholm alternative; see Lemma A.37 and Corollary A.38 in Appendix.

Definition 2.10. A generator $Q(t)$ is said to be *weakly irreducible* if, for each fixed $t \geq 0$, the system of equations

$$\begin{aligned} f(t)Q(t) &= 0, \\ \sum_{i=1}^{m} f_i(t) &= 0 \end{aligned} \tag{2.9}$$

has only the trivial (zero) solution.

In the subsequent development, we often need to treat nonhomogeneous systems of linear equations. Consider

$$f(t)Q(t) = g(t), \tag{2.10}$$

where for each $0 < t \leq T$, $Q(t)$ is a weakly irreducible generator (an $m \times m$ matrix), $f(t)$, $g(t)$ are unknown and known vectors, respectively. Zero is an eigenvalue of the matrix $Q(t)$ and the null space of $Q(t)$ is spanned by $\mathbb{1}$. Then by the Fredholm alternative (see Corollary A.38), (2.10) has a solution iff $g(t)\mathbb{1} = 0$, where $\mathbb{1} = (1, \ldots, 1)' \in \mathbb{R}^{m \times 1}$.

Define $Q_c(t) = (\mathbb{1} \vdots Q(t)) \in \mathbb{R}^{m\times(m+1)}$. Consider (2.10) together with the condition $f(t)\mathbb{1} = \sum_{i=1}^m f_i(t) = \overline{f}$. This may be written as $f(t)Q_c(t) = g_c(t)$ where $g_c(t) = (\overline{f} \vdots g(t))$. Since for each t, (2.9) has a unique solution, it follows that $Q_c(t)Q_c'(t)$ is a matrix with full rank; therefore, the equation

$$f(t)[Q_c(t)Q_c'(t)] = g_c(t)Q_c'(t) \tag{2.11}$$

has a unique solution. This observation will be used repeatedly in what follows.

Remark 2.11. The difference between weak irreducibility and strong irreducibility is that the former only requires the unique solution $\nu(t)$ to be nonnegative and the latter requires $\nu(t)$ to be strictly positive. We keep the nonnegativity requirement $\nu(t) \geq 0$ in the definition of the weak irreducibility to remind the reader of its probabilistic meaning. In fact, this requirement is superfluous, which can be seen from Lemma A.4.

2.6 Gaussian Processes and Diffusions

A random vector $(x_1, x_2, \ldots, x_r)$ is *Gaussian* if its characteristic function has the form

$$\phi(y) = \exp\left(\iota\langle y, a\rangle - \frac{1}{2}\langle Ay, y\rangle\right),$$

where a is a constant vector in $\mathbb{R}^r$, $\langle y, a\rangle$ is the usual inner product, ι denotes the pure imaginary number satisfying $\iota^2 = -1$, and A is a symmetric nonnegative definite $r \times r$ matrix. A stochastic process $x(t)$, $t \geq 0$, is a *Gaussian process*, if its finite-dimensional distributions are Gaussian, that is, for any $0 \leq t_1 < t_2 < \cdots < t_k$ and $k = 1, 2, \ldots$, $(x(t_1), x(t_2), \ldots, x(t_k))$ is a Gaussian vector.

If for any $0 \leq t_1 < t_2 < \cdots < t_k$ and $k = 1, 2, \ldots$,

$$(x(t_1) - x(0)),\ (x(t_2) - x(t_1)),\ \ldots,\ (x(t_k) - x(t_{k-1}))$$

are independent, then we say $x(\cdot)$ is a *process with independent increments*. The following lemma provides a sufficient condition for a process to be Gaussian. The proof of the lemma can be found in Skorohod [197, p. 7].

Lemma 2.12. *Assume that the process $x(\cdot)$ has independent increments and continuous sample paths with probability one. Then $x(\cdot)$ is a Gaussian process.*

An $\mathbb{R}^d$-valued Gaussian random process $w(t)$, $t \geq 0$, is called a *Brownian motion*, if

(a) $w(0) = 0$, w.p.1;

(b) $w(\cdot)$ is a process with independent increments;

(c) $w(\cdot)$ has continuous sample paths (in $C([0,\infty);\mathbb{R}^d)$) with probability one;

(d) the increments $w(t)-w(s)$ have Gaussian distribution with $Ew(t) = 0$ and $\text{Cov}(w(t)) = \overline{\sigma}t$ for some nonnegative definite $d \times d$ matrix $\overline{\sigma}$, where $\text{Cov}(\zeta)$ denotes the covariance of ζ.

When $\overline{\sigma} = I$, the identity matrix, $w(\cdot)$ is a standard Brownian motion. In view of Lemma 2.12, a Brownian motion is necessarily a Gaussian process. For an $\mathbb{R}^d$-valued Brownian motion $w(t)$, let $\mathcal{F}_t = \sigma\{w(s) : s \leq t\}$. Let $\sigma(\cdot)$ denote an $\mathcal{F}_t$-measurable process taking values in $\mathbb{R}^{d\times d}$ such that $\int_0^t |\sigma(s)|^2 ds < \infty$ for all $t \geq 0$. Using $w(\cdot)$ and $\sigma(\cdot)$, one can define a stochastic integral $\int_0^t \sigma(s)dw(s)$ such that it is a martingale with mean 0 and quadratic variation $\int_0^t \sigma(s)\sigma'(s)ds$.

Given $\mathcal{F}_t$-measurable processes $b(\cdot)$ and $\sigma(\cdot)$, a process $\zeta(\cdot)$ defined as

$$\zeta(t) := \zeta(0) + \int_0^t b(s)ds + \int_0^t \sigma(s)dw(s)$$

is called a *diffusion*. Let $C^{2,1}$ denote the class of real-valued functions on (a subset of) $\mathbb{R}^r \times [0,\infty)$ whose second-order mixed partial derivatives with respect to x and first-order partial derivatives with respect to t are continuous. Define an operator $\mathcal{L}$ on $C^{2,1}$ by

$$\mathcal{L}f(t,x) := \frac{\partial f(t,x)}{\partial t} + \sum_{i=1}^r b_i(t)\frac{\partial f(t,x)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^r a_{ij}(t)\frac{\partial^2 f(t,x)}{\partial x_i \partial x_j}, \quad (2.12)$$

where $a(t) = (a_{ij}(t)) = \sigma(t)\sigma'(t)$. Then for all $f \in C^{2,1}$, the process

$$f(t,\zeta(t)) - f(0,\zeta(0)) - \int_0^t \mathcal{L}f(s,\zeta(s))ds$$

is a local martingale. Moreover, $\zeta(\cdot)$ is a Markov process (provided $\sigma(\cdot)$ and $b(\cdot)$ are suitable non-random Borel functions) in the sense that

$$P(x(t) \in A|\mathcal{F}_s) = P(x(t) \in A|x(s))$$

for all $0 \leq s \leq t$ and for any Borel set A. The operator $\mathcal{L}$ is naturally the generator of the diffusion $\zeta(\cdot)$.

2.7 Switching Diffusions

Let $w(\cdot)$ be an $\mathbb{R}^d$-valued standard Brownian motion defined in the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. Suppose that $b(\cdot,\cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^r$ and that $\sigma(\cdot,\cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}^r \times \mathbb{R}^d$. A switching diffusion or a regime-switching diffusion is a two-component process $(x(\cdot), \alpha(\cdot))$, satisfying

$$\begin{aligned} dx(t) &= b(x(t), \alpha(t))dt + \sigma(x(t), \alpha(t))dw(t), \\ (x(0), \alpha(0)) &= (x, \alpha), \end{aligned} \tag{2.13}$$

and for $i \neq j$,

$$P\{\alpha(t+\Delta) = j | \alpha(t) = i, x(s), \alpha(s), s \leq t\} = q_{ij}(x(t))\Delta + o(\Delta). \tag{2.14}$$

For the two-component process $(x(t), \alpha(t))$, we call $x(t)$ the continuous component and $\alpha(t)$ the discrete component, in accordance with their sample path properties. Note that in the above, the switching process $\alpha(t)$ itself is not a Markov chain. The two-component process $(x(t), \alpha(t))$ is jointly Markovian, however. A special case is the so-called Markovian switching diffusion. That is, the switching process $\alpha(t)$ is independent of the Brownian motion and itself a Markov chain.

For $(x(t), \alpha(t))$, there is an associated operator. For each $i \in \mathcal{M}$ and each $f(\cdot, i) \in C^2$, where C^2 denotes the class of real-valued functions whose partial derivatives with respect to the variable x up to the second-order are continuous, we have

$$\begin{aligned} \mathcal{L}f(x,\iota) &= \nabla f'(x,\iota)b(x,\iota) + \frac{1}{2}\mathrm{tr}(\nabla^2 f(x,\iota)A(x,\iota)) + Q(x)f(x,\cdot)(\iota) \\ &= \sum_{i=1}^{r} b_i(x,\iota)\frac{\partial f(x,\iota)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{r} a_{ij}(x,\iota)\frac{\partial^2 f(x,\iota)}{\partial x_i \partial x_j} \\ &\quad + Q(x)f(x,\cdot)(\iota), \end{aligned} \tag{2.15}$$

where $\nabla f(x,\iota)$ and $\nabla^2 f(x,\iota)$ denote the gradient and Hessian of $f(x,\iota)$ with respect to x, respectively,

$$Q(x)f(x,\cdot)(\iota) = \sum_{j=1}^{m} q_{\iota j} f(x,j), \quad \text{and}$$

$$A(x,\iota) = (a_{ij}(x,\iota)) = \sigma(x,\iota)\sigma'(x,\iota) \in \mathbb{R}^{r\times r}.$$

The evolution of the discrete component $\alpha(\cdot)$ can be represented by a stochastic integral with respect to a Poisson random measure (see, e.g.,

Ghosh, Arapostathis, and Marcus [72] and Skorohod [198]). In fact, for $x \in \mathbb{R}^r$ and $i, j \in \mathcal{M}$ with $j \neq i$, let $\Delta_{ij}(x)$ be the consecutive (w.r.t. the lexicographic ordering on $\mathcal{M} \times \mathcal{M}$), left-closed, right-open intervals of the real line, each having length $q_{ij}(x)$. Define a function $h : \mathbb{R}^r \times \mathcal{M} \times \mathbb{R} \mapsto \mathbb{R}$ by

$$h(x, i, z) = \sum_{j=1}^{m} (j - i) I_{\{z \in \Delta_{ij}(x)\}}. \tag{2.16}$$

That is, with the partition $\{\Delta_{ij}(x) : i, j \in \mathcal{M}\}$ used and for each $i \in \mathcal{M}$, if $z \in \Delta_{ij}(x)$, $h(x, i, z) = j - i$; otherwise $h(x, i, z) = 0$. Then (2.14) is equivalent to

$$d\alpha(t) = \int_{\mathbb{R}} h(x(t), \alpha(t-), z) \mathfrak{p}(dt, dz), \tag{2.17}$$

where $\mathfrak{p}(dt, dz)$ is a Poisson random measure with intensity $dt \times m(dz)$, and m is the Lebesgue measure on $\mathbb{R}$. The Poisson random measure $\mathfrak{p}(\cdot, \cdot)$ is independent of the Brownian motion $w(\cdot)$.

Similar to the case of diffusions, for each $f(\cdot, \imath) \in C^2$, $\imath \in \mathcal{M}$, a result known as the generalized Itô lemma (see Björk [14], Mao and Yuan [153], or Skorohod [198]) reads

$$f(x(t), \alpha(t)) - f(x(0), \alpha(0)) = \int_0^t \mathcal{L} f(x(s), \alpha(s)) ds + M_1(t) + M_2(t), \tag{2.18}$$

where

$$\begin{aligned} M_1(t) &= \int_0^t \big\langle \nabla f(x(s), \alpha(s)), \sigma(x(s), \alpha(s)) dw(s) \big\rangle, \\ M_2(t) &= \int_0^t \int_{\mathbb{R}} \big[f(x(s), \alpha(0) + h(x(s), \alpha(s), z)) \\ &\qquad - f(x(s), \alpha(s)) \big] \mu(ds, dz), \end{aligned}$$

and

$$\mu(ds, dz) = \mathfrak{p}(ds, dz) - ds \times m(dz)$$

is a martingale measure.

2.8 Notes

A reference of basic probability theory is Chow and Teicher [30], and a reference on stochastic processes is Gihman and Skorohod [73]. The results mentioned in this chapter and more detailed discussions regarding martingales and diffusions can be found in Elliott [55]; discussion on stochastic

differential equations and diffusion processes can also be found in Ikeda and Watanabe [91] and Khasminskii [114]. The well-known book of Chung [31] provides us with a classical treatment of continuous-time Markov chains. The connection between generators of Markov processes and martingales is illustrated, for example, in Ethier and Kurtz [59]. For a complete account of piecewise-deterministic processes, see Davis [41], Rishel [181], and Vermes [211]. For a comprehensive study of switching diffusion processes including recurrence, ergodicity, stability etc., we refer the reader to Yin and Zhu [244]; see also the references therein.

3
Markovian Models

3.1 Introduction

With the motivation of bridging the gap of theory and practice, this chapter presents a number of Markovian models and examples from a diverse range of applications. Markov chains with stationary transition probabilities have been studied extensively; they are contained in many classical books, for example, Chung [31], Taylor and Karlin [204] among others. However, the homogeneity or stationarity is often violated in applications, and the generator $Q(t)$ of the underlying Markov chain is frequently time dependent. The results for stationary cases alone are no longer adequate in handling these situations. In the discussion to follow, much emphasis is on finite-state Markov chains and on Markov chains with nonstationary transition probabilities; many examples are modifications of classical work for stationary Markov chains. In various applications involving complex systems, the issue of different time scales naturally arises. This often results from aggregating the states, decomposing a large-scale system into a number of subsystems with manageable size, using multiple step size in optimization procedures and other consideration in modeling. To formulate such problems, one introduces a small parameter $\varepsilon > 0$ yielding a two-time scale, the original time scale t and the stretched time scale t/ε. Consequently, one has to face singularly perturbed Markovian systems, in which the generator is given by $Q^\varepsilon(t)$ for $\varepsilon > 0$ small enough so that the process is subject to rapid variation and/or weak and strong interactions. For motivational

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purposes of the subsequent studies, a number of models and/or examples involving singularly perturbed Markov chains are also presented.

This chapter is arranged as follows. It begins with the birth and death processes. In Section 3.3, we treat a number of Markov chains with finite-state spaces, including queueing systems, seasonal variation, and system reliability. Section 3.4 gives examples arising from the context of stochastic optimization involving Markovian structures. Section 3.5 deals with jump linear systems, in particular, the optimal control formulation of linear quadratic control problems with Markovian jumps and large-scale systems via aggregation and decomposition. One of the main ideas that underlies the basis of the asymptotic results throughout the book is the time-scale separation. To give motivation, some heuristic arguments and interpretation of such separations are provided in Section 3.6. Finally, this chapter concludes with further discussion and notes in Section 3.7.

3.2 Birth and Death Processes

In the study of physical and/or biological sciences, one needs to analyze the random evolution of a certain population. Its size (an integer-valued process) is a family of random variables $\{x(t) : t \geq 0\}$. The reproduction and distinction of the population are conveniently modeled by assuming $x(\cdot)$ to be a Markov chain known as a birth and death process.

Dealing with homogeneous cases, one normally (see Karlin and Taylor [105]) makes the following postulates about the transition probability $P(t) = (p_{ij}(t))$ of the underlying processes

1. $p_{i,i+1}(h) = \lambda_i h + o(h)$ as $h \to 0$ for $i \geq 0$;
2. $p_{i,i-1}(h) = \mu_i h + o(h)$ as $h \to 0$ for $i \geq 1$;
3. $p_{ii}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$ as $h \to 0$ for $i \geq 0$;
4. $p_{ij}(0) = \delta_{ij}$;
5. $\mu_0 = 0$, $\lambda_0 > 0$, μ_i, $\lambda_i > 0$ for all $i \geq 1$.

Note that the term $o(h)$ above may depend on i, i.e., $o(h) = o_i(h)$. The quantities λ_i and μ_i are known as the birth and death rates, respectively. Following from the basic postulates, the infinitesimal generator of the Markov chain is

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \dots \\ 0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

which is time independent and hence the chain is known as having stationary transition probabilities according to Chung [31]. This agrees with the intuitive notion that Q represents the rate or "derivative" in an appropriate sense.

One can readily derive the differential equations satisfied by the transition probabilities. They are

$$\begin{aligned}\frac{dp_{i0}(t)}{dt} &= -\lambda_0 p_{i0}(t) + \mu_1 p_{i1}(t),\\ \frac{dp_{ij}(t)}{dt} &= \lambda_{j-1} p_{i,j-1}(t) - (\lambda_j + \mu_j) p_{ij}(t)\\ &\qquad +\mu_{j+1} p_{i,j+1}(t), \quad j \geq 1.\end{aligned}$$

and

$$\begin{aligned}\frac{dp_{0j}(t)}{dt} &= -\lambda_0 p_{0j}(t) + \lambda_0 p_{1j}(t),\\ \frac{dp_{ij}(t)}{dt} &= \mu_i p_{i-1,j}(t) - (\lambda_i + \mu_i) p_{ij}(t)\\ &\qquad +\lambda_i p_{i+1,j}(t), \quad i \geq 1.\end{aligned}$$

More details of the derivations can be found in Karlin and Taylor [105, pp. 135-137].

There are many possible variations and/or specifications of the birth and death processes. For instance, if $\lambda_i = 0$ for all i, the process becomes the pure death process. If $\mu_i = 0$ for all i, the underlying process is the pure birth process. A pure birth process with $\lambda_i = \lambda$ for all i is known as a Poisson process.

For chains with nonstationary transition probabilities, using the definition of generators given in Chapter 2, for the birth and death processes, we simply assume that the generators are given by $Q(t)$ that satisfies the q-Property and

$$q_{ij}(t) = \begin{cases} -\lambda_0(t), & \text{for } j = i = 0,\\ -(\lambda_i(t) + \mu_i(t)), & \text{for } j = i \text{ and } i \geq 1,\\ \mu_i(t), & \text{for } j = i - 1 \text{ and } i \geq 1,\\ \lambda_i(t), & j = i + 1 \text{ and } i \geq 0.\end{cases}$$

One of the widely used models for daily-life congestion and machine performance random systems is the Markovian queueing formulation. A queueing process (with a single server) is one in which customers arrive at some designated place where a service is rendered. For example, the queue in a supermarket checkout counter can be modeled as a birth and death process. Under stationary assumptions, the system is simplified to $\lambda_i(t) = \lambda$

and $\mu_i(t) = \mu$ for all $i \geq 0$. Using the Markovian framework, the inter-arrival and service times follow exponential distributions with arrival and service rates λ and μ, respectively. Suppose that $\lambda > 0$ and $\mu > 0$. Then the stationary distribution ν_i, $i \geq 0$ of the queueing process can be easily calculated. In fact, $\nu_i = (1 - (\lambda/\mu))(\lambda/\mu)^i$ provided $\lambda < \mu$. For more detailed account on this and related issues as well as other examples, we refer the reader to Karlin and Taylor [105], Sharma [194], and Taylor and Karlin [204] among others.

3.3 Finite-State Space Models

This section contains several examples from a diverse range of applications. It begins with a variation of the single-sever queueing model with limited number of waiting rooms, treats singularly perturbed queues, proceeds with discussions of system reliability issues, deals with competing risk problems, continues with singularly perturbed Cox processes, studies random evolutions, and presents a seasonal variation model.

3.3.1 Queues with Finite Capacity

Originated from the classical example of $M/M/1/(N+1)$ (see, for example, Sharma [194]), the queueing system considered below is a modification and generalization of that of [194]. Here $M/M/1/(N+1)$ is the so-called Kendall's notation for a queueing system with a single server. It indicates the Markovian interarrival and service times with limited capacity, namely, N-waiting rooms. That is, the maximum number of customers in the system is $(N+1)$ including the one being served. Under the Markovian framework, the inter-arrival and service distributions of the $M/M/1/(N+1)$ queueing system follow exponential distributions with mean $1/\lambda$ and $1/\mu$, respectively. The problem under consideration is a finite-state Markov chain with stationary transition probabilities.

Very often the mean arrival and service rates are not constants, but rather varying with respect to the time elapsed. Thus a more reasonable model to reflect reality is that both λ and μ are functions of t. Let the generator of the Markov chain be given by

$$Q(t) = \begin{pmatrix} -\lambda(t) & \lambda(t) & & & \\ \mu(t) & -(\lambda(t)+\mu(t)) & \lambda(t) & & \\ & \ddots & \ddots & \ddots & \\ & & \mu(t) & -(\lambda(t)+\mu(t)) & \lambda(t) \\ & & & \mu(t) & -\mu(t) \end{pmatrix}. \quad (3.1)$$

Compared with the classical $M/M/1/(N+1)$ model, the generator Q is no longer a constant. Its corresponding forward equation is

$$\frac{dp(t)}{dt} = p(t)Q(t),$$

where $p(t) = (p_0(t), \ldots, p_N(t))$ is a row vector representing the probability distribution of $x(t)$, i.e., $p_i(t) \geq 0$ and $\sum_{i=0}^{N} p_i(t) = 1$.

Singularly Perturbed Queues. To proceed, consider a singular perturbation problem for the queueing model above. Suppose that $\varepsilon > 0$ is a small parameter. Let $Q(t)$ be a matrix satisfying the q-Property and define $Q^{\varepsilon}(t) = Q(t)/\varepsilon$. We are interested in the limit behavior of the system

$$\frac{dp^{\varepsilon}(t)}{dt} = p^{\varepsilon}(t)Q^{\varepsilon}(t), \quad p^{\varepsilon}(0) = p^0.$$

The interpretation of the model is that the rates of the interarrival and service are changing rapidly for small ε. Consequently, the entire system is expected to reach a quasi-stationary regime in a very short period of time. For other queueing-related problems, see Knessel [124], and Knessel and Morrison [125], among many others.

Uniform Acceleration of Markov Queues. Consider an $M_t/M_t/1/m$ queue with a finite number of waiting buffers, and the first-in first-out service discipline. Suppose that the arrival process is non-homogeneous Poisson with intensity function (arrival rate function) $\lambda(t)$, and that the service time is exponentially distributed with time-dependent rate $\mu(t)$. Let $\alpha(t)$ be the queue length at time t. Then $\alpha(t)$ is a nonstationary Markov chain with generator given by (3.1). Our objective is to seek an approximation to the probability $P(\alpha(t) = k)$ with $0 \leq k \leq m$. Denote

$$p(t) = (P(\alpha(t) = 0), \ldots, P(\alpha(t) = m)).$$

Then we have

$$\frac{dp(t)}{dt} = p(t)Q(t). \tag{3.2}$$

Considering the above problem, Massey and Whitt [157] introduced a small parameter $\varepsilon > 0$ to the generator $Q(t)$. Assume that the rate of change of the generator $Q(t)$ varies slowly in time that the process $\alpha(t)$ can achieve equilibrium before there is any significant change in the rate. Then we can replace $Q(t)$ by $Q(\varepsilon t)$. In this replacement, we focus on $\alpha(t)$ in the neighborhood of time 0. Let $p^{\varepsilon}(t)$ be the probability distribution corresponding to the generator $Q(\varepsilon t)$. Then

$$\frac{dp^{\varepsilon}(t)}{dt} = p^{\varepsilon}(t)Q(\varepsilon t). \tag{3.3}$$

To get the uniform acceleration, Massey and Whitt [157] consider the limits as $t \to \infty$ and $\varepsilon \to 0$ simultaneously. Let $\tau = \varepsilon t$ and $\alpha^\varepsilon(\cdot)$ be the new process associated with the time scale τ. Then the corresponding probability distribution $p^\varepsilon(\tau)$ will solve the forward equation

$$\frac{dp^\varepsilon(\tau)}{d\tau} = p^\varepsilon(\tau)\frac{Q(\tau)}{\varepsilon}. \tag{3.4}$$

Studying such a singularly perturbed model is the objective of this book.

A Queueing System with Weak and Strong Interactions. Let us consider a queueing system consisting of two types of customers. Denote by $x_1(t)$ and $x_2(t)$ the queue lengths of type I and type II customers, respectively. Assume the maximum queue length for both type I and type II customers to be 2, i.e., $x_1(t) \in \{0,1,2\}$ and $x_2(t) \in \{0,1,2\}$. Suppose that the events of interarrival and service of type I customers occur more frequently than that of type II customers. Formulate the queueing system as a finite-state Markov chain, i.e., the process $(x_1(\cdot), x_2(\cdot))$ is Markovian with state space

$$\mathcal{M} = \Big\{(0,0),(1,0),(2,0),(0,1),(1,1),(2,1),(0,2),(1,2),(2,2)\Big\}$$

and generator $Q^\varepsilon(t) = \widetilde{Q}(t)/\varepsilon + \widehat{Q}(t)$, where

$$\widetilde{Q}(t) = \text{diag}\left(\widetilde{Q}^1(t), \widetilde{Q}^2(t), \widetilde{Q}^3(t)\right),$$

$$\widetilde{Q}^1(t) = \widetilde{Q}^2(t) = \widetilde{Q}^3(t) = \begin{pmatrix} -\lambda_1(t) & \lambda_1(t) & 0 \\ \lambda_1(t) & -(\lambda_1(t)+\mu_1(t)) & \mu_1(t) \\ 0 & \mu_1(t) & -\mu_1(t) \end{pmatrix},$$

and

$$\widehat{Q}(t) = \begin{pmatrix} -\lambda_2(t)I_3 & \lambda_2(t)I_3 & 0 \\ \lambda_2(t)I_3 & -(\lambda_2(t)+\mu_2(t))I_3 & \mu_2(t)I_3 \\ 0 & \mu_2(t)I_3 & -\mu_2(t)I_3 \end{pmatrix}$$

with I_3 being the 3×3 identity matrix. Assume that for some $T > 0$, $\lambda_i(t) \geq 0$ and $\mu_i(t) \geq 0$ for all $t \in [0,T]$ and each $i = 1,2$. The arrival and service rates for type I customers are $\lambda_1(t)/\varepsilon$ and $\mu_1(t)/\varepsilon$, respectively. Those rates for type II customers are $\lambda_2(t)$ and $\mu_2(t)$.

For a prototype example, consider the transactions taking place in a bank, which usually have different types of customers. Some customers come to the bank just for depositing a check and others may wish to open a new account. Apparently, the transaction of opening an account requires much longer time than depositing a check. If we consider check-depositing

customers as type I customers and account-opening customers as those of type II, then such a queueing system can be formulated using Q^ε defined above. Again, the small parameter $\varepsilon > 0$ is merely a convenient device for separating the different arrival and service rates. How large $\varepsilon > 0$ should be will depend on the actual situation. In any case, the emphasis is this: It is the relative rates that count. The asymptotic results to be presented in the sequel provide hints and guidance for real-world applications.

In the chapters to follow, we will discuss singular perturbation problems extensively in a more general setting and will derive a number of asymptotic results for the transition probabilities of the Markov chains. A crucial notion is the concept of quasi-stationary distribution. As will be seen in the analysis of the subsequent chapters, this "equilibrium" is different from the case of Markov chains with stationary transition probabilities since the limiting probabilities are time dependent.

3.3.2 System Reliability

This subsection studies system reliability. It begins with a basic model, and proceeds with the discussion of the redundancy formulation and burn-in phenomenon.

A System with Parallel Components. Consider a system consisting of two independent components. The system is functioning if at least one of the components is functioning. Assume that each component has two possible states, functioning, denoted by 1, and out of order, denoted by 0. Therefore, the system as a whole has four states, $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. For example, $(0,1)$ means that the first component failed whereas the second one is in good condition. Let $\lambda_i(t)$ and $\mu_i(t)$ denote the failure rate and repair rate of component i for $i = 1, 2$, respectively. The generator of the Markov chain can be written as

$$Q(t) = \begin{pmatrix} q_{11}(t) & \mu_2(t) & \mu_1(t) & 0 \\ \lambda_2(t) & q_{22}(t) & 0 & \mu_1(t) \\ \lambda_1(t) & 0 & q_{33}(t) & \mu_2(t) \\ 0 & \lambda_1(t) & \lambda_2(t) & q_{44}(t) \end{pmatrix},$$

where $q_{ii}(t)$ are the combinations of $\lambda_i(t)$ and $\mu_i(t)$, $i = 1, 2$, such that the sum in each row of the matrix above is equal to 0. Representation of the probability distribution is obtainable by solving the forward equation via the use of fundamental solutions of the differential equation.

Standby Systems. Very often, certain units, components, or subsystems in a system may be more important for the system's reliability than others. To ensure the entire system's reliability, one may either use units with high reliability or introduce redundancy–standby systems. Owing to cost considerations, the latter often appears to be more preferable. For various

terminology, such as active redundance, cold standby, partly loaded, consult the work of Hoyland and Rausand [88]. We now present a simple standby model. Suppose that a system has two units, one on-line and the other as a backup. The operating component (unit) fails after some time that has an exponential distribution with parameter $\lambda(t)$, and is then replaced by the standby unit. There is a repair facility in which the repair time is exponentially distributed with parameter $\mu(t)$. Let $x(t)$ denote the number of units functioning at time t. The infinitesimal generator of the chain is

$$Q(t) = \begin{pmatrix} -\mu(t) & \mu(t) & 0 \\ \lambda(t) & -(\lambda(t)+\mu(t)) & \mu(t) \\ 0 & \lambda(t) & -\lambda(t) \end{pmatrix}. \tag{3.5}$$

Given $t \geq 0$, if $\lambda(t) + \mu(t) > 0$, then $Q(t)$ is weakly irreducible; if $\lambda(t) > 0$ and $\mu(t) > 0$, then $Q(t)$ is irreducible. The quasi-stationary distribution is given by

$$\nu_i(t) = \frac{(\mu(t)/\lambda(t))^{i-1}}{1 + (\mu(t)/\lambda(t)) + (\mu(t)/\lambda(t))^2}, \quad \text{for } i = 1, 2, 3.$$

If in addition, $\lambda(t) = \lambda > 0$ and $\mu(t) = \mu > 0$, the quasi-stationary distribution becomes the stationary distribution.

Burn-in Phenomenon. It is common that a newly manufactured or newly repaired unit or device has greater chance of failing early in its usage. Such a phenomenon is usually called the burn-in phenomenon. Suppose that a manufacturing system consists of two components, one on-line and one backup. As discussed in Taylor and Karlin [204], the assumption of exponentially distributed operating time does not reflect reality well due to the burn-in phenomenon. To simplify the discussion below, suppose that $\lambda(t) = \lambda$ and $\mu(t) = \mu$ in (3.5), i.e., the Markov chain is stationary. Introduce the hazard rate function $r(t) = f(t)/(1 - F(t))$, where $f(t)$ is the probability density function of the failure time, and $F(t)$ is the corresponding distribution function. In [204] the following mixed exponential distribution is introduced: $f(t) = p\alpha e^{-\alpha t} + q\beta e^{-\beta t}$, and $p > 0$, $q > 0$ with $p + q = 1$. The probability density function is a convex combination of two exponential distributions. That is, with probability p the unit beginning operation will have an exponential up time with parameter α, and likewise with probability $q = 1 - p$ the unit beginning operation will have an exponential up time with parameter β. The rationale behind this is that the hazard rate is initially high and eventually decays to a constant level. This system has the following five states,

0: both units are down;
1_α: one unit is up, and the current up time has parameter α;
1_β: one unit is up, and the current up time has parameter β;
2_α: two units are up, and the current up time has parameter α;
2_β: two units are up, and the current up time has parameter β.

Arrange the states as $\{0, 1_\alpha, 1_\beta, 2_\alpha, 2_\beta\}$. Then we can write the generator of the Markov chain as

$$Q = \begin{pmatrix} -\lambda & p\lambda & q\lambda & 0 & 0 \\ \alpha & -(\lambda+\alpha) & 0 & \lambda & 0 \\ \beta & 0 & -(\lambda+\beta) & 0 & \lambda \\ 0 & p\alpha & q\alpha & -\alpha & 0 \\ 0 & p\beta & q\beta & 0 & -\beta \end{pmatrix}.$$

Assuming $\alpha > 0$ and $\beta > 0$, then Q is irreducible. Therefore there exists a unique stationary distribution that can be determined by solving the system of equations

$$\nu Q = 0 \text{ and } \nu \mathbb{1} = 1.$$

3.3.3 Competing Risk Theory

Suppose that there are a number of different risks (diseases, accidents, etc.) competing for the lives of individuals. For each of the individuals, one of these risks will "win," and this individual will die. The competing risk theory aims to study these risks and their prevention. These risks can often be modeled by using a Markovian assumption; the work of Thompson [205] collects a number of examples and applications in safety and reliability analysis.

Consider the following scenario. Suppose that the risks can be modeled by a finite-state Markov chain. Assume some of them are much riskier than others; the transition rates are quite different. Thus one can split up the different states into two groups, very risky group and not so risky group. These groups are not isolated, and there are transitions among them. To model this situation, introduce a small parameter $\varepsilon > 0$, and let the generator be $Q^\varepsilon(t) = \widetilde{Q}(t)/\varepsilon + \widehat{Q}(t)$, where $\widetilde{Q}(t)$ is a block-diagonal matrix of the form

$$\widetilde{Q}(t) = \begin{pmatrix} Q_0(t) & 0 \\ 0 & 0 \end{pmatrix},$$

$Q_0(t)$ is itself a generator, and $\widehat{Q}(t)$ is another generator. All these matrix-valued functions have appropriate dimensions. The matrix $Q_0(t)$ corresponds to the not-so-risky states. The small parameter ε indicates that the rates of jumps within the not-so-risky states are larger than those of the risky groups.

The results to be presented in Chapter 4 describe the asymptotic behavior of the corresponding probability distribution. Intuitively, owing to the presence of the small parameter $\varepsilon > 0$, the matrix $\widetilde{Q}(t)$ has the dominating effect in the transitions. Since the right-hand corner of $\widetilde{Q}(t)$ is a 0 matrix, the probability of absorbing into the states corresponding to this part of

the generator is fairly close to 1. A generalization of the model above calls for the modeling of $Q(t)$ of the form

$$\widetilde{Q}(t) = \begin{pmatrix} \widetilde{Q}^1(t) & & & & \\ & \widetilde{Q}^2(t) & & & \\ & & \ddots & & \\ & & & \widetilde{Q}^l(t) & \\ & & & & 0 \end{pmatrix}.$$

This shows that there are $(l+1)$ groups of different competing risks. Among them, there are strong and weak interactions.

3.3.4 Two-Time-Scale Cox Processes

In 1955, D. Cox introduced the notion of doubly stochastic Poisson processes that are more commonly referred to as Cox processes nowadays. A Cox process $N(\cdot)$ is a conditional Poisson process with stochastic intensity $\lambda(t, \alpha(t))$ in that for almost every given path of the process $\alpha(\cdot)$, $N(\cdot)$ is a Poisson process with intensity $\lambda(t, \alpha(t))$. That is, the intensity of the process is modulated by an "outside" force or process, which influences the evolution of the point process $N(\cdot)$.

The notion of Cox process is a convenient way to describe and to produce general random sets. By choosing different $\alpha(\cdot)$, one may get a large class of conditional Poisson processes, which are widely used in ecology (to represent the position of the members of a biological production in a plane habitat with variation in fertility or attractiveness of different parts of the habitat; see Kingman [122, Chapter 6]), in system reliability theory describing failures of complex systems (see Di Masi and Kabanov [47, 48]), and in optical detection and communication theory; see Snyder [200], which contains many interesting examples in information and communication systems. For general discussion on the Cox processes as well as further details, we refer to [122, 200] and the large number of references therein; for related problems in point processes and queues, see Brémaud [20].

In what follows, we focus on the case that the modulating process is a finite-state Markov chain. Frequently, the problem of interest is a singularly perturbed one. Let $\varepsilon > 0$ be a small parameter. Denote the underlying conditional Poisson process by $N^\varepsilon(t)$. This process is modulated by a nonhomogeneous Markov chain $\alpha^\varepsilon(\cdot)$ with finite-state space $\mathcal{M}$. The generator of the Markov chain is given by $Q^\varepsilon(t) = Q(t)/\varepsilon$, where $Q(t)$ is itself a generator. To study the Cox process, one needs to first have a handle on the asymptotic properties of the Markov chain $\alpha^\varepsilon(\cdot)$. Chapter 4 deals with such a problem. It will be shown that under simple conditions the probability distribution of $\alpha^\varepsilon(t)$ admits an asymptotic expansion. The leading term in the expansion is the quasi-stationary distribution. Furthermore, let

the compensator of $N^\varepsilon(\cdot)$ be $G^\varepsilon(\cdot)$. As in [48], it satisfies the differential equation

$$\frac{dG^\varepsilon(t)}{dt} = \sum_{i=1}^{m} a_i I_{\{\alpha^\varepsilon(t)=i\}}, \quad G^\varepsilon(0) = G_0$$

for some $a_i > 0$ with $i = 1, \ldots, m$. This can be written in an integral form

$$G^\varepsilon(t) = G_0 + \sum_{i=1}^{m} \int_0^t a_i I_{\{\alpha^\varepsilon(s)=i\}} ds.$$

In fact, it is a weighted occupation measure. The result of Chapter 5 infers the convergence of the process $G^\varepsilon(t)$ in the mean square sense. Further investigation shows that the problem can be studied even when the generator $Q(t)$ is only measurable.

3.3.5 Random Evolutions

A wide range of physical models belong to the random evolution category. Consider the following scenarios. A particle moves in a straight line with a constant velocity, until a random collision happens; it then changes the velocity and again moves in a straight line. A radio signal propagates through a turbulent medium; a population of bacteria evolves in an environment that is subject to random disturbances; all these reflect an abstract notion, namely, random evolution. There is a large amount of literature in dealing with such problems; an extensive survey is in Hersh [85].

A random evolution is an operator M satisfying a linear differential equation

$$\frac{dM(s,t)}{ds} = -V(\alpha(s), \omega)M(s,t), \quad t \geq s \geq 0,$$

where ω is a sample point belonging to the sample space Ω and s indicates the starting time. As a usual practice, we suppress the ω-dependence in what follows. Denote the expected value of the solution above (with initial condition $\alpha(0) = \alpha$) by $u(t, \alpha) = E_\alpha[M(0,t)]$. If $\alpha(t)$ is Markovian, $u(t, \alpha)$ satisfies the equation

$$\frac{du}{dt} = V(\alpha)u + Qu, \tag{3.6}$$

where Q is the generator of the Markov chain $\alpha(t)$. This is known as a generalized Feynman-Kac formula in potential theory.

Various applications call for the investigation of "speed up" in the random evolution. To do so, introduce a small parameter $\varepsilon > 0$. In lieu of (3.6), consider the differential equation

$$\frac{du}{dt} = V(\alpha(t))u + \frac{1}{\varepsilon}Qu;$$

see [85] for the formulation and more detail. It is clear that the probability distribution of the random process $\alpha(t)$ is an important matter. Chapter 4 deals with this issue for a time-dependent generator $Q^\varepsilon(t) = Q(t)/\varepsilon$. It is conceivable that the effort in this direction should provide further insight into the study of random evolution.

3.3.6 Seasonal Variation Models

Many problems in real life display a periodic variation or a seasonal behavior that repeats itself after a certain period of time. A typical example is the seasonal temperature variations in our surrounding environment. Similar formulations also arise in financial engineering, production planning, inventory and operation management, etc. A distinct feature of these models is the nonstationary property due to their periodic nature.

To exploit modeling opportunities, we present a simple example in what follows. Let us concentrate on the temperature variation of a given city. Intuitively, the probability of the next day's temperature reaching 70°F given the current and the past temperatures depends mainly on the current temperature. It is thus reasonable to adopt a Markovian model. Previously, a similar formulation was considered by Hillier and Leiberman [86, p. 587]. Nevertheless, their setup is essentially time homogeneous. To proceed, we consider a formulation with a finite horizon that includes the whole year. The graph in Figure 3.1 represents a sample path of a daily temperature variation of the city.

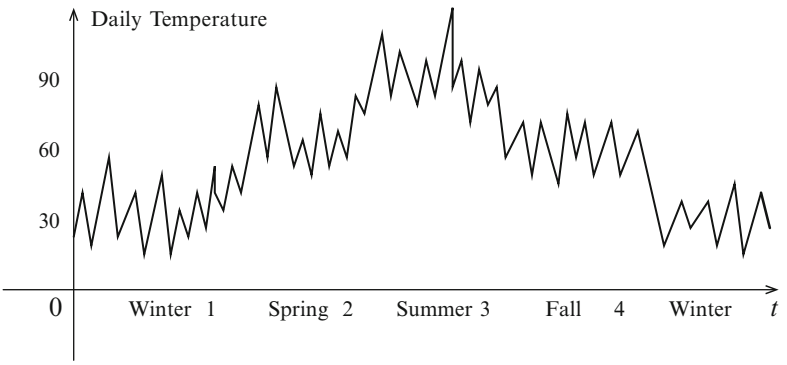

FIGURE 3.1. A Sample Path of Daily Temperature

Clearly, the temperature variation depends largely on the seasonal changes that occur around certain periods of time during the year, which is highly nonstationary in nature. One may approximate the seasonal average by a piecewise-constant function; a typical case is depicted in Figure 3.2.

Although it is a simplified model, it gives us insight from a modeling point of view. To describe the daily temperature process, let $\alpha^\varepsilon(t)$ be a

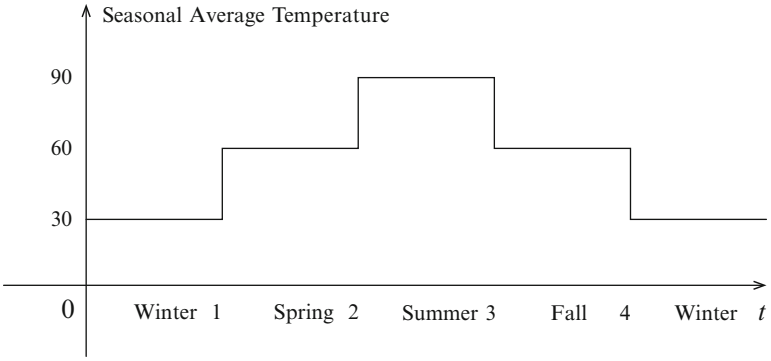

FIGURE 3.2. Average Temperature

Markov chain representing the temperature process of the city. Suppose that the state space of $\alpha^\varepsilon(\cdot)$ is

$$\mathcal{M} = \{20, 30, 40, 50, 60, 70, 80, 90, 100\},$$

and that the generator is

$$Q^\varepsilon(t) = \frac{1}{\varepsilon}\text{diag}\left(\widetilde{Q}^1, \widetilde{Q}^2, \widetilde{Q}^3\right) + \begin{pmatrix} -\bar{\lambda}_1(t)I_3 & \bar{\lambda}_1(t)I_3 & 0 \\ \bar{\mu}_2(t)I_3 & -(\bar{\lambda}_2(t)+\bar{\mu}_2(t))I_3 & \bar{\lambda}_2(t)_3 I_3 \\ 0 & \bar{\mu}_3(t)I_3 & -\bar{\mu}_3(t)I_3 \end{pmatrix},$$

where

$$\widetilde{Q}^i = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \quad \text{for } i = 1, 2, 3,$$

I_3 is the 3×3 identity matrix, $\varepsilon = 1/20$, and $\bar{\lambda}_1(t)$, $\bar{\lambda}_2(t)$, $\bar{\mu}_2(t)$, and $\bar{\mu}_3(t)$ are given by graphs in Figures 3.3–3.6, respectively.

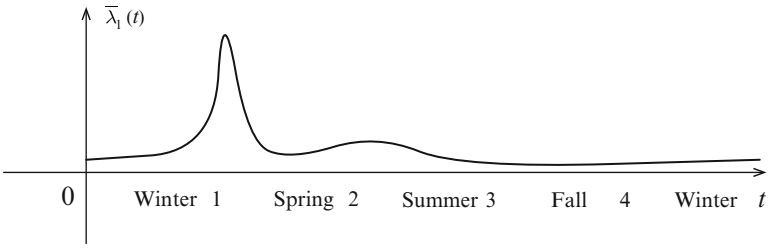

FIGURE 3.3. Jump Rate from Winter to Spring

Let $\mathcal{M}_1 = \{20, 30, 40\}$ denote the group of temperatures in Winter, $\mathcal{M}_2 = \{50, 60, 70\}$ in Spring and Fall, and $\mathcal{M}_3 = \{80, 90, 100\}$ in Summer.

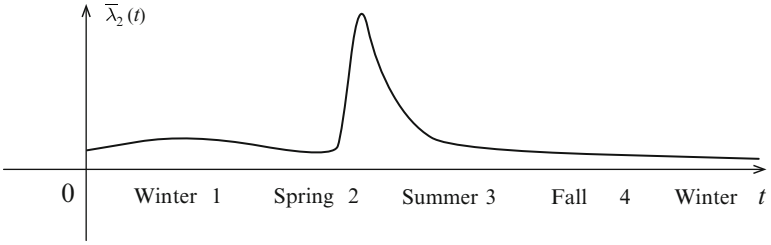

FIGURE 3.4. Jump Rate from Spring to Summer

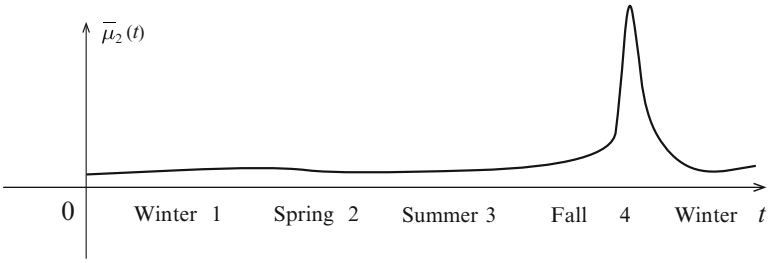

FIGURE 3.5. Jump Rate from Fall to Winter

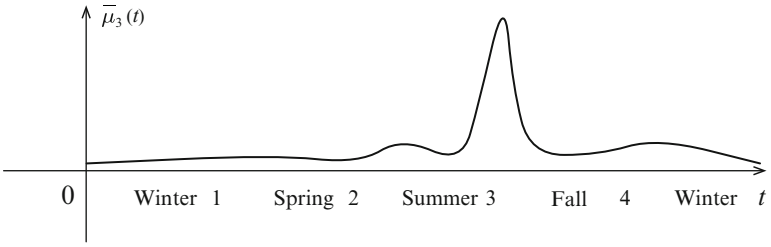

FIGURE 3.6. Jump Rate from Summer to Fall

The $\mathcal{M}_i$, for $i = 1, 2, 3$, subdivide the entire state space $\mathcal{M}$ into three different groups. Within each season, the temperature changes frequently within a certain range (i.e., within some $\mathcal{M}_i$), and less so from one group ($\mathcal{M}_i$) to others ($\mathcal{M}_j$, $j \neq i$). Thus the weak and strong interaction formulation naturally reflects such a situation.

In view of the construction of the Markov chain in Chapter 2, it is very likely that $\alpha^\varepsilon(\cdot)$ will jump from $\mathcal{M}_1$ to $\mathcal{M}_2$ near $t = 1$ owing to the behavior of $\overline{\lambda}_1(t)$ (see Figure 3.3). Similarly the functions $\overline{\lambda}_2(t)$, $\overline{\mu}_2(t)$, and $\overline{\mu}_3(t)$ affect the behavior of the Markov chain at $t = 2$, $t = 4$, and $t = 3$, respectively.

To proceed, define

$$\nu^1(t) = \nu^2(t) = \nu^3(t) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

These are the quasi-stationary distributions for the groups $\mathcal{M}_i$, with $i = 1, 2, 3$, respectively. Using these quasi-stationary distributions, it is easy to obtain the corresponding averages of temperature: 30, 60, 90 degrees.

By grouping the states in accordance with the preceding strategy, the aggregated states act as if each of them were a single state in a Markov chain. One can then get an aggregated process $\overline{\alpha}^{\varepsilon}(t) = \alpha(t)$ that is independent of ε. This aggregated process is Markovian and the generator is

$$\overline{Q}(t) = \begin{pmatrix} -\bar{\lambda}_1(t) & \bar{\lambda}_1(t) & 0 \\ \bar{\mu}_2(t) & -(\bar{\lambda}_2(t)+\bar{\mu}_2(t)) & \bar{\lambda}_2(t) \\ 0 & \bar{\mu}_3(t) & -\bar{\mu}_3(t) \end{pmatrix}.$$

The formulation of singularly perturbed Markovian models with weak and strong interactions provides us with an appropriate mathematical framework. The detailed studies of the asymptotic properties of the aggregated Markov chains are given in Chapters 4 and 5.

3.4 Stochastic Optimization Problems

This section is devoted to stochastic optimization methods with the driving processes being Markovian. Simulated annealing schemes are discussed first and then continuous-time stochastic approximation algorithms are considered.

3.4.1 Simulated Annealing

We use the formulation in Chiang and Chow [28] as follows. Consider a Markov chain $x(t)$ generated by $Q(t) = (q_{ij}(t))$. Let $B = (b_{ij})$ be a matrix satisfying $b_{ij} \geq 0$ for $i \neq j$, and $U_{ij} : \mathcal{M} \times \mathcal{M} \mapsto [0, \infty]$ be a cost function that measures the degree of reachability from one state to another. Assume without loss of generality that $U_{ii} = 0$, and $U_{ij} = \infty$ if $b_{ij} = 0$. Suppose that

$$q_{ij}(t) = \begin{cases} (b_{ij})(\lambda(t))^{U_{ij}}, & \text{if } i \neq j, \\ -\sum_{k \neq i} q_{ik}(t) & \text{if } i = j, \end{cases}$$

where $\lambda(t)$ is a function with $\lim_{t\to\infty} \lambda(t) = 0$.

Such Markov models have been used frequently in simulated annealing (for example, see Kirkpatrick, Gebatt, and Vecchi [123], and Geman and Geman [70] among others), where $U_{ij} = (u_j - u_i)^+$ is determined by a potential function u and $T(t) = (-\log \lambda(t))^{-1}$ is said to be a "temperature"

at time t. Much effort is expended to obtain asymptotic results of the related optimization problem. It can be established that

$$\lim_{t\to\infty} P(x(t) \in G) = 1,$$

where $G = \{i : \ u_i = \min u\}$. The underlying Markov chain plays an important role in studying such a global optimization problem.

To treat the underlying problems, one of the approaches is to examine the Kolmogorov forward equation

$$\frac{dF_i(t)}{dt} = \sum_{j=0}^{N} q_{ji}(t)F_j(t), \quad i = 0, \ldots, N,$$

where $F_i(t) = P(x(t) = i)$. Observe that the annealing cooling schedule $T(t) \to 0$ as $t \to \infty$. Therefore, the problem is closely related to a singular perturbation problem (roughly, $T(t)$ can be treated as a small parameter). Chiang and Chow [28] used the result on the eigenvalue distributions of a generator derived in Wentzel [217] to study the asymptotic properties of the simulated annealing problems.

3.4.2 Continuous-Time Stochastic Approximation

In many optimization problems, one is interested in finding zeros and/or in locating minimal values of a function $f(x)$. Frequently, however, either the function has a very complex form or its exact expression is not known explicitly. Consequently, $f(x)$ is not at our disposal, and only noise-corrupted measurements or observations of the form $f(x)$ plus some noise are available at selected values of x. One can only rely on the measurements or use some form of Monte Carlo method (see various examples in Kushner and Yin [145]). Although discrete procedures are often used, one needs to consider continuous-time algorithms when the sampling rate is high.

Let x, $\xi \in \mathbb{R}^n$ and $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a continuous function satisfying certain conditions. A continuous-time stochastic approximation algorithm takes the form

$$\frac{dx(t)}{dt} = a(t)(f(x(t)) + \xi(t)), \tag{3.7}$$

where

$$a(t) > 0, \ a(t) \to 0 \text{ as } t \to \infty \text{ and } \int_0^\infty a(t)dt = \infty.$$

Typical step size sequences take the form

$$a(t) = \frac{1}{t^\gamma} \quad \text{with} \quad \frac{1}{2} < \gamma \le 1.$$

In the discussion to follow, assume that the driving process $\xi(t)$, satisfying $E\xi(t) = 0$, is a finite-state Markov chain with generator $Q(t) = Q$. Suppose

Q is irreducible. Then the process $\xi(t)$ is a stationary ϕ-mixing process with exponential mixing rate (see Billingsley [13, p. 168]). It is ergodic (see Karlin and Taylor [105, p. 488]) with

$$\frac{1}{t}\int_0^t \xi(s)ds \to 0 \quad \text{w.p.1.}$$

Suppose that x^* is the unique root of the function $f(x)$. Without loss of generality, assume $x^* = 0$ in what follows. Then under suitable conditions, one can show that $x(t) \to 0$ w.p.1 and that $(1/\sqrt{a(t)})x(t)$ converges in distribution to a normal random variable.

Ever since the introduction of the stochastic approximation method, there have been continuing efforts to improve the asymptotic performance and efficiency. Recently, an averaging approach was proposed independently by B. T. Polyak and D. Ruppert independently. The basic idea is to generate a sequence of rough estimates using slowly varying step sizes first and then form a new sequence of estimates by taking a simple arithmetic averaging. It was demonstrated that such an approach leads to asymptotic optimality (see Kushner and Yin [145, Chapter 11]). This method uses large step sizes, hence forcing the iterates to reach the vicinity of the true parameter faster. It is in a simple form; no complicated estimation procedure (for determining the optimal step size) is needed. Moreover, owing to the use of slowly varying step sizes, the iterates move faster than the usual algorithm with smaller step sizes.

A continuous-time version of the averaging approach was examined in Yin and Gupta [224]. The algorithm is a differential-integral equation of the form

$$\begin{aligned} \frac{dx(t)}{dt} &= \frac{1}{t^\gamma} f(x(t)) + \frac{1}{t^\gamma}\xi(t), \quad \frac{1}{2} < \gamma < 1 \\ \overline{x}(t) &= \frac{1}{t}\int_0^t x(s)ds. \end{aligned} \tag{3.8}$$

There is a natural analog of the averaging procedure to a singularly perturbed system. To exploit this connection, define

$$\overline{u}(t) = \sqrt{t}\overline{x}(t) = \frac{1}{\sqrt{t}}\int_0^t x(s)ds.$$

Using the same time scale for both $x(t)$ and $\overline{u}(t)$, we arrive at

$$\begin{aligned} \left(\frac{1}{t^{1-\gamma}}\right)\frac{dx(t)}{dt} &= \frac{1}{t}(f(x(t)) + \xi(t)), \\ \frac{d\overline{u}(t)}{dt} &= -\frac{1}{2t}\overline{u}(t) + \frac{1}{\sqrt{t}}x(t). \end{aligned}$$

Since our interest is focused on the case of $t \to \infty$, loosely speaking, one can treat $1/t^{1-\gamma}$ as a small parameter ε. In [224], it was shown that $\overline{x}(t) \to 0$ w.p.1. Moreover, defining $z(t,\tau) = \tau\sqrt{t}\overline{x}(t\tau)$ for each $\tau \in [0,1]$, we proved that $z(t,\cdot)$ converges weakly to a Brownian motion and $\overline{u}(t) \sim N(0,\Sigma^*)$, i.e., it is asymptotically normal with mean 0 and the optimal covariance Σ^* (the best possible covariance in an appropriate sense). The procedure is therefore asymptotically optimal. Algorithms with constant step size of the form

$$\frac{dx^\varepsilon(t)}{dt} = \varepsilon(f(x^\varepsilon(t)) + \xi(t)),$$

and procedures with additional averaging in the observations

$$x(t) = \overline{x}(t) + \frac{1}{t^\gamma}\int_0^t f(x(s))ds + \frac{1}{t^\gamma}\int_0^t \xi(s)ds,$$
$$\overline{x}(t) = \frac{1}{t}\int_0^t x(s)ds,$$

can also be considered. The setup of the problem in [224] was much more general than the one presented here. For various issues related to stochastic approximation problems, we refer to Kushner and Yin [145] and the references therein.

3.4.3 Systems with Markovian Disturbances

Originating from the study in control of singularly perturbed systems (see Kokotovic [126], Kokotovic, Bensoussan, and Blankenship [127], Kokotovic, Khalil, and O'Reilly [129], Kushner [140], and Sethi and Zhang [192]), this example focuses on systems involving singularly perturbed Markov chains.

Let the state $x^\varepsilon(t) \in \mathbb{R}^n$ and the control $u(t) \in \Gamma \subset \mathbb{R}^{n_1}$ such that

$$\frac{dx^\varepsilon(t)}{dt} = f(x^\varepsilon(t), \alpha^\varepsilon(t), u(t)),\ x^\varepsilon(0) = x,\ t \geq 0, \tag{3.9}$$

where $\varepsilon > 0$ is a small parameter, $f(\cdot)$ is an $\mathbb{R}^n$-valued function, and $\alpha^\varepsilon(t)$, $t \geq 0$, is a Markov chain defined on a probability space $(\Omega, \mathcal{F}, P)$ taking values in

$$\mathcal{M} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l.$$

Our objective is to find a control process $u(t)$ as a function of $x^\varepsilon(s)$ and $\alpha^\varepsilon(s)$, $s \leq t$ that minimizes a finite horizon cost function

$$J^\varepsilon(x,\alpha,u(\cdot)) = E\int_0^T G(x^\varepsilon(t), \alpha^\varepsilon(t), u(t))dt$$

for some $0 < T < \infty$ and a running cost function $G(\cdot)$. Suppose that the generator of the random process $\alpha^\varepsilon(\cdot)$ takes the form

$$Q^\varepsilon(t) = \frac{1}{\varepsilon}\text{diag}\left(\widetilde{Q}^1(t), \ldots, \widetilde{Q}^l(t)\right) + \widehat{Q}(t),$$

where for each $k = 1, \ldots, l$, $\widetilde{Q}^k(t)$ is the generator corresponding to the states in $\mathcal{M}_k$. Such a system is suitable for certain applications in manufacturing and queueing systems; it describes the interconnections and interactions of a number of subsystems involved.

The singularly perturbed system given above can be viewed as one in which the state $x^\varepsilon(t)$ is driven by a process $\alpha^\varepsilon(t)$ with weak and strong interactions. Owing to the nonlinearity of the system and the singularity of $\alpha^\varepsilon(\cdot)$, it is difficult to obtain the optimal solutions in closed form. Moreover, the state space of $\alpha^\varepsilon(\cdot)$ is often very large, which makes the underlying problem even more difficult. Nevertheless, one may seek an alternative approach and may look for nearly optimal solutions. The idea is as follows. For small $\varepsilon > 0$, we disregard the detailed variation of the process $\alpha^\varepsilon(t)$ at each time t and concentrate on an average system in which $\alpha^\varepsilon(\cdot)$ is aggregated so that all the states in $\mathcal{M}_k$ can be replaced by a single state k. Using an optimal solution to the limit of the averaged problem, one may construct a solution to the original problem that is nearly optimal (as $\varepsilon \to 0$). For a detailed account on the problem mentioned above, see Chapter 8 (see also Yin and Zhang [234]), where the techniques of martingale averaging developed in Kushner [140] are used.

3.5 Linear Systems with Jump Markov Disturbance

Owing to a wide range of applications of jump linear systems, there has been a growing interest in control theory and optimization of such systems. In this section, we present several variations of the linear quadratic (LQ) control problem.

3.5.1 Linear Quadratic Control Problems

In the classical setting, feedback control design of linear systems is based on a plant with fixed parameters. This, however, does not allow one to treat situations in which the real systems differ from the assumed nominal model. The closed-loop stability may not be preserved if the real system is different from the nominal plant. To take this into account, efforts have been made to design robust controls such that stability requirements are met simultaneously for a set of plants.

Let $\alpha(t)$ be a finite-state Markov chain with generator $Q(t)$. Consider the linear system

$$\begin{aligned} \frac{dx(t)}{dt} &= A(t, \alpha(t))x(t) + B(t, \alpha(t))u(t), \\ x(0) &= x, \end{aligned} \tag{3.10}$$

where $x(t) \in \mathbb{R}^r$, $u(t) \in \mathbb{R}^m$, $A(t,\alpha) \in \mathbb{R}^{r\times r}$, and $B(t,\alpha) \in \mathbb{R}^{r\times m}$. The objective is to find the optimal control $u(\cdot)$ so that the expected quadratic cost function

$$\begin{aligned} J(x,\alpha,u(\cdot)) &= E\Big\{ \int_0^T [x'(t)\Phi(t,\alpha(t))x(t) \\ &\quad +u'(t)\Psi(t,\alpha(t))u(t)]dt + x'(T)\widetilde{\Phi}(\alpha(T))x(T)\Big\} \end{aligned} \tag{3.11}$$

is minimized. For general notion of optimal control theory and related topics, see Fleming and Rishel [63]. In the formulation above, $A(\cdot,\alpha)$, $B(\cdot,\alpha)$, $\Phi(\cdot,\alpha)$, and $\Psi(\cdot,\alpha)$ are continuous functions for each α. Moreover, Φ, Ψ, and $\widetilde{\Phi}$ are symmetric nonnegative matrices. Some of the recent work on this and related problems can be found in Ji and Chizeck [98]. This problem can be referred to as an LQ problem with Markovian jumps.

A slight variation of the problem mentioned above is based on the linear quadratic Gaussian (LQG) formulation. Let the system equation be

$$dx(t) = [A(t,\alpha(t))x(t) + B(t,\alpha(t))u(t)]dt + D(t,\alpha(t))dw(t), \tag{3.12}$$

where $w(t)$ is an $\mathbb{R}^l$-valued standard Brownian motion, $D(t,\alpha(t)) \in \mathbb{R}^{r\times l}$. The problem now becomes an LQG problem with Markov jumps. Compared with the classical LQ and LQG problems, complication arises due to the jump processes. Many current research activities are on the control, stabilization, and related matters of the underlying systems.

3.5.2 Singularly Perturbed LQ Systems with Wide-Band Noise

In the LQG problem above, if one considers the system consisting of (3.10) and (3.11) with $\alpha(t)$ replaced by $\alpha^\varepsilon(t)$ generated by $Q^\varepsilon(t) = Q(t)/\varepsilon$, then one has a singularly perturbed LQG system. The system undergoes rather rapid changes and variations for small ε. The jumps occur frequently and the system is subject to "irregular" or singular perturbations.

Another variation of the problem is to replace the white noise by a wide-band noise. This is originated from the jump LQG formulation. Consider the system

$$\frac{dx^\varepsilon(t)}{dt} = A(t,\alpha^\varepsilon(t))x^\varepsilon(t) + B(t,\alpha^\varepsilon(t))u(t) + \frac{1}{\varepsilon}D(t,\alpha^\varepsilon(t))\xi^\varepsilon(t). \tag{3.13}$$

The other quantities remain the same as (3.12) while $\xi^\varepsilon(t)$ is a wide-band noise process. Note that in general the wide-band noise depends on another small parameter $\delta > 0$, so the process to be considered is $\xi^\delta(t)$. Here we

consider a special case, namely, $\varepsilon = \delta$ for simplicity. A wide-band noise is one such that it approximates the "white noise." To illustrate, assume

$$\xi^\varepsilon(t) = \xi\left(\frac{t}{\varepsilon^2}\right),$$

where $\xi(\cdot)$ is a right-continuous stationary process with correlation $R(\cdot)$ and spectral density $S(\cdot)$. Let $R^\varepsilon(s)$ be the correlation of $(1/\varepsilon)\xi^\varepsilon(\cdot)$, i.e.,

$$R^\varepsilon(s) = \frac{1}{\varepsilon^2} E\xi^\varepsilon(t+s)\xi^{\varepsilon,\prime}(t),$$

and assume that the corresponding power spectral density (Fourier transform)

$$S^\varepsilon(\tau) = \int_{-\infty}^{\infty} e^{i\tau s} R^\varepsilon(s) ds$$

exists. If $\xi^\varepsilon(\cdot)$ is wide-band, then $S^\varepsilon(\tau)$ is effectively band limited, i.e., $S^\varepsilon(\tau) = 0$ for τ outside a certain interval, and the length of this interval is wide enough. Owing to the scaling, the spectral density of $(1/\varepsilon)\xi\left(t/\varepsilon^2\right)$ is $S^\varepsilon(\tau) = S(\varepsilon^2\tau)$. Moreover, $S^\varepsilon(\tau) = 0$ for all τ satisfying $|\tau| > \varepsilon^{-2}\tau_0$ for some $\tau_0 > 0$. The bandwidth is of the order $(1/\varepsilon^2)$. As ε gets smaller and smaller, the bandwidth gets wider and wider. As $\varepsilon \to 0$ the bandwidth of $S^\varepsilon(\tau)$ tends to infinity, and the spectral density tends to that of the white or Gaussian noise. The motivation for using wide-band noise is that in various applications an exact Gaussian white noise may not exist, but it can be approximated by a physical random process. The optimal control problem of (3.11) subject to (3.13) is quite difficult especially when the underlying system is large. However, one may wish to find nearly optimal controls instead. Interested readers are referred to Kushner [140]; see also Yin and Zhang [233, 234]. Further applications of nearly optimal controls and the corresponding numerical methods will be discussed in Chapters 7–10.

3.5.3 Large-Scale Systems: Decomposition and Aggregation

Consider the following system

$$dx(t) = [A(t,\alpha(t))x(t) + B(t,\alpha(t))u(t)]dt + D(t,\alpha(t))dw(t),$$

where $w(\cdot)$ is a standard Brownian motion, or consider

$$\frac{dx(t)}{dt} = A(t,\alpha(t))x(t) + B(t,\alpha(t))u(t) + D(t,\alpha(t))v(t),$$

where $v(\cdot)$ is a bounded (deterministic) disturbance. In either of the models above, $\alpha(t) \in \mathcal{M} = \{1,\ldots,m\}$, $t \geq 0$, is a finite-state Markov chain

characterizing the piecewise-deterministic behavior of the parameter process. One then works with an appropriate cost function $J(\cdot)$, and aims at deriving the optimality.

Recent interest for such jump linear systems stems from the fact they can be used to describe unpredictable structural changes. Owing to the various source of uncertainties, in many real-world applications, the parameter process is of very high dimension. This brings about much of the difficulty in analyzing such systems. In addition, the systems may be quite sensitive to small perturbations of the parameter values. The large dimensionality together with the sensitivity makes the actual computation infeasible. An immediate question is how can one resolve the problem and render a reasonable solution?

It turns out that the idea of hierarchical approach is useful in this regard. In fact, almost all complex systems in nature exhibit a hierarchical structure (see Simon [195]). If one can effectively take advantage of the structural properties, a large-dimensional (complex) system can be decomposed into a number of subsystems such that each of them with a simpler structure can be handled relatively easily. With this in mind, one introduces a small parameter $\varepsilon > 0$ to make the system under consideration display a two-time-scale behavior (see Phillips and Kokotovic [175] and Pan and Başar [165]).

Note that under the two-time-scale framework, the underlying Markov chain becomes a singularly perturbed one, i.e., $\alpha(t) = \alpha^\varepsilon(t)$ where the generator of $\alpha^\varepsilon(t)$ is given by $Q^\varepsilon(t)$. To analyze the system or to design the optimal controls, the foremost task is to study the structure of the Markov chain $\alpha^\varepsilon(\cdot)$ through its probability distribution. In what follows, consider two models. In both models, the generator $Q^\varepsilon(t)$ consists of two parts, a rapidly changing part and a slowly varying one, i.e.,

$$Q^\varepsilon(t) = \frac{1}{\varepsilon}\widetilde{Q}(t) + \widehat{Q}(t).$$

In the first model,

$$\widetilde{Q}(t) = \begin{pmatrix} \widetilde{Q}^1(t) & & & \\ & \widetilde{Q}^2(t) & & \\ & & \ddots & \\ & & & \widetilde{Q}^l(t) \end{pmatrix},$$

where $\widetilde{Q}^k(t) \in \mathbb{R}^{m_k \times m_k}$ for $k = 1, 2, \ldots, l$, and $\sum_{k=1}^{l} m_k = m$. Assume that $\widetilde{Q}^k(t)$ and $\widehat{Q}(t)$ are generators. Treating such nonstationary models, Chapter 4 obtains asymptotic expansion under mild conditions. Then Chapter 5 continues the discussion, derives further properties of the model, and establishes the convergence of an aggregated chain. Furthermore, in Chapters 7 and 8, we deal with nearly optimal control of large-scale

systems under such a Markovian structure. The key idea is decomposition/aggregation, which leads to an effective reduction of dimensionality.

In the second model, assume that the fast changing part also involves transient states presented by

$$\widetilde{Q}(t) = \begin{pmatrix} \widetilde{Q}^1(t) & & & \\ & \ddots & & \\ & & \widetilde{Q}^l(t) & \\ \widetilde{Q}^1_*(t) & \cdots & \widetilde{Q}^l_*(t) & \widetilde{Q}_*(t) \end{pmatrix}$$

such that for each $k = 1, \ldots, l$, $\widetilde{Q}^k(t)$ is an generator with dimension $m_k \times m_k$, $\widetilde{Q}_*(t)$ is an $m_* \times m_*$ stable matrix, $\widetilde{Q}^k_*(t) \in \mathbb{R}^{m_* \times m_k}$, $m_1 + m_2 + \cdots + m_l + m_* = m$, and $\widetilde{Q}(t)$ and $\widehat{Q}(t)$ are themselves generators. This model is inspired by the work of Phillips and Kokotovic [175] and Pan and Başar [165]; unlike these references, our main concern is on nonstationary (time-dependent) Markov chains. Chapter 4 studies analytic properties of this model. Although the inclusion of the transient states gives rise to seemingly more complex formulation, we demonstrate that the problem can still be dealt with. Our approach does provide generality for various situations.

3.6 Time-Scale Separation

Focusing on singularly perturbed Markov chains, one of our main concerns is time-scale separation, i.e., the models under consideration have two-time scales. The underlying Markovian models involve fast and slow motions that interact through weak and strong components. A main assumption in this book is that the corresponding generator can be separated into two generators having different scales of jump rates. This section gives motivation and interpretation on the time-scale separation. For simplicity, consider a model with time-independent generator

$$Q^\varepsilon = \frac{1}{\varepsilon}\widetilde{Q} + \widehat{Q}, \tag{3.14}$$

where $\widetilde{Q}$ and $\widehat{Q}$ are themselves time-invariant generators. Here ε is a small parameter that separates the time-scale. The generator $\widetilde{Q}$ dictates the fast motion of the Markov chain and $\widehat{Q}$ governs the slow motion. As will be seen in Chapter 4, the probability distribution of the underlying Markov chain will quickly reach a stationary regime determined by $\widetilde{Q}$, then the influence of $\widehat{Q}$ takes over subsequently.

To apply the results of this book in a practical scenario, it is important to determine if a given problem fits the models in this book. The small parameter ε should be interpreted as the parameter that separates different scales in the sense of order of magnitude. For a given generator Q, if we can

write it as the sum of two generators of the form (3.14), where $\widetilde{Q}$ and $\widehat{Q}$ are of the same order of magnitude, then $\varepsilon > 0$ takes care of the separation of scales. Typically, it works well in practice when ε is sufficiently small (i.e., less than a "small" constant, e.g., 0.5). For instance, if all the elements in $\widetilde{Q}/\varepsilon$ are around 10, and that of $\widehat{Q}$ are near 1, then $\varepsilon = 0.1$. In the queueing example of Section 3.3, if the interarrival and service rates of type I customers are 0.5 customer/minute and the corresponding rates of type II customers are 0.05 customer/minute, then $\varepsilon = 0.1$.

If one has certain knowledge about the behavior of the physical model, then one can use the information to determine the rates of different time scales. Alternatively, one may numerically decompose a given generator to determine its structure. To illustrate, consider the following example.

Let a generator Q be given by

$$Q = \begin{pmatrix} -12 & 1 & 10 & 1 \\ 0 & -11 & 1 & 10 \\ 21 & 1 & -22 & 0 \\ 1 & 30 & 2 & -33 \end{pmatrix};$$

the corresponding state space is $\mathcal{M} = \{s_1, s_2, s_3, s_4\}$. We demonstrate how such a generator may be decomposed into the form (3.14).

Step 1. Separate the entries of the matrix in accordance with their order of magnitude.

Apparently, the numbers in $\{1, 2\}$ are at different scale (order of magnitude) from the numbers in $\{10, -11, -12, 21, -22, 30, -33\}$, so we write Q as

$$Q = \begin{pmatrix} -12 & 0 & 10 & 0 \\ 0 & -11 & 0 & 10 \\ 21 & 0 & -22 & 0 \\ 0 & 30 & 0 & -33 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix}.$$

Step 2. Make each matrix a generator.

This step requires moving the entries so that each of the two matrices satisfies the condition of a generator, i.e., a matrix with nonnegative off-diagonal elements, non-positive diagonal elements, and zero row sums. For the Q matrix given above, by rearrangements,

$$Q = \begin{pmatrix} -10 & 0 & 10 & 0 \\ 0 & -10 & 0 & 10 \\ 21 & 0 & -21 & 0 \\ 0 & 30 & 0 & -30 \end{pmatrix} + \begin{pmatrix} -2 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & -3 \end{pmatrix}.$$

Step 3. Permute rows and columns.

By exchanging rows and columns, make the dominating matrix $\widetilde{Q}$ be one having a desired form, e.g., block diagonal form (corresponding to irreducible blocks). In this example, if we exchange the order s_2 and s_3 in $\mathcal{M}$, i.e., taking $\mathcal{M} = \{s_1, s_3, s_2, s_4\}$, then the corresponding generator is

$$Q = \begin{pmatrix} -10 & 10 & 0 & 0 \\ 21 & -21 & 0 & 0 \\ 0 & 0 & -10 & 10 \\ 0 & 0 & 30 & -30 \end{pmatrix} + \begin{pmatrix} -2 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & -3 \end{pmatrix}.$$

Let $\varepsilon = 0.1$. Then we can write

$$Q = \frac{1}{\varepsilon} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 2.1 & -2.1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 3 & -3 \end{pmatrix} + \begin{pmatrix} -2 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & -3 \end{pmatrix}.$$

Note that the above procedure may provide more than one representations of the decomposition. For instance, we can also write

$$Q = \frac{1}{\varepsilon} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 3 & -3 \end{pmatrix} + \begin{pmatrix} -2 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & -3 \end{pmatrix}$$

with $\varepsilon = 0.1$. Up to now, with only elementary row and column operations and some rearrangements, we succeed to reduce the matrix Q under consideration to the form (3.14).

This procedure is applicable to time-dependent generators as well. It can be used to incorporate generators with a block of transient states. The reduction to the "canonical form" may also be done by using the decomposition procedure outlined by Avramovic. The idea is to compute the eigenvalues of the generator and to use that to determine the structure of the fast part $\widetilde{Q}$. This is a more involved approach. We refer the reader to the paper of Phillips and Kokotovic [175] for more details.

3.7 Notes

This chapter presents a number of examples involving Markov chains. The problems arising in manufacturing systems (see, for example, Sethi and Zhang [192], and also Yin and Zhang [235]) are the original motivation of

our study. More details on related problems in Markov decision processes and stochastic dynamic systems will be discussed in Chapters 7–10.

We include relatively simple examples in this chapter. Our purpose is to give an overview and to illustrate the needed study for the underlying properties of Markov chains. For general definition and basic properties of Markov processes, see Chung [31], Davis [41], Dynkin [51], Karlin and Taylor [105] among others. Queueing problems and more finite-state models and variations of the formulations than those presented here can be found in Sharma [194]. Research in queueing system is very active; we are only able to include a handful of examples here and many references can be found in the most recent literature; for example Serfozo [190] and references therein. Reliability theory is discussed in detail in Hoyland and Rausand [88]. Most of these references are concerned primarily with homogeneous or stationary Markov chains. The study of the simulated annealing is related to the asymptotic behavior of the generator (see, for example, Chiang and Chow [28] and Wentzel [217]). A comprehensive study on stochastic optimization and related matters is in Kushner and Yin [145]. Optimal control of Markov decision processes and discrete stochastic dynamic programming can be found in Puterman [179] among others. For recent work on continuous-time Markov decision processes, see Guo and Hernández-Lerma [78]. Other topics not covered in this chapter include, for example, the formulation of random environment and the corresponding queueing system in a random environment (see Neuts [162]). In addition, in this book, we are only able to cover a handful of application examples. A work exclusively dealt with applications in queueing systems, and financial market models, insurance risk problems etc. is Yin, Zhang, and Zhang [232]. For some recent work on two-time-scale systems, we refer the reader to Kushner [140], and Kabanov and Pergamenshchikov [100].

In this book, we mainly deal with continuous-time Markov chains. Related issues and examples for discrete-time models can be found in the classical work of Karlin and Taylor [105], Revuz [180] or the recent publication of Meyn and Tweedie [159]. For discrete-time singularly perturbed Markov chains, see Abbad, Filar, and Bielecki [1], Bielecki and Filar [11], and the references therein. A comprehensive study on singularly perturbed, discrete-time, Markov systems is Yin and Zhang [238].

Part II

Two-Time-Scale Markov Chains

4

Asymptotic Expansions of Solutions for Forward Equations

4.1 Introduction

This chapter is concerned with the analysis of the probability distributions of two-time-scale Markov chains. We aim to approximate the solution of forward equation by means of sequences of functions so that the desired accuracy is reached. As alluded to in Chapter 1, we devote our attention to nonstationary Markov chains with time-varying generators. A key feature here is time-scale separation. By introducing a small parameter $\varepsilon > 0$, the generator and hence the corresponding Markov chain have "two times," a usual running time t and a fast time t/ε. The main approach that we are using is the matched asymptotic expansions from singular perturbation theory. We first construct a sequence of functions that well approximate the solution of the forward equation when t is large enough (outside the initial layer of $O(\varepsilon)$). By adopting the notion of singular perturbation theory, this part of the approximation will be called outer expansions. We demonstrate that it is a good approximation as long as t is not in a neighborhood of 0 of the order $O(\varepsilon)$. Nevertheless, this sequence of functions does not satisfy the given initial condition and the approximation breaks down when $t \leq O(\varepsilon)$. To circumvent these difficulties, we construct another sequence of functions by magnifying the asymptotic behavior of the solution near 0 using the stretched fast time $\tau = t/\varepsilon$. Following the traditional terminology in singular perturbation theory, we call this sequence of functions initial-layer corrections (or sometimes, boundary-layer corrections). It effectively yields

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corrections to the outer expansions and makes sure that the approximation is good in a neighborhood of $O(\varepsilon)$. By combining the outer expansions and the initial-layer corrections, we obtain a sequence of matched asymptotic expansions. The entire process is constructive. Our aims in this chapter include:

- Construct the outer expansions and the initial-layer corrections. This construction is often referred to as formal expansions.
- Justify the sequence of approximations obtained by deriving the desired error bounds. To achieve this, we show that (i) the outer solutions are sufficiently smooth, (ii) the initial-layer terms all decay exponentially fast, and (iii) the error is of the desired order. Thus not only is convergence of the asymptotic expansions proved, but also the error bound is obtained.
- Demonstrate that the error bounds hold uniformly. We would like to mention that in the usual singular perturbation theory, for example, in treating a linear system of differential equations, it is required that the system matrix be stable (i.e., all eigenvalues have negative real parts). In our setup, even for a homogeneous Markov chain, the generator (the system matrix in the equation) has an eigenvalue 0, so is not invertible. Thus, the stability requirement is violated. Nevertheless, using Markov properties, we are still able to obtain the desired asymptotic expansions.

Before proceeding further, we present a lemma. Let $Q(t) \in \mathbb{R}^{m\times m}$ be a generator, and let $\alpha(t)$ be a finite-state Markov chain with state space $\mathcal{M} = \{1, \ldots, m\}$ and generator $Q(t)$. Denote by

$$p(t) = (P(\alpha(t) = 1), \ldots, P(\alpha(t) = m)) \in \mathbb{R}^{1\times m}$$

the row vector of the probability distribution of the underlying chain at time t. Then in view of Theorem 2.5, $p(\cdot)$ is a solution of the forward equation

$$\begin{aligned} &\frac{dp(t)}{dt} = pQ(t) = p(t)Q(t), \\ &p(0) = p^0 \text{ such that } p_i^0 \geq 0 \text{ for each } i, \text{ and } \sum_{i=1}^{m} p_i^0 = 1, \end{aligned} \tag{4.1}$$

where $p^0 = (p_1^0, \ldots, p_m^0)$ and p_i^0 denotes the ith component of p^0. Therefore, studying the probability distribution is equivalent to examining the solution of (4.1). Note that the forward equation is linear, so the solution is unique. As a result, the following lemma is immediate. This lemma will prove useful in subsequent study.

Lemma 4.1. *The solution $p(t)$ of (4.1) satisfies the conditions*

$$0 \le p_i(t) \le 1 \text{ and } \sum_{i=1}^{m} p_i(t) = 1. \tag{4.2}$$

Remark 4.2. For the reader whose interests are mainly in differential equations, we point out that the initial condition $\sum_{i=1}^{m} p_i^0 = 1$ in (4.1) is not restrictive since if $p^0 = 0$, then $p(t) = 0$ is the only solution to (4.1). If $p_i^0 > 0$ for some i, one may divide both sides of (4.1) by $\sum_{i=1}^{m} p_i^0$ (> 0) and consider $\widetilde{p}(t) = p(t)/\sum_{i=1}^{m} p_i^0$ in lieu of $p(t)$.

To achieve our goal, we first treat a simple case, namely, the case that the generator is weakly irreducible. Once this is established, we proceed to the more complex case that the generator has several weakly irreducible classes, the inclusion of absorbing states, and the inclusion of transient states.

The rest of the chapter is arranged as follows. Section 4.2 begins with the study of the situation in which the generator is weakly irreducible. Although it is a simple case, it outlines the main ideas behind the construction of asymptotic expansions. This section begins with the construction of formal expansions, proves the needed regularity, and ascertains the error estimates. Section 4.3 develops asymptotic expansions of the underlying probability distribution for the chains with recurrent states. As will be seen in the analysis to follow, extreme care must be taken to handle two-time-scale Markov chains with fast and slow components. One of the key issues is the selection of appropriate initial conditions to make the series a "matched" asymptotic expansions, in which the separable form of our asymptotic expansion appears to be advantageous compared with the two-time-scale expansions. For easy reference, a subsection is also provided as a user's guide.

Using the methods of matched asymptotic expansion, Section 4.4 extends the results to include absorbing states. It demonstrates that similar techniques can be used. We also demonstrate that the techniques and methods of Section 4.3 are rather general and can be applied to a wide variety of cases. Section 4.5 continues the study of problems involving transient states. By treating chains having recurrent states, chains including absorbing states, and chains including transient states, we are able to characterize the probability distributions of the underlying singularly perturbed chains of general cases with finite-state spaces, and hence provide comprehensive pictures through these "canonical" models.

While Sections 4.3–4.5 cover most practical concerns of interest for the finite-state-space cases, the rest of the chapter makes several remarks on Markov chains with countable-state spaces and two-time-scale diffusions. In Section 4.6.1, we extend the results to processes with countable-state spaces in which $\widetilde{Q}(t)$ is a block-diagonal matrix with infinitely many blocks each of which is finite-dimensional. Then Section 4.6.2 treats the problem in which $\widetilde{Q}(t)$ itself is an infinite-dimensional matrix. In this case,

further conditions are necessary. As in the finite-dimensional counterpart, sufficient conditions that ensure the validity of the asymptotic expansions are provided. The essential ingredients include Fredholm-alternative-like conditions and the notion of weak irreducibility. Finally, we mention related results of singularly perturbed diffusions in Section 4.7. Additional notes and remarks are given in Section 4.8.

4.2 Irreducible Case

We begin with the case concerning weakly irreducible generators. Let $Q(t) \in \mathbb{R}^{m\times m}$ be a generator, $\varepsilon > 0$ be a small parameter, and suppose that $\alpha^\varepsilon(t)$ is a finite-state Markov chain with state space $\mathcal{M} = \{1, \ldots, m\}$ generated by $Q^\varepsilon(t) = Q(t)/\varepsilon$. The row vector $p^\varepsilon(t) = (P(\alpha^\varepsilon(t) = 1), \ldots, P(\alpha^\varepsilon(t) = m)) \in \mathbb{R}^{1\times m}$ denotes the probability distribution of the underlying chain at time t. Then by virtue of Theorem 2.5, $p^\varepsilon(\cdot)$ is a solution of the forward equation

$$\begin{aligned} &\frac{dp^\varepsilon(t)}{dt} = p^\varepsilon Q^\varepsilon(t) = \frac{1}{\varepsilon} p^\varepsilon(t) Q(t), \\ &p^\varepsilon(0) = p^0 \text{ such that } p_i^0 \ge 0 \text{ for each } i, \text{ and } \sum_{i=1}^{m} p_i^0 = 1, \end{aligned} \tag{4.3}$$

where $p^0 = (p_1^0, \ldots, p_m^0)$ and p_i^0 denotes the ith component of p^0. Therefore, studying the probability distribution is equivalent to examining the solution of (4.3). Now, Lemma 4.1 continues to hold for the solution $p^\varepsilon(t)$.

As discussed in Chapters 1 and 3, the equation in (4.3) arises from various applications involving a rapidly fluctuating Markov chain governed by the generator $Q(t)/\varepsilon$. As ε gets smaller and smaller, the Markov chain fluctuates more and more rapidly. Normally, the fast-changing process $\alpha^\varepsilon(\cdot)$ in an actual system is difficult to analyze. The desired limit properties, however, provide us with an alternative. We can replace the actual process by its "average" in the system under consideration. This approach has significant practical value. A fundamental question common to numerous applications involving two-time-scale Markov chains is to understand the asymptotic properties of $p^\varepsilon(\cdot)$, namely, the limit behavior as $\varepsilon \to 0$. If $Q(t) = Q$, a constant matrix, and if Q is irreducible (see Definition 2.7), then for each $t > 0$, $p^\varepsilon(t) \to \nu$, the familiar stationary distribution. For the time-varying counterpart, it is reasonable to expect that the corresponding distribution will converge to a probability distribution that mimics the main features of the distribution of stationary chains, meanwhile preserving the time-varying nature of the nonstationary system. A candidate bearing such characteristics is the quasi-stationary distribution $\nu(t)$. Recall

that $\nu(t)$ is said to be a quasi-stationary distribution (see Definition 2.8) if $\nu(t) = (\nu_1(t), \ldots, \nu_m(t)) \geq 0$ and it satisfies the equations

$$\nu(t)Q(t) = 0 \text{ and } \sum_{i=1}^{m} \nu_i(t) = 1. \tag{4.4}$$

If $Q(t) \equiv Q$, a constant matrix, then an analytic solution of (4.3) is obtainable, since the fundamental matrix solution (see Hale [79]) takes the simple form $\exp(Qt)$; the limit behavior of $p^\varepsilon(t)$ is derivable through the solution $p^0 \exp(Qt/\varepsilon)$. For time-dependent $Q(t)$, although the fundamental matrix solution still exists, it does not have a simple form. The complex integral representation is not very informative in the asymptotic study of $p^\varepsilon(t)$, except in the case $m = 2$. In this case, $\alpha^\varepsilon(\cdot)$ is a two-state Markov chain and the constraint $p_1^\varepsilon(t) + p_2^\varepsilon(t) = 1$ reduces the current problem to a scalar one. Therefore, a closed-form solution is possible. However, such a technique cannot be generalized to $m > 2$. Let $0 < T < \infty$ be a finite real number. We divide the interval $[0, T]$ into two parts. One part is for t very close to 0 (in the range of an ε-layer), and the other is for t bounded away from 0. The behavior of $p^\varepsilon(\cdot)$ differs significantly in these two regions. Such a division led us to the utilization of the matched asymptotic expansion. Not only do we prove the convergence of $p^\varepsilon(t)$ as $\varepsilon \to 0$, but we also obtain an asymptotic series. The procedure involves constructing the regular part (outer expansion) for t to be away from 0, as well as the initial-layer corrections for small t, and to match these expansions by a proper choice of initial conditions.

In what follows, in addition to obtaining the zeroth-order approximation, i.e., the convergence of $p^\varepsilon(\cdot)$ to its quasi-stationary distribution, we derive higher-order approximations and error bounds. A consequence of the findings is that the convergence of the probability distribution and related occupation measures of the corresponding Markov chain takes place in an appropriate sense. The asymptotic properties of a suitably scaled occupation time and the corresponding central limit theorem for $\alpha^\varepsilon(\cdot)$ (based on the expansion) will be studied in Chapter 5.

4.2.1 Asymptotic Expansions

To proceed, we make the following assumptions.

(**A4.1**) Given $0 < T < \infty$, for each $t \in [0, T]$, $Q(t)$ is weakly irreducible, that is, the system of equations

$$\begin{aligned} f(t)Q(t) &= 0, \\ \sum_{i=1}^{m} f_i(t) &= 1 \end{aligned} \tag{4.5}$$

has a unique nonnegative solution.

(**A4.2**) For some n, $Q(\cdot)$ is $(n+1)$-times continuously differentiable on $[0,T]$, and $(d^{n+1}/dt^{n+1})Q(\cdot)$ is Lipschitz on $[0,T]$.

Remark 4.3. Condition (A4.2) requires that the matrix $Q(t)$ be sufficiently smooth. This is necessary for obtaining the desired asymptotic expansion. To validate the asymptotic expansion, we need to estimate the remainder term. Thus for the nth-order approximation, we need the $(n+1)$st-order smoothness.

To proceed, we first state a lemma. Its proof is in Lemma A.2 in the appendix.

Lemma 4.4. *Consider the matrix differential equation*

$$\frac{dP(s)}{ds} = P(s)A, \quad P(0) = I, \tag{4.6}$$

where $P(s) \in \mathbb{R}^{m\times m}$. Suppose $A \in \mathbb{R}^{m\times m}$ is a generator of a (homogeneous or stationary) finite-state Markov chain and is weakly irreducible. Then $P(s) \to \overline{P}$ as $s \to \infty$ and

$$\left|\exp(As) - \overline{P}\right| \le K\exp(-\widetilde{\kappa}s) \quad \text{for some } \widetilde{\kappa} > 0, \tag{4.7}$$

where $\overline{P} = \mathbb{1}(\overline{\nu}_1, \cdots, \overline{\nu}_m) \in \mathbb{R}^{m\times m}$, and $(\overline{\nu}_1, \ldots, \overline{\nu}_m)$ is the quasi-stationary distribution of the Markov process with generator A.

Recall that $\mathbb{1} = (1,\ldots,1)' \in \mathbb{R}^{m\times 1}$ and $(\overline{\nu}_1,\ldots,\overline{\nu}_m) \in \mathbb{R}^{1\times m}$. Thus $\mathbb{1}(\overline{\nu}_1,\ldots,\overline{\nu}_m)$ is the usual matrix product. Recall that an $m\times m$ matrix $P(s)$ is said to be a solution of (4.6) if each row of $P(s)$ satisfies the equation. In the lemma above, if A is a constant matrix that is irreducible, then $(\overline{\nu}_1,\ldots,\overline{\nu}_m)$ becomes the familiar stationary distribution. In general, A could be time-dependent, e.g., $A = A(t)$. As shown in Lemma A.4, by assuming the existence of the solution $\nu(t)$ to (4.5), it follows that $\nu(t) \ge 0$; that is, the nonnegativity assumption is redundant. We seek asymptotic expansions of the form

$$p^\varepsilon(t) = \Phi_n^\varepsilon(t) + \Psi_n^\varepsilon\left(\frac{t}{\varepsilon}\right) + e_n^\varepsilon(t),$$

where $e_n^\varepsilon(t)$ is the remainder,

$$\Phi_n^\varepsilon(t) = \varphi_0(t) + \varepsilon\varphi_1(t) + \cdots + \varepsilon^n\varphi_n(t), \tag{4.8}$$

and

$$\Psi_n^\varepsilon\left(\frac{t}{\varepsilon}\right) = \psi_0\left(\frac{t}{\varepsilon}\right) + \varepsilon\psi_1\left(\frac{t}{\varepsilon}\right) + \cdots + \varepsilon^n\psi_n\left(\frac{t}{\varepsilon}\right), \tag{4.9}$$

with the functions $\varphi_i(\cdot)$ and $\psi_i(\cdot)$ to be determined in the sequel. We now state the main result of this section.

Theorem 4.5. *Suppose that* (A4.1) *and* (A4.2) *are satisfied. Denote the unique solution of* (4.3) *by* $p^\varepsilon(\cdot)$. *Then two sequences of functions* $\varphi_i(\cdot)$ *and* $\psi_i(\cdot)$, $0 \le i \le n$, *can be constructed such that*

(a) $\varphi_i(\cdot)$ *is* $(n+1-i)$*-times continuously differentiable on* $[0,T]$;

(b) *for each* i, *there is a* $\kappa_0 > 0$ *such that*

$$\left|\psi_i\left(\frac{t}{\varepsilon}\right)\right| \le K\exp\left(-\frac{\kappa_0 t}{\varepsilon}\right);$$

(c) *the following estimate holds:*

$$\sup_{t\in[0,T]}\left|p^\varepsilon(t) - \sum_{i=0}^{n}\varepsilon^i\varphi_i(t) - \sum_{i=0}^{n}\varepsilon^i\psi_i\left(\frac{t}{\varepsilon}\right)\right| \le K\varepsilon^{n+1}. \tag{4.10}$$

Remark 4.6. The method described in what follows gives an explicit construction of the functions $\varphi_i(\cdot)$ and $\psi_i(\cdot)$ for $i \le n$. Thus the proof to be presented is constructive. Our plan is first to obtain these sequences, and then validate properties (a) and (b) above and derive an error bound in (c) by showing that the remainder

$$\left|p^\varepsilon(t) - \sum_{i=0}^{n}\varepsilon^i\varphi_i(t) - \sum_{i=0}^{n}\varepsilon^i\psi_i\left(\frac{t}{\varepsilon}\right)\right|$$

is of order $O(\varepsilon^{n+1})$ uniformly in t.

It will be seen from the subsequent development that $\varphi_0(t)$ is equal to the quasi-stationary distribution, that is, $\varphi_0(t) = \nu(t)$. In particular, if $n = 0$ in the above theorem, we have the following result.

Corollary 4.7. *Suppose* $Q(\cdot)$ *is continuously differentiable on* $[0,T]$, *which satisfies* (A4.1), *and* $(d/dt)Q(\cdot)$ *is Lipschitz on* $[0,T]$. *Then for all* $t > 0$,

$$\lim_{\varepsilon\to 0} p^\varepsilon(t) = \nu(t) = \varphi_0(t), \tag{4.11}$$

i.e., $p^\varepsilon(\cdot)$ *converges to the quasi-stationary distribution.*

Remark 4.8. The theorem manifests the convergence of $p^\varepsilon(\cdot)$ to $\varphi_0(\cdot)$, as well as the rate of convergence. In addition to the zeroth-order approximation, we have the first-order approximation, the second-order approximation, and so on. In fact, the difference $p^\varepsilon(\cdot) - \varphi_0(\cdot)$ is characterized by the initial-layer term $\psi_0(\cdot)$ and the associated error bound.

If the initial condition is chosen to be exactly equal to $p^0 = \varphi_0(0)$, then in the expansion, the zeroth-order initial layer $\psi_0(\cdot)$ will vanish. This cannot be expected in general, however. Even if $\psi_0(\cdot) = 0$, the rest of the initial-layer terms $\psi_i(\cdot)$, $i \ge 1$ will still be there.

To proceed, we define an operator $\mathcal{L}^\varepsilon$ by

$$\mathcal{L}^\varepsilon f = \varepsilon \frac{df}{dt} - fQ, \tag{4.12}$$

for any smooth row-vector-valued function $f(\cdot)$. Then $\mathcal{L}^\varepsilon f = 0$ iff f is a solution to the differential equation in (4.3). The proof of Theorem 4.5 is divided into the following steps.

1. Construct the asymptotic series, i.e., find $\varphi_i(\cdot)$ and $\psi_i(\cdot)$, for $i \leq n$. For the purpose of evaluating the remainder, we need to calculate two extra terms $\varphi_{n+1}(\cdot)$ and $\psi_{n+1}(\cdot)$. This will become clear when we carry out the error analysis.
2. Obtain the regularity of $\varphi_i(\cdot)$ and $\psi_i(\cdot)$ by proving that $\varphi_i(\cdot)$ is $(n+1-i)$-times continuously differentiable on $[0,T]$ and that $\psi_i(\cdot)$ decays exponentially fast.
3. Carry out the error analysis and justify that the remainder has the desired property.

4.2.2 Outer Expansion

We begin with the construction of $\Phi_n^\varepsilon(\cdot)$ in the asymptotic expansion. We call it the outer expansion or the regular part of expansion. Consider the differential equation

$$\mathcal{L}^\varepsilon \Phi_{n+1}^\varepsilon = 0$$

where $\mathcal{L}^\varepsilon$ is given by (4.12).

By equating the coefficients of ε^k, for $k = 1, \ldots, n+1$, we obtain

$$\begin{aligned}
\varepsilon^0 : &\quad \varphi_0(t)Q(t) = 0, \\
\varepsilon^1 : &\quad \varphi_1(t)Q(t) = \frac{d\varphi_0(t)}{dt}, \\
&\quad \cdots \\
\varepsilon^k : &\quad \varphi_k(t)Q(t) = \frac{d\varphi_{k-1}(t)}{dt}, \quad \text{for } k = 1, \ldots, n+1.
\end{aligned} \tag{4.13}$$

Remark 4.9. First, one has to make sure that the equations above have solutions, that is, a consistency condition needs to be verified. For each $t \in [0,T]$, denote the null space of $Q(t)$ by $N(Q(t))$. Note that the irreducibility of $Q(t)$ implies that

$$\operatorname{rank}(Q(t)) = m - 1,$$

thus

$$\dim(N(Q(t))) = 1.$$

It is easily seen that $N(Q(t))$ is spanned by the vector $\mathbb{1}$. By virtue of the Fredholm alternative (see Corollary A.38), the second equation in (4.13) has a solution only if its right-hand side, namely, $(d/dt)\varphi_0(t)$ is orthogonal to $N(Q(t))$. Since $N(Q(t))$ is spanned by $\mathbb{1}$,

$$\varphi_0(t)\mathbb{1} = 1$$

and

$$\frac{d\varphi_0(t)}{dt}\mathbb{1} = \frac{d\left(\varphi_0(t)\mathbb{1}\right)}{dt} = 0,$$

the orthogonality is easily verified. Similar arguments hold for the rest of the equations. The consistency in fact is rather crucial. Without such a condition, one would not be able to solve the equations in (4.13). This point will be made again when we deal with weak and strong interaction models in Section 4.3.

Recall that the components of $p^\varepsilon(\cdot)$ are probabilities (see (4.2)). In what follows, we show that all these $\varphi_i(\cdot)$ can be determined by (4.13) and (4.2).

Note that $\operatorname{rank}(Q(t)) = m - 1$. Thus $Q(t)$ is singular, and each equation in (4.13) is not uniquely solvable. For example, the first equation (4.13) cannot be solved uniquely. Nevertheless, this equation together with the constraint $\sum_{i=1}^m \varphi_0^i(t) = 1$ leads to a unique solution, namely, the quasi-stationary distribution.

In fact, a direct consequence of (A4.3) and (A4.4) is that the weak irreducibility of $Q(t)$ is uniform in the sense that for any $t \in [0,T]$, if *any* column of $Q(t)$ is replaced by $\mathbb{1} \in \mathbb{R}^{m\times 1}$, the resulting determinant $\Delta(t)$ satisfies $|\Delta(t)| > 0$, since (4.5) has only one solution, and $\sum_{j=1}^m q_{ij}(t) = 0$ for each $i = 1, \ldots, m$. Moreover, there is a number $c > 0$ such that $|\Delta(t)| \geq c > 0$. Thus, in view of the uniform continuity of $Q(t)$, $|\Delta(t)| \geq c > 0$ on $[0,T]$. We can replace any equation in the first m equations of the system $\varphi_0(t)Q(t) = 0$ by the equation $\sum_{i=1}^m \varphi_0^i(t) = 1$. The corresponding determinant $\Delta(t)$ of the resulting coefficient matrix satisfies $|\Delta(t)| \geq c > 0$, for some $c > 0$ and all $t \in [0,T]$. To illustrate, we may suppose without loss of generality that the mth equation is the one that can be replaced. Then we have

$$\begin{aligned}
&q_{11}(t)\varphi_0^1(t) + \cdots + q_{m1}(t)\varphi_0^m(t) = 0,\\
&q_{12}(t)\varphi_0^1(t) + \cdots + q_{m2}(t)\varphi_0^m(t) = 0,\\
&\quad\cdots\\
&q_{1,m-1}(t)\varphi_0^1(t) + \cdots + q_{m,m-1}(t)\varphi_0^m(t) = 0,\\
&\varphi_0^1(t) + \cdots + \varphi_0^m(t) = 1.
\end{aligned} \tag{4.14}$$

The determinant of the coefficient matrix in (4.14) is

$$\Delta(t) = \begin{vmatrix} q_{11}(t) & q_{21}(t) & \cdots & q_{m1}(t) \\ q_{12}(t) & q_{22}(t) & \cdots & q_{m2}(t) \\ \vdots & \vdots & \cdots & \vdots \\ q_{1,m-1}(t) & q_{2,m-1}(t) & \cdots & q_{m,m-1}(t) \\ 1 & 1 & \cdots & 1 \end{vmatrix} \tag{4.15}$$

and satisfies $|\Delta(t)| \geq c > 0$. Now by Cramer's rule, for each $0 \leq i \leq m$,

$$\varphi_0^i(t) = \frac{1}{\Delta(t)} \begin{vmatrix} q_{11}(t) & \cdots & 0 & \cdots & q_{m1}(t) \\ q_{12}(t) & \cdots & 0 & \cdots & q_{m2}(t) \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ q_{1,m-1}(t) & \cdots & 0 & \cdots & q_{m,m-1}(t) \\ 1 & \cdots & \underbrace{1}_{i\text{th column}} & \cdots & 1 \end{vmatrix},$$

that is, the ith column of $\Delta(t)$ in (4.15) is replaced by $(0, \ldots, 0, 1)' \in \mathbb{R}^{m \times 1}$. By the assumption of $Q(\cdot)$, it is plain that $\varphi_0(\cdot)$ is $(n+1)$-times continuously differentiable on $[0, T]$.

The foregoing method can be used to solve other equations in (4.13) analogously. Owing to the smoothness of $\varphi_0(\cdot)$, $(d/dt)\varphi_0(t)$ exists, and we can proceed to obtain $\varphi_1(\cdot)$. Repeat the procedure above, and continue inductively. For each $k \geq 1$,

$$\begin{aligned} &\sum_{i=1}^{m} \varphi_k^i(t) q_{ij}(t) = \frac{d\varphi_{k-1}^j(t)}{dt} \text{ for } j = 1, \ldots, m, \\ &\sum_{i=1}^{m} \varphi_k^i(t) = 0. \end{aligned} \tag{4.16}$$

Note that $\varphi_{k-1}^j(\cdot)$ has been found so $(d/dt)\varphi_{k-1}^j(t)$ is a known function. After a suitable replacement of one of the first m equations by the last equation in (4.16), the determinant $\Delta(t)$ of the resulting coefficient matrix satisfies $|\Delta(t)| \geq c > 0$. We obtain for each $0 \leq i \leq m$,

$$\varphi_k^i(t) = \frac{1}{\Delta(t)} \begin{vmatrix} q_{11}(t) & \cdots & \dfrac{d\varphi_{k-1}^1(t)}{dt} & \cdots & q_{m1}(t) \\ q_{12}(t) & \cdots & \dfrac{d\varphi_{k-1}^2(t)}{dt} & \cdots & q_{m2}(t) \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ q_{1,m-1}(t) & \cdots & \dfrac{d\varphi_{k-1}^{m-1}(t)}{dt} & \cdots & q_{m,m-1}(t) \\ 1 & \cdots & \underbrace{0}_{i\text{th column}} & \cdots & 1 \end{vmatrix}.$$

Hence $\varphi_k(\cdot)$ is $(n+1-k)$-times continuously differentiable on $[0,T]$. Thus we have constructed a sequence of functions $\varphi_k(t)$ that are $(n+1-k)$-times continuously differentiable on $[0,T]$ for $k=0,1,\ldots,n+1$.

Remark 4.10. The method used above is convenient for computational purposes. An alternative way of obtaining the sequence $\varphi_k(t)$ is as follows. For example, to solve

$$\varphi_0(t)Q(t)=0,\quad \sum_{j=1}^{m}\varphi_0^j(t)=1,$$

define $Q_c(t)=(\mathbb{1}\vdots Q(t))\in\mathbb{R}^{m\times(m+1)}$. Then the equation above can be written as

$$\varphi_0(t)Q_c(t)=(1,0,\ldots,0).$$

Note that $Q_c(t)Q_c'(t)$ has full rank m owing to weak irreducibility. Thus the solution of the equation is

$$\varphi_0(t)=(1,0,\ldots,0)Q_c'(t)[Q_c(t)Q_c'(t)]^{-1}.$$

We can obtain all other $\varphi_k(t)$ for $k=1,\ldots,n+1$, similarly.

The regular part $\Phi_n^\varepsilon(\cdot)$ is a good approximation to $p^\varepsilon(\cdot)$ when t is bounded away from 0. When t approaches 0, an initial layer (or a boundary layer) develops and the approximation breaks down. To accommodate this situation, an initial-layer correction, i.e., a sequence of functions $\psi_k(t/\varepsilon)$ for $k=0,1,\ldots,n+1$ needs to be constructed.

4.2.3 Initial-Layer Correction

This section is on the construction of the initial-layer terms. The presentation consists of two parts. We obtain the sequence $\{\psi_k(\cdot)\}$ in the first subsection, and derive the exponential decay property in the second subsection.

Construction of $\psi_k(\cdot)$. Following usual practice in singular perturbation theory, define the stretched (or rescaled) time variable by

$$\tau=\frac{t}{\varepsilon}. \tag{4.17}$$

Note that $\tau\to\infty$ as $\varepsilon\to 0$ for any given $t>0$.

Consider the differential equation

$$\mathcal{L}^\varepsilon\Psi_{n+1}^\varepsilon=\sum_{i=0}^{n+1}\varepsilon^i\mathcal{L}^\varepsilon\psi_i=0.$$

Using the stretched time variable τ, we arrive at

$$\frac{d\Psi^{\varepsilon}_{n+1}(\tau)}{d\tau} = \Psi^{\varepsilon}_{n+1}(\tau)Q(\varepsilon\tau).$$

Owing to the smoothness of $Q(\cdot)$, a truncated Taylor expansion about $\tau = 0$ leads to

$$Q(t) = Q(\varepsilon\tau) = \sum_{i=0}^{n+1} \frac{(\varepsilon\tau)^i}{i!} \frac{d^i Q(0)}{dt^i} + R_{n+1}(\varepsilon\tau),$$

where

$$R_{n+1}(t) = \frac{t^{n+1}}{(n+1)!} \left(\frac{d^{n+1}Q(\xi)}{dt^{n+1}} - \frac{d^{n+1}Q(0)}{dt^{n+1}} \right),$$

for some $0 < \xi < t$. In view of (A4.2),

$$R_{n+1}(t) = O(t^{n+2}) \text{ uniformly in } t \in [0, T].$$

Drop the term $R_{n+1}(t)$ and use the first $n+2$ terms to get

$$\frac{d\Psi^{\varepsilon}_{n+1}(\tau)}{d\tau} = \Psi^{\varepsilon}_{n+1}(\tau) \left(\sum_{i=0}^{n+1} \frac{(\varepsilon\tau)^i}{i!} \frac{d^i Q(0)}{dt^i} \right).$$

Similar to the previous section, for $k = 1, \ldots, n+1$, equating coefficients of ε^k, we have

$$\begin{aligned}
\varepsilon^0 :&\ \frac{d\psi_0(\tau)}{d\tau} = \psi_0(\tau)Q(0), \\
\varepsilon^1 :&\ \frac{d\psi_1(\tau)}{d\tau} = \psi_1(\tau)Q(0) + \tau\psi_0(\tau)\frac{dQ(0)}{dt}, \\
&\ \cdots \\
\varepsilon^k :&\ \frac{d\psi_k(\tau)}{d\tau} = \psi_k(\tau)Q(0) + r_k(\tau),
\end{aligned} \tag{4.18}$$

where $r_k(\tau)$ is a function having the form

$$\begin{aligned}
r_k(\tau) &= \frac{\tau^k}{k!}\psi_0(\tau)\frac{d^k Q(0)}{dt^k} + \cdots + \tau\psi_{k-1}(\tau)\frac{dQ(0)}{dt} \\
&= \sum_{i=1}^{k} \frac{\tau^i}{i!}\psi_{k-i}(\tau)\frac{d^i Q(0)}{dt^i}.
\end{aligned} \tag{4.19}$$

These equations together with appropriate initial conditions allow us to determine the $\psi_k(\cdot)$'s. For constructing $\varphi_k(\cdot)$, a number of algebraic equations are solved, whereas when determining ψ_k, one has to solve a number of differential equations instead. Two points are worth mentioning in

connection with (4.18). First the time-varying differential equation is replaced by one with constant coefficients; the solution thus can be written explicitly. The second point is on the selection of the initial conditions for $\psi_k(\cdot)$, with $k = 0, 1, \ldots, n+1$. We choose the initial conditions so that the initial data of the asymptotic expansion will "match" that of the differential equation (4.3). To be more specific,

$$\varphi_0(0) + \psi_0(0) = p^0, \text{ and}$$

$$\varphi_k(0) + \psi_k(0) = 0 \text{ for } k = 1, 2, \ldots, n+1.$$

Corresponding to ε^0, solving

$$\frac{d\psi_0(\tau)}{d\tau} = \psi_0(\tau)Q(0),$$

$$\psi_0(0) = p^0 - \varphi_0(0),$$

where p^0 is the initial data given in (4.3), one has

$$\psi_0(\tau) = (p^0 - \varphi_0(0)) \exp(Q(0)\tau). \tag{4.20}$$

Continuing in this fashion, for $k = 1, \ldots, n+1$, we obtain

$$\frac{d\psi_k(\tau)}{d\tau} = \psi_k(\tau)Q(0) + r_k(\tau),$$

$$\psi_k(0) = -\varphi_k(0).$$

In the equations above, we purposely separated $Q(0)$ from the term $r_k(\tau)$. As a result, the equations are linear systems with a constant matrix $Q(0)$ and time-varying forcing terms. This is useful for our subsequent investigation.

For $k = 1, 2, \ldots$, the solutions are given by

$$\begin{aligned} \psi_k(\tau) \; &= -\varphi_k(0) \exp(Q(0)\tau) \\ &\quad + \int_0^\tau r_k(s) \exp(Q(0)(\tau - s))\, ds. \end{aligned} \tag{4.21}$$

The construction of $\psi_k(\cdot)$ for $k = 0, 1, \ldots, n+1$, and hence the construction of the asymptotic series is complete.

4.2.4 Exponential Decay of $\psi_k(\cdot)$

This subsection concerns the exponential decay of $\psi_k(\cdot)$. At first glance, it seems to be troublesome since $Q(0)$ has a zero eigenvalue. Nevertheless, probabilistic argument helps us to derive the desired property. Two key points in the proof below are the utilization of orthogonality and repeated application of the approximation of $\exp(Q(0)\tau)$ in Lemma 4.4.

By virtue of Assumption (A4.1), the finite-state Markov chain generated by $Q(0)$ is weakly irreducible. Identifying $Q(0)$ with the matrix A in Lemma 4.4 yields that

$$\exp(Q(0)\tau) \to \overline{P} \text{ as } \tau \to \infty,$$

where $\overline{P} = \mathbb{1}\overline{\nu}$, and $\overline{\nu} = (\overline{\nu}_1, \ldots, \overline{\nu}_m)$ is the quasi-stationary distribution corresponding to the constant matrix $Q(0)$.

Proposition 4.11. *Under the conditions of Theorem 4.5, for each $0 \le k \le n+1$, there exist a nonnegative real polynomial $c_{2k}(\tau)$ of degree $2k$ and a positive number $\kappa_{0,0} > 0$ such that*

$$|\psi_k(\tau)| \le c_{2k}(\tau)\exp(-\kappa_{0,0}\tau). \tag{4.22}$$

Proof: First of all, note that

$$\sum_{i=1}^{m} p_i^0 = 1 \text{ and } \sum_{i=1}^{m} \varphi_0^i(0) = 1.$$

It follows that

$$\sum_{i=1}^{m} \psi_0^i(0) = \sum_{i=1}^{m} p_i^0 - \sum_{i=1}^{m} \varphi_0^i(0) = 0.$$

That is, $\psi_0(0)$ is orthogonal to $\mathbb{1}$. Consequently, $\psi_0(0)\overline{P} = 0$ and by virtue of Lemma 4.4 (with $A = Q(0)$), for some $\kappa_{0,0} := \widetilde{\kappa} > 0$,

$$\begin{aligned} |\psi_0(\tau)| &= |\psi_0(0)\exp(Q(0)\tau)| \\ &\le \left|\psi_0(0)\overline{P}\right| + \left|\psi_0(0)(\exp(Q(0)\tau) - \overline{P})\right| \\ &= \left|\psi_0(0)(\exp(Q(0)\tau) - \overline{P})\right| \le K\exp(-\kappa_{0,0}\tau). \end{aligned} \tag{4.23}$$

Note that

$$Q(t)\mathbb{1} = 0 \text{ for all } t \ge 0.$$

Differentiating this equation repeatedly leads to

$$\frac{d^k Q(t)}{dt^k}\mathbb{1} = \frac{d^k(Q(t)\mathbb{1})}{dt^k} = 0.$$

Hence, it follows that

$$\frac{d^k Q(0)}{dt^k}\mathbb{1} = 0 \quad \text{and} \quad \frac{d^k Q(0)}{dt^k}\overline{P} = 0,$$

for each $0 \le k \le n+1$. Owing to Lemma 4.4 and (4.21),

$$\begin{aligned}
|\psi_1(\tau)| \le\ & |\varphi_1(0)\exp(Q(0)\tau)| \\
& + \left| \int_0^\tau \psi_0(s) \frac{dQ(0)}{dt} \left(\overline{P} + \left(\exp(Q(0)(\tau - s) - \overline{P} \right) \right) sds \right| \\
\le\ & K \exp(-\kappa_{0,0}\tau) \\
& + \int_0^\tau |\psi_0(s)| \left| \frac{dQ(0)}{dt} \left(\exp(Q(0)(\tau - s)) - \overline{P} \right) \right| sds \\
\le\ & K \exp(-\kappa_{0,0}\tau) + K \int_0^\tau \exp(-\kappa_{0,0}s) \exp(-\kappa_{0,0}(\tau - s)) sds \\
\le\ & K \exp(-\kappa_{0,0}\tau) + K\tau^2 \exp(-\kappa_{0,0}\tau) \le c_2(\tau) \exp(-\kappa_{0,0}\tau),
\end{aligned}$$

for some nonnegative polynomial $c_2(\tau)$ of degree 2.

Note that $r_k(s)$ is orthogonal to $\overline{P}$. By induction, for any k with $k = 1, \ldots, n+1$,

$$\begin{aligned}
& |\psi_k(\tau)| \\
& \le |\varphi_k(0)\exp(Q(0)\tau)| + \int_0^\tau \left| r_k(s) \left(\exp(Q(0)(\tau - s)) - \overline{P} \right) \right| ds \\
& \le K \exp(-\kappa_{0,0}\tau) + \sum_{i=1}^{k} \frac{1}{i!} \int_0^\tau s^i |\psi_{k-i}(s)| \\
& \qquad \times \left| \frac{d^i Q(0)}{dt^i} \left(\exp(Q(0)(\tau - s)) - \overline{P} \right) \right| ds \\
& \le K \exp(-\kappa_{0,0}\tau) + K \sum_{i=1}^{2k-1} \int_0^\tau s^i \exp(-\kappa_{0,0}\tau) ds \\
& \le K \exp(-\kappa_{0,0}\tau) + K \sum_{i=1}^{2k} \tau^i \exp(-\kappa_{0,0}\tau) \le c_{2k}(\tau) \exp(-\kappa_{0,0}\tau),
\end{aligned}$$

where $c_{2k}(\tau)$ is a nonnegative polynomial of degree $2k$. This completes the proof of the proposition. □

Since n is a finite integer, the growth of $c_{2k}(\tau)$ for $0 \le k \le n+1$ is much slower than exponential. Thus the following corollary is in force.

Corollary 4.12. *For each* $0 \le k \le n+1$, *with* $\kappa_{0,0}$ *given in Proposition* 4.11,

$$|\psi_k(\tau)| \le K \exp(-\kappa_0 \tau), \text{ for some } \kappa_0 \text{ with } 0 < \kappa_0 < \kappa_{0,0}.$$

4.2.5 Asymptotic Validation

Recall that $\mathcal{L}^\varepsilon f = \varepsilon(d/dt)f - fQ$. Then we have the following lemma.

Lemma 4.13. *Suppose that for some* $0 \le k \le n+1$,

$$\sup_{t\in[0,T]} |\mathcal{L}^\varepsilon v^\varepsilon(t)| = O\left(\varepsilon^{k+1}\right) \text{ and } v^\varepsilon(0) = 0.$$

Then

$$\sup_{t\in[0,T]} |v^\varepsilon(t)| = O\left(\varepsilon^k\right).$$

Proof: Let $\eta^\varepsilon(\cdot)$ be a function satisfying $\sup_{t\in[0,T]} |\eta^\varepsilon(t)| = O\left(\varepsilon^{k+1}\right)$. Consider the differential equation

$$\begin{aligned} &\mathcal{L}^\varepsilon v^\varepsilon(t) = \eta^\varepsilon(t), \\ &v^\varepsilon(0) = 0. \end{aligned} \tag{4.24}$$

Then the solution of (4.24) is given by

$$v^\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t \eta^\varepsilon(s) X^\varepsilon(t,s) ds,$$

where $X^\varepsilon(t,s)$ is a principal matrix solution. Recall that (see Hale [79, p. 80]) a fundamental matrix solution of the differential equation is an invertible matrix each row of which is a solution of the equation; a principal matrix solution is a fundamental matrix solution with initial value the identity matrix. In view of Lemma 4.1,

$$|X^\varepsilon(t,s)| \le K \quad \text{for all } t,s \in [0,T].$$

Therefore, we have the inequalities

$$\sup_{t\in[0,T]} |v^\varepsilon(t)| \le \frac{K}{\varepsilon} \sup_{t\in[0,T]} \int_0^t |\eta^\varepsilon(s)| ds \le K\varepsilon^k.$$

The proof of the lemma is thus complete. □

Recall that the vector-valued "error" or remainder $e_n^\varepsilon(t)$ is defined by

$$e_n^\varepsilon(t) = p^\varepsilon(t) - \sum_{i=0}^{n} \varepsilon^i \varphi_i(t) - \sum_{i=0}^{n} \varepsilon^i \psi_i\left(\frac{t}{\varepsilon}\right), \tag{4.25}$$

where $p^\varepsilon(\cdot)$ is the solution of (4.3), and $\varphi_i(\cdot)$ and $\psi_i(\cdot)$ are constructed in (4.13) and (4.18). It remains to show that $e_n^\varepsilon(t) = O\left(\varepsilon^{n+1}\right)$. To do so, we utilize Lemma 4.13 as a bridge. It should be pointed out, however, that to get the correct order for the remainder, a trick involving "back up one step" is needed. The details follow.

Proposition 4.14. *Assume* (A4.1) *and* (A4.2)*, for each* $k = 0, \ldots, n$*,*

$$\sup_{t\in[0,T]} |e_k^\varepsilon(t)| = O(\varepsilon^{k+1}).$$

Proof: We begin with

$$e_1^\varepsilon(t) = p^\varepsilon(t) - \varphi_0(t) - \varepsilon\varphi_1(t) - \psi_0\left(\frac{t}{\varepsilon}\right) - \varepsilon\psi_1\left(\frac{t}{\varepsilon}\right). \tag{4.26}$$

We will use the exponential decay property given in $\psi_i(\tau)$ Corollary 4.12. Clearly, $e_1^\varepsilon(0) = 0$, and hence the condition of Lemma 4.13 on the initial data is satisfied. By virtue of the exponential decay property of $\psi_i(\cdot)$ in conjunction with the defining equations of $\varphi_i(\cdot)$ and $\psi_i(\cdot)$,

$$\begin{aligned}
\mathcal{L}^\varepsilon e_1^\varepsilon(t) \; &= -\Bigg[\varepsilon\frac{d\varphi_0(t)}{dt} - \varphi_0(t)Q(t) + \varepsilon^2\frac{d\varphi_1(t)}{dt} - \varepsilon\varphi_1(t)Q(t) \\
&\qquad + \varepsilon\frac{d}{dt}\psi_0\left(\frac{t}{\varepsilon}\right) - \psi_0\left(\frac{t}{\varepsilon}\right)Q(t) + \varepsilon^2\frac{d}{dt}\psi_1\left(\frac{t}{\varepsilon}\right) \\
&\qquad - \varepsilon\psi_1\left(\frac{t}{\varepsilon}\right)Q(t)\Bigg] \\
&= -\varepsilon^2\frac{d\varphi_1(t)}{dt} + \psi_0\left(\frac{t}{\varepsilon}\right)\left[Q(t) - Q(0) - t\frac{dQ(0)}{dt}\right] \\
&\qquad + \varepsilon\psi_1\left(\frac{t}{\varepsilon}\right)[Q(t) - Q(0)].
\end{aligned}$$

For the term involving $\psi_0(t/\varepsilon)$, using a Taylor expansion on $Q(t)$ yields that for some $\xi \in (0, t)$

$$\left|Q(t) - Q(0) - t\frac{dQ(0)}{dt}\right| = \left|\frac{1}{2}\left(\frac{d^2Q(\xi)}{dt^2}\right)t^2\right| \le Kt^2.$$

Owing to the exponential decay property of $\psi_i(\cdot)$, the fact that $\varphi_1(\cdot)$ is n-times continuously differentiable on $[0, T]$, and the above estimate, we have

$$|\mathcal{L}^\varepsilon e_1^\varepsilon(t)| \le K\left(\varepsilon^2 + (\varepsilon t + t^2)\exp\left(-\frac{\kappa_0 t}{\varepsilon}\right)\right).$$

Moreover, for any $k = 0, 1, 2 \ldots, n+1$, it is easy to see that

$$t^k \exp\left(-\frac{\kappa_0 t}{\varepsilon}\right) = \varepsilon^k \left(\frac{t}{\varepsilon}\right)^k \exp\left(-\frac{\kappa_0 t}{\varepsilon}\right) \le K\varepsilon^k. \tag{4.27}$$

This implies $\mathcal{L}^\varepsilon e_1^\varepsilon(t) = O(\varepsilon^2)$ uniformly in t. Thus, $e_1^\varepsilon(t) = O(\varepsilon)$ by virtue of Lemma 4.13 and the bound is uniform in $t \in [0, T]$.

We now go back one step to show that the zeroth-order approximation also possesses the correct error estimate, that is, $e_0^\varepsilon(t) = O(\varepsilon)$. Note that the desired order seems to be difficult to obtain directly, and as a result the back-tracking is employed.

Note that

$$e_1^\varepsilon(t) = e_0^\varepsilon(t) - \varepsilon\varphi_1(t) - \varepsilon\psi_1\left(\frac{t}{\varepsilon}\right). \tag{4.28}$$

However, the smoothness of $\varphi_1(\cdot)$ and the exponential decay of $\psi_1(\cdot)$ imply that

$$\varepsilon\varphi_1(t) + \varepsilon\psi_1\left(\frac{t}{\varepsilon}\right) = O(\varepsilon) \quad \text{uniformly in } t. \tag{4.29}$$

Thus $e_0^\varepsilon(t) = O(\varepsilon)$ uniformly in t.

Proceeding analogously, we obtain

$$\begin{aligned}
&\mathcal{L}^\varepsilon e_{n+1}^\varepsilon \\
&= \mathcal{L}^\varepsilon \left(p^\varepsilon(t) - \sum_{i=0}^{n+1} \varepsilon^i \varphi_i(t) - \sum_{i=0}^{n+1} \varepsilon^i \psi_i\left(\frac{t}{\varepsilon}\right)\right) \\
&= -\varepsilon \left(\sum_{i=0}^{n+1} \varepsilon^i \frac{d\varphi_i(t)}{dt} + \sum_{i=0}^{n+1} \varepsilon^i \frac{d}{dt}\psi_i\left(\frac{t}{\varepsilon}\right)\right) \\
&\quad + \left(\sum_{i=0}^{n+1} \varepsilon^i \varphi_i(t) + \sum_{i=0}^{n+1} \varepsilon^i \psi_i\left(\frac{t}{\varepsilon}\right)\right) Q(t) \\
&= -\varepsilon^{n+2} \frac{d\varphi_{n+1}(t)}{dt} + \left[\sum_{i=0}^{n+1} \varepsilon^i \varphi_i(t) Q(t) - \sum_{i=0}^{n} \varepsilon^{i+1} \varphi_{i+1}(t) Q(t)\right] \\
&\quad + \sum_{i=0}^{n+1} \varepsilon^i \psi_i\left(\frac{t}{\varepsilon}\right) Q(t) - \sum_{i=0}^{n+1} \varepsilon^i \left[\psi_i\left(\frac{t}{\varepsilon}\right) Q(0) + r_i\left(\frac{t}{\varepsilon}\right)\right].
\end{aligned} \tag{4.30}$$

Note that the term in the fifth line above is

$$\sum_{i=0}^{n+1} \varepsilon^i \varphi_i(t) Q(t) - \sum_{i=1}^{n+1} \varepsilon^i \varphi_i(t) Q(t) = \varphi_0(t) Q(t) = 0.$$

Using (4.19), we represent $r_i(t)$ in terms of $(d^i/dt^i)Q(0)$, etc. For the term involving $\psi_0(t/\varepsilon)$, using a truncated Taylor expansion up to order $(n+1)$

for $Q(t)$, by virtue of the Lipschitz continuity of $(d^{n+1}/dt^{n+1})Q(\cdot)$, there is a $\xi \in (0,t)$ such that

$$\left|Q(t) - \sum_{i=0}^{n+1} \frac{t^i}{i!}\frac{d^iQ(0)}{dt^i}\right| = \frac{1}{(n+1)!}\left|t^{n+1}\frac{d^{n+1}Q(\xi)}{dt^{n+1}} - t^{n+1}\frac{d^{n+1}Q(0)}{dt^{n+1}}\right|$$

$$\leq Kt^{n+1}\xi \leq Kt^{n+2}.$$

For all the other terms involving $\psi_i(t/\varepsilon)$, for $i = 1, \ldots, n+1$ in (4.30), we proceed as in the calculation of $\mathcal{L}^\varepsilon e_1^\varepsilon$. As a result, the last two terms in (4.30) are bounded by

$$\psi_0\left(\frac{t}{\varepsilon}\right)O(t^{n+2}) + \varepsilon\psi_1\left(\frac{t}{\varepsilon}\right)O(t^{n+1}) + \cdots + \varepsilon^{n+1}\psi_{n+1}\left(\frac{t}{\varepsilon}\right)O(t),$$

which in turn leads to the bound

$$K(t^{n+2} + \varepsilon t^{n+1} + \cdots + \varepsilon^{n+1}t)\exp\left(-\frac{\kappa_0 t}{\varepsilon}\right) \leq K\varepsilon^{n+2},$$

in accordance with (4.27). Moreover, it is clear that $e_{n+1}^\varepsilon(0) = 0$. In view of the fact that $\varphi_{n+1}(\cdot)$ is continuously differentiable on $[0,T]$ and $Q(\cdot)$ is $(n+1)$-times continuously differentiable on $[0,T]$, by virtue of Lemma 4.13, we infer that $e_{n+1}^\varepsilon(t) = O(\varepsilon^{n+1})$ uniformly in t. Since

$$e_{n+1}^\varepsilon(t) = e_n^\varepsilon(t) + O(\varepsilon^{n+1}),$$

it must be that $e_n^\varepsilon(t) = O(\varepsilon^{n+1})$. The proof of Proposition 4.14 is complete, and so is the proof of Theorem 4.5. □

Remark 4.15. In the estimate given above, we actually obtained

$$\mathcal{L}^\varepsilon e_k^\varepsilon(t) = O\left(\varepsilon^{k+1} + (\varepsilon t^k + \cdots + \varepsilon^k t)\exp\left(-\frac{\kappa_0 t}{\varepsilon}\right)\right). \tag{4.31}$$

This observation will be useful when we consider the unbounded interval $[0,\infty)$.

The findings reported are very useful for further study of the limit behavior of the corresponding Markov chain problems of central limit type, which will be discussed in the next chapter. In many applications, a system is governed by a Markov chain, which consists of both slow and fast motions. An immediate question is this: Can we still develop an asymptotic series expansion? This question will be dealt with in Section 4.3.

Suppose that in lieu of (A4.2), we assume that $Q(\cdot)$ is piecewise $(n+1)$-times continuously differentiable on $[0,T]$, and $(d^{n+1}/dt^{n+1})Q(\cdot)$ is piecewise Lipschitz, that is, there is a partition of $[0,T]$, namely,

$$t_0 = 0 < t_1 < t_2 < \cdots \leq t_k = T$$

such that $Q(\cdot)$ is $(n+1)$-times continuously differentiable and (d^{n+1}/dt^{n+1}) $Q(\cdot)$ is Lipschitz on each subinterval $[t_i, t_{i+1})$. Then the result obtained still holds. In this case, in addition to the initial layers, one also has a finite number of inner-boundary layers. In each interval $[t_i, t_{i+1} - \eta]$ for $\eta > 0$, the expansion is similar to that presented in Theorem 4.5.

4.2.6 Examples

As a further illustration, we consider two examples in this section. The first example is concerned with a stationary Markov chain, i.e., $Q(t) = Q$ is a constant matrix. The second example deals with an analytically solvable case for a two-state Markov chain with nonstationary transition probabilities. Although they are simple, these examples give us insight into the asymptotic behavior of the underlying systems.

Example 4.16. Let $\alpha^\varepsilon(t)$ be an m-state Markov chain with a constant generator $Q(t) = Q$ that is irreducible. This is an analytically solvable case, with

$$p^\varepsilon(t) = p^0 \exp\left(\frac{Qt}{\varepsilon}\right).$$

Using the technique of asymptotic expansion, we obtain

$$\varphi_0(t) + \psi_0\left(\frac{t}{\varepsilon}\right) = \varphi_0 + (p^0 - \varphi_0)\exp\left(\frac{Qt}{\varepsilon}\right),$$

$$\text{with } \exp\left(\frac{Qt}{\varepsilon}\right) \to \overline{P}, \text{ as } \varepsilon \to 0,$$

where

$$\varphi_0(t) = (\nu_1, \ldots, \nu_m) \text{ and } \overline{P} = \mathbb{1}\varphi_0.$$

Note that $p^0\overline{P} = \varphi_0$, and hence

$$(p^0 - \varphi_0)\exp\left(\frac{Qt}{\varepsilon}\right) = (p^0 - \varphi_0)\left[\exp\left(\frac{Qt}{\varepsilon}\right) - \overline{P}\right].$$

Moreover,

$$\varphi_i(t) \equiv 0, \quad \psi_i\left(\frac{t}{\varepsilon}\right) \equiv 0 \quad \text{for } i \geq 1.$$

In this case, $\varphi_0(t) \equiv \varphi_0$, a constant vector, which is the equilibrium distribution of Q; the series terminates. Moreover, the solution consists of two terms, one of them the equilibrium distribution (the zeroth-order approximation) and the other the zeroth-order initial-layer correction term. Since φ_0 is the quasi-stationary distribution,

$$\varphi_0 Q = 0 \quad \text{and} \quad \varphi_0 \exp\left(\frac{Qt}{\varepsilon}\right) = \varphi_0.$$

Hence the analytic solution and the asymptotic expansion coincide.

In particular, let Q be a two-dimensional matrix, i.e.,

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}.$$

Then setting

$$y_0^\varepsilon(t) = \varphi_0(t) + \psi_0(t/\varepsilon),$$

we have

$$p_1^\varepsilon(t) = y_{0,1}^\varepsilon(t) = \frac{\mu}{\lambda+\mu} + \left(p_1^0 - \frac{\mu}{\lambda+\mu}\right)\exp\left(-\frac{(\lambda+\mu)t}{\varepsilon}\right),$$

$$p_2^\varepsilon(t) = y_{0,2}^\varepsilon(t) = \frac{\lambda}{\lambda+\mu} + \left(p_2^0 - \frac{\lambda}{\lambda+\mu}\right)\exp\left(-\frac{(\lambda+\mu)t}{\varepsilon}\right).$$

Therefore,

$$\varphi_0(t) = \left(\frac{\mu}{\lambda+\mu}, \frac{\lambda}{\lambda+\mu}\right),$$

$$\psi_0\left(\frac{t}{\varepsilon}\right) = \left(\left(p_1^0 - \frac{\mu}{\lambda+\mu}\right), \left(p_2^0 - \frac{\lambda}{\lambda+\mu}\right)\right)\exp\left(-\frac{(\lambda+\mu)t}{\varepsilon}\right),$$

$$\varphi_i(t) \equiv 0 \text{ and } \psi_i\left(\frac{t}{\varepsilon}\right) \equiv 0 \quad \text{for } i \geq 1.$$

Example 4.17. Consider a two-state Markov chain with generator

$$Q(t) = \begin{pmatrix} -\lambda(t) & \lambda(t) \\ \mu(t) & -\mu(t) \end{pmatrix}$$

where $\lambda(t) \geq 0$, $\mu(t) \geq 0$ and $\lambda(t) + \mu(t) > 0$ for each $t \in [0, T]$. Therefore $Q(\cdot)$ is weakly irreducible. For the following discussion, assume $Q(\cdot)$ to be sufficiently smooth. Although it is time-varying, a closed-form solution is obtainable. Since $p_1^\varepsilon(t) + p_2^\varepsilon(t) = 1$ for each t, (4.3) can be solved explicitly and the solution is given by

$$p_1^\varepsilon(t) = p_1^0 \exp\left(-\frac{1}{\varepsilon}\int_0^t (\lambda(s)+\mu(s))ds\right) + \int_0^t \frac{\mu(u)}{\varepsilon}\exp\left(-\frac{1}{\varepsilon}\int_u^t (\lambda(s)+\mu(s))ds\right)du,$$

$$p_2^\varepsilon(t) = p_2^0 \exp\left(-\frac{1}{\varepsilon}\int_0^t (\lambda(s)+\mu(s))ds\right) + \int_0^t \frac{\lambda(u)}{\varepsilon}\exp\left(-\frac{1}{\varepsilon}\int_u^t (\lambda(s)+\mu(s))ds\right)du.$$

Following the approach in the previous sections, we construct the first a few terms in the asymptotic expansion. By considering (4.13) together with (4.2), a system of the form

$$\lambda(t)\varphi_0^1(t) - \mu(t)\varphi_0^2(t) = 0,$$

$$\varphi_0^1(t) + \varphi_0^2(t) = 1$$

is obtained. The solution of the system yields that

$$\varphi_0(t) = \left(\frac{\mu(t)}{\lambda(t)+\mu(t)}, \frac{\lambda(t)}{\lambda(t)+\mu(t)}\right).$$

To find $\varphi_1(\cdot)$, consider

$$\lambda(t)\varphi_1^1(t) - \mu(t)\varphi_1^2(t) = \frac{\dot\lambda(t)\mu(t) - \dot\mu(t)\lambda(t)}{(\lambda(t)+\mu(t))^2},$$

$$\varphi_1^1(t) + \varphi_1^2(t) = 0,$$

where $\dot\lambda = (d/dt)\lambda$ and $\dot\mu = (d/dt)\mu$. Solving this system of equations gives us

$$\varphi_1(t) = \left(\frac{\dot\lambda(t)\mu(t) - \dot\mu(t)\lambda(t)}{(\lambda(t)+\mu(t))^3}, \frac{\lambda(t)\dot\mu(t) - \mu(t)\dot\lambda(t)}{(\lambda(t)+\mu(t))^3}\right).$$

To get the inner expansion, consider the differential equation

$$\frac{d\psi_0(\tau)}{d\tau} = \psi_0(\tau)Q(0),$$

$$\psi_0(0) = p^0 - \varphi_0(0),$$

with $\tau = t/\varepsilon$. We obtain

$$\psi_0(\tau) = (p^0 - \varphi_0(0))\exp(Q(0)\tau),$$

where

$$\exp(Q(0)\tau) = \frac{1}{\lambda(0)+\mu(0)} \times \begin{pmatrix} \mu(0) + \lambda(0)e^{-(\lambda(0)+\mu(0))\tau} & \lambda(0) - \lambda(0)e^{-(\lambda(0)+\mu(0))\tau} \\ \mu(0) - \mu(0)e^{-(\lambda(0)+\mu(0))\tau} & \lambda(0) + \mu(0)e^{-(\lambda(0)+\mu(0))\tau} \end{pmatrix}.$$

Similarly $\psi_1(\cdot)$ can be obtained from (4.21) with the exponential matrix given above.

It is interesting to note that either $\lambda(t)$ or $\mu(t)$ can be equal to 0 for some t as long as $\lambda(t) + \mu(t) > 0$. For example, if we take $\mu(\cdot)$ to be the repair rate of a machine in a manufacturing model, then $\mu(t) = 0$ corresponds to the repair workers taking breaks or waiting for parts on order to arrive. The minors of $Q(t)$ are $\lambda(t)$, $-\lambda(t)$, $\mu(t)$, and $-\mu(t)$. As long as not all of them are zero at the same time, the weak irreducibility condition will be met.

4.2.7 Two-Time-Scale Expansion

The asymptotic expansion derived in the preceding sections is separable in the sense that it is the sum of a regular part and initial-layer corrections. Naturally, one is interested in the relationship between such an expansion and the so-called two-time-scale expansion (see, for example, Smith [199]). To answer this question, we first obtain the two-time-scale asymptotic expansion for the forward equation (4.3), proceed with the exploration of the relationships between these two expansions, and conclude with a discussion of the connection between these two methods.

Two-Time-Scale Expansion. Following the literature on asymptotic expansion (e.g., Kevorkian and Cole [108, 109] and Smith [199] among others), consider two scales t and $\tau = t/\varepsilon$, both as "times." One of them is in a normal time scale and the other is a stretched one. We seek asymptotic expansions of the form

$$y^{\varepsilon}(t,\tau) = \sum_{i=0}^{n} \varepsilon^i y_i(t,\tau), \tag{4.32}$$

where $\{y_i(t,\tau)\}_{i=0}^{n}$ is a sequence of two-time-scale functions. Treating t and τ as independent variables, one has

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{1}{\varepsilon}\frac{\partial}{\partial \tau}. \tag{4.33}$$

Formally substituting (4.32) into (4.3) and equating coefficients of like powers of ε^i results in

$$\begin{aligned}
&\frac{\partial y_0(t,\tau)}{\partial \tau} = y_0(t,\tau)Q(t),\\
&\frac{\partial y_1(t,\tau)}{\partial \tau} = y_1(t,\tau)Q(t) - \frac{\partial y_0(t,\tau)}{\partial t},\\
&\quad\cdots\\
&\frac{\partial y_i(t,\tau)}{\partial \tau} = y_i(t,\tau)Q(t) - \frac{\partial y_{i-1}(t,\tau)}{\partial t}, \quad 1 \le i \le n.
\end{aligned} \tag{4.34}$$

The initial conditions are

$$\begin{aligned} y_0(t,0) &= p^0 \quad \text{and} \\ y_i(t,0) &= 0, \quad \text{for } 1 \le i \le n. \end{aligned} \tag{4.35}$$

Holding t constant and solving the first equation in (4.34) (with the first equation in (4.35) as the initial condition) yields

$$y_0(t,\tau) = p^0 \exp(Q(t)\tau). \tag{4.36}$$

By virtue of (A4.4), $(\partial/\partial t)y_0(t,\tau)$ exists and

$$\frac{\partial y_0(t,\tau)}{\partial t} = p^0 \exp(Q(t)\tau)\left(\frac{dQ(t)}{dt}\right)\tau.$$

As a result, $(\partial/\partial t)y_0(t,\tau)$ is orthogonal to $\mathbb{1}$. We continue the procedure recursively. It can be verified that for $1 \le i \le n$,

$$y_i(t,\tau) = -\int_0^\tau \frac{\partial y_{i-1}(t,s)}{\partial t} \exp(Q(t)(\tau - s))ds. \tag{4.37}$$

Furthermore, for $i = 1, \ldots, n$, $(\partial/\partial t)y_i(t,\tau)$ exists and is continuous; it is also orthogonal to $\mathbb{1}$. It should be emphasized that in the equations above, t is viewed as being "frozen," and as a consequence, $Q(t)$ is a "constant" matrix.

Parallel to the previous development, one can show that for all $1 \le i \le n$,

$$|y_i(t,\tau)| \le K(t) \exp(-\kappa_0(t)\tau).$$

Compared with the separable expansions presented before, note the t-dependence of $K(\cdot)$ and $\kappa_0(\cdot)$ above. Furthermore, the asymptotic series is valid. We summarize this as the following theorem.

Theorem 4.18. *Under the conditions of Theorem 4.5, a sequence of functions* $\{y_i(t,\tau)\}_{i=0}^n$ *can be constructed so that*

$$\sup_{t\in[0,T]} \left| p^\varepsilon(t) - \sum_{i=0}^n \varepsilon^i y_i(t,\tau) \right| = O(\varepsilon^{n+1}).$$

Example 4.19. We return to Example 4.16. It is readily verified that the zeroth-order two-time-scale expansion coincides with that of the analytic solution, in fact, with

$$y_0(t,\tau) = p^0 \exp\left(\frac{Qt}{\varepsilon}\right) \text{ and } y_i(t,\tau) \equiv 0 \text{ for all } i \ge 1.$$

Relationship between the Two Methods. Now we have two different asymptotic expansions. Do they in some sense produce similar asymptotic results? Note that each term in $y_i(t,\tau)$ contains the regular part $\varphi_i(t)$ as well as the initial-layer corrections. Examining the zeroth-order approximation leads to

$$\exp(Q(t)\tau) \to \overline{P}(t) \text{ as } \tau \to \infty$$

via the same argument employed in the proof of Lemma 4.4. The matrix has identical rows, and is given by $\overline{P}(t) = \mathbb{1}\nu(t)$. In fact, owing to $p^0\mathbb{1} = \sum_{i=1}^m p_i^0 = 1$, we have

$$y_0(t,\tau) = \nu(t) + p^0\left(\exp(Q(t)\tau) - \overline{P}(t)\right) = \nu(t) + \widetilde{y}_0(t,\tau), \tag{4.38}$$

where $\widetilde{y}_0(t,\tau)$ decays exponentially fast as $\tau \to \infty$ for $t < \tau$.

In view of (4.38), the two methods produce the same limit as $\tau \to \infty$, namely, the quasi-stationary distribution. To explore further, we study a special case (a two-state Markov chain) so as to keep the notation simple. Consider the two-state Markov chain model Example 4.17. In view of (4.38) and the formulas in Example 4.17, we have

$$y_0(t,\tau) = \nu(t) + \widetilde{y}_0(t,\tau) = \varphi_0(t) + \widetilde{y}_0(t,\tau).$$

Owing to (4.37), direct calculation yields that

$$\begin{aligned} y_1(t,\tau) \ &= -\int_0^\tau \frac{d\varphi_0(t)}{dt}\exp(Q(t)(\tau-s))ds \\ &\quad -\int_0^\tau \frac{\partial\widetilde{y}_0(t,\tau)}{\partial t}\exp(Q(t)(\tau-s))ds. \end{aligned}$$

It can be verified that the second term on the right-hand side of the equal sign above decays exponentially fast, while the first term yields $\varphi_1(t)$ plus another term tending to 0 exponentially fast as $\tau \to \infty$. Using the result of Example 4.17 yields

$$\begin{aligned} &-\int_0^\tau \frac{d\varphi_0(t)}{dt}\exp(Q(t)(\tau-s))ds \\ &\quad = \frac{d\varphi_0(t)}{dt}\left(\frac{1-\exp(-(\lambda(t)+\mu(t))\tau)}{\lambda(t)+\mu(t)}\right)\begin{pmatrix}\lambda(t) & -\lambda(t)\\ -\mu(t) & \mu(t)\end{pmatrix} \\ &\quad = \varphi_1(t) - \frac{d\varphi_0(t)}{dt}\left(\frac{\exp(-(\lambda(t)+\mu(t))\tau)}{\lambda(t)+\mu(t)}\right)\begin{pmatrix}\lambda(t) & -\lambda(t)\\ -\mu(t) & \mu(t)\end{pmatrix}. \end{aligned}$$

Thus, it follows that

$$y_1(t,\tau) = \varphi_1(t) + \widetilde{y}_1(t,\tau),$$

where

$$\widetilde{y}_1(t,\tau) = -\int_0^\tau \frac{\partial \widetilde{y}_0(t,\tau)}{\partial t} \exp(Q(t)(\tau - s))ds$$
$$-\frac{d\varphi_0(t)}{dt}\left(\frac{\exp(-(\lambda(t)+\mu(t))\tau)}{\lambda(t)+\mu(t)}\right)\begin{pmatrix} \lambda(t) & -\lambda(t) \\ -\mu(t) & \mu(t) \end{pmatrix}.$$

Similarly, we can obtain

$$y_i(t,\tau) = \varphi_i(t) + \widetilde{y}_i(t,\tau), \text{ for } 1 \le i \le n,$$

where $\widetilde{y}_i(t,\tau)$ decay exponentially fast as $\tau \to \infty$ for all $t < \tau$. This establishes the connection between these two different expansions.

Comparison and Additional Remark. A moment of reflection reveals that:

- The conditions required to obtain the asymptotic expansions are the same.
- Except for the actual forms, there is no significant difference between these two methods.
- No matter which method is employed, in one way or another the results for stationary Markov chains are used. In the separable expansion, this is accomplished by using $Q(0)$, and in the two-time-scale expansion, this is carried out by holding t constant and hence treating $Q(t)$ as a constant matrix.
- Although the two-time-scale expansion admits a seemingly more general form, the separable expansion is more transparent as far as the quasi-stationary distribution is concerned.
- When a more complex problem, for example the case of weak and strong interactions, is encountered, the separable expansion becomes more advantageous.
- To study asymptotic normality, etc., in the sequel, the separable expansion will prove to be more convenient than the two-time-scale expansion.

In view of the items mentioned above, we choose to use the separable form of the expansion throughout.

4.3 Markov Chains with Multiple Weakly Irreducible Classes

This section presents the asymptotic expansions of two-time-scale Markov chains with slow and fast components subject to weak and strong interactions. We assume that all the states of the Markov chain are recurrent. In contrast to Section 4.2, the states belong to multiple weakly

irreducible classes. As was mentioned in the introductory chapter, such time-scale separation stems from various applications in production planning, queueing networks, random fatigue, system reliability, competing risk theory, control and optimization of large-scale dynamical systems, and related fields. The sunderlying models in which some components change very rapidly whereas others vary relatively slowly, are more complex than those of Section 4.2. The weak and strong interactions of the systems are modeled by assuming the generator of the underlying Markov chain to be of the form

$$Q^\varepsilon(t) = \frac{1}{\varepsilon}\widetilde{Q}(t) + \widehat{Q}(t), \tag{4.39}$$

where $\widetilde{Q}(t)$ governs the rapidly changing part and $\widehat{Q}(t)$ describes the slowly changing components. They have the appropriate forms to be mentioned in the sequel.

This section extends the results in Section 4.2 to incorporate the cases in which the generator $\widetilde{Q}(t)$ is not irreducible. Our study focuses on the forward equation, similar to (4.3); now the forward equation takes the form

$$\frac{dp^\varepsilon(t)}{dt} = p^\varepsilon(t)\left(\frac{1}{\varepsilon}\widetilde{Q}(t) + \widehat{Q}(t)\right), \quad p^\varepsilon(0) = p^0 \tag{4.40}$$

such that

$$p_i^0 \geq 0 \text{ for each } i \text{ and } \sum_{i=1}^{m} p_i^0 = 1.$$

To illustrate, we present a simple example below.

Example 4.20. Consider a two-machine flowshop with machines that are subject to breakdown and repair. The production capacity of the machines is described by a finite-state Markov chain. If the machine is up, then it can produce parts with production rate $u(t)$; its production rate is zero if the machine is under repair. For simplicity, suppose each of the machines is either in operating condition (denoted by 1) or under repair (denoted by 0). Then the capacity of the workshop becomes a four-state Markov chain with state space $\{(1,1),(0,1),(1,0),(0,0)\}$. Suppose that the first machine breaks down much more often than the second one. To reflect this situation, consider a Markov chain $\alpha^\varepsilon(\cdot)$ generated by $Q^\varepsilon(t)$ in (4.39), with $\widetilde{Q}(\cdot)$ and $\widehat{Q}(\cdot)$ given by

$$\widetilde{Q}(t) = \begin{pmatrix} -\lambda_1(t) & \lambda_1(t) & 0 & 0 \\ \mu_1(t) & -\mu_1(t) & 0 & 0 \\ 0 & 0 & -\lambda_1(t) & \lambda_1(t) \\ 0 & 0 & \mu_1(t) & -\mu_1(t) \end{pmatrix}$$

and

$$\widehat{Q}(t) = \begin{pmatrix} -\lambda_2(t) & 0 & \lambda_2(t) & 0 \\ 0 & -\lambda_2(t) & 0 & \lambda_2(t) \\ \mu_2(t) & 0 & -\mu_2(t) & 0 \\ 0 & \mu_2(t) & 0 & -\mu_2(t) \end{pmatrix},$$

where $\lambda_i(\cdot)$ and $\mu_i(\cdot)$ are the rates of repair and breakdown, respectively. The matrices $\widetilde{Q}(t)$ and $\widehat{Q}(t)$ are themselves generators of Markov chains. Note that

$$\widetilde{Q}(t) = \operatorname{diag}\left(\begin{pmatrix} -\lambda_1(t) & \lambda_1(t) \\ \mu_1(t) & -\mu_1(t) \end{pmatrix}, \begin{pmatrix} -\lambda_1(t) & \lambda_1(t) \\ \mu_1(t) & -\mu_1(t) \end{pmatrix} \right)$$

is a block-diagonal matrix, representing the fast motion, and $\widehat{Q}(t)$ governs the slow components. In order to obtain any meaningful results for controlling and optimizing the performance of the underlying systems, the foremost task is to determine the asymptotic behavior (as $\varepsilon \to 0$) of the probability distribution of the underlying chain.

In this example, a first glance reveals that $\widetilde{Q}(t)$ is reducible, hence the results in Section 4.2 are not applicable. However, closer scrutiny indicates that $\widetilde{Q}(t)$ consists of two irreducible submatrices. One expects that the asymptotic expansions may still be established. Our main objective is to develop asymptotic expansions of such systems and their variants. The corresponding procedure is, however, much more involved compared with the irreducible cases.

Examining (4.39), it is seen that the asymptotic properties of the underlying Markov chains largely depend on the structure of the matrix $\widetilde{Q}(t)$. In accordance with the classification of states, we may consider three different cases: the chains with recurrent states only, the inclusion of absorbing states, and the inclusion of transient states. We treat the recurrent-state cases in this section, and then extend the results to notationally more involved cases including absorbing states and transient states in the following two sections.

Suppose $\alpha^{\varepsilon}(\cdot)$ is a finite-state Markov chain with generator given by (4.39), where both $\widetilde{Q}(t)$ and $\widehat{Q}(t)$ are generators of appropriate Markov chains. In view of the results in Section 4.2, it is intuitively clear that the structure of the generator $\widetilde{Q}(t)$ governs the fast-changing part of the Markov chain. As mentioned in the previous section, our study of the finite-state-space cases is naturally divided into the recurrent cases, the inclusion of absorbing states, and the inclusion of transient states of the generator $\widetilde{Q}(t)$. In accordance with classical results (see Chung [31] and Karlin and Taylor [105, 106]), one can always decompose the states of a finite-state Markov chain into recurrent (including absorbing) and transient classes. Inspired by Seneta's approach to nonnegative matrices (see Seneta [189]), we aim to put the matrix $\widetilde{Q}(t)$ into some sort of "canonical" form so that a

systematic study can be carried out. In a finite-state Markov chain, not all states are transient, and it has at least one recurrent state. Similar to the argument of Iosifescu [95, p. 94] (see also Goodman [75], Karlin and McGregor [104], Keilson [107] among others), if there are no transient states, then after suitable permutations and rearrangements (i.e., by appropriately relabeling the states), $\widetilde{Q}(t)$ can be put into the block-diagonal form

$$\begin{aligned}\widetilde{Q}(t) &= \begin{pmatrix} \widetilde{Q}^1(t) & & & \\ & \widetilde{Q}^2(t) & & \\ & & \ddots & \\ & & & \widetilde{Q}^l(t) \end{pmatrix} \\ &= \operatorname{diag}\left(\widetilde{Q}^1(t), \ldots, \widetilde{Q}^l(t)\right), \end{aligned} \tag{4.41}$$

where $\widetilde{Q}^k(t) \in \mathbb{R}^{m_k \times m_k}$ are weakly irreducible, for $k = 1, 2, \ldots, l$, and $\sum_{k=1}^{l} m_k = m$. Here and hereinafter, $\widetilde{Q}^k(t)$, (a superscript without parentheses) denotes the kth block matrix in $\widetilde{Q}(t)$. The rest of this section deals with the generator $Q^\varepsilon(t)$ given by (4.39) with $\widetilde{Q}(t)$ taking the form (4.41). Note that an example of the recurrent case is that of the irreducible (or weakly irreducible) generators treated in Section 4.2.

Let $\mathcal{M}_k = \{s_{k1}, \ldots, s_{km_k}\}$ for $k = 1, \ldots, l$ denote the states corresponding to $\widetilde{Q}^k(t)$ and let $\mathcal{M}$ denote the state space of the underlying chains given by

$$\begin{aligned}\mathcal{M} &= \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l \\ &= \left\{s_{11}, \ldots, s_{1m_1}, \ldots, s_{l1}, \ldots, s_{lm_l}\right\}. \end{aligned}$$

Since $\widetilde{Q}^k(t) = (\widetilde{q}^k_{ij}(t))_{m_k \times m_k}$ and $\widehat{Q}(t) = (\widehat{q}_{ij}(t))_{m \times m}$ are generators, for $k = 1, 2, \ldots, l$, we have

$$\begin{aligned}&\sum_{j=1}^{m_k} \widetilde{q}^k_{ij}(t) = 0, \quad \text{for } i = 1, \ldots, m_k, \quad \text{and} \\ &\sum_{j=1}^{m} \widehat{q}_{ij}(t) = 0, \quad \text{for } i = 1, \ldots, m. \end{aligned}$$

The slow and fast components are coupled through weak and strong interactions in the sense that the underlying Markov chain fluctuates rapidly within a single group $\mathcal{M}_k$ and jumps less frequently between groups $\mathcal{M}_k$ and $\mathcal{M}_j$ for $k \neq j$. The states in $\mathcal{M}_k$, $k = 1, \ldots, l$, are not isolated or independent of each other. More precisely, if we consider the states in $\mathcal{M}_k$ as a single "state," then these "states" are coupled through the matrix $\widehat{Q}(t)$, and transitions from $\mathcal{M}_k$ to $\mathcal{M}_j$, $k \neq j$ are possible.

In fact, $\widehat{Q}(\cdot)$, together with the quasi-stationary distributions of $\widetilde{Q}^k(t)$, determines the transition rates among states in $\mathcal{M}_k$, for $k = 1, \ldots, l$.

Consider the forward equation (4.40). Our goal here is to develop an asymptotic series for the solution $p^\varepsilon(\cdot)$ of (4.40). Working with the interval $[0, T]$ for some $T < \infty$, we will need the following conditions:

(**A4.3**) For each $t \in [0, T]$ and $k = 1, 2, \ldots, l$, $\widetilde{Q}^k(t)$ is weakly irreducible.

(**A4.4**) For some positive integer n, $\widetilde{Q}(\cdot)$ and $\widehat{Q}(\cdot)$ are $(n+1)$-times and n-times continuously differentiable on $[0, T]$, respectively. Moreover, $(d^{n+1}/dt^{n+1})\widetilde{Q}(\cdot)$ and $(d^n/dt^n)\widehat{Q}(\cdot)$ are Lipschitz on $[0, T]$.

Compared with the irreducible models in Section 4.2, the main difficulty in this chapter lies in the interactions among different blocks. In constructing the expansion in Section 4.2, for $i = 1, \ldots, n$, the two sets of functions $\{\varphi_i(\cdot)\}$ and $\{\psi_i(\cdot)\}$ are obtained independently except the initial conditions $\psi_i(0) = -\varphi_i(0)$. For Markov chains with weak and strong interactions, $\varphi_i(\cdot)$ and $\psi_i(\cdot)$ are highly intertwined. The essence is to find $\varphi_i(\cdot)$ and $\psi_i(\cdot)$ jointly and recursively. In the process of construction, one of the crucial and delicate points is to select the "right" initial conditions. This is done by demanding that $\psi_i(\tau)$ decay to 0 as $\tau \to \infty$. For future use, we define a differential operator $\mathcal{L}^\varepsilon$ on $\mathbb{R}^{1\times m}$-valued functions by

$$\mathcal{L}^\varepsilon f = \varepsilon \frac{df}{dt} - f(\widetilde{Q} + \varepsilon \widehat{Q}). \tag{4.42}$$

Then it follows that $\mathcal{L}^\varepsilon f = 0$ iff f is a solution to the differential equation in (4.40). We are now in a position to derive the asymptotic expansion.

4.3.1 Asymptotic Expansions

As in Section 4.2, we seek expansions of the form

$$y_n^\varepsilon(t) = \Phi_n^\varepsilon(t) + \Psi_n^\varepsilon(t) = \sum_{i=0}^{n} \varepsilon^i \varphi_i(t) + \sum_{i=0}^{n} \varepsilon^i \psi_i\left(\frac{t}{\varepsilon}\right). \tag{4.43}$$

For the purpose of estimating the remainder (or error), the terms $\varphi_{n+1}(\cdot)$ and $\psi_{n+1}(\cdot)$ are needed. Set $\mathcal{L}^\varepsilon y_{n+1}^\varepsilon(t) = 0$. Parallel to the approach in

Section 4.2, equating like powers of ε^i (for $i = 0, 1, \ldots, n+1$) leads to the equations for the regular part:

$$\begin{aligned}
&\varepsilon^0: \ \varphi_0(t)\widetilde{Q}(t) = 0,\\
&\varepsilon^1: \ \varphi_1(t)\widetilde{Q}(t) = \frac{d\varphi_0(t)}{dt} - \varphi_0(t)\widehat{Q}(t),\\
&\qquad \cdots\\
&\varepsilon^i: \ \varphi_i(t)\widetilde{Q}(t) = \frac{d\varphi_{i-1}(t)}{dt} - \varphi_{i-1}(t)\widehat{Q}(t).
\end{aligned} \tag{4.44}$$

As discussed in Section 4.2, the approximation above is good for t away from 0. When t is sufficiently close to 0, an initial layer of thickness ε develops. Thus for the singular part of the expansion we enlarge the picture near 0 using the stretched variable τ defined by $\tau = t/\varepsilon$. Identifying the initial-layer terms in $\mathcal{L}^\varepsilon y^\varepsilon_{n+1} = 0$, we obtain

$$\begin{aligned}
&\frac{d}{d\tau}\left(\psi_0(\tau) + \varepsilon\psi_1(\tau) + \cdots + \varepsilon^{n+1}\psi_{n+1}(\tau)\right)\\
&\quad = \left(\psi_0(\tau) + \varepsilon\psi_1(\tau) + \cdots + \varepsilon^{n+1}\psi_{n+1}(\tau)\right)\left(\widetilde{Q}(\varepsilon\tau) + \varepsilon\widehat{Q}(\varepsilon\tau)\right).
\end{aligned}$$

By means of the Taylor expansion, we have

$$\begin{aligned}
\widetilde{Q}(\varepsilon\tau) &= \widetilde{Q}(0) + \varepsilon\tau\frac{d\widetilde{Q}(0)}{dt} + \cdots\\
&\qquad + \frac{(\varepsilon\tau)^{n+1}}{(n+1)!}\frac{d^{n+1}\widetilde{Q}(0)}{dt^{n+1}} + \widetilde{R}_{n+1}(\varepsilon\tau),\\
\varepsilon\widehat{Q}(\varepsilon\tau) &= \varepsilon\widehat{Q}(0) + \varepsilon^2\tau\frac{d\widehat{Q}(0)}{dt} + \cdots\\
&\qquad + \frac{\varepsilon(\varepsilon\tau)^n}{n!}\frac{d^n\widehat{Q}(0)}{dt^n} + \widehat{R}_n(\varepsilon\tau),
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{R}_{n+1}(t) &= \frac{t^{n+1}}{(n+1)!}\left(\frac{d^{n+1}\widetilde{Q}(\xi)}{dt^{n+1}} - \frac{d^{n+1}\widetilde{Q}(0)}{dt^{n+1}}\right),\\
\widehat{R}_n(t) &= \frac{\varepsilon t^n}{n!}\left(\frac{d^n\widehat{Q}(\zeta)}{dt^n} - \frac{d^n\widehat{Q}(0)}{dt^n}\right),
\end{aligned}$$

for some $0 \le \xi \le t$ and $0 \le \zeta \le t$. Note that in view of (A4.4),

$$\widetilde{R}_{n+1}(t) = O(t^{n+2}) \text{ and } \widehat{R}_n(t) = O(\varepsilon t^{n+1}).$$

Equating coefficients of like powers of ε^i, for $i = 0, 1, \ldots, n+1$, and using the Taylor expansion above, we obtain

$$\begin{aligned}
\varepsilon^0 :\ & \frac{d\psi_0(\tau)}{d\tau} = \psi_0(\tau)\widetilde{Q}(0), \\
\varepsilon^1 :\ & \frac{d\psi_1(\tau)}{d\tau} = \psi_1(\tau)\widetilde{Q}(0) \\
& \qquad + \psi_0(\tau)\left(\widehat{Q}(0) + \tau\frac{d\widetilde{Q}(0)}{dt}\right), \\
& \cdots \\
\varepsilon^i :\ & \frac{d\psi_i(\tau)}{d\tau} = \psi_i(\tau)\widetilde{Q}(0) + \sum_{j=0}^{i-1}\psi_{i-j-1}(\tau) \\
& \qquad \times\left(\frac{\tau^j}{j!}\frac{d^j\widehat{Q}(0)}{dt^j} + \frac{\tau^{j+1}}{(j+1)!}\frac{d^{j+1}\widetilde{Q}(0)}{dt^{j+1}}\right).
\end{aligned} \tag{4.45}$$

In view of the essence of matched asymptotic expansion, we have necessarily at $t = 0$ that

$$\sum_{i=0}^{n+1}\varepsilon^i\left(\varphi_i(0) + \psi_i(0)\right) = p^0. \tag{4.46}$$

This equation implies

$$p^0 = \varphi_0(0) + \psi_0(0) \text{ and } \varphi_i(0) + \psi_i(0) = 0,$$

for $i \geq 1$. Moreover, note that $p^\varepsilon(t)\mathbb{1} = 1$ for all $t \in [0, T]$. Sending $\varepsilon \to 0$ in the asymptotic expansion, one necessarily has to have the following conditions: For all $t \in [0, T]$,

$$\varphi_0(t)\mathbb{1} = 1 \text{ and } \varphi_i(t)\mathbb{1} = 0,\ i \geq 1. \tag{4.47}$$

Our task now is to determine the functions $\varphi_i(\cdot)$ and $\psi_i(\cdot)$.

Determining $\varphi_0(\cdot)$ and $\psi_0(\cdot)$. Write $v = (v^1, \ldots, v^l)$ for a vector $v \in \mathbb{R}^{1\times m}$, where v^k denotes the subvector corresponding to the kth block of the partition. Meanwhile, a superscript with parentheses denotes a sequence. Thus v_n^k denotes the kth subblock of the corresponding partitioned vector of the sequence v_n.

Let us start with the first equation in (4.44). In view of (4.47), we have

$$\begin{aligned}
&\varphi_0(t)\widetilde{Q}(t) = 0, \\
&\sum_{i=1}^{m}\varphi_0^i(t) = 1.
\end{aligned} \tag{4.48}$$

Note that the system above depends only on the generator $\widetilde{Q}(t)$. However, by itself, the system is not uniquely solvable. Since for each $t \in [0,T]$ and $k = 1,\ldots,l$, $\widetilde{Q}^k(t)$ is weakly irreducible, it follows that $\text{rank}(\widetilde{Q}^k(t)) = m_k - 1$ and $\text{rank}(\widetilde{Q}(t)) = m - l$. Therefore, to get a unique solution, we need to supply l auxiliary equations. Where can we find these equations? Upon dividing the system (4.48) into l subsystems, one can apply the Fredholm alternative (see Lemma A.37 and Corollary A.38) and use the orthogonality condition to choose l additional equations to replace l equations in the system represented by the first equation in (4.48).

Since for each k, $\widetilde{Q}^k(t)$ is weakly irreducible, there exists a unique quasi-stationary distribution $\nu^k(t)$. Therefore any solution to the equation

$$\varphi_0^k(t)\widetilde{Q}^k(t) = 0$$

can be written as the product of $\nu^k(t)$ and a scalar "multiplier," say $\vartheta_0^k(t)$. It follows from the second equation in (4.48) that $\sum_{k=1}^{l} \vartheta_0^k(t) = 1$. These $\vartheta_0^k(t)$'s can be interpreted as the probabilities of the "grouped states" (or "aggregated states") $\mathcal{M}_k$.

As will be seen in the sequel, $\vartheta_0^k(t)$ becomes an important spinoff in the process of construction. Effort will subsequently be devoted to finding the unique solution $(\vartheta_0^1(t),\ldots,\vartheta_0^l(t))$. Let $\mathbb{1}_{m_k} = (1,\ldots,1)' \in \mathbb{R}^{m_k \times 1}$.

Lemma 4.21. *Under* (A4.3) *and* (A4.4)*, for each* $k = 1,\ldots,l$*, the solution of the equation*

$$\begin{aligned} &\varphi_0^k(t)\widetilde{Q}^k(t) = 0, \\ &\varphi_0^k(t)\mathbb{1}_{m_k} = \vartheta_0^k(t), \end{aligned} \tag{4.49}$$

can be uniquely expressed as $\varphi_0^k(t) = \nu^k(t)\vartheta_0^k(t)$*, where* $\nu^k(t)$ *is the quasi-stationary distribution corresponding to* $\widetilde{Q}^k(t)$*. Moreover,* $\varphi_0^k(t)$ *is* $(n+1)$*-times continuously differentiable on* $[0,T]$*, provided that* $\vartheta_0^k(\cdot)$ *is* $(n+1)$*-times continuously differentiable.*

Proof: For each k, let us regard $\vartheta_0^k(\cdot)$ as a known function temporarily. For $t \in [0,T]$, let $\widetilde{Q}_c^k(t) = (\mathbb{1}_{m_k} \vdots \widetilde{Q}^k(t))$. Then the solution can be written as

$$\varphi_0^k(t) = (\vartheta_0^k(0) \vdots 0'_{m_k})\widetilde{Q}_c^{k,\prime}(t)\left(\widetilde{Q}_c^k(t)\widetilde{Q}_c^{k,\prime}(t)\right)^{-1},$$

where $0_{m_k} = (0,\ldots,0)' \in \mathbb{R}^{m_k \times 1}$. Moreover, $\varphi_0(\cdot)$ is $(n+1)$-times continuously differentiable. The lemma is thus concluded. □

Remark 4.22. This lemma indicates that for each k, the subvector $\varphi_0^k(\cdot)$ is a multiple of the quasi-stationary distribution $\nu^k(\cdot)$ for each $k = 1,\ldots,l$. The multipliers $\vartheta_0^k(\cdot)$ are to be determined. Owing to the interactions

among different "aggregated states" corresponding to the block matrices, piecing together quasi-stationary distributions does not produce a quasi-stationary distribution for the entire system (i.e., $(\nu^1(t), \ldots, \nu^k(t))$ is not a quasi-stationary distribution for the entire system). Therefore, the leading term in the asymptotic expansion is proportional to (or a "multiple" of) the quasi-stationary distributions of the Markov chains generated by $\widetilde{Q}^k(t)$, for $k = 1, \ldots, l$. The multiplier $\vartheta_0^k(t)$ reflects the interactions of the Markov chain among the "aggregated states." The probabilistic meaning of the leading term $\varphi_0(\cdot)$ is in the sense of total probability. Intuitively, $\vartheta_0^k(t)$ is the corresponding probability of the chain belonging to $\mathcal{M}_k$, and $\varphi_0^k(t)$ is the probability distribution of the chain belonging to $\mathcal{M}_k$ and the transitions taking place within this group of states.

We proceed to determining $\vartheta_0^k(\cdot)$ for $k = 1, \ldots, l$. Define an $m \times l$ matrix

$$\widetilde{\mathbb{1}} = \begin{pmatrix} \mathbb{1}_{m_1} & & & \\ & \mathbb{1}_{m_2} & & \\ & & \ddots & \\ & & & \mathbb{1}_{m_l} \end{pmatrix} = \operatorname{diag}(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_l}).$$

A crucial observation is that $\widetilde{Q}(t)\widetilde{\mathbb{1}} = 0$, that is, $\widetilde{Q}(t)$ and $\widetilde{\mathbb{1}}$ are orthogonal. Thus postmultiplying by $\widetilde{\mathbb{1}}$ leads to

$$\mathcal{L}^\varepsilon \left(\sum_{i=0}^{n+1} \varepsilon^i \varphi_i(t) \widetilde{\mathbb{1}} \right) = 0.$$

Recall that

$$\varphi_0^k(t) = \vartheta_0^k(t)\nu^k(t) \text{ and } \varphi_0^k(t)\mathbb{1} = \vartheta_0^k(t).$$

Equating the coefficients of ε in the above equation yields

$$\frac{d}{dt}(\vartheta_0^1(t), \ldots, \vartheta_0^l(t)) = (\vartheta_0^1(t), \ldots, \vartheta_0^l(t))\overline{Q}(t), \tag{4.50}$$

where

$$\begin{aligned} \overline{Q}(t) &= \begin{pmatrix} \nu^1(t) & & & \\ & \nu^2(t) & & \\ & & \ddots & \\ & & & \nu^l(t) \end{pmatrix} \widehat{Q}(t)\widetilde{\mathbb{1}} \\ &= \operatorname{diag}(\nu^1(t), \ldots, \nu^l(t))\widehat{Q}(t)\widetilde{\mathbb{1}}. \end{aligned} \tag{4.51}$$

Remark 4.23. Intuitively, $\overline{Q}(t)$ is the "average" of $\widehat{Q}(t)$ weighted by the collection of quasi-stationary distributions $(\nu^1(t), \ldots, \nu^l(t))$. In fact, (4.50)

is merely a requirement that the equations in (4.44) be consistent in the sense of Fredholm. This can be seen as follows. Denote by $N(\widetilde{Q}(t))$ the null space of the matrix $Q(t)$. Since $\text{rank}(\widetilde{Q}(t)) = m - l$, the dimension of $N(\widetilde{Q}(t))$ is l. Observe that $\widetilde{\mathbb{1}} = \text{diag}(\widetilde{\mathbb{1}}_{m_1}, \ldots, \widetilde{\mathbb{1}}_{m_l})$ where

$$\begin{aligned}
\widetilde{\mathbb{1}}_{m_1} &= (\underbrace{1, \ldots, 1}_{m_1}, \underbrace{0, \ldots, 0}_{m_2+\cdots+m_l})', \\
\widetilde{\mathbb{1}}_{m_2} &= (\underbrace{0, \ldots, 0}_{m_1}, \underbrace{1, \ldots, 1}_{m_2}, \underbrace{0, \ldots, 0}_{m_3+\cdots+m_l})', \\
&\cdots \\
\widetilde{\mathbb{1}}_{m_l} &= (\underbrace{0, \ldots, 0}_{m_1+\cdots+m_{l-1}}, \underbrace{1, \ldots, 1}_{m_l})'
\end{aligned} \tag{4.52}$$

are linearly independent and span the null space of $\widetilde{Q}(t)$. The equations in (4.44) have solutions only if the right-hand side of each equation is orthogonal to $\widetilde{\mathbb{1}}$. Hence, (4.50) must hold.

Next we determine the initial value $\vartheta_0(0)$. Assuming that the asymptotic expansion of $p^\varepsilon(\cdot)$ is given by $y_n^\varepsilon(\cdot)$ (see (4.43)), then it is necessary that

$$\varphi_0(0)\widetilde{\mathbb{1}} = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} p^\varepsilon(\delta)\widetilde{\mathbb{1}}. \tag{4.53}$$

We will refer to such a condition as an initial-value *consistency* condition. Moreover, in view of (4.40) and $\widetilde{Q}(t)\widetilde{\mathbb{1}} = 0$,

$$p^\varepsilon(t)\widetilde{\mathbb{1}} = p^0\widetilde{\mathbb{1}} + \int_0^\delta p^\varepsilon(s)\widehat{Q}(s)ds\widetilde{\mathbb{1}}.$$

Since $p^\varepsilon(\cdot)$ and $\widehat{Q}(\cdot)$ are both bounded, it follows that

$$\lim_{\delta\to 0}\left(\limsup_{\varepsilon\to 0}\int_0^\delta p^\varepsilon(s)\widehat{Q}(s)ds\widetilde{\mathbb{1}}\right) = 0.$$

Therefore, the initial-value consistency condition (4.53) yields

$$\varphi_0(0)\widetilde{\mathbb{1}} = \lim_{\delta\to 0}\left(\lim_{\varepsilon\to 0} p^\varepsilon(\delta)\widetilde{\mathbb{1}}\right) = p^0\widetilde{\mathbb{1}}.$$

Note that $(\vartheta_0^1(0), \ldots, \vartheta_0^l(0)) = \varphi_0(0)\widetilde{\mathbb{1}}$. So the initial value for $\vartheta_0(t)$ should be

$$(\vartheta_0^1(0), \ldots, \vartheta_0^l(0)) = p^0\widetilde{\mathbb{1}}.$$

Using this initial condition and solving (4.50) yields that

$$(\vartheta_0^1(t), \ldots, \vartheta_0^l(t)) = p^0\widetilde{\mathbb{1}}X(t, 0),$$

where $X(t,s)$ is the principal matrix solution of (4.50) (see Hale [79]). Since the smoothness of $X(\cdot,\cdot)$ depends solely on the smoothness properties of $\widetilde{Q}(t)$ and $\widehat{Q}(t)$, $(\vartheta_0^1(\cdot),\dots,\vartheta_0^l(\cdot))$ is $(n+1)$-times continuously differentiable on $[0,T]$. Up to now, we have shown that $\varphi_0(\cdot)$ can be constructed that is $(n+1)$-times continuously differentiable on $[0,T]$. Set $\vartheta_0(t) = (\vartheta_0^1(t),\dots,\vartheta_0^l(t))$. We now summarize the discussion above as follows:

Proposition 4.24. *Assume conditions* (A4.3) *and* (A4.4). *Then for* $t \in [0,T]$, $\varphi_0(t)$ *can be obtained uniquely by solving the following system of equations:*

$$\begin{aligned} &\varphi_0^k(t)\widetilde{Q}^k(t) = 0,\\ &\varphi_0^k(t)\mathbb{1}_{m_k} = \vartheta_0^k(t),\\ &\frac{d\vartheta_0(t)}{dt} = \vartheta_0(t)\overline{Q}(t),\\ &\text{with } \vartheta_0(0) = p^0\widetilde{\mathbb{1}}, \end{aligned} \tag{4.54}$$

such that $\varphi_0(\cdot)$ *is* $(n+1)$*-times continuously differentiable.* □

We next consider the initial-layer term $\psi_0(\cdot)$. First note that solving (4.45) for each $i = 0, 1\dots, n+1$ leads to

$$\begin{aligned} \psi_0(\tau) &= \psi_0(0)\exp(\widetilde{Q}(0)\tau),\\ &\dots\\ \psi_i(\tau) &= \psi_i(0)\exp(\widetilde{Q}(0)\tau)\\ &\quad + \sum_{j=0}^{i-1}\int_0^\tau \psi_{i-j-1}(s)\left(\frac{s^j}{j!}\frac{d^j\widehat{Q}(0)}{dt} + \frac{s^{j+1}}{(j+1)!}\frac{d^{j+1}\widetilde{Q}(0)}{dt^{j+1}}\right)\\ &\qquad \times \exp(\widetilde{Q}(0)(\tau-s))ds. \end{aligned} \tag{4.55}$$

Once again, to match the asymptotic expansion requires that (4.46) hold and hence

$$p^0 = p^\varepsilon(0) = \varphi_0(0) + \psi_0(0).$$

Solving the first equation in (4.45) together with the above initial condition, one obtains

$$\psi_0(\tau) = (p^0 - \varphi_0(0))\exp(\widetilde{Q}(0)\tau). \tag{4.56}$$

Note that in Proposition 4.25 to follow, we still use $\kappa_{0,0}$ as a positive constant, which is generally a different constant from that in Section 4.2.

Proposition 4.25. *Assume conditions* (A4.3) *and* (A4.4). *Then* $\psi_0(\cdot)$ *can be obtained uniquely by* (4.56). *In addition, there is a positive number* $\kappa_{0,0}$ *such that*

$$|\psi_0(\tau)| \leq K \exp(-\kappa_{0,0}\tau),\ \tau \geq 0.$$

Proof: We prove only the exponential decay property, since the rest is obvious. Let $\nu^k(0)$ be the stationary distribution corresponding to the generator $\widetilde{Q}^k(0)$. Define

$$\begin{aligned} \pi &= \widetilde{\mathbb{1}} \begin{pmatrix} \nu^1(0) & & & \\ & \nu^2(0) & & \\ & & \ddots & \\ & & & \nu^l(0) \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{1}_{m_1}\nu^1(0) & & & \\ & \mathbb{1}_{m_2}\nu^2(0) & & \\ & & \ddots & \\ & & & \mathbb{1}_{m_l}\nu^l(0) \end{pmatrix}, \end{aligned} \tag{4.57}$$

where

$$\mathbb{1}_{m_k}\nu^k(0) = \begin{pmatrix} \nu_1^k(0) & \cdots & \nu_{m_k}^k(0) \\ & \vdots & \\ \nu_1^k(0) & \cdots & \nu_{m_k}^k(0) \end{pmatrix}.$$

Noting the block-diagonal structure of $\widetilde{Q}(0)$, we have

$$\exp(\widetilde{Q}(0)\tau) = \begin{pmatrix} \exp(\widetilde{Q}^1(0)\tau) & & & \\ & \exp(\widetilde{Q}^2(0)\tau) & & \\ & & \ddots & \\ & & & \exp(\widetilde{Q}^l(0)\tau) \end{pmatrix}.$$

It is easy to see that

$$(p^0 - \varphi_0(0))\widetilde{\mathbb{1}} = p^0\widetilde{\mathbb{1}} - \varphi_0(0))\widetilde{\mathbb{1}} = p^0\widetilde{\mathbb{1}} - \vartheta_0(0) = 0.$$

Owing to the choice of initial condition, $(p^0 - \varphi_0(0))$ is orthogonal to π, and by virtue of Lemma 4.4, for each $k = 1, \ldots, l$ and some $\kappa_{0,k} > 0$,

$$\left|\exp(\widetilde{Q}^k(0)\tau) - \mathbb{1}_{m_k}\nu^k(0)\right| \leq K \exp(-\kappa_{0,k}\tau),$$

we have

$$\begin{aligned}|\psi_0(\tau)| &= \left|(p^0 - \varphi_0(0))[\exp(\widetilde{Q}(0)\tau) - \pi]\right| \\ &\le K \sup_{k\le l} \left|\exp(\widetilde{Q}^k(0)\tau) - \mathbb{1}_{m_k}\nu^k(0)\right| \\ &\le K\exp(-\kappa_{0,0}\tau),\end{aligned}$$

where $\kappa_{0,0} = \min_{k\le l}\kappa_{0,k} > 0$. □

Determining $\varphi_i(\cdot)$ and $\psi_i(\cdot)$ for $i \ge 1$. In contrast to the situation encountered in Section 4.2, the sequence $\{\varphi_i(\cdot)\}$ cannot be obtained without the involvement of $\{\psi_i(\cdot)\}$. We thus obtain the sequences pairwise. While the determination of $\varphi_0(\cdot)$ and $\psi_0(\cdot)$ is similar to that of Section 4.2, the solutions for the rest of the functions show distinct features resulting from the underlying weak and strong interactions. With known

$$b_0(t) = \frac{d\varphi_0(t)}{dt} - \varphi_0(t)\widehat{Q}(t),$$

we proceed to solve the second equation in (4.44) together with the constraint $\sum_{i=1}^m \varphi_1^i(t) = 0$ due to (4.47). Partition the vectors $\varphi_1(t)$ and $b_0(t)$ as

$$\varphi_1(t) = (\varphi_1^1(t), \ldots, \varphi_1^l(t)),$$

$$b_0(t) = (b_0^1(t), \ldots, b_0^l(t)).$$

In view of the definition of $\overline{Q}(t)$ in (4.51) and $\varphi_0^k(t) = \nu^k(t)\vartheta_0^k(t)$, it follows that $b_0(t)\widetilde{\mathbb{1}} = 0$, thus,

$$b_0^k(t)\mathbb{1}_{m_k} = 0,\ k = 1, \ldots, l.$$

Let $\vartheta_1^k(t)$ denote the function such that $\sum_{k=1}^l \vartheta_1^k(t) = 0$ because $\varphi_1(t)\mathbb{1} = 0$. Then for each $k = 1, \ldots, l$, the solution to

$$\begin{aligned}\varphi_1^k(t)\widetilde{Q}^k(t) &= b_0^k(t), \\ \varphi_1^k(t)\mathbb{1}_{m_k} &= \vartheta_1^k(t),\end{aligned} \tag{4.58}$$

can be expressed as

$$\varphi_1^k(t) = \widetilde{b}_0^k(t) + \vartheta_1^k(t)\nu^k(t), \tag{4.59}$$

where $\widetilde{b}_0^k(t)$ is a solution to the following equation:

$$\widetilde{b}_0^k(t)\widetilde{Q}^k(t) = b_0^k(t),$$

$$\widetilde{b}_0^k(t)\mathbb{1}_{m_k} = 0,$$

or equivalently,

$$\widetilde{b}_0^k(t)(\mathbb{1}_{m_k}\dot{:}\widetilde{Q}^k(t)) = (0\dot{:}b_0^k(t)).$$

The procedure for solving this equation is similar to that for $\varphi_0(\cdot)$.

Analogously to the previous treatment, we proceed to determine $\vartheta_1^k(t)$ by solving the system of equations

$$\mathcal{L}^{\varepsilon}\left(\sum_{i=0}^{n+1}\varepsilon^i\varphi_i(t)\widetilde{\mathbb{1}}\right) = 0. \tag{4.60}$$

Using the conditions

$$\widetilde{b}_0^k(t)\mathbb{1}_{m_k} = 0 \text{ and } \nu^k(t)\mathbb{1}_{m_k} = 1,$$

we have

$$\varphi_1(t)\widetilde{\mathbb{1}} = (\vartheta_1^1(t), \ldots, \vartheta_1^l(t))$$

and

$$\varphi_1(t)\widehat{Q}(t)\widetilde{\mathbb{1}} = (\vartheta_1^1(t), \ldots, \vartheta_1^l(t))\overline{Q}(t) + (\widetilde{b}_0^1(t), \ldots, \widetilde{b}_0^l(t))\widehat{Q}(t)\widetilde{\mathbb{1}},$$

where $\overline{Q}(t)$ was defined in (4.51).

By equating the coefficients of ε^2 in (4.60), we obtain a system of linear inhomogeneous equations

$$\begin{aligned}\frac{d}{dt}(\vartheta_1^1(t), \ldots, \vartheta_1^l(t)) = (\vartheta_1^1(t), \ldots, \vartheta_1^l(t))\overline{Q}(t)\\ +(\widetilde{b}_0^1(t), \ldots, \widetilde{b}_0^l(t))\widehat{Q}(t)\widetilde{\mathbb{1}},\end{aligned} \tag{4.61}$$

with initial conditions

$$\vartheta_1^k(0), \text{ for } k = 1, 2, \ldots, l \text{ such that } \sum_{k=1}^{l}\vartheta_1^k(0) = 0.$$

Again, as observed in Remark 4.23, equation (4.61) comes from the consideration in the sense of Fredholm since the functions on the right-hand sides in (4.44) must be orthogonal to $\widetilde{\mathbb{1}}$.

The initial conditions $\vartheta_1^k(0)$ for $k = 1, \ldots, l$ have not been completely specified yet. We do this later to ensure the matched asymptotic expansion. Once the $\vartheta_1^k(0)$'s are given, the solution of the above equation is

$$(\vartheta_1^1(t), \ldots, \vartheta_1^l(t)) = (\vartheta_1^1(0), \ldots, \vartheta_1^l(0))X(t,0) + \int_0^t (\widetilde{b}_0^1(s), \ldots, \widetilde{b}_0^l(s))\widehat{Q}(s)\widetilde{\mathbb{1}}X(t,s)ds.$$

Thus if the initial value $\vartheta_1^k(0)$ is given, then $\vartheta_1^k(\cdot)$, $k = 1, \ldots, l$ can be found, and so can $\varphi_1(\cdot)$. Moreover, $\varphi_1(\cdot)$ is n-times continuously differentiable on $[0, T]$. The problem boils down to finding the initial condition of $\vartheta_1(0)$.

So far, with the proviso of specified initial conditions $\vartheta_1^k(0)$, for $k = 1, \ldots, l$, the construction of $\varphi_1(\cdot)$ has been completed, and its smoothness has been established. Compared with the determination of $\varphi_0(\cdot)$, the multipliers $\vartheta_1^k(\cdot)$ can no longer be determined using the information about the regular part alone because its initial values have to be determined in conjunction with that of the singular part. This will be seen as follows.

In view of (4.55),

$$\begin{aligned}\psi_1(\tau) &= \psi_1(0)\exp(\widetilde{Q}(0)\tau) \\ &\quad + \int_0^\tau \psi_0(0)\exp(\widetilde{Q}(0)s)\widehat{Q}(0)\exp(\widetilde{Q}(0)(\tau - s))ds \\ &\quad + \int_0^\tau s\psi_0(0)\exp(\widetilde{Q}(0)s)\frac{d\widetilde{Q}(0)}{dt}\exp(\widetilde{Q}(0)(\tau - s))ds.\end{aligned} \tag{4.62}$$

Recall that $\psi_1(0)$ has not been specified yet.

Similar to Section 4.2, for each $t \in [0, T]$, $\widetilde{Q}(t)\widetilde{\mathbb{1}} = 0$. Therefore,

$$\left(\frac{d^i\widetilde{Q}(t)}{dt^i}\right)\widetilde{\mathbb{1}} = 0 \quad \text{and} \quad \left(\frac{d^i\widetilde{Q}(0)}{dt^i}\right)\pi = 0,$$

for $i = 1, \ldots, n+1$, where π is defined in (4.57). This together with $\psi_0(0)\pi = 0$ yields

$$\begin{aligned}&\left|\int_0^\tau s\psi_0(0)\exp(\widetilde{Q}(0)s)\frac{d\widetilde{Q}(0)}{dt}\exp(\widetilde{Q}(0)(\tau - s))ds\right| \\ &\leq \int_0^\tau s\left|\psi_0(0)[\exp(\widetilde{Q}(0)s) - \pi]\right| \\ &\qquad \times\left|\frac{d\widetilde{Q}(0)}{dt}[\exp(\widetilde{Q}(0)(\tau - s)) - \pi]\right|ds \\ &\leq K\tau^2\exp(-\kappa_{0,0}\tau).\end{aligned} \tag{4.63}$$

To obtain the desired property, we need only work with the first two terms on the right side of the equal sign of (4.62). Noting the exponential decay property of $\psi_0(\tau) = \psi_0(0)\exp(\widetilde{Q}(0)\tau)$, we have

$$\int_0^\infty \left|\psi_0(0)\exp(\widetilde{Q}(0)s)\right| ds < \infty,$$

that is, the improper integral converges absolutely. Set

$$\overline{\psi}_0 = \left(\int_0^\infty \psi_0(0)\exp(\widetilde{Q}(0)s)ds\right)\widehat{Q}(0) \in \mathbb{R}^{1\times m}. \tag{4.64}$$

Consequently,

$$\begin{aligned}
&\lim_{\tau\to\infty} \psi_1(0)\exp(\widetilde{Q}(0)\tau) = \psi_1(0)\pi \quad \text{and} \\
&\lim_{\tau\to\infty} \int_0^\tau \psi_0(0)\exp(\widetilde{Q}(0)s)\widehat{Q}(0)\exp(\widetilde{Q}(0)(\tau-s))ds \\
&\quad = \left(\int_0^\infty \psi_0(0)\exp(\widetilde{Q}(0)s)ds\right)\widehat{Q}(0)\pi \\
&\quad := \overline{\psi}_0\pi.
\end{aligned} \tag{4.65}$$

Recall that $\pi = \mathrm{diag}(\mathbb{1}_{m_1}\nu^1(0), \ldots, \mathbb{1}_{m_l}\nu^l(0))$. Partitioning the vector $\overline{\psi}_0$ as $(\overline{\psi}_0^1, \ldots, \overline{\psi}_0^l)$ for $k = 1, \ldots, l$, we have

$$\begin{aligned}
\psi_1(0)\pi &= \left(\left(\psi_1^1(0)\mathbb{1}_{m_1}\right)\nu^1(0), \ldots, \left(\psi_1^l(0)\mathbb{1}_{m_l}\right)\nu^l(0)\right) \\
\overline{\psi}_0\pi &= \left(\left(\overline{\psi}_0^1\mathbb{1}_{m_1}\right)\nu^1(0), \ldots, \left(\overline{\psi}_0^l\mathbb{1}_{m_l}\right)\nu^l(0)\right).
\end{aligned} \tag{4.66}$$

Our expansion requires that $\lim_{\tau\to\infty}\psi_1(\tau) = 0$. As a result,

$$\psi_1(0)\pi + \overline{\psi}_0\pi = 0, \tag{4.67}$$

which implies, by virtue of (4.66),

$$\psi_1^k(0)\mathbb{1}_{m_k} = -\overline{\psi}_0^k\mathbb{1}_{m_k},$$

for $k = 1, \ldots, l$. Solving these equations and in view of

$$\vartheta_1^k(0) = \varphi_1^k(0)\mathbb{1}_{m_k},$$

we choose

$$\vartheta_1^k(0) = -\psi_1^k(0)\mathbb{1}_{m_k} = \overline{\psi}_0^k\mathbb{1}_{m_k} \text{ for } k = 1, \ldots, l.$$

Substituting these into (4.59), we obtain $\varphi_1(\cdot)$. Finally, we use $\psi_1(0) = -\varphi_1(0)$. The process of choosing initial conditions for $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ is complete. Furthermore,

$$|\psi_1(\tau)| \le K \exp(-\kappa_{1,0}\tau) \quad \text{for some } 0 < \kappa_{1,0} < \kappa_{0,0}.$$

This procedure can be applied to $\varphi_i(\cdot)$ and $\psi_i(\cdot)$ for $i = 2, \ldots, n+1$. We proceed recursively to solve for $\varphi_i(\cdot)$ and $\psi_i(\cdot)$ jointly. Using exactly the same methods as the solution for $\varphi_1(\cdot)$, we define

$$\vartheta_i^k(t) = \varphi_i^k(t)\mathbb{1}_{m_k},$$

for each $k = 1, \ldots, l$ and $i = 2, \ldots, n+1$. Similar to $\widetilde{b}_0^k(\cdot)$, we define $\widetilde{b}_i^k(\cdot)$. and write

$$\widetilde{b}_i(t) = (\widetilde{b}_i^1(t), \ldots, \widetilde{b}_i^l(t)).$$

Proceeding inductively, suppose that $\vartheta_i^k(0)$ is selected and in view of (4.55), it has been shown that

$$|\psi_i(\tau)| \le K \exp(-\kappa_{i,0}\tau), \quad i \le n \tag{4.68}$$

for some $0 < \kappa_{i,0} < \kappa_{i-1,0}$. Solve

$$\psi_{i+1}(0)\pi = -\left(\sum_{j=0}^{i} \int_0^\infty \frac{s^j}{j!} \psi_{i-j}(s) ds \frac{d^j \widehat{Q}(0)}{dt^j} \right) \pi := -\overline{\psi}_i \pi$$

to obtain $\psi_{i+1}^k(0)\mathbb{1}_{m_k} = -\overline{\psi}_i^k \mathbb{1}_{m_k}$. Set

$$\vartheta_{i+1}^k(0) = -\psi_{i+1}^k(0)\mathbb{1}_{m_k} = \overline{\psi}_i^k \mathbb{1}_{m_k}, \text{ for } k = 1, \ldots, l.$$

Finally choose $\psi_{i+1}(0) = -\varphi_{i+1}(0)$. We thus have determined the initial conditions for $\varphi_i(\cdot)$. Exactly the same arguments as in Proposition 4.25 lead to

$$|\psi_{i+1}(\tau)| \le K \exp(-\kappa_{i+1,0}\tau) \text{ for some } 0 < \kappa_{i+1,0} < \kappa_{i,0}.$$

Proposition 4.26. *Assume* (A4.3) *and* (A4.4). *Then the following assertions hold:*

(a) *The sequences of row-vector-valued functions $\varphi_i(\cdot)$ and $\vartheta_i(\cdot)$ for $i = 1, 2, \ldots, n$ can be obtained by solving the system of algebraic differential equations*

$$\begin{aligned} \varphi_i(t)\widetilde{Q}(t) &= \frac{d\varphi_{i-1}(t)}{dt} - \varphi_{i-1}(t)\widehat{Q}(t), \\ \vartheta_i^k(t) &= \varphi_i^k(t)\mathbb{1}_{m_k}, \\ \frac{d\vartheta_i(t)}{dt} &= \vartheta_i(t)\overline{Q}(t) + \widetilde{b}_{i-1}(t)\widehat{Q}(t)\widetilde{\mathbb{1}}. \end{aligned} \tag{4.69}$$

(b) *For $i = 1, \ldots, n$, the initial conditions are selected as follows:*

- *For $k = 1, 2, \ldots, l$, find $\psi_i^k(0)\mathbb{1}_{m_k}$ from the equation*

$$\psi_i(0)\pi = -\left(\sum_{j=0}^{i-1} \int_0^\infty \frac{s^j}{j!} \psi_{i-j-1}(s) ds \frac{d^j \widehat{Q}(0)}{dt^j} \right) \pi := -\overline{\psi}_{i-1}\pi.$$

- *Choose*

$$\vartheta_i^k(0) = -\psi_i^k(0)\mathbb{1}_{m_k} = \overline{\psi}_{i-1}^k \mathbb{1}_{m_k}, \text{ for } k = 1, \ldots, l.$$

- *Choose $\psi_i(0) = -\varphi_i(0)$.*

(c) *There is a positive real number $0 < \kappa_0 < \kappa_{i,0}$ (given in (4.68)) for $i = 0, 1, \ldots, n+1$ such that*

$$|\psi_i(\tau)| \leq K \exp(-\kappa_0 \tau).$$

(d) *The choice of initial conditions yields that $\vartheta_i^k(\cdot)$ is $(n+1-i)$-times continuously differentiable on $[0, T]$ and hence $\varphi_i(\cdot)$ is $(n+1-i)$-times continuously differentiable on $[0, T]$.* □

4.3.2 Analysis of Remainder

The objective here is to carry out the error analysis and validate the asymptotic expansion. Since the details are quite similar to those of Section 4.2, we make no attempt to spell them out. Only the following lemma and proposition are presented.

Lemma 4.27. *Suppose that* (A4.3) *and* (A4.4) *are satisfied. Let $\eta^\varepsilon(\cdot)$ be a function such that*

$$\sup_{t \in [0,T]} |\eta^\varepsilon(t)| = O(\varepsilon^{k+1}) \text{ for } k \leq n$$

and let $\mathcal{L}^\varepsilon$ be an operator defined in (4.42). If $f^\varepsilon(\cdot)$ is a solution to the equation

$$\mathcal{L}^\varepsilon f^\varepsilon(t) = \eta^\varepsilon(t) \text{ with } f^\varepsilon(0) = 0,$$

then $f^\varepsilon(\cdot)$ satisfies

$$\sup_{t \in [0,T]} |f^\varepsilon(t)| = O(\varepsilon^k).$$

Proof: Note that using $Q^\varepsilon(t) = \widetilde{Q}(t)/\varepsilon + \widehat{Q}(t)$, the differential equation can be written as

$$\frac{df^\varepsilon(t)}{dt} = f^\varepsilon(t) Q^\varepsilon(t) + \frac{\eta^\varepsilon(t)}{\varepsilon}.$$

We can then proceed as in the proof of Lemma 4.13. □

Lemma 4.27 together with detailed computation similar to that of Section 4.2 yields the following proposition.

Proposition 4.28. *For each* $i = 0, 1, \ldots, n$, *define*

$$e_i^\varepsilon(t) = p^\varepsilon(t) - y_i^\varepsilon(t). \tag{4.70}$$

Under conditions (A4.3) *and* (A4.4),

$$\sup_{0\le t\le T} |e_i^\varepsilon(t)| = O(\varepsilon^{i+1}).$$

4.3.3 Computational Procedure: User's Guide

Since the constructions of $\varphi_i(\cdot)$ and $\psi_i(\cdot)$ are rather involved, and the choice of initial conditions is tricky, we summarize the procedure below. This procedure, which can be used as a user's guide for developing the asymptotic expansion, comprises two main stages.

Step 1: Initialization: finding $\varphi_0(\cdot)$ and $\psi_0(\cdot)$.

1. Obtain the unique solution $\varphi_0(\cdot)$ via (4.54).
2. Obtain the unique solution $\psi_0(\cdot)$ via (4.55) and the initial condition $\psi_0(0) = p^0 - \varphi_0(0)$.

Step 2. Iteration: finding $\varphi_i(\cdot)$ and $\psi_i(\cdot)$ for $1 \le i \le n$.
While $i \le n$, do the following:

1. Find $\varphi_i(\cdot)$ the solution of (4.69) with temporarily unspecified $\vartheta_i^k(0)$ for $k = 1, \ldots, l$.
2. Obtain $\psi_i(\cdot)$ from (4.55) with temporarily unspecified $\psi_i(0)$.
3. Use the equation

$$\psi_i(0)\pi = -\Big(\sum_{j=0}^{i-1}\int_0^\infty \frac{s^j}{j!}\psi_{i-j-1}(s)ds\frac{d^j\widehat{Q}(0)}{dt^j}\Big)\pi := -\overline{\psi}_{i-1}\pi$$

   to obtain $\psi_i^k(0)\mathbb{1}_{m_k} = -\overline{\psi}_{i-1}^k\mathbb{1}_{m_k}$.
4. Set $\vartheta_i^k(0) = -\psi_i^k(0)\mathbb{1}_{m_k} = \overline{\psi}_{i-1}^k\mathbb{1}_{m_k}$. By now, $\varphi_i(\cdot)$ has been determined uniquely.
5. Choose $\psi_i(0) = -\varphi_i(0)$. By now, $\psi_i(\cdot)$ has also been determined uniquely.
6. Set $i = i + 1$.
7. If $i > n$, stop.

4.3.4 Summary of Results

While the previous subsection gives the computational procedure, this subsection presents the main theorem. It establishes the validity of the asymptotic expansion.

Theorem 4.29. *Suppose conditions* (A4.3) *and* (A4.4) *are satisfied. Then the asymptotic expansion*

$$y_n^\varepsilon(t) = \sum_{i=0}^{n} \left(\varepsilon^i \varphi_i(t) + \varepsilon^i \psi_i \left(\frac{t}{\varepsilon} \right) \right)$$

can be constructed as in the computational procedure such that

(a) $\varphi_i(\cdot)$ *is* $(n+1-i)$*-times continuously differentiable on* $[0,T]$*;*

(b) $|\psi_i(t)| \le K \exp(-\kappa_0 t)$ *for some* $\kappa_0 > 0$*;*

(c) $|p^\varepsilon(t) - y_n^\varepsilon(t)| = O(\varepsilon^{n+1})$ *uniformly in* $t \in [0,T]$.

Remark 4.30 In general, in view of Proposition 4.11, the error bound is of the form $c_{2n}(t)\exp(-\kappa_0 t)$, where $c_{2n}(t)$ is a polynomial of degree $2n$. The exponential constant κ_0 typically depends on n. The larger n is, the smaller κ_0 will be to account for the polynomial $c_{2n}(t)$.

The following result is a corollary to Theorem 4.29 and will be used in Chapters 5 and 7. Denote the jth component of $\nu^k(t)$ by $\nu_j^k(t)$.

Corollary 4.31. *Assume, in addition to the conditions in Theorem 4.29 with* $n = 0$*, that* $\widetilde{Q}(t) = \widetilde{Q}$ *and* $\widehat{Q}(t) = \widehat{Q}$ *are time independent. Then there exist positive constants* K *and* κ_0 *(both independent of* ε *and* t*) such that*

$$\left| P(\alpha^\varepsilon(t) = s_{kj}) - \nu_j^k(t)\vartheta^k(t) \right| \le K\left(\varepsilon(t+1) + \exp\left(-\frac{\kappa_0 t}{\varepsilon} \right) \right), \tag{4.71}$$

where $\vartheta^k(t)$ *satisfies*

$$\frac{d}{dt}(\vartheta^1(t), \ldots, \vartheta^l(t)) = (\vartheta^1(t), \ldots, \vartheta^l(t))\overline{Q},$$

with $(\vartheta^1(0), \ldots, \vartheta^l(0)) = (P(\alpha^\varepsilon(0) \in \mathcal{M}_1), \ldots, P(\alpha^\varepsilon(0) \in \mathcal{M}_l))$.

Proof: By a slight modification of the analysis of remainder in Section 4.3, we can obtain (4.71) with a constant K independent of ε and t. The second part of the lemma follows from the uniqueness of the solution to the ordinary differential equation (4.71). □

Remark 4.32. We mention an alternative approach to establishing the asymptotic expansion. In lieu of the constructive procedure presented previously, one may wish to write $\varphi_i(t)$ as a sum of solutions of the homogeneous part and the inhomogeneous part. For instance, one may set

$$\varphi_i(t) = v_i(t)\mathrm{diag}(\nu^1(t), \ldots, \nu^l(t)) + U_i(t), \tag{4.72}$$

where $v_i(t) \in \mathbb{R}^l$ and $U_i(t)$ is a particular solution of the inhomogeneous equation. For $i \geq 0$, the equation

$$\varphi^{(i+1)}(t)\widetilde{Q}(t) = \frac{d\varphi_i(t)}{dt} - \varphi_i(t)\widehat{Q}(t)$$

and $\widetilde{Q}(t)\widetilde{\mathbb{1}} = 0$ lead to

$$0 = \left(\frac{d\varphi_i(t)}{dt} - \varphi_i(t)\widehat{Q}(t)\right)\widetilde{\mathbb{1}}.$$

Substituting (4.72) into the equation above, and noting that $\nu^k(t)\mathbb{1}_{m_k} = 1$ for $k = 1, \ldots, l$, and that $\text{diag}(\nu^1(t), \ldots, \nu^l(t))\widetilde{\mathbb{1}} = I_l$, the $l \times l$ identity matrix, one obtains

$$\frac{dv_i(t)}{dt} = v_i(t)\overline{Q}(t) + U_i(t)\widehat{Q}(t)\widetilde{\mathbb{1}} - \left(\frac{dU_i(t)}{dt}\right)\widetilde{\mathbb{1}}.$$

One then proceeds to determine $v_i(0)$ via the matching condition. The main ideas are similar, and the details are slightly different.

4.3.5 An Example

Consider Example 4.20 again. Note that the conditions in (A4.3) and (A4.4) require that

$$\lambda_1(t) + \mu_1(t) > 0 \text{ for all } t \in [0, T],$$

and the jump rates $\lambda(t)$ and $\mu(t)$ be smooth enough.

The probability distribution of the state process is given by $p^\varepsilon(t)$ satisfying

$$\frac{dp^\varepsilon(t)}{dt} = p^\varepsilon(t)Q^\varepsilon(t),$$

$$p^\varepsilon(0) = p^0 \quad \text{such that}$$

$$p_i^0 \geq 0 \text{ and } \sum_{i=1}^{4} p_i^0 = 1.$$

To solve this set of equations, note that

$$\frac{d}{dt}(p_1^\varepsilon(t) + p_2^\varepsilon(t)) = -\lambda_2(t)(p_1^\varepsilon(t) + p_2^\varepsilon(t)) + \mu_2(t)(p_3^\varepsilon(t) + p_4^\varepsilon(t)),$$

$$\frac{d}{dt}(p_1^\varepsilon(t) + p_3^\varepsilon(t)) = -\frac{\lambda_1(t)}{\varepsilon}(p_1^\varepsilon(t) + p_3^\varepsilon(t)) + \frac{\mu_1(t)}{\varepsilon}(p_2^\varepsilon(t) + p_4^\varepsilon(t)),$$

$$\frac{d}{dt}(p_2^\varepsilon(t) + p_4^\varepsilon(t)) = \frac{\lambda_1(t)}{\varepsilon}(p_1^\varepsilon(t) + p_3^\varepsilon(t)) - \frac{\mu_1(t)}{\varepsilon}(p_2^\varepsilon(t) + p_4^\varepsilon(t)),$$

$$\frac{d}{dt}(p_3^\varepsilon(t) + p_4^\varepsilon(t)) = \lambda_2(t)(p_1^\varepsilon(t) + p_2^\varepsilon(t)) - \mu_2(t)(p_3^\varepsilon(t) + p_4^\varepsilon(t)).$$

To proceed, define functions $a_{12}(t)$, $a_{13}(t)$, $a_{24}(t)$, and $a_{34}(t)$ as follows:

$$\begin{aligned}
a_{12}(t) = {} & (p_1^0 + p_2^0) \exp\left(- \int_0^t (\lambda_2(s) + \mu_2(s)) ds \right) \\
& + \int_0^t \mu_2(u) \exp\left(- \int_u^t (\lambda_2(s) + \mu_2(s)) ds \right) du, \\
a_{13}(t) = {} & (p_1^0 + p_3^0) \exp\left(-\frac{1}{\varepsilon} \int_0^t (\lambda_1(s) + \mu_1(s)) ds \right) \\
& + \int_0^t \frac{\mu_1(u)}{\varepsilon} \exp\left(-\frac{1}{\varepsilon} \int_u^t (\lambda_1(s) + \mu_1(s)) ds \right) du, \\
a_{24}(t) = {} & (p_2^0 + p_4^0) \exp\left(-\frac{1}{\varepsilon} \int_0^t (\lambda_1(s) + \mu_1(s)) ds \right) \\
& + \int_0^t \frac{\lambda_1(u)}{\varepsilon} \exp\left(-\frac{1}{\varepsilon} \int_u^t (\lambda_1(s) + \mu_1(s)) ds \right) du, \\
a_{34}(t) = {} & (p_3^0 + p_4^0) \exp\left(- \int_0^t (\lambda_2(s) + \mu_2(s)) ds \right) \\
& + \int_0^t \lambda_2(u) \exp\left(- \int_u^t (\lambda_2(s) + \mu_2(s)) ds \right) du.
\end{aligned}$$

Then using the fact that $p_1^\varepsilon(t) + p_2^\varepsilon(t) + p_3^\varepsilon(t) + p_4^\varepsilon(t) = 1$, we have

$$\begin{aligned}
p_1^\varepsilon(t) + p_2^\varepsilon(t) &= a_{12}(t), \\
p_1^\varepsilon(t) + p_3^\varepsilon(t) &= a_{13}(t), \\
p_2^\varepsilon(t) + p_4^\varepsilon(t) &= a_{24}(t), \\
p_3^\varepsilon(t) + p_4^\varepsilon(t) &= a_{34}(t).
\end{aligned} \tag{4.73}$$

Note also that

$$\begin{aligned}
\frac{dp_1^\varepsilon(t)}{dt} = {} & - \left(\frac{\lambda_1(t)}{\varepsilon} + \frac{\mu_1(t)}{\varepsilon} + \lambda_2(t) + \mu_2(t) \right) p_1^\varepsilon(t) \\
& + \frac{\mu_1(t)}{\varepsilon} a_{12}(t) + \mu_2(t) a_{13}(t).
\end{aligned}$$

The solution to this equation is

$$\begin{aligned}
p_1^\varepsilon(t) = {} & p_1^0 \exp\left(- \int_0^t \left(\frac{\lambda_1(s) + \mu_1(s)}{\varepsilon} + \lambda_2(s) + \mu_2(s) \right) ds \right) \\
& + \int_0^t \left(\frac{\mu_1(u)}{\varepsilon} a_{12}(u) + \mu_2(u) a_{13}(u) \right) \\
& \times \exp\left(- \int_u^t \left(\frac{\lambda_1(s) + \mu_1(s)}{\varepsilon} + \lambda_2(s) + \mu_2(s) \right) ds \right) du.
\end{aligned}$$

Consequently, in view of (4.73), it follows that

$$\begin{aligned} p_2^\varepsilon(t) &= a_{12}(t) - p_1^\varepsilon(t), \\ p_3^\varepsilon(t) &= a_{13}(t) - p_1^\varepsilon(t), \\ p_4^\varepsilon(t) &= a_{24}(t) - p_2^\varepsilon(t). \end{aligned}$$

In this example, the zeroth-order term is given by

$$\varphi_0(t) = (\nu^1(t)\vartheta_0^1(t), \nu^2(t)\vartheta_0^2(t)),$$

where the quasi-stationary distributions are given by

$$\nu^1(t) = \nu^2(t) = \left(\frac{\mu_1(t)}{\lambda_1(t) + \mu_1(t)}, \frac{\lambda_1(t)}{\lambda_1(t) + \mu_1(t)} \right),$$

and the multipliers $(\vartheta_0^1(t), \vartheta_0^2(t))$ are determined by the differential equation

$$\frac{d}{dt}(\vartheta_0^1(t), \vartheta_0^2(t)) = (\vartheta_0^1(t), \vartheta_0^2(t)) \begin{pmatrix} -\lambda_2(t) & \lambda_2(t) \\ \mu_2(t) & -\mu_2(t) \end{pmatrix},$$

with initial value $(\vartheta_0^1(0), \vartheta_0^2(0)) = (p_1^0 + p_2^0, p_3^0 + p_4^0)$.

The inner expansion term $\psi_0(\tau)$ is given by

$$\frac{d\psi_0(\tau)}{d\tau} = \psi_0(\tau)\widetilde{Q}(0), \ \psi_0(0) = p^0 - \varphi_0(0).$$

By virtue of Theorem 4.29,

$$p^\varepsilon(t) - \varphi_0(t) - \psi_0\left(\frac{t}{\varepsilon}\right) = O(\varepsilon),$$

provided that $Q^\varepsilon(t)$ is continuously differentiable on $[0, T]$. Noting the exponential decay of $\psi_0(t/\varepsilon)$, we further have

$$p^\varepsilon(t) = \varphi_0(t) + O\left(\varepsilon + \exp\left(-\frac{\kappa_0 t}{\varepsilon}\right)\right).$$

In particular, for any $t > 0$,

$$\lim_{\varepsilon \to 0} p^\varepsilon(t) = \varphi_0(t).$$

Namely, $\varphi_0(t)$ is the limit distribution of the Markov chain generated by $Q^\varepsilon(t)$.

4.4 Inclusion of Absorbing States

While the case of recurrent states was considered in the previous section, this section concerns the asymptotic expansion in which the Markov chain generated by $Q^\varepsilon(t)$ in which $\widetilde{Q}(t)$ includes components corresponding to absorbing states. By rearrangement, the matrix $\widetilde{Q}(t)$ takes the form

$$\widetilde{Q}(t) = \begin{pmatrix} \widetilde{Q}^1(t) & & & & \\ & \widetilde{Q}^2(t) & & & \\ & & \ddots & & \\ & & & \widetilde{Q}^l(t) & \\ & & & & 0_{m_a \times m_a} \end{pmatrix}, \tag{4.74}$$

where $\widetilde{Q}^k(t) \in \mathbb{R}^{m_k \times m_k}$ for $k = 1, 2, \ldots, l$, $0_{m_a \times m_a}$ is an $m_a \times m_a$ zero matrix, and

$$m_1 + m_2 + \cdots + m_l + m_a = m.$$

Let $\mathcal{M}_a = \{s_{a1}, \ldots, s_{am_a}\}$ denote the set of absorbing states. We may, as in Section 4.3, represent the state space as

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l \cup \mathcal{M}_a \\ &= \left\{ s_{11}, \ldots, s_{1m_1}, \ldots, s_{l1}, \ldots, s_{lm_l}, s_{a1}, \ldots, s_{am_a} \right\}. \end{aligned}$$

Following the development of Section 4.3, suppose that $\alpha^\varepsilon(\cdot)$ is a Markov chain generated by $Q^\varepsilon(\cdot) = \widetilde{Q}(\cdot)/\varepsilon + \widehat{Q}(\cdot)$. Compared with Section 4.3, the difference is that now the dominant part in the generator includes absorbing states corresponding to the $m_a \times m_a$ matrix $0_{m_a \times m_a}$. As in the previous case, our interest is to obtain an asymptotic expansion of the probability distribution.

Remark 4.33. The motivation of the current study stems from the formulation of competitive risk theory discussed in Section 3.3. The idea is that within the m states, there are several groups. Some of them are much riskier than the others (in the sense of frequency of the occurrence of the corresponding risks). The different rates (sensitivity) of risks are modeled by the use of a small parameter $\varepsilon > 0$.

Denote by $p^\varepsilon(\cdot)$ the solution of (4.40). The objective here is to obtain an asymptotic expansion

$$y_n^\varepsilon = \sum_{i=0}^{n} \varepsilon^i \varphi_i(t) + \sum_{i=0}^{n} \varepsilon^i \psi_i\left(\frac{t}{\varepsilon}\right).$$

Since the techniques employed are essentially the same as in the previous section, it will be most instructive here to highlight the main ideas. Thus, we only note the main steps and omit most of the details.

Assume conditions (A4.3) and (A4.4) for the current matrices $\widetilde{Q}^k(t)$, $\widetilde{Q}(t)$, and $\widehat{Q}(t)$. For $t \in [0,T]$, substituting the expansion above into (4.40) and equating coefficients of ε^i, for $i = 1, \ldots, n+1$, yields

$$\begin{aligned} &\varphi_0(t)\widetilde{Q}(t) = 0, \\ &\varphi_i(t)\widetilde{Q}(t) = \frac{d\varphi_{i-1}(t)}{dt} - \varphi_{i-1}(t)\widehat{Q}(t), \end{aligned} \tag{4.75}$$

and (with the use of the stretched variable $\tau = t/\varepsilon$)

$$\begin{aligned} \frac{d\psi_0(\tau)}{d\tau} &= \psi_0(\tau)\widetilde{Q}(0), \\ \frac{d\psi_i(\tau)}{d\tau} &= \psi_i(\tau)\widetilde{Q}(0) + \sum_{j=0}^{i-1} \psi_{i-j-1}(\tau) \\ &\qquad \times \left(\frac{\tau^j}{j!}\frac{d^j\widehat{Q}(0)}{dt^j} + \frac{\tau^{j+1}}{(j+1)!}\frac{d^{j+1}\widetilde{Q}(0)}{dt^{j+1}} \right). \end{aligned} \tag{4.76}$$

For each $i \geq 0$, we use the following notation for the partitioned vectors:

$$\varphi_i(t) = (\varphi_i^1(t), \ldots, \varphi_i^l(t), \varphi_i^a(t)),$$

$$\psi_i(\tau) = (\psi_i^1(\tau), \ldots, \psi_i^l(\tau), \psi_i^a(\tau)).$$

In the above $\varphi_i^a(t)$ and $\psi_i^a(\tau)$ are vectors in $\mathbb{R}^{1\times m_a}$.

To determine the outer- and the initial-layer expansions, let us start with $i = 0$. For each $t \in [0,T]$, the use of the partitioned vector $\varphi_0(t)$ leads to

$$\varphi_0^k(t)\widetilde{Q}^k(t) = 0, \text{ for } k = 1, \ldots, l.$$

Note that $\varphi_0^a(t)$ does not show up in any of these equations owing to the $0_{m_a\times m_a}$ matrix in $\widetilde{Q}(t)$. It will have to be obtained from the equation in (4.75) corresponding to $i = 1$. Put another way, $\varphi_0^a(t)$ is determined mainly by the matrix $\widehat{Q}(t)$.

Similar to Section 4.3, $\varphi_0^k(t) = \vartheta_0^k(t)\nu^k(t)$, where $\nu^k(t)$ are the quasi-stationary distributions corresponding to the generators $\widetilde{Q}^k(t)$ for $k = 1, \ldots, l$ and $\vartheta_0^k(t)$ are the corresponding multipliers. Define

$$\widetilde{\mathbb{1}}_a = \begin{pmatrix} \mathbb{1}_{m_1} & & & \\ & \ddots & & \\ & & \mathbb{1}_{m_l} & \\ & & & I_{m_a} \end{pmatrix},$$

where I_{m_a} is an $m_a \times m_a$ identity matrix. Clearly, $\widetilde{\mathbb{1}}_a$ is orthogonal to $\widetilde{Q}(t)$ for each $t \in [0,T]$. As a result, multiplying (4.75) by $\widetilde{\mathbb{1}}_a$ from the right with $i = 1$ leads to

$$\begin{aligned} \frac{d\varphi_0(t)}{dt}\widetilde{\mathbb{1}}_a &= \varphi_0(t)\widehat{Q}(t)\widetilde{\mathbb{1}}_a, \\ (\vartheta_0(0), \varphi_0^a(0)) &= p^0\widetilde{\mathbb{1}}_a, \end{aligned} \tag{4.77}$$

where $\vartheta_0(0) = (\vartheta_0^1(0), \ldots, \vartheta_0^l(0))$.

The above initial condition is a consequence of the initial-value consistency condition in (4.53). It is readily seen that

$$\sum_{k=1}^{l} \vartheta_0^k(0) = 1 - \varphi_0^a(0)\mathbb{1}_{m_a} = 1 - p^{0,a}\mathbb{1}_{m_a},$$

where $p^0 = (p^{0,1}, \ldots, p^{0,l}, p^{0,a})$.

We write

$$\varphi_0(t) = (\vartheta_0^1(t), \ldots, \vartheta_0^l(t), \varphi_0^a(t))\mathrm{diag}(\nu^1(t), \ldots, \nu^l(t), I_{m_a}).$$

Define

$$\overline{Q}(t) = \mathrm{diag}(\nu^1(t), \ldots, \nu^l(t), I_{m_a})\widehat{Q}(t)\widetilde{\mathbb{1}}_a. \tag{4.78}$$

Then (4.77) is equivalent to

$$\begin{aligned} \frac{d}{dt}(\vartheta_0(t), \varphi_0^a(t)) &= (\vartheta_0(t), \varphi_0^a(t))\overline{Q}(t), \\ (\vartheta_0(0), \varphi_0^a(0)) &= p^0\widetilde{\mathbb{1}}_a. \end{aligned}$$

This is a linear system of differential equations. Therefore it has a unique solution given by

$$(\vartheta_0(t), \varphi_0^a(t)) = p^0\widetilde{\mathbb{1}}_a X(t,0),$$

where $X(t,0)$ is the principal matrix solution of the homogeneous equation. Thus $\varphi_0(t)$ has been found and is $(n+1)$-times continuously differentiable.

Remark 4.34. Note that in $\varphi_0(t)$, the term $\varphi_0^a(t)$ corresponds to the set of absorbing states $\mathcal{M}_a$. Clearly, these states cannot be aggregated to a single state as in the case of recurrent states. Nevertheless, the function $\varphi_0^a(t)$ tends to be stabilized in a neighborhood of a constant for t large enough. To illustrate, let us consider a stationary case, that is, both $\widetilde{Q}(t) = \widetilde{Q}$ and $\widehat{Q}(t) = \widehat{Q}$ are independent of t. Partition $\widehat{Q}$ as blocks of submatrices

$$\widehat{Q} = \begin{pmatrix} \widehat{Q}^{11} & \widehat{Q}^{12} \\ \widehat{Q}^{21} & \widehat{Q}^{22} \end{pmatrix},$$

where $\widehat{Q}^{22}$ is an $m_a \times m_a$ matrix. Assume that the eigenvalues of $\widehat{Q}^{22}$ have negative real parts. Then, in view of the definition of $\overline{Q}(t) = \overline{Q}$ in (4.78), it follows that

$$\varphi_0^a(t) \to \text{ a constant as } t \to \infty.$$

Using the partition $\psi_0(\tau) = (\psi_0^1(\tau), \ldots, \psi_0^l(\tau), \psi_0^a(\tau))$, consider the zeroth-order initial-layer term given by

$$\begin{aligned} \frac{d\psi_0(\tau)}{d\tau} &= \frac{d}{d\tau}(\psi_0^1(\tau), \ldots, \psi_0^l(\tau), \psi_0^a(\tau)) \\ &= \psi_0(\tau)\widetilde{Q}(0) = (\psi_0^1(\tau)\widetilde{Q}^1(\tau), \ldots, \psi_0^l(\tau)\widetilde{Q}^l(0), 0_{m_a}). \end{aligned}$$

We obtain

$$\psi_0^k(\tau) = \psi_0^k(0)\exp(\widetilde{Q}^k(0)\tau), \text{ for } k = 1, \ldots, l, \text{ and}$$

$$\psi_0^a(\tau) = \text{constant}.$$

Noting that $p^{0,a} = \varphi_0^a(0)$ and choosing $\psi_0(0) = p^0 - \varphi_0(0)$ lead to $\psi_0^a(\tau) = 0_{m_a}$. Thus

$$\psi_0(\tau) = (\psi_0^1(0)\exp(\widetilde{Q}^1(0)\tau), \ldots, \psi_0^l(0)\exp(\widetilde{Q}^l(0)\tau), 0_{m_a}).$$

Similar to the result in Section 4.3, the following lemma holds. The proof is analogous to that of Proposition 4.25.

Lemma 4.35. *Define*

$$\pi_a = \text{diag}(\mathbb{1}_{m_1}\nu^1(0), \ldots, \mathbb{1}_{m_l}\nu^l(0), I_{m_a}).$$

Then there exist positive constants K and $\kappa_{0,0}$ such that

$$|\exp(\widetilde{Q}(0)\tau) - \pi_a| \le K\exp(-\kappa_{0,0}\tau).$$

By virtue of the lemma above and the orthogonality $(p^0 - \varphi_0(0))\pi_a = 0$, we have

$$\begin{aligned} |\psi_0(\tau)| &= |(p^0 - \varphi_0(0))(\exp(\widetilde{Q}(0)\tau) - \pi_a)| \\ &\le K\exp(-\kappa_{0,0}\tau) \end{aligned}$$

for some $K > 0$ and $\kappa_{0,0} > 0$ given in Lemma 4.35; that is, $\psi_0(\tau)$ decays exponentially fast. Therefore, $\psi_0(\tau)$ has the desired property.

We continue in this fashion and proceed to determine the next term $\varphi_1(t)$ as well as $\psi_1(t/\varepsilon)$. Let

$$b_0(t) = \frac{d\varphi_0(t)}{dt} - \varphi_0(t)\widehat{Q}(t) \quad \text{with}$$

$$b_0(t) = (b_0^1(t), \ldots, b_0^l(t), b_0^a(t)).$$

It is easy to check that $b_0^a(t) = 0_{m_a}$. The equation $\varphi_1(t)\widetilde{Q}(t) = b_0(t)$ then leads to

$$\begin{aligned} &\varphi_1^k(t)\widetilde{Q}^k(t) = b_0^k(t), \text{ for } k = 1, \ldots, l, \\ &b_0^a(t) = 0_{m_a}. \end{aligned} \tag{4.79}$$

The solutions of the l inhomogeneous equations in (4.79) above are of the form

$$\varphi_1^k(t) = \vartheta_1^k(t)\nu^k(t) + \widetilde{b}_0^k(t), \ k = 1, \ldots, l,$$

where $\vartheta_1^k(t)$ for $k = 1, \ldots, l$ are scalar multipliers. Again, $\varphi_1^a(t)$ cannot be obtained from the equation above, it must come from the contribution of the matrix-valued function $\widehat{Q}(t)$.

Note that

$$\widetilde{b}_0^k(t)\widetilde{Q}^k(t) = b_0^k(t) \quad \text{and} \quad \widetilde{b}_0^k(t)\mathbb{1}_{m_k} = 0.$$

Using the equation

$$\varphi_2(t)\widetilde{Q}(t) = \frac{d\varphi_1(t)}{dt} - \varphi_1(t)\widehat{Q}(t),$$

one obtains

$$0 = \varphi_2(t)\widetilde{Q}(t)\widetilde{\mathbb{1}}_a = \frac{d\varphi_1(t)}{dt}\widetilde{\mathbb{1}}_a - \varphi_1(t)\widehat{Q}(t)\widetilde{\mathbb{1}}_a,$$

which in turn implies that

$$\begin{aligned} \frac{d}{dt}(\vartheta_1(t), \varphi_1^a(t)) = (\vartheta_1(t), \varphi_1^a(t))\overline{Q}(t) \\ +(\widetilde{b}_0(t), 0_{m_a})\widehat{Q}(t)\widetilde{\mathbb{1}}_a, \end{aligned} \tag{4.80}$$

where

$$\vartheta_1(t) = (\vartheta_1^1(t), \ldots, \vartheta_1^l(t)) \text{ and } \widetilde{b}_0(t) = (\widetilde{b}_0^1(t), \ldots, \widetilde{b}_0^l(t)).$$

Let $X(t,s)$ denote the principal matrix solution to the homogeneous differential equation

$$\frac{dy(t)}{dt} = y(t)\overline{Q}(t).$$

Then the solution to (4.80) can be represented by $X(t,s)$ as follows:

$$\begin{aligned}(\vartheta_1(t), \varphi_1^a(t)) &= (\vartheta_1(0), \varphi_1^a(0))X(t,0) \\ &\quad + \int_0^t (\widetilde{b}_0(s), 0_{m_a})\widehat{Q}(s)\widetilde{\mathbb{1}}_a X(t,s)ds.\end{aligned}$$

Note that the initial conditions $\varphi_1^a(0)$ and $\vartheta_1^k(0)$ for $k = 1, \ldots, l$ need to be determined using the initial-layer terms just as in Section 4.3.

Using (4.76) with $i = 1$, one obtains an equation that has the same form as that of (4.62). That is,

$$\begin{aligned}\psi_1(\tau) &= \psi_1(0)\exp(\widetilde{Q}(0)\tau) \\ &\quad + \int_0^\tau \psi_0(0)\exp(\widetilde{Q}(0)s)\widehat{Q}(0)\exp(\widetilde{Q}(0)(\tau - s))ds \\ &\quad + \int_0^\tau s\psi_0(0)\exp(\widetilde{Q}(0)s)\frac{d\widetilde{Q}(0)}{dt}\exp(\widetilde{Q}(0)(\tau - s))ds.\end{aligned}$$

As in Section 4.3, with the use of π_a, it can be shown that $|\psi_1(\tau)| \leq K\exp(-\kappa_{1,0}\tau)$ for some $K > 0$ and $0 < \kappa_{1,0} < \kappa_{0,0}$. By requiring that $\psi_1(\tau)$ decay to 0 as $\tau \to \infty$, we obtain the equation

$$\psi_1(0)\pi_a = -\overline{\psi}_0\pi_a, \tag{4.81}$$

where

$$\overline{\psi}_0 = \int_0^\infty \psi_0(0)\exp(\widetilde{Q}(0)s)ds\widehat{Q}(0).$$

Owing to (4.81) and the known form of $\psi_0(\tau)$,

$$\begin{aligned}\overline{\psi}_0 &= (\overline{\psi}_0^1, \ldots, \overline{\psi}_0^l, \overline{\psi}_0^a) \\ &= (p^{0,1} - \varphi_0^1(0), \ldots, p^{0,l} - \varphi_0^l, 0_{m_a})\left(\int_0^\infty \exp(\widetilde{Q}(0)s)ds\right)\widehat{Q}(0),\end{aligned}$$

which is a completely known vector. Thus the solution to (4.81) is

$$\psi_1^k(0)\mathbb{1}_{m_k} = -\overline{\psi}_0^k\mathbb{1}_{m_k} \text{ for } k = 1, \ldots, l, \text{ and } \psi_1^a(0) = -\overline{\psi}_0^a.$$

To obtain the desired matching property for the inner-outer expansions, choose

$$\vartheta_1^k(0) = -\psi_1^k(0)\mathbb{1}_{m_k} = \overline{\psi}_0^k\mathbb{1}_{m_k} \text{ for } k = 1, \ldots, l,$$

$$\varphi_1^a(0) = -\psi_1^a(0) = \overline{\psi}_0^a.$$

In general, for $i = 2, \ldots, n$, the initial conditions are selected as follows: For $k = 1, 2, \ldots, l$, find $\psi_i^k(0)\mathbb{1}_{m_k}$ from the equation

$$\psi_i(0)\pi_a = -\left(\sum_{j=0}^{i-1} \int_0^\infty \frac{s^j}{j!}\psi_{i-j-1}(s)ds \frac{d^j\widehat{Q}(0)}{dt^j} \right)\pi_a := -\overline{\psi}_{i-1}\pi_a.$$

Choose

$$\vartheta_i^k(0) = -\psi_i^k(0)\mathbb{1}_{m_k} = \overline{\psi}_{i-1}^k\mathbb{1}_{m_k},$$

for $k = 1, \ldots, l$,

$$\phi_i^a(0) = -\overline{\psi}_{i-1}^a, \quad \text{and } \psi_i(0) = -\varphi_i(0).$$

Proceeding inductively, we then construct all $\varphi_i(t)$ and $\psi_i(\tau)$. Moreover, we can verify that there exists $0 < \kappa_{i,0} < \kappa_{i-1,0} < \kappa_{0,0}$ such that $|\psi_i(\tau)| \leq K\exp(-\kappa_{i,0}\tau)$. This indicates that the inclusion of absorbing states is very similar to the case of all recurrent states. In the zeroth-order outer expansion, there is a component $\varphi_0^a(t)$ that "takes care of" the absorbing states. Note, however, that starting from the leading term (zeroth-order approximation), the matching will be determined not only by the multipliers $\vartheta_i(0)$ but also by the vector $\psi_i(0)$ associated with the absorbing states. We summarize the results in the following theorem.

Theorem 4.36. *Consider $\widetilde{Q}(t)$ given by (4.74), and suppose conditions (A4.3) and (A4.4) are satisfied for the matrix-valued functions $\widetilde{Q}^k(\cdot)$ for $k = 1, \ldots, l$ and $\widehat{Q}(\cdot)$. An asymptotic expansion*

$$y_n^\varepsilon(t) = \sum_{i=0}^{n} \left(\varepsilon^i\varphi_i(t) + \varepsilon^i\psi_i\left(\frac{t}{\varepsilon}\right) \right)$$

exists such that

(a) *$\varphi_i(\cdot)$ is $(n+1-i)$-times continuously differentiable on $[0, T]$;*

(b) *$|\psi_i(t)| \leq K\exp(-\kappa_0 t)$ for some $0 < \kappa_0 < \kappa_{i,0}$;*

(c) *$|p^\varepsilon(t) - y_n^\varepsilon(t)| = O(\varepsilon^{n+1})$ uniformly in $t \in [0, T]$.*

Finally, at the end of this section, we give a simple example to illustrate the result.

Example 4.37. Let us consider a Markov chain generated by

$$Q^\varepsilon = \frac{1}{\varepsilon}\widetilde{Q} + \widehat{Q},$$

where

$$\widetilde{Q} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \widehat{Q} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Not being irreducible, the chain generated by $\widetilde{Q}$ includes an absorbing state. In this example, $\overline{Q} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$. Let $p^0 = (p_1^0, p_2^0, p^{0,a})$ denote the initial distribution of $\alpha^\varepsilon(\cdot)$. Then solving the forward equation (4.40) gives us

$$p^\varepsilon(t) = (p_1^\varepsilon(t), p_2^\varepsilon(t), p_3^\varepsilon(t)),$$

where

$$\begin{aligned} p_1^\varepsilon(t) &= \frac{p_1^0 + p_2^0 + p^{0,a}}{2} \\ &\quad - \left(\frac{-p_1^0 + p_2^0 - p^{0,a}}{2} + \frac{p^{0,a}}{2-\varepsilon}\right)\exp\left(-\frac{2t}{\varepsilon}\right) - \left(\frac{(1-\varepsilon)p^{0,a}}{2-\varepsilon}\right)\exp(-t), \\ p_2^\varepsilon(t) &= \frac{p_1^0 + p_2^0 + p^{0,a}}{2} \\ &\quad + \left(\frac{-p_1^0 + p_2^0 - p^{0,a}}{2} + \frac{p^{0,a}}{2-\varepsilon}\right)\exp\left(-\frac{2t}{\varepsilon}\right) - \left(\frac{p^{0,a}}{2-\varepsilon}\right)\exp(-t), \\ p_3^\varepsilon(t) &= p^{0,a}\exp(-t). \end{aligned}$$

Computing $\varphi_0(t)$ yields

$$\begin{aligned} \varphi_0(t) &= \left(\frac{p_1^0 + p_2^0 + p^{0,a}}{2}, \frac{p_1^0 + p_2^0 + p^{0,a}}{2}, 0\right) \\ &\quad + \left(-\frac{p^{0,a}}{2}, -\frac{p^{0,a}}{2}, p^{0,a}\right)\exp(-t). \end{aligned}$$

It is easy to see that for $t > 0$,

$$\lim_{\varepsilon \to 0} |p^\varepsilon(t) - \varphi_0(t)| = 0.$$

The limit behavior of the underlying Markov chain as $\varepsilon \to 0$ is determined by $\varphi_0(t)$ (for $t > 0$). Moreover, when t is large, the influence from $\widehat{Q}$ corresponding to the absorbing state (the vector multiplied by $\exp(-t)$) can be ignored because $\exp(-t)$ goes to 0 exponentially fast as $t \to \infty$.

4.5 Inclusion of Transient States

If a Markov chain has transient states, then, relabeling the states through suitable permutations, one can decompose the states into several groups of recurrent states, each of which is weakly irreducible, and a group of transient states. Naturally, we consider the generator $\widetilde{Q}(t)$ in $Q^\varepsilon(t)$ having the form

$$\widetilde{Q}(t) = \begin{pmatrix} \widetilde{Q}^1(t) & & & \\ & \ddots & & \\ & & \widetilde{Q}^l(t) & \\ \widetilde{Q}^1_*(t) & \cdots & \widetilde{Q}^l_*(t) & \widetilde{Q}_*(t) \end{pmatrix} \tag{4.82}$$

such that for each $t \in [0,T]$, and each $k = 1,\dots,l$, $\widetilde{Q}^k(t)$ is a generator with dimension $m_k \times m_k$, $\widetilde{Q}_*(t)$ is an $m_* \times m_*$ matrix, $\widetilde{Q}^k_*(t) \in \mathbb{R}^{m_* \times m_k}$, and

$$m_1 + m_2 + \cdots + m_l + m_* = m.$$

We continue our study of singularly perturbed chains with weak and strong interactions by incorporating the transient states into the model. Let $\alpha^\varepsilon(\cdot)$ be a Markov chain generated by $Q^\varepsilon(\cdot)$, with $Q^\varepsilon(t) \in \mathbb{R}^{m\times m}$ given by (4.39) with $\widetilde{Q}(t)$ given by (4.82). The state space of the underlying Markov chain is given by

$$\mathcal{M} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l \cup \mathcal{M}_*$$

where $\mathcal{M}_k = \{s_{k1},\dots,s_{km_k}\}$ are the states corresponding to the recurrent states and $\mathcal{M}_* = \{s_{*1},\dots,s_{*m_*}\}$ are those corresponding to the transient states.

Since $\widetilde{Q}(t)$ is a generator, for each $k = 1,\dots,l$, $\widetilde{Q}^k(t)$ is a generator. Thus the matrix $\widetilde{Q}^k_*(t) = (\widetilde{q}^k_{*,ij})$ satisfies $\widetilde{q}^k_{*,ij} \geq 0$ for each $i = 1,\dots,m_*$ and $j = 1,\dots,m_k$, and $\widetilde{Q}_*(t) = (\widetilde{q}_{*,ij})$ satisfies

$$\widetilde{q}_{*,ij}(t) \geq 0 \text{ for } i \neq j, \widetilde{q}_{*,ii}(t) < 0, \text{ and } \widetilde{q}_{*,ii}(t) \leq -\sum_{j\neq i} \widetilde{q}_{*,ij}(t).$$

Roughly, the block matrix $(\widetilde{Q}^1_*(t),\dots,\widetilde{Q}^l_*(t),\widetilde{Q}_*(t))$ is "negatively dominated" by the matrix $\widetilde{Q}_*(t)$. Thus it is natural to assume that $\widetilde{Q}_*(t)$ is a stable matrix (or Hurwitz, i.e., all its eigenvalues have negative real parts). Comparing with the setups of Sections 4.3 and 4.4, the difference in $\widetilde{Q}(t)$ is the additional matrices $\widetilde{Q}^k_*(t)$ for $k = 1,\dots,l$ and $\widetilde{Q}_*(t)$. Note that $\widetilde{Q}^k_*(t)$ are nonsquare matrices, and $\widetilde{Q}(t)$ no longer has block-diagonal form.

The formulation here is inspired by the work of Phillips and Kokotovic [175] and Delebecque and Quadrat [44]; see also the recent work of Pan and Başar [164], in which the authors treated time-invariant $\widetilde{Q}$ matrix of a similar form. Sections 4.3 and 4.4 together with this section essentially include

generators of finite-state Markov chains of the most practical concerns. It ought to be pointed out that just as one cannot in general simultaneously diagonalize two matrices, for Markov chains with weak and strong interactions, one cannot put both $\widetilde{Q}(t)$ and $\widehat{Q}(t)$ into the forms mentioned above simultaneously. Although the model to be studied in this section is slightly more complex compared with the block-diagonal $\widetilde{Q}(t)$ in (4.41), we demonstrate that an asymptotic expansion of the probability distribution can still be obtained by using the same techniques of the previous sections. Moreover, it can be seen from the expansion that the underlying Markov chain stays in the transient states only with very small probability. In some cases, for example $\widehat{Q}(t) = 0$, these transient states can be ignored; see Remark 4.40 for more details.

To incorporate the transient states, we need the following conditions. The main addition is the assumption that $\widetilde{Q}_*(t)$ is stable.

(**A4.5**) For each $t \in [0, T]$ and $k = 1, \ldots, l$, $\widetilde{Q}(t)$, $\widehat{Q}(t)$, and $\widetilde{Q}^k(t)$ satisfy (A4.3) and (A4.4).

(**A4.6**) For each $t \in [0, T]$, $\widetilde{Q}_*(t)$ is Hurwitz (i.e., all of its eigenvalues have negative real parts).

Remark 4.38. Condition (A4.6) indicates the inclusion of transient states. Since $\widetilde{Q}_*(t)$ is Hurwitz, it is nonsingular. Thus the inverse matrix $\widetilde{Q}_*^{-1}(t)$ exists for each $t \in [0, T]$.

Let $p^\varepsilon(\cdot)$ denote the solution to (4.40) with $\widetilde{Q}(t)$ specified in (4.82). We seek asymptotic expansions of $p^\varepsilon(\cdot)$ having the form

$$y_n^\varepsilon(t) = \sum_{i=0}^{n} \varepsilon^i \varphi_i(t) + \sum_{i=0}^{n} \varepsilon^i \psi_i\left(\frac{t}{\varepsilon}\right).$$

The development is very similar to that of Section 4.3, so no attempt is made to give verbatim details. Instead, only the salient features will be brought out.

Substituting $y_n^\varepsilon(t)$ into the forward equation and equating coefficients of ε^i for $i = 1, \ldots, n$ lead to the equations

$$\begin{aligned} &\varphi_0(t)\widetilde{Q}(t) = 0, \\ &\varphi_i(t)\widetilde{Q}(t) = \frac{d\varphi_{i-1}(t)}{dt} - \varphi_{i-1}(t)\widehat{Q}(t), \end{aligned} \tag{4.83}$$

and with the change of time scale $\tau = t/\varepsilon$,

$$\begin{aligned}
\frac{d\psi_0(\tau)}{d\tau} &= \psi_0(\tau)\widetilde{Q}(0), \\
\frac{d\psi_i(\tau)}{d\tau} &= \psi_i(\tau)\widetilde{Q}(0) + \sum_{j=0}^{i-1} \psi_{i-j-1}(\tau) \\
&\qquad \times \left(\frac{\tau^j}{j!} \frac{d^j \widehat{Q}(0)}{dt^j} + \frac{\tau^{j+1}}{(j+1)!} \frac{d^{j+1}\widetilde{Q}(0)}{dt^{j+1}} \right).
\end{aligned} \tag{4.84}$$

As far as the expansions are concerned, the equations have exactly the same form as that of Section 4.3. Note, however, that the partitioned vector $\varphi_i(t)$ has the form

$$\varphi_i(t) = (\varphi_i^1(t), \ldots, \varphi_i^l(t), \varphi_i^*(t)), \ i = 0, 1, \ldots, n,$$

where $\varphi_i^k(t)$, $k = 1, \ldots, l$, is an m_k row vector and $\varphi_i^*(t)$ is an m_* row vector. A similar partition holds for the vector $\psi_i(t)$. To construct these functions, we begin with $i = 0$. Writing $\varphi_0(t)\widetilde{Q}(t) = 0$ in terms of the corresponding partition, we have

$$\varphi_0^k(t)\widetilde{Q}^k(t) + \varphi_0^*(t)\widetilde{Q}_*^k(t) = 0, \text{ for } k = 1, \ldots, l, \text{ and}$$

$$\varphi_0^*(t)\widetilde{Q}_*(t) = 0.$$

Since $\widetilde{Q}_*(t)$ is stable, it is nonsingular. The last equation above implies $\varphi_0^*(t) = 0_{m_*} = (0, \ldots, 0) \in \mathbb{R}^{1\times m_*}$. Consequently, as in the previous section, for each $k = 1, \ldots, l$, the weak irreducibility of $\widetilde{Q}^k(t)$ implies that $\varphi_0^k(t) = \vartheta_0^k(t)\nu^k(t)$, for some scalar function $\vartheta_0^k(t)$. Equivalently,

$$\varphi_0(t) = (\vartheta_0^1(t)\nu^1(t), \ldots, \vartheta_0^l(t)\nu^l(t), 0_{m_*}).$$

Comparing the equation above with the corresponding expression of $\varphi_0(t)$ in Section 4.3, the only difference is the addition of the m_*-dimensional row vector 0_{m_*}.

Remark 4.39. Note that the dominant term in the asymptotic expansion is $\varphi_0(t)$, in which the probabilities corresponding to the transient states are 0. Thus, the probability corresponding to $\alpha^\varepsilon(t) \in \{$ transient states $\}$ is negligibly small.

Define

$$\widetilde{\mathbb{1}}_*(t) = \begin{pmatrix} \mathbb{1}_{m_1} & & & \\ & \ddots & & \\ & & \mathbb{1}_{m_l} & \\ a_{m_1}(t) & \cdots & a_{m_l}(t) & 0_{m_*\times m_*} \end{pmatrix} \tag{4.85}$$

where $a_{m_k}(t) = -\widetilde{Q}_*^{-1}(t)\widetilde{Q}_*^k(t)\mathbb{1}_{m_k}$ for $k = 1, \ldots, l$, and $0_{m_* \times m_*}$ is the zero matrix in $\mathbb{R}^{m_* \times m_*}$.

It is readily seen that

$$\widetilde{Q}(t)\widetilde{\mathbb{1}}_*(t) = 0 \text{ for each } t \in [0, T].$$

In view of (4.83), it follows that

$$\begin{aligned} &\frac{d}{dt}(\vartheta_0^1(t), \ldots, \vartheta_0^l(t), 0_{m_*}) \\ &\qquad = (\vartheta_0^1(t), \ldots, \vartheta_0^l(t), 0_{m_*})\overline{Q}(t), \end{aligned} \tag{4.86}$$

where

$$\overline{Q}(t) = \operatorname{diag}(\nu^1(t), \ldots, \nu^l(t), 0_{m_* \times m_*})\widehat{Q}(t)\widetilde{\mathbb{1}}_*(t).$$

We write $\widehat{Q}(t)$ as follows:

$$\widehat{Q}(t) = \begin{pmatrix} \widehat{Q}^{11}(t) & \widehat{Q}^{12}(t) \\ \widehat{Q}^{21}(t) & \widehat{Q}^{22}(t) \end{pmatrix},$$

where for each $t \in [0, T]$,

$$\widehat{Q}^{11}(t) \in \mathbb{R}^{(m-m_*)\times(m-m_*)},\ \widehat{Q}^{12}(t) \in \mathbb{R}^{(m-m_*)\times m_*},$$

$$\widehat{Q}^{21}(t) \in \mathbb{R}^{m_*\times(m-m_*)},\ \text{and } \widehat{Q}^{22}(t) \in \mathbb{R}^{m_*\times m_*}.$$

Let

$$\overline{Q}_*(t) = \operatorname{diag}(\nu^1(t), \ldots, \nu^l(t)) \left(\widehat{Q}^{11}(t)\widetilde{\mathbb{1}} + \widehat{Q}^{12}(t)(a_{m_1}(t), \ldots, a_{m_l}(t))\right).$$

Then $\overline{Q}(t) = \operatorname{diag}(\overline{Q}_*(t), 0_{m_* \times m_*})$. Moreover, the differential equation (4.86) becomes

$$\frac{d}{dt}(\vartheta_0^1(t), \ldots, \vartheta_0^l(t)) = (\vartheta_0^1(t), \ldots, \vartheta_0^l(t))\overline{Q}_*(t).$$

Remark 4.40. Note that the submatrix $\widehat{Q}^{12}(t)$ in $\widehat{Q}(t)$ determines the jump rates of the underlying Markov chain from a recurrent state in $\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l$ to a transient state in $\mathcal{M}_*$. If the magnitude of the entries of $\widehat{Q}^{12}(t)$ is small, then the transient state can be safely ignored because the contribution of $\widehat{Q}^{12}(t)$ to $\overline{Q}(t)$ is small. On the other hand, if $\widehat{Q}^{12}(t)$ is not negligible, then one has to be careful to include the corresponding terms in $\overline{Q}(t)$.

We now determine the initial value $\vartheta_0^k(0)$. In view of the asymptotic expansions $y_n^\varepsilon(t)$ and the initial-value consistency condition in (4.53), it is necessary that for $k = 1, \ldots, l$,

$$\vartheta_0^k(0) = \varphi_0^k(0)\mathbb{1}_{m_k} = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} p^{\varepsilon,k}(\delta)\mathbb{1}_{m_k}, \tag{4.87}$$

where $p^\varepsilon(t) = (p^{\varepsilon,1}(t), \ldots, p^{\varepsilon,l}(t), p^{\varepsilon,*}(t))$ is a solution to (4.40). Here $p^{\varepsilon,k}(t)$ has dimensions compatible with $\varphi_0^k(0)$ and $\psi_0^k(0)$. Similarly, we write the partition of the initial vector as $p^0 = (p^{0,1}, \ldots, p^{0,l}, p^{0,*})$. The next theorem establishes the desired *consistency* of the initial values. Its proof is placed in Appendix A.4.

Theorem 4.41. *Assume* (A4.5) *and* (A4.6). *Then for* $k = 1, \ldots, l$,

$$\lim_{\delta\to 0}\left(\limsup_{\varepsilon\to 0}\left|p^{\varepsilon,k}(\delta)\mathbb{1}_{m_k} - \left(p^{0,k}\mathbb{1}_{m_k} - p^{0,*}\widetilde{Q}_*^{-1}(0)\widetilde{Q}_*^k(0)\mathbb{1}_{m_k}\right)\right|\right) = 0.$$

Remark 4.42. In view of this theorem, the initial value should be given as

$$\vartheta_0^k(0) = p^{0,k}\mathbb{1}_{m_k} - p^{0,*}\widetilde{Q}_*^{-1}(0)\widetilde{Q}_*^k(0)\mathbb{1}_{m_k}. \tag{4.88}$$

Therefore, in view of (4.88), to make sure that the initial condition satisfies the probabilistic interpretation, it is necessary that

$$\vartheta_0^k(t) \geq 0 \text{ for } t \in [0,T] \text{ and } k = 1, \ldots, l \text{ and } \sum_{k=1}^{l} \vartheta_0^k(0) = 1.$$

In view of the structure of the $\widetilde{Q}(0)$ matrix, for each $k = 1, \ldots, l$, all components of the vector $\widetilde{Q}_*^k(0)\mathbb{1}_{m_k}$ are nonnegative. Note that the solution of the differential equation

$$\frac{dy(t)}{dt} = y(t)\widetilde{Q}(0),$$

$$y(0) = p^0$$

is $p^0 \exp(\widetilde{Q}(0)t)$. This implies that all components of $p^{0,*}\exp(\widetilde{Q}_*(0)t)$ are nonnegative. By virtue of the stability of $\widetilde{Q}_*(0)$,

$$-\widetilde{Q}_*^{-1}(0) = \int_0^\infty \exp(\widetilde{Q}_*(0)t)dt.$$

Thus all components of $-p^{0,*}\widetilde{Q}_*^{-1}(0)$ are nonnegative, and as a result, the inner product

$$-p^{0,*}\widetilde{Q}_*^{-1}(0)\widetilde{Q}_*^k(0)\mathbb{1}_{m_k}$$

is nonnegative. It follows that for each $k = 1, \dots, l$, $\vartheta_0^k(0) \geq p^{0,k}\mathbb{1}_{m_k} \geq 0$. Moreover,

$$\begin{aligned}\sum_{k=1}^{l} \vartheta_0^k(0) &= \sum_{k=1}^{l} p^{0,k}\mathbb{1}_{m_k} - p^{0,*}\widetilde{Q}_*^{-1}(0)\Big(\sum_{k=1}^{l}\widetilde{Q}_*^k(0)\mathbb{1}_{m_k}\Big) \\ &= (1 - p^{0,*}\mathbb{1}_{m_*}) - p^{0,*}\widetilde{Q}_*^{-1}(0)(-\widetilde{Q}_*(0)\mathbb{1}_{m_*}) = 1.\end{aligned} \tag{4.89}$$

Before treating the terms in $\psi_0(\cdot)$, let us give an estimate on $\exp(\widetilde{Q}(0)t)$.

Lemma 4.43. *Set*

$$\pi_* = \begin{pmatrix} \mathbb{1}_{m_1}\nu^1(0) & & & \\ & \ddots & & \\ & & \mathbb{1}_{m_l}\nu^l(0) & \\ a_{m_1}(0)\nu^1(0) & \cdots & a_{m_l}(0)\nu^l(0) & \mathbb{1}_{m_*}0_{m_*} \end{pmatrix}.$$

Then there exist positive constants K and $\kappa_{0,0}$ such that

$$\Big|\exp(\widetilde{Q}(0)\tau) - \pi_*\Big| \leq K\exp(-\kappa_{0,0}\tau), \tag{4.90}$$

for $\tau \geq 0$.

Proof: To prove (4.90), it suffices to show for any m-row vector y^0,

$$\Big|y^0(\exp(\widetilde{Q}(0)\tau) - \pi_*)\Big| \leq K|y^0|\exp(-\kappa_0\tau).$$

Given $y^0 = (y^{0,1}, \dots, y^{0,l}, y^{0,*}) \in \mathbb{R}^{1\times m}$, let

$$y(\tau) = (y^1(\tau), \dots, y^l(\tau), y^*(\tau)) = y^0\exp(\widetilde{Q}(0)\tau).$$

Then, $y(\tau)$ is a solution to

$$\frac{dy(\tau)}{d\tau} = y(\tau)\widetilde{Q}(0),\ y(0) = y^0.$$

It follows that

$$y^*(\tau) = y^{0,*}\exp(\widetilde{Q}_*(0)\tau)$$

and for $k = 1, \dots, l$,

$$y^k(\tau) = y^{0,k}\exp(\widetilde{Q}^k(0)\tau) + \int_0^\tau y^*(s)\widetilde{Q}_*^k(0)\exp(\widetilde{Q}^k(0)(\tau - s))ds.$$

For each $k = 1, \ldots, l$, we have

$$\begin{aligned} y^k(\tau) - \left(y^{0,k} \mathbb{1}_{m_k} \nu^k(0) + y^{0,*} \int_0^\infty \exp(\widetilde{Q}_*(0)s) ds \widetilde{Q}_*^k(0) \mathbb{1}_{m_k} \nu^k(0) \right) \\ = y^{0,k} \left(\exp(\widetilde{Q}^k(0)\tau) - \mathbb{1}_{m_k} \nu^k(0) \right) \\ + y^{0,*} \int_0^\tau \exp(\widetilde{Q}_*(0)s) \widetilde{Q}_*^k(0) \left(\exp(\widetilde{Q}^k(0)(\tau - s)) - \mathbb{1}_{m_k} \nu^k(0) \right) ds \\ - y^{0,*} \int_\tau^\infty \exp(\widetilde{Q}_*(0)s) \widetilde{Q}_*^k(0) \mathbb{1}_{m_k} \nu^k(0) ds. \end{aligned}$$

By virtue of the stability of $\widetilde{Q}_*(0)$, the last term above is bounded above by $K|y^{0,*}| \exp(-\kappa_* \tau)$. Recall that by virtue of Lemma 4.4, for some $\kappa_{0,k} > 0$,

$$\left| \exp(\widetilde{Q}^k(0)\tau) - \mathbb{1}_{m_k} \nu^k(0) \right| \leq K \exp(-\kappa_{0,k}\tau).$$

Choose $\kappa_{0,0} = \min(\kappa_*, \min_k\{\kappa_{0,k}\})$. The terms in the second and the third lines above are bounded by $K|y^0| \exp(-\kappa_{0,0}\tau)$. The desired estimate thus follows. □

Next consider the first equation in the initial-layer expansions:

$$\frac{d\psi_0(\tau)}{d\tau} = \psi_0(\tau)\widetilde{Q}(0).$$

The solution to this equation can be written as

$$\psi_0(\tau) = \psi_0(0) \exp(\widetilde{Q}(0)\tau).$$

To be able to match the asymptotic expansion, choose

$$\psi_0(0) = p^0 - \varphi_0(0).$$

Thus,

$$\begin{aligned} \psi_0(\tau) &= (p^0 - \varphi_0(0)) \exp(\widetilde{Q}(0)\tau) \\ &= (p^0 - \varphi_0(0)) \left(\exp(\widetilde{Q}(0)\tau) - \pi_* \right) + (p^0 - \varphi_0(0))\pi_*. \end{aligned}$$

By virtue of the choice of $\varphi_0(0)$, it is easy to show that

$$(p^0 - \varphi_0(0))\pi_* = 0.$$

Therefore, in view of Lemma 4.43, $\psi_0(\cdot)$ decays exponentially fast in that for some constants K and $\kappa_{0,0} > 0$ given in Lemma 4.43,

$$|\psi_0(\tau)| \leq K \exp(-\kappa_{0,0}\tau), \ \tau \geq 0.$$

We have obtained $\varphi_0(\cdot)$ and $\psi_0(\cdot)$. To proceed, set

$$b_0(t) = \frac{d\varphi_0(t)}{dt} - \varphi_0(t)\widehat{Q}(t)$$

and

$$b_0(t) = (b_0^1(t), \ldots, b_0^l(t), b_0^*(t)).$$

Note that $b_0(t)$ is a completely known function.

In view of the second equation in (4.83),

$$\begin{aligned} &\varphi_1^k(t)\widetilde{Q}^k(t) + \varphi_1^*(t)\widetilde{Q}_*^k(t) = b_0^k(t) \text{ for } k = 1, \ldots, l, \\ &\varphi_1^*(t)\widetilde{Q}_*(t) = b_0^*(t). \end{aligned} \tag{4.91}$$

Solving the last equation in (4.91) yields

$$\varphi_1^*(t) = b_0^*(t)\widetilde{Q}_*^{-1}(t).$$

Putting this back into the first l equations of (4.91) leads to

$$\varphi_1^k(t)\widetilde{Q}^k(t) = b_0^k(t) - b_0^*(t)\widetilde{Q}_*^{-1}(t)\widetilde{Q}_*^k(t). \tag{4.92}$$

Again, the right side is a known function. In view of the choice of $\varphi_0(\cdot)$ and (4.86), we have $b_0(t)\widetilde{\mathbb{1}}_*(t) = 0$. This implies

$$\begin{aligned} &b_0^k(t)\mathbb{1}_{m_k} - b_0^*(t)\widetilde{Q}_*^{-1}(t)\widetilde{Q}_*^i(t)\mathbb{1}_{m_k} \\ &\quad = b_0^k(t)\mathbb{1}_{m_k} + b_0^*(t)a_{m_k}(t) = 0. \end{aligned}$$

Therefore, (4.92) has a particular solution $\widetilde{b}_0^k(t)$ with

$$\widetilde{b}_0^k(t)\mathbb{1}_{m_k} = 0, \text{ for } k = 1, \ldots, l.$$

As in the previous section, we write the solution of $\varphi_1^k(t)$ as a sum of the homogeneous solution and a solution of the inhomogeneous equation $\widetilde{b}_0^k(t)$, that is,

$$\varphi_1^k(t) = \vartheta_1^k(t)\nu^k(t) + \widetilde{b}_0^k(t) \text{ for } k = 1, \ldots, l.$$

In view of

$$\widetilde{Q}(t)\widetilde{\mathbb{1}}_*(t) = 0 \text{ and}$$

$$\widetilde{b}_0^k(t)\mathbb{1}_{m_k} = 0,$$

using the equation

$$\varphi_2(t)\widetilde{Q}(t) = \frac{d\varphi_1(t)}{dt} - \varphi_1(t)\widehat{Q}(t),$$

we obtain that

$$\begin{aligned}
&\frac{d}{dt}(\vartheta_1^1(t), \ldots, \vartheta_1^l(t), 0) \\
&\quad = (\vartheta_1^1(t), \ldots, \vartheta_1^l(t), 0)\overline{Q}(t) + \widetilde{b}_0(t)\widehat{Q}(t)\widetilde{\mathbb{1}}_*(t) \\
&\qquad -\left(\frac{d\widetilde{b}_0^*(t)}{dt}\right)\left(a_{m_1}(t), \ldots, a_{m_l}(t), 0'_{m_*}\right).
\end{aligned} \tag{4.93}$$

The initial value $\vartheta_1(0)$ will be determined in conjunction with the initial value of $\psi_1(\cdot)$ next.

Note that in comparison with the differential equation governing $\vartheta_1(t)$ in Section 4.3, the equation (4.93) has an extra term involving the derivative of $\widetilde{b}_0^*(t)$.

To determine $\psi_1(\cdot)$, solving the equation in (4.84) with $i = 1$, we have

$$\begin{aligned}
\psi_1(\tau) = \ & \psi_1(0)\exp(\widetilde{Q}(0)\tau) \\
& + \int_0^\tau \psi_0(0)\exp(\widetilde{Q}(0)s)\widehat{Q}(0)\exp(\widetilde{Q}(0)(\tau - s))ds \\
& + \int_0^\tau s\psi_0(0)\exp(\widetilde{Q}(0)s)\left(\frac{d\widetilde{Q}(0)}{dt}\right)\exp(\widetilde{Q}(0)(\tau - s))ds.
\end{aligned}$$

Choose the initial values of $\psi_1(0)$ and $\vartheta_1^k(0)$ as follows:

$$\begin{aligned}
\psi_1(0) &= -\varphi_1(0), \\
\vartheta_1^k(0) &= -\psi_1^k(0)\mathbb{1}_{m_k}, \\
\psi_1(0)\pi_* &= -\left(\int_0^\infty \psi_0(0)\exp(\widetilde{Q}(0)s)ds\right)\widehat{Q}(0)\pi_* \\
&\qquad -\left(\int_0^\infty s\psi_0(0)\exp(\widetilde{Q}(0)s)ds\right)\frac{d\widetilde{Q}(0)}{dt}\pi_* \\
&:= -\overline{\psi}_0\pi_*.
\end{aligned} \tag{4.94}$$

Write $\overline{\psi}_0 = (\overline{\psi}_0^1, \ldots, \overline{\psi}_0^l, \overline{\psi}_0^*)$. Then the definition of π_* implies that

$$\psi_1^k(0)\mathbb{1}_{m_k} + \psi_1^*(0)a_{m_k}(0) = -(\overline{\psi}_0^k\mathbb{1}_{m_k} + \overline{\psi}_0^* a_{m_k}(0)).$$

Recall that

$$\varphi_1^*(0) + \psi_1^*(0) = 0$$

and

$$\varphi_1^*(t) = b_0^*(t)\widetilde{Q}_*^{-1}(t).$$

It follows that

$$\psi_1^k(0)\mathbb{1}_{m_k} = -(\overline{\psi}_0^k \mathbb{1}_{m_k} + \overline{\psi}_0^* a_{m_k}(0)) + b_0^*(0)\widetilde{Q}_*^{-1}(0)a_{m_k}(0).$$

Moreover, it can be verified that $|\psi_1(\tau)| \leq K\exp(-\kappa_{1,0}\tau)$ for some $0 < \kappa_{1,0} < \kappa_{0,0}$.

Remark 4.44. Note that there is an extra term

$$\left(\int_0^\infty s\psi_0(0)\exp(\widetilde{Q}(0)s)ds\right)\frac{d\widetilde{Q}(0)}{dt}\pi_*$$

involved in the equation determining $\vartheta_1(0)$ in (4.94). This term does not vanish as in Section 4.3 because generally $((d/dt)\widetilde{Q}(0))\pi_* \neq 0$.

To obtain the desired asymptotic expansion, continue inductively. For each $i = 2, \ldots, n$, we first obtain the solution of $\varphi_i(t)$ with the "multiplier" given by the solution of the differential equation but with unspecified condition $\vartheta_i(0)$; solve $\psi_i(t)$ with the as yet unavailable initial condition $\psi_i(0) = -\varphi_i(0)$. Next jointly prove the exponential decay properties of $\psi_i(\tau)$ and obtain the solution $\vartheta_i(0)$. The equation to determine $\vartheta_i(0)$ with transient states becomes

$$\begin{aligned}&\psi_i(0)\pi_* \\ &\quad = -\left(\sum_{j=0}^{i-1}\int_0^\infty \psi_{i-j-1}(s)\left(\frac{s^j}{j!}\frac{d^j\widehat{Q}(0)}{dt^j} + \frac{s^{j+1}}{(j+1)!}\frac{d^{j+1}\widetilde{Q}(0)}{dt^{j+1}}\right)ds\right)\pi_*.\end{aligned}$$

In this way, we have constructed the asymptotic expansion with transient states. In addition, we can show that $\varphi_i(\cdot)$ are smooth and $\psi_i(\cdot)$ satisfies $|\psi_i(\tau)| \leq K\exp(-\kappa_{i,0}\tau)$ for some $0 < \kappa_{i,0} < \kappa_{i-1,0} < \kappa_{0,0}$. Similarly as in the case with all recurrent states, we establish the following theorem.

Theorem 4.45. *Suppose* (A4.5) *and* (A4.6) *hold. Then an asymptotic expansion*

$$y_n^\varepsilon(t) = \sum_{i=0}^n \left(\varepsilon^i\varphi_i(t) + \varepsilon^i\psi_i\left(\frac{t}{\varepsilon}\right)\right)$$

can be constructed such that for $i = 0, \ldots, n$,

(a) $\varphi_i(\cdot)$ *is* $(n+1-i)$*-times continuously differentiable on* $[0,T]$*;*

(b) $|\psi_i(t)| \le K \exp(-\kappa_0 t)$ *for some* $K > 0$ *and* $0 < \kappa_0 < \kappa_{i,0}$*;*

(c) $|p^\varepsilon(t) - y_n^\varepsilon(t)| = O(\varepsilon^{n+1})$ *uniformly in* $t \in [0,T]$.

Example 4.46. Let $\widetilde{Q}(t) = \widetilde{Q}$, a constant matrix such that

$$\widetilde{Q} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix} \text{ and } \widehat{Q} = 0.$$

In this example,

$$\widetilde{Q}^1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \widetilde{Q}_* = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad \text{and} \quad \widetilde{Q}_*^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The last two rows in $\widetilde{Q}$ represent the jump rates corresponding to the transient states. The matrix $\widetilde{Q}^1$ is weakly irreducible and $\widetilde{Q}_*$ is stable. Solving the forward equation gives us

$$p^\varepsilon(t) = (p_1^\varepsilon(t), p_2^\varepsilon(t), p_3^\varepsilon(t), p_4^\varepsilon(t)),$$

where

$$\begin{aligned}
p_1^\varepsilon(t) = \frac{1}{2} + \frac{1}{2}\Bigg[& (-p_3^0 - p_4^0)\exp\left(-\frac{t}{\varepsilon}\right) \\
& + (p_1^0 - p_2^0 + p_3^0 - p_4^0)\exp\left(-\frac{2t}{\varepsilon}\right) \\
& + (-p_3^0 + p_4^0)\exp\left(-\frac{3t}{\varepsilon}\right)\Bigg], \\
p_2^\varepsilon(t) = \frac{1}{2} + \frac{1}{2}\Bigg[& (-p_3^0 - p_4^0)\exp\left(-\frac{t}{\varepsilon}\right) \\
& + (-p_1^0 + p_2^0 - p_3^0 + p_4^0)\exp\left(-\frac{2t}{\varepsilon}\right) \\
& + (p_3^0 - p_4^0)\exp\left(-\frac{3t}{\varepsilon}\right)\Bigg], \\
p_3^\varepsilon(t) = \frac{1}{2}\Bigg[& (p_3^0 + p_4^0)\exp\left(-\frac{t}{\varepsilon}\right) + (p_3^0 - p_4^0)\exp\left(-\frac{3t}{\varepsilon}\right)\Bigg], \\
p_4^\varepsilon(t) = \frac{1}{2}\Bigg[& (p_3^0 + p_4^0)\exp\left(-\frac{t}{\varepsilon}\right) + (-p_3^0 + p_4^0)\exp\left(-\frac{3t}{\varepsilon}\right)\Bigg].
\end{aligned}$$

It is easy to see that $\varphi_0(t) = (1/2, 1/2, 0, 0)$ and

$$|p^\varepsilon(t) - \varphi_0(t)| \le K \exp\left(-\frac{t}{\varepsilon}\right).$$

The limit behavior of the underlying Markov chain as $\varepsilon \to 0$ is determined by $\varphi_0(t)$ for $t > 0$. It is clear that the probability of the Markov chain staying at the transient states is very small for small ε.

Remark 4.47. The model discussed in this section has the extra ingredient of including transient states as compared with that of Section 4.3. The main feature is embedded in the last few rows of the $\widetilde{Q}(t)$ matrix. One of the crucial points here is that the matrix $\widetilde{Q}_*(t)$ in the right corner is Hurwitzian. This stability condition guarantees the exponential decay properties of the boundary layers. As far as the regular part (or the outer) expansion is concerned, we have that the last subvector $\varphi_0^*(t) = 0$. The determination of the initial conditions $\vartheta_i(0)$ uses the same technique as before, namely, matching the outer terms and inner layers. The procedure involves recursively solving a sequence of algebraic and differential equations. Although the model is seemingly more general, the methods and techniques involved in obtaining the asymptotic expansion and proof of the results are essentially the same as in the previous section. The notation is slightly more complex, nevertheless.

4.6 Remarks on Countable-State-Space Cases

4.6.1 Countable-State Spaces: Part I

This section presents an extension of the singularly perturbed Markov chains with fast and slow components and finite-state spaces. In this section, the generator $\widetilde{Q}(\cdot)$ is a block-diagonal matrix consisting of infinitely many blocks each of which is of finite dimension. The generator $Q^\varepsilon(t)$ still has the form (4.39). However,

$$\widetilde{Q}(t) = \begin{pmatrix} \widetilde{Q}^1(t) & & & & \\ & \widetilde{Q}^2(t) & & & \\ & & \ddots & & \\ & & & \widetilde{Q}^k(t) & \\ & & & & \ddots \end{pmatrix}, \tag{4.95}$$

where $\widetilde{Q}^k(t) \in \mathbb{R}^{m_k \times m_k}$ is a generator of an appropriate Markov chain with finite-state space, and $\widehat{Q}(t)$ is an infinite-dimensional matrix and is a generator of a Markov chain having a countable-state space, that is,

$\widehat{Q}(t) = (\widehat{q}_{ij}(t))$ such that

$$\widehat{q}_{ij}(t) \geq 0 \text{ for } i \neq j, \text{ and } \sum_j \widehat{q}_{ij}(t) = 0.$$

We aim at deriving asymptotic results under the current setting. To do so, assume that the following condition holds:

(**A4.7**) For $t \in [0, T]$, $\widetilde{Q}^k(t)$, for $k = 1, 2, \ldots$, are weakly irreducible.

Parallel to the development of Section 4.3, the solution of $\varphi_i(\cdot)$ can be constructed similar to that of Theorem 4.29 as in (4.44) and (4.45). In fact, we obtain $\varphi_0(\cdot)$ from (4.49) and (4.50) with $l = \infty$; the difference is that now we have an infinite number of equations. Similarly, for all $k = 1, 2, \ldots$ and $i = 0, 1, \ldots, n+1$, $\varphi_i(\cdot)$ can be obtained from

$$\begin{aligned}
&\varphi_0(t)\widetilde{Q}(t) = 0, \text{ if } i = 0 \\
&\varphi_i(t)\widetilde{Q}(t) = \frac{d\varphi_{i-1}(t)}{dt} - \varphi_{i-1}(t)\widehat{Q}(t), \text{ if } i \geq 1 \\
&\varphi_i^k(t)\mathbb{1}_{m_k} = \vartheta_i^k(t), \\
&\frac{d\vartheta_i(t)}{dt} = \vartheta_i(t)\overline{Q}(t) + \widetilde{b}_{i-1}(t)\widehat{Q}(t)\widetilde{\mathbb{1}}.
\end{aligned} \tag{4.96}$$

The problem is converted to one that involves infinitely many algebraic differential equations. The same technique as presented before still works.

Nevertheless, the boundary layer corrections deserve more attention. Let us start with $\psi_0(\cdot)$, which is the solution of the abstract Cauchy problem

$$\begin{aligned}
&\frac{d\psi_0(\tau)}{d\tau} = \psi_0(\tau)\widetilde{Q}(0), \\
&\psi_0(0) = p^0 - \varphi_0(0).
\end{aligned} \tag{4.97}$$

To continue our study, one needs the notion of semigroup (see Dunford and Schwartz [52], and Pazy [172]). Recall that for a Banach space $\mathbb{B}$, a one-parameter family $T(t)$, $0 \leq t < \infty$, of bounded linear operators from $\mathbb{B}$ into $\mathbb{B}$ is a semigroup of bounded linear operators on $\mathbb{B}$ if (i) $T(0) = I$ and (ii) $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$.

Let $\mathbb{R}^\infty$ be the sequence space with a canonical element $x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty$. Let $A = (a_{ij})$ satisfying $A : \mathbb{R}^\infty \mapsto \mathbb{R}^\infty$, equipped with the l_1-norm

$$|A|_1 = \sup_j \sum_i |a_{ij}|;$$

(see Hutson and Pym [90, p. 74]) Using the definition of semigroup above, the solution of (4.97) is

$$\psi_0(\tau) = T(\tau)\psi_0(0),$$

where $T(\tau)$ is a one-parameter family of semigroups generated by $\widetilde{Q}(0)$. Moreover, since $\widetilde{Q}(0)$ is a bounded linear operator, $\exp(\widetilde{Q}(0)\tau)$ still makes sense. Thus $T(\tau)\psi_0(0) = \psi_0(0)\exp(\widetilde{Q}(0)\tau)$, where

$$\begin{aligned} T(\tau) &= \exp(\widetilde{Q}(0)\tau) = \sum_{j=0}^{\infty} \frac{\left(\widetilde{Q}(0)\tau\right)^j}{j!} \\ &= \operatorname{diag}\left(\exp\left(\widetilde{Q}^1(0)\tau\right), \ldots, \exp\left(\widetilde{Q}^k(0)\tau\right), \ldots\right). \end{aligned}$$

Therefore, the solution has the same form as in the previous section. Under (A4.7), exactly the same argument as in the proof of Lemma 4.4 yields that for each $k = 1, 2, \ldots,$

$$\exp(\widetilde{Q}^k(0)\tau) \to \mathbb{1}_{m_k}\nu^k(0) \text{ as } \tau \to \infty$$

and the convergence takes place at an exponential rate, that is,

$$\left|\exp(\widetilde{Q}^k(0)\tau) - \mathbb{1}_{m_k}\nu^k(0)\right| \le K\exp(-\kappa_k\tau),$$

for some $\kappa_k > 0$. In order to obtain a valid asymptotic expansion, another piece of assumption is needed. That is, these κ_k, for all $k = 1, 2, \ldots$, are uniformly bounded below by a positive constant κ_0.

(**A4.8**) There exists a positive number $\kappa_0 = \min_k\{\kappa_k\} > 0$.

Set

$$\widetilde{\mathbb{1}} = \operatorname{diag}\left(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_k}, \ldots\right) \text{ and } \nu(0) = \operatorname{diag}\left(\nu^1(0), \ldots, \nu^k(0), \ldots\right).$$

In view of (A4.8)

$$\begin{aligned} \left|\exp(\widetilde{Q}(0)\tau) - \widetilde{\mathbb{1}}\nu(0)\right|_1 &\le \sup_k \left|\exp(\widetilde{Q}^k(0)\tau) - \mathbb{1}_{m_k}\nu^k(0)\right| \\ &\le K\exp(-\kappa_0\tau). \end{aligned} \tag{4.98}$$

The exponential decay property of $\psi_0(\cdot)$ is thus established. Likewise, it can be proved that all $\psi_i(\cdot)$ for $i = 1, \ldots, n+1$, satisfy the exponential decay property. From here on, we can proceed as in the previous section to get the error estimate and verify the validity of the asymptotic expansion. In short the following theorem is obtained.

Theorem 4.48. *Suppose conditions* (A4.7) *and* (A4.8) *are satisfied. Then the results in Theorem* 4.29 *hold for the countable-state-space model with* $\widetilde{Q}(\cdot)$ *given by* (4.95).

4.6.2 Countable-State Spaces: Part II

The aim of this section is to develop further results on singularly perturbed Markov chains with fast and slow components whose generators are infinite-dimensional matrices but in different form from that described in Section 4.6.1. The complexity as well as difficulty increase. A number of technical issues also arise. One idea arises almost immediately: to approximating the underlying system via a Galerkin-kind procedure, that is, to approximate an infinite-dimensional system by finite-dimensional truncations. Unfortunately, this does not work in the setting of this section. We will return to this question at the end of this section.

To proceed, as in the previous sections, the first step invariably involves the solution of algebraic differential equations in the constructions of the approximating functions. One of the main ideas used is the Fredholm alternative. There are analogues to the general setting in Banach spaces for compact operators. Nevertheless, the infinite-dimensional matrices are in fact more difficult to handle.

Throughout this section, we treat the class of generators with $|Q(t)|_1 < \infty$ only. We use $\mathbb{1}$ to denote the column vector with all components equal to 1. Consider $(\mathbb{1}\vdots Q(t))$ as an operator for a generator $Q(t)$ of a Markov chain with state space $\mathcal{M} = \{1, 2, \ldots\}$. To proceed, we first give the definitions of irreducibility and quasi-stationary distribution. Set $Q_c(t) := (\mathbb{1}\vdots Q(t))$.

Definition 4.49. The generator $Q(t)$ is said to be weakly irreducible at $t_0 \in [0, T]$, for $w \in \mathbb{R}^\infty$, if the equation $wQ_c(t_0) = 0$ has only the zero solution. If $Q(t)$ is weakly irreducible for each $t \in [0, T]$, then it is said to be weakly irreducible on $[0, T]$.

Definition 4.50. A quasi-stationary distribution $\nu(t)$ (with respect to $Q(t)$) is a solution to (2.8) with the finite summation replaced by $\sum_{i=1}^{\infty} \nu_i(t) = 1$ that satisfies $\nu(t) \geq 0$.

As was mentioned before, the Fredholm alternative plays an important role in our study. For infinite-dimensional systems, we state another definition to take this into account.

Definition 4.51. A generator $Q(t)$ satisfies the F-Property if $wQ_c(t) = b$ has a unique solution for each $b \in \mathbb{R}^\infty$.

Note that for all weakly irreducible generators of finite dimension (i.e., generators for Markov chains with finite-state space), the F-Property above is automatically satisfied.

Since $\mathbb{1} \in l_\infty$ (l_∞ denotes the sequence space equipped with the l_∞ norm) for each $t \in [0, T]$, $Q(t) \in \mathbb{R}^\infty \times \mathbb{R}^\infty$. Naturally, we use the norm

$$|(z\vdots A)|_{\infty,1} = \max\left\{\sup_{z_j} |z_j|, \sup_j \sum_{i=1}^{\infty} |a_{ij}(t)|\right\}.$$

It is easily seen that

$$|Q_c(t)|_{\infty,1} \le \max\left\{1, \sup_j \sum_i |q_{ij}(t)|\right\}.$$

If a generator $Q(t)$ satisfies the F-Property, then it is weakly irreducible. In fact if $Q(t)$ satisfies the F-Property on $t \in [0,T]$, then $yQ_c(t) = 0$ has a unique solution $y = 0$.

By the definition of the generator, in particular the q-Property, $Q_c(t)$ is a bounded linear operator for each $t \in [0,T]$. If $Q_c(t)$ is bijective (i.e., one-to-one and onto), then it has a bounded inverse. This, in turn, implies that $Q_c(t)$ exhibits the F-Property. Roughly, the F-Property is a generalization of the conditions in dealing with finite-dimensional spaces. Recall from Section 4.2 that although $fQ(t) = b$ is not solvable uniquely, by adding an equation $f\mathbb{1} = c$, the system has a unique solution.

Owing to the inherited difficulty caused by the infinite dimensionality, the irreducibility and smoothness of $Q(\cdot)$ are not sufficient to guarantee the existence of asymptotic expansions. Stronger conditions are needed. In the sequel, for ease of presentation, we consider the model with $\widetilde{Q}(\cdot)$ irreducible and both $\widetilde{Q}(\cdot)$ and $\widehat{Q}(\cdot)$ infinite-dimensional.

For each t, we denote the spectrum of $Q(t)$ by $\sigma(Q(t))$. In view of Pazy [172] and Hutson and Pym [90], we have

$$\sigma(Q(t)) = \sigma_d(Q(t)) \cap \sigma_c(Q(t)) \cap \sigma_r(Q(t)),$$

where $\sigma_d(Q(t))$, $\sigma_c(Q(t))$, and $\sigma_r(Q(t))$ denote the discrete, continuous, and residue spectrum of $Q(t)$, respectively. The well-known linear operator theory implies that for a compact operator A, $\sigma_r(A) = \emptyset$, and the only possible candidate for $\sigma_c(A)$ is 0. Keeping this in mind, we assume that the following condition holds.

(**A4.9**) The following condition holds.

- (a) The smoothness condition (A4.4) is satisfied.
- (b) The generator $\widetilde{Q}(t)$ exhibits the F-Property.
- (c) $\sup_{t\in[0,T]} |\widetilde{Q}(t)|_1 < \infty$ and $\sup_{t\in[0,T]} |\widehat{Q}(t)| < \infty$.
- (d) The eigenvalue 0 of $\widetilde{Q}(t)$ has multiplicity 1 and 0 is not an accumulation point of the eigenvalues.
- (e) $\sigma_r(\widetilde{Q}(t)) = \emptyset$.

Remark 4.52. Item (a) above requires that the smoothness condition be satisfied and Item (b) requires the operator $(\mathbb{1}\vdots\widetilde{Q}(t))$ satisfy a Fredholm-alternative-like condition. Finally, (d) indicates the spectrum of $(\mathbb{1}\vdots\widetilde{Q}(t))$ is like a compact operator. Recall that for a compact linear operator, 0 is in its spectrum, and the only possible accumulation point is 0. Our conditions

mimic such a condition. It will be used when we prove the exponential decay property of the initial-layer terms.

Theorem 4.53. *Under condition* (A4.9)*, the results in Theorem* 4.29 *hold for Markov chains with countable-state space.*

Proof: The proof is very similar to its finite-dimensional counterpart. We only point out the difference here.

As far as the regular part is concerned, we get the same equation (4.44). One thing to note is that we can no longer use Cramer's rule to solve the systems of equations. Without such an explicit representation of the solution, the smoothness of $\varphi_i(\cdot)$ needs to be proved by examining (4.44) directly. For example,

$$\sum_{i=1}^{\infty} \varphi_{0,i}(t) = 1,$$

$$\varphi_0(t)\widetilde{Q}(t) = 0,$$

can be rewritten as

$$\varphi_0(t) \left(\mathbb{1} \vdots \widetilde{Q}(t) \right) = (1, 0, \ldots). \tag{4.99}$$

Since $\widetilde{Q}(t)$ satisfies the F-Property, this equation has a unique solution.

To verify the differentiability, consider also

$$\varphi_0(t+\delta) \left(\mathbb{1} \vdots \widetilde{Q}(t+\delta) \right) = (1, 0, \ldots).$$

Examining the difference quotient leads to

$$\begin{aligned} 0 &= \frac{\varphi_0(t+\delta) \left(\mathbb{1} \vdots \widetilde{Q}(t+\delta) \right) - \varphi_0(t) \left(\mathbb{1} \vdots \widetilde{Q}(t) \right)}{\delta} \\ &= \frac{[\varphi_0(t+\delta) - \varphi_0(t)] \left(\mathbb{1} \vdots \widetilde{Q}(t+\delta) \right)}{\delta} \\ &\quad + \frac{\varphi_0(t) \left((\mathbb{1} \vdots \widetilde{Q}(t+\delta)) - (\mathbb{1} \vdots \widetilde{Q}(t)) \right)}{\delta}. \end{aligned}$$

Taking the limit as $\delta \to 0$ and by virtue of the smoothness of $\widetilde{Q}(\cdot)$, we have

$$\lim_{\delta \to 0} \frac{[\varphi_0(t+\delta) - \varphi_0(t)] \left(\mathbb{1} \vdots \widetilde{Q}(t+\delta) \right)}{\delta} = -\varphi_0(t) \left(0 \vdots \frac{d\widetilde{Q}(t)}{dt} \right).$$

That is $(d/dt)\varphi_0(t)$ exists and is given by the solution of

$$\frac{d\varphi_0(t)}{dt}\left(\mathbb{1}\vdots\widetilde{Q}(t)\right) = -\varphi_0(t)\left(0\vdots\frac{d\widetilde{Q}(t)}{dt}\right).$$

Again by the F-Property, there is a unique solution for this equation. Higher-order derivatives of $\varphi_0(\cdot)$ and smoothness of $\varphi_i(\cdot)$ can be proved in a similar way.

As far as the initial-layer terms are concerned, since $\widetilde{Q}(0)$ is a bounded linear operator, the semigroup interpretation $\exp(\widetilde{Q}(0)\tau)$ makes sense. It follows from Theorem 1.4 of Pazy [172, p. 104] that the equation

$$\frac{d\psi_0(\tau)}{d\tau} = \psi_0(\tau)\widetilde{Q}(0), \quad \psi_0(0) = p_0 - \varphi_0(0)$$

has a unique solution.

To show that $\psi_0(\cdot)$ decays exponentially fast, we use an argument that is analogous to the finite-dimensional counterpart. Roughly, since the multiplicity of the eigenvalue 0 is 1, the subspace generated by the corresponding eigenvector v_0 is one-dimensional. Similar to the situation of Section 4.2, $\lim_{\tau\to\infty}\exp(\widetilde{Q}(0)\tau)$ exists and the limit must have identical rows. Denote the limit by $\overline{P}$. It then follows that

$$\left|\exp(\widetilde{Q}(0)\tau) - \overline{P}\right| \le K\exp(-\kappa_0\tau).$$

The meaning should be very clear. Upon "subtracting" the subspace generated by v_0, it ought to behave like $\exp(-\kappa_0\tau)$. A similar argument works for $i = 1, \ldots, n+1$, so the $\psi_i(\cdot)$ decay exponentially fast. □

4.6.3 A Remark on Finite-Dimensional Approximation

Concerning the cases in Section 4.6.2, a typical way of dealing with infinite-dimensional Markov chains is to make a finite-dimensional approximation. Let $Q(t) = (q_{ij}(t))$, $t \ge 0$, denote a generator of a Markov chain with countable-state space. We consider an $N \times N$, $N = 1, 2, \ldots$, truncation matrix $Q_N(t) = (q_{ij}(t))_{i,j=1}^N$. Then $Q_N(t)$ is a subgenerator in the sense that $\sum_{j=1}^N q_{ij}(t) \le 0$, $i = 1, 2, \ldots, N$.

A first glance seems to indicate that the idea of subgenerator provides a way to treat the problem of approximating an infinite-dimensional generator by finite-dimensional matrices. In fact, Reuter and Ledermann used such an idea to derive the existence and uniqueness of the solution to the forward equation (see Bharucha-Reid [10]). Dealing with singularly perturbed chains with countable-state space, one would be interested in knowing whether a Galerkin-like approximation would work in the sense that an asymptotic expansion of a finite-dimensional system would provide

an approximation to the probability distribution. To be more precise, let $\alpha^\varepsilon(\cdot)$ denote the Markov chain generated by $Q(t)/\varepsilon$ and let

$$p^\varepsilon(t) = (P(\alpha^\varepsilon(t) = 1), \ldots, P(\alpha^\varepsilon(t) = k), \ldots).$$

Consider the following approximation via N-dimensional systems

$$\frac{dp^{\varepsilon,N}(t)}{dt} = \frac{1}{\varepsilon} p^{\varepsilon,N}(t) Q_N(t),\ p^{\varepsilon,N}(0) = p^0. \tag{4.100}$$

Using the techniques presented in the previous sections, we can find outer and inner expansions to approximate $p^{\varepsilon,N}(t)$. The questions are these: For small ε and large N, can we approximate $p^\varepsilon(t)$ by $p^{\varepsilon,N}(t)$? Can we approximate $p^{\varepsilon,N}(t)$ by $y_n^{\varepsilon,N}(t)$, where $y_n^{\varepsilon,N}(t)$ is an expansion of the form (4.43) when subgenerators are used? More importantly, can we use $y_n^{\varepsilon,N}(t)$ to approximate $p^\varepsilon(t)$?

Although $p_i^\varepsilon(t)$ can be approximated by its truncation $p_i^{\varepsilon,N}(t)$ for large N and $p^{\varepsilon,N}(t)$ can be expanded as $y_n^{\varepsilon,N}(t)$ for small ε, the approximation of $y_n^{\varepsilon,N}(t)$ to $p^\varepsilon(t)$ does not work in general because the limits as $\varepsilon \to 0$ and $N \to \infty$ are not interchangeable. This can be seen by considering the following example.

Let

$$Q(t) = Q = \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2^2} & \cdots \\ \frac{1}{2} & -1 & \frac{1}{2^2} & \cdots \\ \frac{1}{2^2} & \frac{1}{2} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Then for any N, the truncation matrix Q_N has only negative eigenvalues. It follows that the solution $p^{\varepsilon,N}(t)$ decays exponentially fast, i.e.,

$$\left| p^{\varepsilon,N}(t) \right| \le C \exp\left(-\frac{\kappa_0 t}{\varepsilon} \right).$$

Thus, all terms in the regular part of $y_n^{\varepsilon,N}$ vanish. It is clear from this example that $y_n^{\varepsilon,N}(t)$ cannot be used to approximate $p^\varepsilon(t)$.

4.7 Remarks on Singularly Perturbed Diffusions

In this section, we present some related results on singular perturbations of diffusions. If in lieu of a discrete state space, one considers a continuous-state space, then naturally the singularly perturbed Markov chains become

singularly perturbed Markov processes. We illustrate the idea of matched asymptotic expansions for singularly perturbed diffusions. In this section, we only summarize the results and refer the reader to Khasminskii and Yin [116] for details of proofs. To proceed, consider the following example.

Example 4.54. This example discusses a model arising from stochastic control, namely, a controlled singularly perturbed system. As pointed out in Kushner [140] and Kokotovic, Bensoussan, and Blankenship [127], many control problems can be modeled by systems of differential equations, where the state variables can be divided into two coupled groups, consisting of "fast" and "slow" variables. A typical system takes the form

$$dx_1^\varepsilon = f_1(x_1^\varepsilon, x_2^\varepsilon, u)dt + \sigma_1(x_1^\varepsilon, x_2^\varepsilon)dw_1, \ x_1^\varepsilon(0) = x_1,$$

$$dx_2^\varepsilon = \frac{1}{\varepsilon} f_2(x_1^\varepsilon, x_2^\varepsilon, u)dt + \frac{1}{\sqrt{\varepsilon}}\sigma_2(x_1^\varepsilon, x_2^\varepsilon)dw_2, \ x_2^\varepsilon(0) = x_2,$$

where $w_1(\cdot)$ and $w_2(\cdot)$ are independent Brownian motions, $f_i(\cdot)$ and $\sigma_i(\cdot)$ for $i = 1, 2$ are suitable functions, u is the control variable, and $\varepsilon > 0$ is a small parameter. The underlying control problem is to minimize the cost function

$$J^\varepsilon(x_1, x_2, u) = E\int_0^T R(x_1^\varepsilon(t), x_2^\varepsilon(t), u)dt,$$

where $R(\cdot)$ is the running cost function. The small parameter $\varepsilon > 0$ signifies the relative rates of x_1^ε and x_2^ε. Such singularly perturbed systems have drawn much attention (see Bensoussan [8], Kushner [140], and the references therein). The system is very difficult to analyze directly; the approach of Kushner [140] is to use weak convergence methods to approximate the total system by the reduced system that is obtained using the differential equation for the slow variable, where the fast variable is fixed at its steady-state value as a function of the slow variable. In order to gain further insight, it is crucial to understand the asymptotic behavior of the rapidly changing process x_2^ε through the transition density given by the solution of the corresponding Kolmogorov-Fokker-Planck equations.

As demonstrated in the example above, a challenge common to many applications is to study the asymptotic behavior of the following problem. Let $\varepsilon > 0$ be a small parameter, and let $X_1^\varepsilon(\cdot)$ and $X_2^\varepsilon(\cdot)$ be real-valued diffusion processes satisfying

$$\begin{cases} dX_1^\varepsilon = a_1(t, X_1^\varepsilon, X_2^\varepsilon)dt + \sigma_1(t, X_1^\varepsilon, X_2^\varepsilon)dw_1, \\ dX_2^\varepsilon = \dfrac{1}{\varepsilon} a_2(t, X_1^\varepsilon, X_2^\varepsilon)dt + \dfrac{1}{\sqrt{\varepsilon}}\sigma_2(t, X_1^\varepsilon, X_2^\varepsilon)dw_2, \end{cases}$$

where the real-valued functions $a_1(t, x_1, x_2)$, $a_2(t, x_1, x_2)$, $\sigma_1(t, x_1, x_2)$, and $\sigma_2(t, x_1, x_2)$ represent the drift and diffusion coefficients, respectively, and $w_1(\cdot)$ and $w_2(\cdot)$ are independent and standard Brownian motions. Define a vector X as $X = (X_1, X_2)'$. Then $X^\varepsilon(\cdot) = (X_1^\varepsilon(\cdot), X_2^\varepsilon(\cdot))'$ is a diffusion process. This is a model treated in Khasminskii [113], in which a probabilistic approach was employed. It was shown that as $\varepsilon \to 0$, the fast component is averaged out and the slow component $X_1^\varepsilon(\cdot)$ has a limit $X_1^0(\cdot)$ such that

$$dX_1^0(t) = \overline{a}_1(X_1^0(t))dt + \overline{\sigma}_1(X_1^0(t))dw_1,$$

where

$$\overline{a}_1(t, x_1) = \int a_1(t, x_1, x_2)\mu(t, x_1, x_2)dx_2,$$

$$\overline{\sigma}_1(t, x_1) = \int \sigma_1(t, x_1, x_2)\mu(t, x_1, x_2)dx_2,$$

and $\mu(\cdot)$ is a limit density of the fast process $X_2^\varepsilon(\cdot)$.

To proceed further, it is necessary to investigate the limit properties of the rapidly changing process $X_2^\varepsilon(\cdot)$. To do so, consider the transition density of the underlying diffusion process. It is known that it satisfies the forward equation

$$\begin{aligned} &\frac{\partial p^\varepsilon}{\partial t} = \frac{1}{\varepsilon}\mathcal{L}_2^* p^\varepsilon + \mathcal{L}_1^* p^\varepsilon, \\ &p^\varepsilon(0, x_1, x_2) = p_0(x_1, x_2) \text{ with } p_0(x_1, x_2) \geq 0 \text{ and} \\ &\int\int p_0(x_1, x_2)dx_1dx_2 = 1, \end{aligned} \tag{4.101}$$

where

$$\mathcal{L}_1^*(t, x_1, x_2)\cdot = \frac{1}{2}\frac{\partial^2}{\partial x_1^2}(\sigma_1^2(t, x_1, x_2)\cdot) - \frac{\partial}{\partial x_1}(a_1(t, x_1, x_2)\cdot),$$

$$\mathcal{L}_2^*(t, x_1, x_2)\cdot = \frac{1}{2}\frac{\partial^2}{\partial x_2^2}(\sigma_2^2(t, x_1, x_2)\cdot) - \frac{\partial}{\partial x_2}(a_2(t, x_1, x_2)\cdot).$$

Similar to the discrete-state-space cases, the basic problems to be addressed are these: As $\varepsilon \to 0$, does the system display certain asymptotic properties? Is there any equilibrium distribution? If $p^\varepsilon(t, x_1, x_2) \to p(t, x_1, x_2)$ for some function $p(\cdot)$, can one get a handle on the error bound (i.e., a bound on $|p^\varepsilon(t, x_1, x_2) - p(t, x_1, x_2)|$)?

To obtain the desired asymptotic expansion in this case, one needs to make sure the quasi-stationary density exists. Note that for diffusions in unbounded domains, the quasi-stationary density may not exist. Loosely for the existence of the quasi-stationary distribution, it is necessary that

the Markov processes corresponding to $\mathcal{L}_2^*$ be positive recurrent for each fixed t. Certain sufficient conditions for the existence of the quasi-stationary density are provided in Il'in and Khasminskii [93]. An alternative way of handling the problem is to concentrate on a compact manifold. In doing so we are able to establish the existence of the quasi-stationary density. To illustrate, we choose the second alternative and suppose the following conditions are satisfied.

For each $t \in [0,T]$, $i,j = 1, 2$, and

- for each $x_2 \in \mathbb{R}$, $a_1(t,\cdot,x_2)$, $\sigma_1^2(t,\cdot,x_2)$ and $p_0(\cdot,x_2)$ are periodic with period 1;
- for each $x_1 \in \mathbb{R}$, $a_2(t,x_1,\cdot)$, $\sigma_2^2(t,x_1,\cdot)$ and $p_0(x_1,\cdot)$ are periodic with period 1.

There is an $n \in \mathbb{Z}_+$ such that for each $i = 1, 2$,

$$a_i(\cdot),\ \sigma_i^2(\cdot) \in C^{n+1,2(n+1),2(n+1)},\ \text{for all } t \in [0,T],\ x_1,x_2 \in [0,1], \quad (4.102)$$

the $(n+1)$st partial with respect to t of $a_i(\cdot,x_1,x_2)$, and $\sigma_i^2(\cdot,x_1,x_2)$ are Lipschitz continuous uniformly in $x_1,x_2 \in [0,1]$. In addition, for each $t \in [0,T]$ and each $x_1,x_2 \in [0,1]$, $\sigma_i^2(t,x_1,x_2) > 0$.

Definition 4.55. A function $\mu(\cdot)$ is said to be a quasi-stationary density for the periodic diffusion corresponding to the Kolmogorov-Fokker-Planck operator $\mathcal{L}_2^*$ if it is periodic in x_1 and x_2 with period 1,

$$0 \le \mu(t,x_1,x_2) \text{ for each } (t,x_1,x_2) \in [0,T] \times [0,1] \times [0,1],$$

and for each fixed t and x_1,

$$\int_0^1 \mu(t,x_1,x_2)dx_2 = 1 \quad \text{and} \quad \mathcal{L}_2^*\mu(t,x_1,x_2) = 0.$$

To proceed, let $\mathcal{H}$ be the space of functions that are bounded and continuous and are Hölder continuous in $(x_1,x_2) \in [0,1]\times[0,1]$ (with Hölder exponent Δ for some $0 < \Delta < 1$), uniformly with respect to t. For each $h_1,\ h_2 \in \mathcal{H}$ define $\langle h_1,h_2\rangle_{\mathcal{H}}$ as

$$\langle h_1,h_2\rangle_{\mathcal{H}} = \int_0^T \int_0^1 \int_0^1 h_1(t,x_1,x_2)h_2(t,x_1,x_2)dx_1dx_2dt.$$

Under the assumptions mentioned above, two sequences of functions $\varphi_i(\cdot)$ (periodic in x_1 and x_2) and $\psi_i(\cdot)$ for $i = 0,\dots,n$ can be found such that

(a) $\varphi_i(\cdot,\cdot,\cdot) \in C^{n+1-i,2(n+1-i),2(n+1-i)}$;

(b) $\psi_i(t/\varepsilon,x_1,x_2)$ decay exponentially fast in that for some $c_1 > 0$ and $c_2 > 0$,

$$\sup_{x_1,x_2\in[0,1]} \left|\psi_i\left(\frac{t}{\varepsilon},x_1,x_2\right)\right| \le c_1 \exp\left(-\frac{c_2 t}{\varepsilon}\right);$$

(c) define $\widetilde{s}_n^\varepsilon$ by

$$\widetilde{s}_n^\varepsilon(t,x_1,x_2)=\sum_{i=0}^{n}\left(\varepsilon^i\varphi_i(t,x_1,x_2)+\varepsilon^i\psi_i\left(\frac{t}{\varepsilon},x_1,x_2\right)\right);$$

for each $h\in\mathcal{H}$, the following error bound holds:

$$\left|\left\langle p^\varepsilon-\widetilde{s}_n^\varepsilon,h\right\rangle_{\mathcal{H}}\right|=O(\varepsilon^{n+1}). \tag{4.103}$$

It is interesting to note that the leading term of the approximation $\varphi_0(\cdot)$ is approximately the probability density of X_1, namely, $v_0(t,x_1)$ multiplied by the conditional density of X_2 given $X_1=x_1$ (i.e., holding x_1 as a parameter), the quasi-stationary density $\mu(t,x_1,x_2)$. The rest of the terms in the regular part of the expansion assume the form

$$\mu(t,x_1,x_2)v_i(t,x_1)+U_i(t,x_1,x_2),$$

where $U_i(\cdot)$ is a particular solution of an inhomogeneous equation. Note the resemblance of the form to that of the Markov-chain cases studied in this chapter. A detailed proof of the assertion is in Khasminskii and Yin [116]. In fact, more complex systems (allowing interaction of X_1^ε and X_2^ε, the mixed partial derivatives of x_1 and x_2 as well as extension to multidimensional systems) are treated in [116]. In addition, in lieu of $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, convergence under the uniform topology can be considered via the use of stochastic representation of solutions of partial differential equations or energy integration methods (see, for example, the related treatment of singularly perturbed switching diffusion systems in Il'in, Khasminskii, and Yin [94]).

4.8 Notes

Two-time-scale Markov chains are dealt with in this chapter using purely analytic methods, which are closely connected with the singular perturbation methods. The literature of singular perturbation for ordinary differential equations is rather rich. For an extensive list of references in singular perturbation methods for ordinary differential equations and various techniques such as initial-layer etc., we refer to Vasi'leva and Butuzov [209], Wasow [215, 216], O'Malley [163], and the references therein. The development of singular perturbation methods has been intertwined with advances in technology and progress in various applications. It can be traced back to the beginning of the twentieth century when Prandtl dealt with fluid motion with small friction (see Prandtl [178]). Nowadays, the averaging principle developed by Krylov, Bogoliubov, and Mitropolskii (see Bogoliubov and

Mitropolskii [18]) has become a popular technique, taught in standard graduate applied mathematics courses and employed widely. General results on singular perturbations can be found in Bensoussan, Lion, and Papanicolaou [7], Bogoliubov and Mitropolskii [18], Eckhaus [54], Erdélyi [58], Il'in [92], Kevorkian and Cole [108, 109], Krylov and Bogoliubov [133], O'Malley [163], Smith [199], Vasil'eava and Butuzov [209, 210], Wasow [215, 216]; applications to control theory and related fields are in Bensoussan [8], Bielecki and Filar [11], Delebecque and Quadrat [44], Delebecque, Quadrat, and Kokotovic [45], Kokotovic [126], Kokotovic, Bensoussan, and Blankenship [127], Kokotovic and Khalil [128], Kokotovic, Khalil, and O'Reilly [129], Kushner [140], Pan and Başar [164, 165, 166], Pervozvanskii and Gaitsgori [174], Phillips and Kokotovic [175], Yin and Zhang [233], among others; the vast literature on applications to different branches of physics are in Risken [182], van Kampen [208]; the survey by Hänggi, Talkner, and Borkovec [80] contains hundreds of references concerning applications in physics; related problems via large deviations theory are in Lerman and Schuss [151]; some recent work of singular perturbations to queueing networks, and heavy traffic, etc., is in Harrison and Reiman [81], Knessel and Morrison [125], and the references therein; applications to manufacturing systems are in Sethi and Zhang [192], Soner [202], Zhang [248], and the references cited there; related problems for stochastic differential equations and diffusion approximations, etc., can be found in Day [42], Friedlin and Wentzell [67], Il'in and Khasminskii [93], Khaminskii [111, 112], Kushner [139], Ludwig [152], Matkowsky and Schuss [158], Naeh, Klosek, Matkowski, and Schuss [160], Papanicolaou [169, 170], Schuss [187, 188], Skorohod [198], Yin [222], Yin and Ramachandran [227], and Zhang [247], among others. Singularly perturbed Markov processes also appear in the context of random evolution, a generalization of the motion of a particle on a fixed line with a random velocity or a random diffusivity; see, for example, Griego and Hersh [76, 77] and Pinsky [177]; an extensive survey can be found in Hersh [85]. A first-order approximation of the distribution of the Cox process with rapid switching is in Di Masi and Kabanov [48]. Recently, modeling communication systems via two-time-scale Markov chains has gained renewed interest; see Tse, Gallager, and Tsitsiklis [206], and the references therein.

It should be pointed out that there is a distinct feature in the problem we are studying compared with the traditional study of singularly perturbed systems. In contrast to many singularly perturbed ordinary differential equations, the matrix $Q(t)$ in (4.3) is singular, and has an eigenvalue 0. Thus the usual stability condition does not hold. To circumvent this difficulty, we utilize the q-Property of the matrix $Q(t)$, which leads to a probabilistic interpretation. The main emphasis in this chapter is on developing approximations to the solutions of the forward equations. The underlying systems arise from a wide range of applications where a finite-state Markov chain is involved and a fast time scale t/ε is used. Asymptotic series of the probability distribution of the Markov chain have been developed by em-

ploying the techniques of matched expansions. An attempt to obtain the asymptotic expansion of (4.3) is initiated in Khasminskii, Yin, and Zhang [119] for time-inhomogeneous Markov chains. The result presented here is a refinement of the aforementioned reference.

Extending the results for irreducible generators, this chapter further discusses two-time-scale Markov chains with weak and strong interactions. The formulations substantially generalize the work of Khasminskii, Yin, and Zhang [120]. Section 4.3 discusses Markovian models with recurrent states belonging to several ergodic classes is a refinement of [120].

Previous work on singularly perturbed Markov chains with weak and strong interactions can be found in Delebecque, Quadrat, and Kokotovic [45], Gaitsgori and Pervozvanskii [69], Pervozvanskii and Gaitsgori [174], and Phillips and Kokotovic [175]. The essence is a decomposition and aggregation point of view. Their models are similar to that considered in this chapter. For example, translating the setup into our setting, the authors of [175] assumed that the Markov chain generated by $\widetilde{Q}/\varepsilon + \widehat{Q}$ has a single ergodic class for ε sufficiently small. Moreover, for each $j = 1, 2, \ldots, l$, the subchain has a single ergodic class. Their formulation requires that $\widetilde{Q}(t) = \widetilde{Q}$ and $\widehat{Q}(t) = \widehat{Q}$, and it requires essentially the irreducibility of $\widetilde{Q}/\varepsilon + \widehat{Q}$ for all $\varepsilon \le \varepsilon_0$ for some $\varepsilon_0 > 0$ small enough in addition to the irreducibility of $\widetilde{Q}^j$ for $j = 1, 2, \ldots, l$. The problem considered in this chapter is nonstationary; the generators are time-varying. The irreducibility is in the weak sense, and only weak irreducibility of each subgenerator (or block matrix) $\widetilde{Q}^j(t)$ for $j = 1, 2, \ldots, l$ is needed. Thus our results generalize the existing theorems to nonstationary cases under weaker assumptions. The condition on $\widetilde{Q}(t)$ exploits the intrinsic properties of the underlying chains. Furthermore, our results also include Markov chains with countable-state spaces. The formulation and development of Section 4.5 are inspired by that of [175] (see also Pan and Başar [164]). This together with the consideration of chains with recurrent states and the inclusion of absorbing states includes most of practical concerns for the rapidly varying part of the generator. Although the forms of the generators with absorbing states and with transient states have more complex structures, the asymptotic expansion of the probability distributions can still be obtained via a similar approach to that of the case of block-diagonal $\widetilde{Q}(\cdot)$. Applications to manufacturing systems are discussed, for example, in Jiang and Sethi [99] and Sethi and Zhang [192] among others. As a complement of the development in this chapter, the work of Il'in, Khasminskii, and Yin [94] deals with the cases that the underlying Markov processes involve both diffusion and pure jump processes; see also Yin and Yang [229]. Previous work of singular perturbation of stochastic systems can be found in Day [42], Friedlin and Wentzel [67], Khasminskii [111, 112, 113], Kushner [139], Ludwig [152], Matkowsky and Schuss [158], Naeh, Klosek, Matkowski, and Schuss [160], Papanicolaou [169, 170], Schuss [187], Yin and Ramachandran [227], and

the references therein. Singular perturbation in connection with optimal control problems are contained in Bensoussan [8], Bielecki and Filar [11], Delebecque and Quadrat [44], Kokotovic [126], Kokotovic, Bensoussan, and Blankenship [127], Kushner [140], Lehoczky, Sethi, Soner, and Taksar [150], Martins and Kushner [156], Pan and Başar [164], Pervozvanskii and Gaitsgori [174], Sethi and Zhang [192], Soner [202], and Yin and Zhang [233] among others. For discrete-time two-time-scale Markov chains, we refer the reader to Yin and Zhang [238] Yin, Zhang, and Badowski [242] among others.

We note that one of the key points that enables us to solve these problems is the Fredholm alternative. This is even more crucial compared with the situation in Section 4.2 for irreducible generators. In Section 4.2, the consistency conditions are readily verified, whereas in the formulation under weak and strong interactions, the verification needs more work and we have to utilize the consistency to obtain the desired solution.

The discussions on Markov chains with countable-state spaces in this chapter focused on simple situations. For more general cases, see Yin and Zhang [230, 231], in which applications to quasi-birth-death queues were considered; see also Altman, Avrachenkov, and Nunez-Queija [4] for a different approach. The discussions on singularly perturbed diffusion processes dealt with mainly forward equations. For related work on singularly perturbed diffusions, see the papers of Khasminskii and Yin [115, 116] and the references therein; one of the motivations for studying singularly perturbed diffusion comes from wear process modeling (see Rishel [181]). For treatments of averaging principles and related backward equations, we refer the reader to Khasminskii and Yin [117, 118]. For a number of applications on queueing systems, financial engineering, and insurance risk, we refer the reader to Yin, Zhang, and Zhang [232] and references therein.

5

Occupation Measures: Asymptotic Properties and Ramification

5.1 Introduction

Chapter 4 deals with the probability distribution of $\alpha^{\varepsilon}(\cdot)$ through the corresponding forward equation and is mainly an analytical approach, whereas the current chapter is largely probabilistic in nature. The central theme of this chapter is limit results of unscaled as well as scaled sequences of occupation measures, which include the law of large numbers for an unscaled sequence, exponential upper bounds, and asymptotic distribution of a suitably scaled sequence of occupation times. It further exploits the deviation of the functional occupation times from its quasi-stationary distribution. We obtain estimates of centered deviations, prove the convergence of a properly scaled and centered sequence of occupation times, characterize the limit process by deriving the limit distribution and providing explicit formulas for the mean and covariance functions, and provide exponential bounds for the normalized process. It is worthwhile to note that the limit covariance function depends on the initial-layer terms in contrast with most of the existing results of central limit type.

The rest of the chapter is arranged as follows. We first study the asymptotic properties of irreducible Markov chains in Section 5.2. In view of the developments in Remarks 4.34 and 4.39, the Markov chain with recurrent states is the most illustrative and representative one. As a result, in the remaining chapters, we mainly treat problems associated with this model. Starting in Section 5.3.1, we consider Markov chains with weak and strong

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interactions with generators consisting of multiple irreducible blocks. After treating aggregation of the Markov states, we study the corresponding exponential bounds. We deal with asymptotic distributions. Then in Section 5.4, we treat Markov chains with generators that are merely measurable. Next, remarks on inclusion of transient and absorbing states are provided in Section 5.5. Applications of the weak convergence results and a related stability problem are provided in Section 5.6. Finally, Section 5.7 concludes the chapter with notes and further remarks.

5.2 The Irreducible Case

The notion of occupation measure is set forth first. We consider a sequence of unscaled occupation measures and establish its convergence in probability to that of the accumulative quasi-stationary distribution. This is followed by exponential bounds of the function occupation time and moment estimates. In addition, asymptotic normality is derived. Although the prelimit process has nonzero mean and is nonstationary, using the results of Section 4.2, the quasi-stationary regime is established after a short period (of order $O(\varepsilon)$). We also calculate explicitly the covariance representation of the limit process, and prove that the process $\alpha^\varepsilon(\cdot)$ satisfies a mixing condition. The tightness of the sequence and the w.p.1 continuity of the sample paths of the limit process are proved by estimating the fourth moment. The limit of the finite-dimensional distributions is then calculated and shown to be Gaussian. By proving a series of lemmas, we derive the desired asymptotic normality.

As was mentioned in previous chapters, the process $\alpha^\varepsilon(\cdot)$ arises from pervasive practical use that involves a rapidly fluctuating finite-state Markov chain. In these applications, the asymptotic behavior of the Markov chain $\alpha^\varepsilon(\cdot)$ has a major influence. Further investigation and understanding of the asymptotic properties of $\alpha^\varepsilon(\cdot)$, in particular, the probabilistic structures, play an important role in the in-depth study.

In Section 4.2, using singular perturbation techniques, we examined the asymptotic properties of $p_i^\varepsilon(t) = P(\alpha^\varepsilon(t) = i)$. It has been proved that $p^\varepsilon(t) = (p_1^\varepsilon(t), \ldots, p_m^\varepsilon(t))$ converges to the quasi-stationary distribution $\nu(t)$ as $\varepsilon \to 0$ for each $t > 0$ and $p^\varepsilon(t)$ admits an asymptotic expansion in terms of ε. To gain further insight, we ask whether there is a limit result for the occupation measure $\int_0^t I_{\{\alpha^\varepsilon(s)=i\}} ds$. If a convergence is expected to take place, then what is the rate of convergence? Does one have a central limit theorem associated with the $\alpha^\varepsilon(\cdot)$-process? The answers to these questions are affirmative. We will prove a number of limit results related to an unscaled sequence, and a suitably scaled and normalized sequence. Owing to the asymptotic expansions, the scaling factor is $\sqrt{\varepsilon}$. The limit process is Gaussian with zero mean, and the covariance of the limit process depends on the asymptotic expansion in an essential way, which reflects

one of the distinct features of the central limit theorem. It appears that it is virtually impossible to calculate the asymptotic covariance of the Gaussian process without the help of the asymptotic expansion, which reveals a salient characteristic of the two-time-scale Markov chain.

A related problem is to examine the exponential bounds on the scaled occupation measure process. This is similar to the estimation of the moment generating function. Such estimates have been found useful in studying hierarchical controls of manufacturing systems. Using the asymptotic expansion and the martingale representation of finite-state Markov chains, we are able to establish such exponential bounds for the scaled occupation measures.

5.2.1 Occupation Measure

Let $(\Omega, \mathcal{F}, P)$ denote the underlying probability space. As in Section 4.2, $\alpha^\varepsilon(\cdot)$ is a nonstationary Markov chain on $(\Omega, \mathcal{F}, P)$ with finite-state space $\mathcal{M} = \{1, \ldots, m\}$ and generator $Q^\varepsilon(t) = Q(t)/\varepsilon$.

For each $i \in \mathcal{M}$, let $\beta_i(\cdot)$ denote a bounded Borel measurable deterministic function and define a sequence of centered (around the quasi-stationary distribution) occupation measures $Z_i^\varepsilon(t)$ as

$$Z_i^\varepsilon(t) = \int_0^t \left(I_{\{\alpha^\varepsilon(s)=i\}} - \nu_i(s)\right) \beta_i(s) ds. \tag{5.1}$$

Set $Z^\varepsilon(t) = (Z_1^\varepsilon(t), \ldots, Z_m^\varepsilon(t))$. It is a measure of the functional occupancy for the process $\alpha^\varepsilon(\cdot)$. Our interest lies in the asymptotic properties of the sequence defined in (5.1). To proceed, we first present some conditions and preliminary results needed in the sequel.

Note that a special choice of $\beta_i(\cdot)$ is $\beta_i(t) = 1$, for $t \in [0, T]$. To insert $\beta_i(\cdot)$ in sequence allows one to treat various situations in some applications. For example, in the manufacturing problem, $\beta_i(t)$ is often given by a function of a control process; see Chapter 8 for further details.

5.2.2 Conditions and Preliminary Results

To proceed, we make the following assumptions.

(**A5.1**) For each $t \in [0, T]$, $Q(t)$ is weakly irreducible.

(**A5.2**) $Q(\cdot)$ is continuously differentiable on $[0, T]$, and its derivative is Lipschitz.

Recall that $p^\varepsilon(t) = (P(\alpha^\varepsilon(t) = 1), \ldots, P(\alpha^\varepsilon(t) = m))$ and let

$$p_{ij}^\varepsilon(t, t_0) = P(\alpha^\varepsilon(t) = j | \alpha^\varepsilon(t_0) = i) \quad \text{for all } i, j \in \mathcal{M}.$$

Use $P^\varepsilon(t, t_0)$ to denote the transition matrix $(p_{ij}^\varepsilon(t, t_0))$. The following lemma is on the asymptotic expansion of $P^\varepsilon(t, t_0)$.

Lemma 5.1. *Assume* (A5.1) *and* (A5.2). *Then there exists a positive constant* κ_0 *such that for each fixed* $0 \le T < \infty$,

$$P^\varepsilon(t, t_0) = P_0(t) + O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t - t_0)}{\varepsilon}\right)\right) \tag{5.2}$$

uniformly in (t_0, t) *where* $0 \le t_0 \le t \le T$ *and*

$$P_0(t) = \begin{pmatrix} \nu(t) \\ \vdots \\ \nu(t) \end{pmatrix}.$$

In addition, assume $Q(\cdot)$ *to be twice continuously differentiable on* $[0, T]$ *with the second derivative being Lipschitz. Then*

$$\begin{aligned} P^\varepsilon(t, t_0) \ &= P_0(t) + \varepsilon P_1(t) \\ &\quad + Q_0\left(\frac{t - t_0}{\varepsilon}, t_0\right) + \varepsilon Q_1\left(\frac{t - t_0}{\varepsilon}, t_0\right) + O(\varepsilon^2) \end{aligned} \tag{5.3}$$

uniformly in (t_0, t), *where* $0 \le t_0 \le t \le T$,

$$P_1(t) = \begin{pmatrix} \varphi_1(t) \\ \vdots \\ \varphi_1(t) \end{pmatrix},$$

$$\frac{dQ_0(\tau, t_0)}{d\tau} = Q_0(\tau, t_0)Q(t_0), \ \tau \ge 0,$$

$$Q_0(0, t_0) = I - P_0(t_0),$$

and

$$\frac{dQ_1(\tau, t_0)}{d\tau} = Q_1(\tau, t_0)Q(t_0) + \tau Q_0(\tau, t_0)\frac{dQ(t_0)}{dt}, \quad \tau \ge 0$$

$$Q_1(0, t_0) = -P_1(t_0),$$

where $\varphi_1(t)$ *is given in* (4.13) (*with* $\tau := (t - t_0)/\varepsilon$). *Furthermore, for* $i = 0, 1$, *the* $P_i(\cdot)$ *are* $(2 - i)$ *times continuously differentiable on* $[0, T]$ *and there exist constants* $K > 0$ *and* $\kappa_0 > 0$ *such that*

$$|Q_i(\tau, t_0)| \le K \exp(-\kappa_0 \tau), \tag{5.4}$$

uniformly for $t_0 \in [0, T]$.

Remark 5.2. Recall that $\nu(t)$ and $\varphi_1(t)$ are row vectors. As a result, $P_0(\cdot)$ and $P_1(\cdot)$ have identical rows. This is a consequence of the convergence of $p^\varepsilon(t)$ to the quasi-stationary distribution and the asymptotic expansions.

Proof of Lemma 5.1: It suffices to verify (5.3) because (5.2) can be derived similarly. The asymptotic expansion of $P^\varepsilon(t, t_0)$ can be obtained as in Section 4.2. Thus only the exponential bound (5.4) needs to be proved. The main task is to verify the uniformity in t_0. To this end, it suffices to treat each row of $Q_i(\tau, t_0)$ separately. For a fixed $i = 0, 1$, let

$$\eta(\tau, t_0) = (\eta_1(\tau, t_0), \ldots, \eta_m(\tau, t_0))$$

denote any row of $Q_i(\tau, t_0)$ and $\eta_0(t_0)$ the corresponding row in $Q_i(0, t_0)$ with

$$Q_0(0, t_0) = I - P_0(t_0) \quad \text{and}$$

$$Q_1(0, t_0) = -P_1(t_0).$$

Then $\eta(\tau, t_0)$ satisfies the differential equation

$$\frac{d\eta(\tau, t_0)}{d\tau} = \eta(\tau, t_0)Q(t_0), \ \tau \geq 0,$$

$$\eta(0, t_0) = \eta_0(t_0).$$

By virtue of the assumptions of Lemma 5.1 and the asymptotic expansion, it follows that $\eta_0(t_0)$ is uniformly bounded and $\eta_0(t_0)\mathbb{1} = 0$.

Define

$$\widehat{\kappa} = -\max\left\{\text{The real parts of eigenvalues of } Q(t), \ t \in [0, T]\right\}.$$

Then Lemma A.6 implies that $\widehat{\kappa} > 0$. In view of Theorem 4.5, it suffices to show that for all $\tau \geq 0$ and for some constant $K > 0$ independent of t_0,

$$|\eta(\tau, t_0)| \leq K \exp\left(-\frac{\widehat{\kappa}\tau}{2}\right). \tag{5.5}$$

To verify (5.5), note that for any $\varsigma_0 \in [0, T]$,

$$\frac{d\eta(\tau, t_0)}{d\tau} = \eta(\tau, t_0)Q(\varsigma_0) + \eta(\tau, t_0)[Q(t_0) - Q(\varsigma_0)].$$

Solving this differential equation by treating $\eta(\tau, t_0)[Q(t_0) - Q(\varsigma_0)]$ as the driving term, we have

$$\begin{aligned} \eta(\tau, t_0) \ &= \eta_0(t_0) \exp\left(Q(\varsigma_0)\tau\right) \\ &\quad + \int_0^\tau \eta(\varsigma, t_0)[Q(t_0) - Q(\varsigma_0)] \exp\left(Q(\varsigma_0)(\tau - \varsigma)\right) d\varsigma. \end{aligned} \tag{5.6}$$

In view of (A5.2), for some $K_0 > 0$,

$$\left|Q(t_0) - Q(\varsigma_0)\right| \le K_0|t_0 - \varsigma_0|.$$

Noting that $\eta_0(t_0)\mathbb{1} = 0$ and that $P_0(t)$ has identical rows, we have

$$\eta_0(t_0)P_0(t) = 0, \qquad \text{for } t \ge 0.$$

Thus the equation in (5.6) is equivalent to

$$\eta(\tau, t_0) = \eta_0(t_0)(\exp(Q(\varsigma_0)s) - P_0(\varsigma_0)) + \int_0^\tau \eta(\varsigma, t_0)[Q(t_0) - Q(\varsigma_0)](\exp(Q(\varsigma_0)(\tau - \varsigma)) - P_0(\varsigma_0))d\varsigma.$$

From Lemma A.2, we have

$$|\eta(\tau, t_0)| \le K_1 \exp(-\widehat{\kappa}\tau) + K_2|t_0 - \varsigma_0| \int_0^\tau |\eta(\varsigma, t_0)| \exp(-\widehat{\kappa}(\tau - \varsigma))\, d\varsigma,$$

for some constants K_1 and K_2 which may depend on ς_0 but are independent of t_0. By Gronwall's inequality (see Hale [79, p. 36]),

$$|\eta(\tau, t_0)| \le K_1 \exp(-(\widehat{\kappa} - K_2|t_0 - \varsigma_0|)\tau), \tag{5.7}$$

for all $t_0 \in [0, T]$ and $\tau > 0$.

If (5.5) does not hold uniformly, then there exist $\tau_n > 0$ and $t_n \in [0, T]$ such that

$$|\eta(\tau_n, t_n)| \ge n \exp\left(-\frac{\widehat{\kappa}\tau_n}{2}\right).$$

Since T is finite, we may assume $t_n \to \varsigma_0$, as $n \to \infty$. This contradicts (5.7) for n large enough satisfying $|t_n - \varsigma_0| < \widehat{\kappa}/(2K_2)$ and $K_1 < n$. Thus the proof is complete. □

Unscaled Occupation Measure

To study the unscaled occupation measure $Z_i^\varepsilon(t)$ in (5.1), we define a related sequence $\{\widehat{Z}^\varepsilon(t)\}$ of $\mathbb{R}^m$-valued processes with its ith component $\widehat{Z}_i^\varepsilon(t)$ given by

$$\widehat{Z}_i^\varepsilon(t) = \int_0^t \left(I_{\{\alpha^\varepsilon(s)=i\}} - P(\alpha^\varepsilon(s) = i)\right)\beta_i(s)ds.$$

Assume the conditions (A5.1) and (A5.2). We claim that for any $\delta > 0$,

$$\lim_{\varepsilon\to 0}\left(\sup_{0\le t\le T} P(|\widehat{Z}^\varepsilon(t)| \ge \delta)\right) = 0 \text{ and} \tag{5.8}$$

$$\lim_{\varepsilon\to 0}\left(\sup_{0\le t\le T} E|\widehat{Z}^\varepsilon(t)|^2\right) = 0. \tag{5.9}$$

Note that (5.8) follows from (5.9) using Tchebyshev's inequality. The verification of (5.9), which mainly depends on a mixing property of the underlying sequence, is almost the same as the moment estimates in the proof of asymptotic normality in Lemma 5.13. The details of the verifications of (5.8) and (5.9) are omitted here.

With (5.9) in hand for any $\delta > 0$, to study the asymptotic properties of $Z^\varepsilon(\cdot)$, it remains to show that

$$\lim_{\varepsilon\to 0}\left(\sup_{0\le t\le T} P(|Z^\varepsilon(t)|\ge\delta)\right) = 0 \text{ and}$$
$$\lim_{\varepsilon\to 0}\left(\sup_{0\le t\le T} E|Z^\varepsilon(t)|^2\right) = 0.$$

In fact, it is enough to work with each component of $Z^\varepsilon(t)$. Note that both $Z^\varepsilon(t)$ and $\widehat{Z}^\varepsilon(t)$ are bounded. This together with the boundedness of $\beta(t)$ and Lemma 5.1 implies that for each $i \in \mathcal{M}$,

$$\begin{aligned}
&\sup_{0\le t\le T} E|Z_i^\varepsilon(t)|^2\\
&\le 2\left(\sup_{0\le t\le T} E|\widehat{Z}_i^\varepsilon(t)|^2 + \sup_{0\le t\le T} E\left|\int_0^t (P(\alpha^\varepsilon(s)=i)-\nu_i(s))\,\beta_i(s)ds\right|^2\right)\\
&\le 2\left(\sup_{0\le t\le T} E|\widehat{Z}_i^\varepsilon(t)|^2 + \int_0^T O(\varepsilon)ds\right)\to 0,
\end{aligned}$$

as $\varepsilon\to 0$, which yields the desired results.

The limit result above is of the law-of-large-numbers type. What has been obtained is that as $\varepsilon\to 0$,

$$\int_0^t I_{\{\alpha^\varepsilon(s)=i\}}ds \to \int_0^t \nu_i(s)ds \quad \text{in probability as } \varepsilon\to 0,$$

for $0 < t \le T$. In fact, a somewhat stronger result on uniform convergence in terms of the second moment is established. To illustrate, suppose that $\alpha^\varepsilon(t) = \alpha(t/\varepsilon)$ such that $\alpha(\cdot)$ is a stationary process with stationary distribution $\overline{\nu} = (\overline{\nu}_1,\ldots,\overline{\nu}_m)$. Then via a change of variable $\varsigma = s/\varepsilon$, we have

$$\begin{aligned}
\frac{1}{t}\int_0^t I_{\{\alpha^\varepsilon(s)=i\}}ds &= \frac{\varepsilon}{t}\int_0^{t/\varepsilon} I_{\{\alpha(\varsigma)=i\}}d\varsigma\\
&= \frac{\varepsilon}{t}\int_0^{t/\varepsilon} I_{\{\alpha(\varsigma)=i\}}d\varsigma \to \overline{\nu}_i \quad \text{in probability as } \varepsilon\to 0,
\end{aligned}$$

for $0 < t \le T$. This is exactly the continuous-time version of the law of large numbers.

Example 5.3. Let us return to the singularly perturbed Cox process of Section 3.3. Recall that the compensator of the singularly perturbed Cox process is given by

$$G^\varepsilon(t) = G_0 + \sum_{i=1}^{m} \int_0^t a_i I_{\{\alpha^\varepsilon(s)=i\}} ds,$$

where $a_i > 0$ for $i = 1, \ldots, m$. Assume that all the conditions in Lemma 5.1 hold. Then Theorem 4.5 implies that $P(\alpha^\varepsilon(t) = i) \to \nu_i(t)$ as $\varepsilon \to 0$. What we have discussed thus far implies that for each $i \in \mathcal{M}$,

$$\int_0^t a_i I_{\{\alpha^\varepsilon(s)=i\}} ds \to \int_0^t a_i \nu_i(s) ds \quad \text{in probability as } \varepsilon \to 0 \text{ and}$$

$$G^\varepsilon(t) \to G(t) = G_0 + \sum_{i=1}^{m} \int_0^t a_i \nu_i(s) ds \quad \text{in probability.}$$

Moreover,

$$\lim_{\varepsilon \to 0} \left(\sup_{0 \le t \le T} E|G^\varepsilon(t) - G(t)|^2 \right) = 0.$$

In the rest of this chapter, we treat suitably scaled occupation measures; the corresponding results for the Cox process can also be derived.

With the limit results in hand, the next question is this: How fast does the convergence take place? The rate of convergence issue together with more detailed asymptotic properties is examined fully in the following sections.

5.2.3 Exponential Bounds

This section is devoted to the derivation of exponential bounds for the normalized occupation measure (or occupation time) $n^\varepsilon(\cdot)$. Given a deterministic process $\beta(\cdot)$, we consider the "centered" and "scaled" functional occupation-time process $n^\varepsilon(t, i)$ defined by

$$\begin{aligned} n^\varepsilon(t,i) &= \frac{1}{\sqrt{\varepsilon}} \int_0^t \left(I_{\{\alpha^\varepsilon(s)=i\}} - \nu_i(s) \right) \beta_i(s) ds \quad \text{and} \\ n^\varepsilon(t) &= (n^\varepsilon(t,1), \ldots, n^\varepsilon(t,m)) \in \mathbb{R}^{1\times m}. \end{aligned} \tag{5.10}$$

In view of Lemma 5.1, we have, for $0 \le s \le t \le T$,

$$P^\varepsilon(t,s) - P_0(t) = O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-s)}{\varepsilon} \right) \right),$$

for some $\kappa_0 > 0$. Note that the big $O(\cdot)$ usually depends on T. Let K_T denote an upper bound of

$$\frac{P^\varepsilon(t,s) - P_0(t)}{\varepsilon + \exp(-\kappa_0(t-s)/\varepsilon)}$$

for $0 \le s \le t \le T$. For convenience, we use the notation $O_1(y)$ to denote a function of y such that $|O_1(y)|/|y| \le 1$. The rationale is that K_T represents the magnitude of the bounding constant and the rest of the bound is in terms of a function with norm bounded by 1. Using this notation and K_T, we write

$$P^\varepsilon(t,s) - P_0(t) = K_T O_1\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-s)}{\varepsilon}\right)\right). \tag{5.11}$$

Let $y(t) = (y_{ij}(t))$ and $z(t) = (z_i(t))$ denote a matrix-valued function and a vector-valued function defined on $[0,T]$, respectively. Their norms are defined by

$$\begin{aligned} |y|_T &= \max_{i,j} \sup_{0\le t\le T} |y_{ij}(t)|, \\ |z|_T &= \max_{i} \sup_{0\le t\le T} |z_i(t)|. \end{aligned} \tag{5.12}$$

For future use, define $\beta(t) = \mathrm{diag}(\beta_1(t), \ldots, \beta_m(t))$. The following theorem is concerned with the exponential bound of $n^\varepsilon(t)$ for ε sufficiently small.

Theorem 5.4. *Assume that* (A5.1) *and* (A5.2) *are satisfied. Then there exist* $\varepsilon_0 > 0$ *and* $K > 0$ *such that for all* $0 < \varepsilon \le \varepsilon_0$, $T \ge 0$, *and any bounded and measurable deterministic function* $\beta(\cdot) = \mathrm{diag}(\beta_1(\cdot), \ldots, \beta_m(\cdot))$, *the following exponential bound holds:*

$$E\exp\left\{\frac{\theta_T}{(T+1)^{\frac{3}{2}}} \sup_{0\le t\le T} |n^\varepsilon(t)|\right\} \le K, \tag{5.13}$$

where θ_T *is a constant satisfying*

$$0 \le \theta_T \le \frac{\min\{1, \kappa_0\}}{K_T |\beta|_T} \tag{5.14}$$

with κ_0 *being the exponential constant in Lemma* 5.1.

Remark 5.5. Note that the constants ε_0 and K are independent of T. This is a convenient feature of the estimate in certain applications. The result is in terms of a fixed but otherwise arbitrary T, which is particularly useful for systems in an infinite horizon.

Proof: The proof is divided into several steps.
Step 1. In the first step, we show that (5.13) holds when the "sup" is absent. Let $\chi^\varepsilon(\cdot)$ denote the indicator vector of $\alpha^\varepsilon(\cdot)$, that is,

$$\chi^\varepsilon(t) = \left(I_{\{\alpha^\varepsilon(t)=1\}}, \ldots, I_{\{\alpha^\varepsilon(t)=m\}}\right) \quad \text{and}$$

$$w^\varepsilon(t) = \chi^\varepsilon(t) - \chi^\varepsilon(0) - \frac{1}{\varepsilon}\int_0^t \chi^\varepsilon(s)Q(s)ds.$$

It is well known (see Elliott [56]) that $w^\varepsilon(t) = (w_1^\varepsilon(t), \ldots, w_m^\varepsilon(t))$, for $t \geq 0$, is a $\sigma\{\alpha^\varepsilon(s) : s \leq t\}$-martingale. In view of a result of Kunita and Watanabe [134] (see also Ikeda and Watanabe [91, p. 55]), one can define a stochastic integral with respect to $w^\varepsilon(t)$. Moreover, the solution of the linear stochastic differential equation

$$d\chi^\varepsilon(t) = \chi^\varepsilon(t)Q^\varepsilon(t)dt + dw^\varepsilon(t)$$

is given by

$$\chi^\varepsilon(t) = \chi^\varepsilon(0)P^\varepsilon(t,0) + \int_0^t (dw^\varepsilon(s))P^\varepsilon(t,s),$$

where $P^\varepsilon(t,s)$ is the principal matrix solution to the equation

$$\frac{dP^\varepsilon(t,s)}{dt} = \frac{1}{\varepsilon}P^\varepsilon(t,s)Q(t), \quad \text{with } P^\varepsilon(s,s) = I$$

representing the transition probability matrix.

Note that for $t \geq s \geq 0$,

$$\chi^\varepsilon(s)P_0(t) = (\chi^\varepsilon(s)\mathbb{1})\nu(t) = \nu(t).$$

Using this and (5.11), we have

$$\begin{aligned}
&\chi^\varepsilon(t) - \nu(t) \\
&= \chi^\varepsilon(0)(P^\varepsilon(t,0) - P_0(t)) + \int_0^t (dw^\varepsilon(s))[(P^\varepsilon(t,s) - P_0(t)) + P_0(t)] \\
&= K_T O_1\left(\varepsilon + \exp\left(-\frac{\kappa_0 t}{\varepsilon}\right)\right) \\
&\quad + K_T \int_0^t (dw^\varepsilon(s))O_1\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-s)}{\varepsilon}\right)\right) + w^\varepsilon(t)P_0(t) \\
&= K_T O_1\left(\varepsilon + \exp\left(-\frac{\kappa_0 t}{\varepsilon}\right)\right) \\
&\quad + K_T \int_0^t (dw^\varepsilon(s))O_1\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-s)}{\varepsilon}\right)\right).
\end{aligned}$$

The last equality above follows from the observation that

$$Q(s)P_0(t) = 0 \text{ for all } t \geq s \geq 0,$$

and

$$\begin{aligned}
w^\varepsilon(t)P_0(t) &= \left(\chi^\varepsilon(t) - \chi^\varepsilon(0) - \frac{1}{\varepsilon}\int_0^t \chi^\varepsilon(r)Q(r)dr\right)P_0(t) \\
&= \nu(t) - \nu(t) - \frac{1}{\varepsilon}\int_0^t \chi^\varepsilon(r)Q(r)P_0(t)dr = 0.
\end{aligned}$$

Recall that $\beta(t) = \text{diag}(\beta_1(t), \ldots, \beta_m(t))$. Then it follows that

$$\begin{aligned}
&\int_0^t (\chi^\varepsilon(s) - \nu(s))\beta(s)ds \\
&= K_T O_1(\varepsilon(t+1)) \\
&\quad + K_T \int_0^t \int_0^s (dw^\varepsilon(r)) O_1\left(\varepsilon + \exp\left(-\frac{\kappa_0(s-r)}{\varepsilon}\right)\right)\beta(s)ds \\
&= K_T O_1(\varepsilon(t+1)) \\
&\quad + K_T \int_0^t (dw^\varepsilon(r)) \left(\int_r^t O_1\left(\varepsilon + \exp\left(-\frac{\kappa_0(s-r)}{\varepsilon}\right)\right)\beta(s)ds\right) \\
&= K_T O_1(\varepsilon(t+1)) \\
&\quad + \varepsilon K_T \int_0^t (dw^\varepsilon(r)) O_1\left((t-r) + \frac{1}{\kappa_0}\left(1 - \exp\left(-\frac{\kappa_0(t-r)}{\varepsilon}\right)\right)\right)|\beta|_T \\
&= K_T O_1(\varepsilon(t+1)) + \varepsilon K_T |\beta|_T \left(T + \frac{1}{\kappa_0}\right) \int_0^t (dw^\varepsilon(r)) b(r,t),
\end{aligned}$$

where $b(s,t)$ is a measurable function and $|b(s,t)| \le 1$ for all s and t. Dividing both sides by $(T+1)$, we obtain

$$\begin{aligned}
&\frac{1}{T+1}\left|\int_0^t (\chi^\varepsilon(s) - \nu(s))\beta(s)ds\right| \\
&\quad = \varepsilon K_T O_1(1) + \varepsilon K_T |\beta|_T \left(\frac{T + (1/\kappa_0)}{T+1}\right)\left|\int_0^t (dw^\varepsilon(s)) b(s,t)\right|.
\end{aligned} \tag{5.15}$$

Therefore, we have

$$\begin{aligned}
&E\exp\left\{\frac{\theta_T}{\sqrt{\varepsilon}(T+1)^{\frac{3}{2}}}\left|\int_0^t (\chi^\varepsilon(s) - \nu(s))\beta(s)ds\right|\right\} \\
&\quad \le E\exp\left\{\frac{1}{\sqrt{\varepsilon}\sqrt{T+1}}\left[\varepsilon O_1(1) + \varepsilon\left|\int_0^t (dw^\varepsilon(s)) b(s,t)\right|\right]\right\}.
\end{aligned}$$

In view of the choice of θ_T, it follows that

$$\begin{aligned}
&E\exp\left\{\frac{\theta_T}{\sqrt{\varepsilon}(T+1)^{\frac{3}{2}}}\left|\int_0^t (\chi^\varepsilon(s) - \nu(s))\beta(s)ds\right|\right\} \\
&\quad \le \exp\left(\frac{\sqrt{\varepsilon}}{\sqrt{T+1}}\right) E\exp\left\{\frac{\sqrt{\varepsilon}}{\sqrt{T+1}}\left|\int_0^t (dw^\varepsilon(s)) b(s,t)\right|\right\} \\
&\quad \le eE\exp\left\{\frac{\sqrt{\varepsilon}}{\sqrt{T+1}}\left|\int_0^t (dw^\varepsilon(s)) b(s,t)\right|\right\}.
\end{aligned} \tag{5.16}$$

Recall that

$$w^\varepsilon(t) = (w_1^\varepsilon(t), \dots, w_m^\varepsilon(t)).$$

It suffices to work out the estimate for each component $w_i^\varepsilon(t)$. That is, it is enough to show that for each $i = 1, \dots, m$,

$$E \exp\left\{ \frac{\sqrt{\varepsilon}}{\sqrt{T+1}} \left| \int_0^t b(s,t) dw_i^\varepsilon(s) \right| \right\} \le K, \tag{5.17}$$

for all measurable functions $b(\cdot,\cdot)$ with $|b(s,t)| \le 1$ and $0 \le t \le T$. For each $t_0 \ge 0$, let $b_0(s) = b(s, t_0)$.

For any nonnegative random variable ξ,

$$\begin{aligned} Ee^\xi &= \sum_{j=0}^\infty \int_{\{j \le \xi < j+1\}} e^\xi dP \\ &\le \sum_{j=0}^\infty \int_{\{j \le \xi < j+1\}} e^{j+1} dP \\ &= \sum_{j=0}^\infty e^{j+1} P(j \le \xi < j+1) \\ &= \sum_{j=0}^\infty e^{j+1} [P(\xi \ge j) - P(\xi \ge j+1)] \\ &\le e + (e-1) \sum_{j=1}^\infty e^j P(\xi \ge j). \end{aligned}$$

By virtue of the inequality above, we have

$$\begin{aligned} &E \exp\left\{ \frac{\sqrt{\varepsilon}}{\sqrt{T+1}} \left| \int_0^t b_0(s) dw_i^\varepsilon(s) \right| \right\} \\ &\le e + (e-1) \sum_{j=1}^\infty e^j P\left(\frac{\sqrt{\varepsilon}}{\sqrt{T+1}} \left| \int_0^t b_0(s) dw_i^\varepsilon(s) \right| \ge j \right). \end{aligned} \tag{5.18}$$

To proceed, let us concentrate on the estimate of

$$P\left(\frac{\sqrt{\varepsilon}}{\sqrt{T+1}} \left| \int_0^t b_0(s) dw_i^\varepsilon(s) \right| \ge j \right).$$

For each $i = 1, \dots, m$, let

$$\widetilde{p}_i(t) = \int_0^t b_0(s) dw_i^\varepsilon(s)$$

and let $\widetilde{q}_i(\cdot)$ denote the only solution to the following equation (see Elliott [55, Chapter 13])

$$\widetilde{q}_i(t) = 1 + \zeta \int_0^t \widetilde{q}_i(s^-) d\widetilde{p}_i(s),$$

where $\widetilde{q}_i(s^-)$ is the left-hand limit of $\widetilde{q}_i$ at s and ζ is a positive constant to be determined later. In what follows, we suppress the i-dependence and write $\widetilde{p}_i(\cdot)$ and $\widetilde{q}_i(\cdot)$ as $\widetilde{p}(\cdot)$ and $\widetilde{q}(\cdot)$ whenever there is no confusion.

Note that $\widetilde{p}(t)$, for $t \geq 0$, is a local martingale. Since

$$\zeta \int_0^t \widetilde{q}(s^-) d\widetilde{p}(s),\ t \geq 0,$$

is a local martingale, we have $E\widetilde{q}(t) \leq 1$ for all $t \geq 0$. Moreover, $\widetilde{q}(t)$ can be written as follows (see Elliott [55, Chapter 13]):

$$\widetilde{q}(t) = \exp\left(\zeta \widetilde{p}(t)\right) \prod_{s \leq t} (1 + \zeta \Delta \widetilde{p}(s)) \exp\left(-\zeta \Delta \widetilde{p}(s)\right), \tag{5.19}$$

where $\Delta \widetilde{p}(s) := \widetilde{p}(s) - \widetilde{p}(s^-)$, with $|\Delta \widetilde{p}(s)| \leq 1$.

Now observe that there exist positive constants ζ_0 and κ_1 such that for $0 < \zeta \leq \zeta_0$ and for all $s > 0$,

$$(1 + \zeta \Delta \widetilde{p}(s)) \exp\left(-\zeta \Delta \widetilde{p}(s)\right) \geq \exp\left(-\kappa_1 \zeta^2\right). \tag{5.20}$$

Combining (5.19) and (5.20), we obtain

$$\widetilde{q}(t) \geq \exp\{\zeta \widetilde{p}(t) - \kappa_1 \zeta^2 N(t)\}, \quad \text{for } 0 < \zeta \leq \zeta_0,\ t > 0,$$

where $N(t)$ is the number of jumps of $\widetilde{p}(s)$ in $s \in [0, t]$. Since $N(t)$ is a monotonically increasing process, we have

$$\widetilde{q}(t) \geq \exp\left\{\zeta \widetilde{p}(t) - \kappa_1 \zeta^2 N(T)\right\}, \quad \text{for } 0 < \zeta \leq \zeta_0.$$

Note also that for each $i = 1, \ldots, m$,

$$\begin{aligned} & P\left(\frac{\sqrt{\varepsilon}}{\sqrt{T+1}} \left| \int_0^t b_0(s) dw_i^{\varepsilon}(s) \right| \geq j \right) \\ &= P\left(|\widetilde{p}(t)| \geq \frac{j\sqrt{T+1}}{\sqrt{\varepsilon}}\right) \\ &\leq P\left(\widetilde{p}(t) \geq \frac{j\sqrt{T+1}}{\sqrt{\varepsilon}}\right) + P\left(-\widetilde{p}(t) \geq \frac{j\sqrt{T+1}}{\sqrt{\varepsilon}}\right). \end{aligned}$$

Consider the first term on the right-hand side of the inequality above. Let $a_j = j(T+1)/(8\kappa_1 \varepsilon)$. Then

$$
\begin{aligned}
&P\left(\widetilde{p}(t) \geq \frac{j\sqrt{T+1}}{\sqrt{\varepsilon}}\right) \\
&\leq P\left(\widetilde{q}(t) \geq \exp\left\{\frac{j\zeta\sqrt{T+1}}{\sqrt{\varepsilon}} - \kappa_1\zeta^2 N(T)\right\}\right) \\
&\leq P\left(\widetilde{q}(t) \geq \exp\left\{\frac{j\zeta\sqrt{T+1}}{\sqrt{\varepsilon}} - \kappa_1\zeta^2 N(T)\right\}, N(T) \leq a_j\right) \\
&\quad + P(N(T) \geq a_j) \\
&\leq P\left(\widetilde{q}(t) \geq \exp\left(\frac{j\zeta\sqrt{T+1}}{\sqrt{\varepsilon}} - \kappa_1\zeta^2 a_j\right)\right) + P(N(T) \geq a_j) \\
&\leq 2\exp\left(-\frac{j\zeta\sqrt{T+1}}{\sqrt{\varepsilon}} + \kappa_1\zeta^2 a_j\right) + P(N(T) \geq a_j).
\end{aligned}
$$

The last inequality follows from the local martingale property (see Elliott [55, Theorem 4.2]).

Now if we choose $\zeta = 4\sqrt{\varepsilon}/\sqrt{T+1}$, then

$$
\exp\left(-\frac{j\zeta\sqrt{T+1}}{\sqrt{\varepsilon}} + \kappa_1\zeta^2 a_j\right) = e^{-2j}.
$$

In view of the construction of Markov chains in Section 2.4, there exists a Poisson process $N_0(\cdot)$ with parameter (i.e., mean) a/ε for some $a > 0$, such that $N(t) \leq N_0(t)$. Assume $a = 1$ without loss of generality (otherwise one may replace ε by εa^{-1}). Using the Poisson distribution of $N_0(t)$, we have

$$
P(N_0(T) \geq k) \leq \frac{(T/\varepsilon)^k}{k!} \text{ for } k \geq 0.
$$

In view of Stirling's formula (see Chow and Teicher [30] or Feller [60]), for ε small enough,

$$
P(N(T) \geq a_j) \leq \frac{(T/\varepsilon)^{\lfloor a_j \rfloor}}{\lfloor a_j \rfloor!} \leq 2\left(\frac{8\kappa_1}{j}\right)^{a_j - 1} := 2\gamma_0^{a_j - 1},
$$

where $\lfloor a_j \rfloor$ is the integer part of a_j and

$$
\gamma_0 = \frac{8e\kappa_1}{j_0} \in (0,1) \quad \text{for } j_0 > \max\{1, 8e\kappa_1\}.
$$

Thus, for $j \geq j_0$,

$$
P\left(\frac{\sqrt{\varepsilon}}{\sqrt{T+1}}\int_0^t b_0(s)dw_i^\varepsilon(s) \geq j\right) \leq 2e^{-2j} + 2\gamma_0^{a_j - 1}.
$$

Repeating the same argument for the martingale $(-\widetilde{p}(\cdot))$, we get for $j \geq j_0$,

$$
P\left(-\frac{\sqrt{\varepsilon}}{\sqrt{T+1}}\int_0^t b_0(s)dw_i^\varepsilon(s) \geq j\right) \leq 2e^{-2j} + 2\gamma_0^{a_j - 1}.
$$

Combining the above two inequalities yields that for $j \geq j_0$,

$$P\left(\frac{\sqrt{\varepsilon}}{\sqrt{T+1}}\left|\int_0^t b_0(s)dw_i^\varepsilon(s)\right| \geq j\right) \leq 4(e^{-2j} + \gamma_0^{a_j - 1}).$$

Then by (5.18),

$$E\exp\left\{\frac{\sqrt{\varepsilon}}{\sqrt{T+1}}\left|\int_0^t b_0(s)dw_i^\varepsilon(s)\right|\right\} \leq K_0 + 4(e-1)\sum_{j=1}^{\infty} e^j(e^{-2j} + \gamma_0^{a_j-1}),$$

where K_0 is the sum corresponding to those terms with $j \leq j_0$. Now choose ε small enough that $e\gamma_0^{1/(8\kappa_1\varepsilon)} \leq 1/2$. Then

$$E\exp\left\{\frac{\sqrt{\varepsilon}}{\sqrt{T+1}}\left|\int_0^t b_0(s)dw_i^\varepsilon(s)\right|\right\} \leq K_0 + 4e\gamma_0^{-1}.$$

Since t_0 is arbitrary, we may take $t_0 = t$ in the above inequality. Then

$$E\exp\left\{\frac{\sqrt{\varepsilon}}{\sqrt{T+1}}\left|\int_0^t b(s,t)dw_i^\varepsilon(s)\right|\right\} \leq K_0 + 4e\gamma_0^{-1}.$$

Combining this inequality with (5.16) leads to

$$E\exp\left\{\frac{\theta_T}{\sqrt{\varepsilon}(T+1)^{\frac{3}{2}}}\left|\int_0^t (\chi^\varepsilon(s) - \nu(s))\beta(s)ds\right|\right\} \leq e(K_0 + 4e\gamma_0^{-1}) := K.$$

Step 2. Recall that

$$n^\varepsilon(t,i) = \frac{1}{\sqrt{\varepsilon}}\int_0^t \left(I_{\{\alpha(\varepsilon,s)=i\}} - \nu_i(s)\right)\beta_i(s)\,ds.$$

Then, for each $i \in \mathcal{M}$, $n^\varepsilon(t,i)$ is nearly a martingale, i.e., for ε small enough,

$$|E[n^\varepsilon(t,i)|\mathcal{F}_s] - n^\varepsilon(s,i)| \leq O(\sqrt{\varepsilon}), \quad \text{for all } \omega \in \Omega \text{ and } 0 \leq s \leq t \leq T. \tag{5.21}$$

Here $O(\sqrt{\varepsilon})$ is deterministic, i.e., it does not depend on the sample point ω. The reason is that it is obtained from the asymptotic expansions. In fact, for all $i_0 \in \mathcal{M}$,

$$E\left[\int_s^t (I_{\{\alpha(\varepsilon,r)=i\}} - \nu_i(r))\beta_i(r)\,dr \big| \alpha(\varepsilon,s) = i_0\right]$$
$$= \int_s^t (E[I_{\{\alpha(\varepsilon,r)=i\}}|\alpha(\varepsilon,s) = i_0] - \nu_i(r))\beta_i(r)\,dr$$
$$= \int_s^t [P(\alpha(\varepsilon,r) = i|\alpha(\varepsilon,s) = i_0) - \nu_i(r_)]\beta_i(r)\,dr$$
$$= \int_s^t O(\varepsilon + \exp(-\kappa_0(r-s)/\varepsilon)\,dr = O(\varepsilon).$$

So, (5.21) follows.

Step 3. We show that for each $a > 0$,

$$E[\exp\{a|n^\varepsilon(t,i)|\}|\mathcal{F}_s] \geq \exp\{a|n^\varepsilon(s,i)|\}(1 + O(\sqrt{\varepsilon})).$$

First of all, note that $\phi(x) = |x|$ is a convex function. There exists a vector function $\phi_0(x)$ bounded by 1 such that

$$\phi(x) \geq \phi(a) + \phi_0(a)\cdot(x-a),$$

for all x and a. Noting that $O(\sqrt{\varepsilon}) = -O(\sqrt{\varepsilon})$, we have

$$E[|n^\varepsilon(t,i)|\,|\mathcal{F}_s] \geq\ |n^\varepsilon(s,i)| + \phi_0(n^\varepsilon(s,i))\cdot E[n^\varepsilon(t,i) - n^\varepsilon(s,i)|\mathcal{F}_s]$$

$$\geq\ |n^\varepsilon(s,i)| + O(\sqrt{\varepsilon}).$$

Moreover, note that e^{ax} is also convex. It follows that

$$E[\exp(a|n^\varepsilon(t,i)|)|\mathcal{F}_s]$$

$$\geq \exp(a|n^\varepsilon(s,i)|) + a\exp\{a|n^\varepsilon(s,i)|\}E[|n^\varepsilon(t,i)| - |n^\varepsilon(s,i)|\,|\mathcal{F}_s]$$

$$\geq \exp(a|n^\varepsilon(s,i)|)(1 + O(\sqrt{\varepsilon})).$$

Step 4. Let $x^\varepsilon(t) = \exp(a|n^\varepsilon(t,i)|)$ for $a > 0$. Then, for any $\mathcal{F}_t$ stopping time $\tau \leq T$,

$$E[x^\varepsilon(T)|\mathcal{F}_\tau] \geq x^\varepsilon(\tau)(1 + O(\sqrt{\varepsilon})). \tag{5.22}$$

Note that $x^\varepsilon(t)$ is continuous. Therefore, it suffices to show the above inequality when τ takes values in a countable set $\{t_1, t_2, \ldots\}$. To this end, note that, for each t_i,

$$E[x^\varepsilon(T)|\mathcal{F}_{t_i}] \geq x^\varepsilon(t_i)(1 + O(\sqrt{\varepsilon})).$$

For all $A \in \mathcal{F}_\tau$, we have $A \cap \{\tau = t_i\} \in \mathcal{F}_{t_i}$. Therefore,

$$\int_{A\cap\{\tau=t_i\}} x^\varepsilon(T)\,dP \geq \left(\int_{A\cap\{\tau=t_i\}} x^\varepsilon(\tau)\,dP\right)(1 + O(\sqrt{\varepsilon})).$$

Thus

$$\int_A x^\varepsilon(T)\,dP \geq \left(\int_A x^\varepsilon(\tau)\,dP\right)(1+O(\sqrt{\varepsilon})),$$

and (5.22) follows.

Step 5. Let $a = \theta/\sqrt{(T+1)^3}$ in Step 3. Then, for ε small enough, there exists K such that

$$P\left(\sup_{t\leq T} x^\varepsilon(t) \geq x\right) \leq \frac{K}{x}, \tag{5.23}$$

for all $x > 0$.

In fact, let $\tau = \inf\{t > 0 \ : \ x^\varepsilon(t) \geq x\}$. We adopt the convention that $\tau = \infty$ if $\{t > 0 \ : \ x^\varepsilon(t) \geq x\} = \emptyset$. Then we have

$$E[x^\varepsilon(T)] \geq (E[x^\varepsilon(T\wedge\tau)])(1+O(\sqrt{\varepsilon})),$$

and write

$$E[x^\varepsilon(T\wedge\tau)] = E[x^\varepsilon(\tau)I_{\{\tau<T\}}] + E[x^\varepsilon(T)I_{\{\tau\geq T\}}] \geq E[x^\varepsilon(\tau)I_{\{\tau<T\}}].$$

Moreover, in view of the definition of τ, we have

$$E\left[x^\varepsilon(\tau)I_{\{\tau<T\}}\right] \geq xP(\tau < T) \geq xP\left(\sup_{t\leq T} x^\varepsilon(t) \geq x\right).$$

It follows that

$$P\left(\sup_{t\leq T} x^\varepsilon(t) \geq x\right) \leq \frac{E[x^\varepsilon(T)]}{(1+O(\sqrt{\varepsilon}))x} \leq \frac{K}{x}.$$

Thus, (5.23) follows.

Finally, to complete the proof of (5.13), note that, for $0 < \kappa < 1$,

$$E\exp\left(\frac{\kappa\theta}{\sqrt{(1+T)^3}}\sup_{t\leq T}|n^\varepsilon(t,i)|\right) = E\left[\sup_{t\leq T}(x^\varepsilon(t))^\kappa\right].$$

It follows that

$$\begin{aligned}
E\left[\sup_{t\leq T}(x^\varepsilon(t))^\kappa\right] &= \int_0^\infty P\left(\sup_{t\leq T}(x^\varepsilon(t))^\kappa \geq x\right)dx \\
&\leq 1 + \int_1^\infty P\left(\sup_{t\leq T}(x^\varepsilon(t))^\kappa \geq x\right)dx \\
&\leq 1 + \int_1^\infty P\left(\sup_{t\leq T} x^\varepsilon(t) \geq x^{1/\kappa}\right)dx \\
&\leq 1 + \int_1^\infty Kx^{-1/\kappa}\,dx < \infty.
\end{aligned}$$

This completes the proof. □

Next we give several corollaries to the theorem. Such estimates are useful for establishing exponential bounds of asymptotic optimal hierarchical controls in manufacturing models (see Sethi and Zhang [192]).

Corollary 5.6. *In Theorem 5.4, if $Q(t) = Q$, a constant matrix, then the following stronger estimate holds:*

$$E \exp\left\{ \frac{\theta_T}{\sqrt{1+T}} \sup_{0\le t\le T} |n^\varepsilon(t)| \right\} \le K. \tag{5.24}$$

Moreover, the constant $\theta_T = \theta$ is independent of T for $T > 0$.

Proof: If $Q(t) = Q$, then $\varphi_1(t)$ in Lemma 5.1 is identically 0. Therefore, the estimate (5.11) can be replaced by

$$P^\varepsilon(t,s) - P_0(t) = K_T O_1\left(\exp\left(-\frac{\kappa_0(t-s)}{\varepsilon}\right)\right).$$

As a result, the estimate in (5.15) can be replaced by

$$\sup_{0\le t\le T}\left|\int_0^t (\chi^\varepsilon(s) - \nu(s))\beta(s)ds\right| = \varepsilon K_T O_1(1) + \varepsilon K_T \sup_{0\le t\le T}\left|\int_0^t O_1(1)dw^\varepsilon(s)\right|.$$

The proof of (5.24) follows in essentially the same way as that of Theorem 5.4 (from equation (5.15) on).

To see that θ_T in (5.24) is independent of T, it suffices to note that in (5.11) the constant K_T is independent of T, which can be seen by examining closely Example 4.16. □

Corollary 5.7. *Under the conditions of Theorem 5.4, there exist constants K_j, such that for $j = 1, 2, \ldots$,*

$$E \sup_{0\le t\le T} |n^\varepsilon(t)|^{2j} \le K_j(1+T)^{3j}. \tag{5.25}$$

Moreover, if $Q(t) = Q$, then for some K_j independent of T and

$$E \sup_{0\le t\le T} |n^\varepsilon(t)|^{2j} \le K_j(1+T)^{j}. \tag{5.26}$$

Proof: Since (5.26) follows from a similar argument to that of Corollary 5.6, it suffices to verify (5.25) using Theorem 5.4. Note that for each $j = 1, 2, \ldots$, there exists K_j^0 such that for all x, we have $x^{2j} \le K_j^0 e^x$. Thus,

$$\left(\frac{\theta_T}{(T+1)^{\frac{3}{2}}} \sup_{0\le t\le T} |n^\varepsilon(t)|\right)^{2j} \le K_j^0 \exp\left\{\frac{\theta_T}{(T+1)^{\frac{3}{2}}} \sup_{0\le t\le T} |n^\varepsilon(t)|\right\}.$$

Taking expectations on both sides of the above inequality yields the desired estimate. □

Corollary 5.8. *Under the conditions of Theorem 5.4, for each $0 < \delta < 1/2$, we have*

$$P\left(\sup_{0\le t\le T}\left|\int_0^t (I_{\{\alpha^\varepsilon(s)=i\}}-\nu_i(s))\beta_i(s)ds\right|\ge\varepsilon^{\frac{1}{2}-\delta}\right) \le K\exp\left\{-\frac{\theta_T}{\varepsilon^\delta(T+1)^{\frac{3}{2}}}\right\}. \tag{5.27}$$

Moreover, if $Q(t)=Q$, then $\theta_T=\theta$ is independent of T and

$$P\left(\sup_{0\le t\le T}\left|\int_0^t (I_{\{\alpha^\varepsilon(s)=i\}}-\nu_i(s))\beta_i(s)ds\right|\ge\varepsilon^{\frac{1}{2}-\delta}\right) \le K\exp\left\{-\frac{\theta}{\varepsilon^\delta\sqrt{1+T}}\right\}. \tag{5.28}$$

Proof: Using Theorem 5.4, we obtain

$$\begin{aligned} &P\left(\sup_{0\le t\le T}\left|\int_0^t (I_{\{\alpha^\varepsilon(s)=i\}}-\nu_i(s))\beta_i(s)ds\right|\ge\varepsilon^{\frac{1}{2}-\delta}\right)\\ &= P\left(\exp\left\{\frac{\theta_T}{(T+1)^{\frac{3}{2}}}\sup_{0\le t\le T}|n^\varepsilon(t)|\right\}\ge\exp\left\{\frac{\theta_T\varepsilon^{\frac{1}{2}-\delta}}{\sqrt{\varepsilon}(T+1)^{\frac{3}{2}}}\right\}\right)\\ &\le K\exp\left\{-\frac{\theta_T}{\varepsilon^\delta(T+1)^{\frac{3}{2}}}\right\}. \end{aligned}$$

This proves (5.27). Similarly, (5.28) follows from Corollary 5.6. □

5.2.4 Asymptotic Normality

Recall that the ith component of $n^\varepsilon(\cdot)$ is given by

$$n^\varepsilon(t,i)=\frac{1}{\sqrt{\varepsilon}}\int_0^t \left(I_{\{\alpha^\varepsilon(s)=i\}}-\nu_i(s)\right)\beta_i(s)ds.$$

It is expected that the sequence of centered and scaled occupation measures will display certain "central limit type" phenomena. The goal here is to study the asymptotic properties of $n^\varepsilon(\cdot)$ as $\varepsilon\to 0$. To be more specific, we show that $n^\varepsilon(\cdot)$ converges to a Gaussian process as ε goes to 0. The following theorem is the main result of this section.

Theorem 5.9. *Suppose that* (A5.1) *is satisfied and $Q(\cdot)$ is twice continuously differentiable in $[0,T]$ with the second derivative being Lipschitz. Then for $t\in[0,T]$, the process $n^\varepsilon(\cdot)$ converges weakly to a Gaussian process $n(\cdot)$ with independent increments such that*

$$En(t)=0 \text{ and } E[n'(t)n(t)]=\int_0^t A(s)ds, \tag{5.29}$$

where $A(t) = (A_{ij}(t))$ *with*

$$A_{ij}(t) = \beta_i(t)\beta_j(t)\left[\nu_i(t)\int_0^\infty q_{0,ij}(r,t)dr + \nu_j(t)\int_0^\infty q_{0,ji}(r,t)dr\right], \tag{5.30}$$

and $Q_0(r,t) = (q_{0,ij}(r,t))$.

Remark 5.10. In view of (5.29) and the independent increment property of $n(t)$, it follows that

$$E[n'(t_1)n(t_2)] = \int_0^{\min\{t_1,t_2\}} A(s)ds. \tag{5.31}$$

The form of the covariance matrix (between t_1 and t_2) reveals the nonstationarity of the limit process $n(\cdot)$. Note that the limit covariance of $n(t)$ given in (5.31) is an integral of the function $A(s)$ defined in (5.30). For simplicity, with a slight abuse of notation, we shall also call $A(t)$ as the covariance. This convention will be used throughout the chapter.

Remark 5.11. The additional assumptions on the second derivative of $Q(\cdot)$ in Theorem 5.9 are required for computing or characterizing the function $A(\cdot)$. It is not crucial for the convergence of $n^\varepsilon(\cdot)$; see Remark 5.44 in Section 5.3.3 for details.

Proof of Theorem 5.9: We divide the proof into several steps, which are presented by a number of lemmas.

Step 1. Show that the limit of the mean of $n^\varepsilon(\cdot)$ is 0.

Lemma 5.12. *For each* $t \in [0,T]$,

$$\lim_{\varepsilon\to 0} En^\varepsilon(t) = 0.$$

Proof: Using Theorem 4.5 and the boundedness of $\beta_i(\cdot)$, for $t \in [0,T]$,

$$\begin{aligned}
En^\varepsilon(t,i) &= \frac{1}{\sqrt{\varepsilon}}\int_0^t (EI_{\{\alpha^\varepsilon(s)=i\}} - \nu_i(s))\beta_i(s)ds \\
&= \frac{1}{\sqrt{\varepsilon}}\int_0^t (P(\alpha^\varepsilon(s)=i) - \nu_i(s))\beta_i(s)ds \\
&= \frac{1}{\sqrt{\varepsilon}}\int_0^t \left[O(\varepsilon) + O\left(\exp\left(-\frac{\kappa_0 s}{\varepsilon}\right)\right)\right]\beta_i(s)ds \\
&= O(\sqrt{\varepsilon}) + \frac{1}{\sqrt{\varepsilon}}\int_0^t O\left(\exp\left(-\frac{\kappa_0 s}{\varepsilon}\right)ds = O(\sqrt{\varepsilon}) \to 0,\right.
\end{aligned}$$

for each $i \in \mathcal{M}$. □

Step 2. Calculate the limit covariance function.

Lemma 5.13. *For each $t \in [0,T]$,*

$$\lim_{\varepsilon\to 0} E(n^{\varepsilon,\prime}(t)n^{\varepsilon}(t)) = \int_0^t A(s)ds, \tag{5.32}$$

where $A(t)$ is given by (5.30).

Proof: For each $i, j \in \mathcal{M}$,

$$\begin{aligned} E\left[n^{\varepsilon}(t,i)n^{\varepsilon}(t,j)\right] &= \frac{1}{\varepsilon}E\Bigg[\left(\int_0^t (I_{\{\alpha^{\varepsilon}(\varsigma)=i\}} - \nu_i(\varsigma))\beta_i(\varsigma)d\varsigma\right) \\ &\quad \times\left(\int_0^t (I_{\{\alpha^{\varepsilon}(r)=j\}} - \nu_j(r))\beta_j(r)dr\right)\Bigg] \\ &= \frac{1}{\varepsilon}E\Bigg[\int_0^t\int_0^t \Big(I_{\{\alpha^{\varepsilon}(\varsigma)=i,\alpha^{\varepsilon}(r)=j\}} - \nu_i(\varsigma)I_{\{\alpha^{\varepsilon}(r)=j\}} \\ &\quad -\nu_j(r)I_{\{\alpha^{\varepsilon}(\varsigma)=i\}} + \nu_i(\varsigma)\nu_j(r)\Big)\beta_i(\varsigma)\beta_j(r)d\varsigma dr\Bigg]. \end{aligned}$$

Let

$$D_1 = \{(\varsigma, r) :\ 0 \le r \le \varsigma \le t\},$$

$$D_2 = \{(\varsigma, r) :\ 0 \le \varsigma \le r \le t\},$$

and let

$$\begin{aligned} \Phi^{\varepsilon}(\varsigma, r) \ &= P(\alpha^{\varepsilon}(\varsigma) = i, \alpha^{\varepsilon}(r) = j) - \nu_i(\varsigma)P(\alpha^{\varepsilon}(r) = j) \\ &\quad -\nu_j(r)P(\alpha^{\varepsilon}(\varsigma) = i) + \nu_i(\varsigma)\nu_j(r). \end{aligned}$$

Then it follows that

$$\begin{aligned} E\left[n^{\varepsilon}(t,i)n^{\varepsilon}(t,j)\right] &= \frac{1}{\varepsilon}\left[\int_0^t\int_0^t \Phi^{\varepsilon}(\varsigma,r)\beta_i(\varsigma)\beta_j(r)d\varsigma dr\right] \\ &= \frac{1}{\varepsilon}\left(\int_{D_1} + \int_{D_2}\right)\Phi^{\varepsilon}(\varsigma,r)\beta_i(\varsigma)\beta_j(r)d\varsigma dr. \end{aligned}$$

Note that if $(\varsigma, r) \in D_1$, then $\varsigma \ge r$ and

$$\begin{aligned} &P(\alpha^{\varepsilon}(\varsigma) = i, \alpha^{\varepsilon}(r) = j) \\ &= P(\alpha^{\varepsilon}(\varsigma) = i | \alpha^{\varepsilon}(r) = j)P(\alpha^{\varepsilon}(r) = j). \end{aligned}$$

Hence, for $(\varsigma, r) \in D_1$ we have

$$\Phi^\varepsilon(\varsigma, r) = [P(\alpha^\varepsilon(\varsigma) = i | \alpha^\varepsilon(r) = j) - \nu_i(\varsigma)] P(\alpha^\varepsilon(r) = j)$$

$$+\nu_j(r)[\nu_i(\varsigma) - P(\alpha^\varepsilon(\varsigma) = i)].$$

Using Theorem 4.5 and Lemma 5.1, for $(\varsigma, r) \in D_1$,

$$\begin{aligned}
\Phi^\varepsilon(\varsigma, r) = & \left(\varepsilon\varphi_1^i(\varsigma) + q_{0,ji}\left(\frac{\varsigma - r}{\varepsilon}, r\right) + \varepsilon q_{1,ji}\left(\frac{\varsigma - r}{\varepsilon}, r\right) + O(\varepsilon^2)\right) \\
& \times \left(\nu_j(r) + \varepsilon\varphi_1^j(r) + \psi_0^j\left(\frac{r}{\varepsilon}\right) + \varepsilon\psi_1^j\left(\frac{r}{\varepsilon}\right) + O(\varepsilon^2)\right) \\
& - \nu_j(r)\left(\varepsilon\varphi_1^i(\varsigma) + \psi_0^i\left(\frac{\varsigma}{\varepsilon}\right) + \varepsilon\psi_1^i\left(\frac{\varsigma}{\varepsilon}\right) + O(\varepsilon^2)\right) \\
= & \ \nu_j(r) q_{0,ji}\left(\frac{\varsigma - r}{\varepsilon}, r\right) + \left[O\left(\varepsilon \exp\left(-\frac{\kappa_0 r}{\varepsilon}\right)\right)\right. \\
& \left. + O\left(\varepsilon \exp\left(-\frac{\kappa_0(\varsigma - r)}{\varepsilon}\right)\right) + O\left(\exp\left(-\frac{\kappa_0 \varsigma}{\varepsilon}\right)\right) + O(\varepsilon^2)\right].
\end{aligned}$$

In the above, φ_ℓ^i and ψ_ℓ^i denote the ith components of the vectors φ_ℓ and ψ_ℓ, respectively. By elementary integration, we have

$$\int_0^t \left(\int_0^\varsigma \exp\left(-\frac{\kappa_0 \varsigma}{\varepsilon}\right) dr\right) d\varsigma = \int_0^t \varsigma \exp\left(-\frac{\kappa_0 \varsigma}{\varepsilon}\right) d\varsigma = O(\varepsilon^2),$$
$$\varepsilon \int_0^t \left(\int_0^\varsigma \exp\left(-\frac{\kappa_0 r}{\varepsilon}\right) dr\right) d\varsigma = \frac{\varepsilon^2}{\kappa_0} \int_0^t \left(1 - \exp\left(-\frac{\kappa_0 \varsigma}{\varepsilon}\right)\right) d\varsigma = O(\varepsilon^2),$$

and

$$\varepsilon \int_0^t \left(\int_0^\varsigma \exp\left(-\frac{\kappa_0(\varsigma - r)}{\varepsilon}\right) dr\right) d\varsigma = \varepsilon \int_0^t \left(\int_0^\varsigma \exp\left(-\frac{\kappa_0 r}{\varepsilon}\right) dr\right) d\varsigma = O(\varepsilon^2).$$

Thus, it follows that

$$\begin{aligned}
& \int_{D_1} \Phi^\varepsilon(\varsigma, r) \beta_i(\varsigma) \beta_j(r) d\varsigma dr \\
& = \int_0^t \left(\int_0^\varsigma q_{0,ji}\left(\frac{\varsigma - r}{\varepsilon}, r\right) \nu_j(r) \beta_i(\varsigma) \beta_j(r) dr\right) d\varsigma + O(\varepsilon^2).
\end{aligned}$$

Exchanging the order of integration leads to

$$\begin{aligned}
&\int_0^t \left(\int_0^\varsigma q_{0,ji}\left(\frac{\varsigma - r}{\varepsilon}, r\right) \nu_j(r)\beta_i(\varsigma)\beta_j(r)dr \right) d\varsigma \\
&\quad = \int_0^t \left(\int_r^t q_{0,ji}\left(\frac{\varsigma - r}{\varepsilon}, r\right) \nu_j(r)\beta_i(\varsigma)\beta_j(r)d\varsigma \right) dr \\
&\quad = \int_0^t \beta_j(r)\nu_j(r) \left(\int_r^t q_{0,ji}\left(\frac{\varsigma - r}{\varepsilon}, r\right) \beta_i(\varsigma)d\varsigma \right) dr.
\end{aligned}$$

Making a change of variables (via $\varsigma - r = \varepsilon s$) yields

$$\int_r^t q_{0,ji}\left(\frac{\varsigma - r}{\varepsilon}, r\right) \beta_i(\varsigma)d\varsigma = \varepsilon \int_0^{(t-r)/\varepsilon} q_{0,ji}(s, r)\beta_i(r + \varepsilon s)ds.$$

We note that $\beta_i(\cdot)$ is bounded and $\beta_i(r + \varepsilon s) \to \beta_i(r)$ in L^1 for each $r \in [0, T]$, as $\varepsilon \to 0$. Since $q_{0,ji}(\cdot)$ decays exponentially fast, as in Lemma 5.1, we have

$$\int_0^{(t-r)/\varepsilon} q_{0,ji}(s, r)\beta_i(r + \varepsilon s)ds \to \beta_i(r) \int_0^\infty q_{0,ji}(s, r)ds.$$

Therefore, we obtain

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{D_1} \Phi^\varepsilon(\varsigma, r)\beta_i(\varsigma)\beta_j(r)d\varsigma dr \\
&\quad = \int_0^t \beta_i(r)\beta_j(r)\nu_j(r) \left(\int_0^\infty q_{0,ji}(s, r)ds \right) dr.
\end{aligned} \tag{5.33}$$

Similarly, we can show that

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{D_2} \Phi^\varepsilon(\varsigma, r)\beta_i(\varsigma)\beta_j(r)d\varsigma dr \\
&\quad = \int_0^t \beta_i(r)\beta_j(r)\nu_i(r) \left(\int_0^\infty q_{0,ij}(s, r)ds \right) dr.
\end{aligned} \tag{5.34}$$

Combining (5.33) and (5.34), we obtain

$$\lim_{\varepsilon \to 0} E\left[n^\varepsilon(t, i)n^\varepsilon(t, j)\right] = \int_0^t A_{ij}(s)ds,$$

with $A(t) = (A_{ij}(t))$ given by (5.30). □

Step 3. Establish a mixing condition for the sequence $\{n^\varepsilon(\cdot)\}$.

Lemma 5.14. *For any $\varsigma \geq 0$ and $\sigma\{\alpha^\varepsilon(s) : s \geq t + \varsigma\}$-measurable η with $|\eta| \leq 1$,*

$$\left| E(\eta | \alpha^\varepsilon(s) : s \leq t) - E\eta \right| \leq K \exp\left(-\frac{\kappa\varsigma}{\varepsilon} \right) \quad w.p.1. \tag{5.35}$$

Remark 5.15. It follows from (5.35) that for any $\sigma\{\alpha^\varepsilon(s) : 0 \le s \le t\}$-measurable ξ with $|\xi| \le 1$ and η given in Lemma 5.14,

$$\Big|E\xi\eta - E\xi E\eta\Big| \le K \exp\left(-\frac{\kappa\varsigma}{\varepsilon}\right). \tag{5.36}$$

We will make crucial use of (5.35) and (5.36) in what follows.

Proof of Lemma 5.14: For any

$$0 \le s_1 \le s_2 \le \cdots \le s_n = t \le t+\varsigma = t_0 \le t_1 \le \cdots \le t_l < \infty,$$

let

$$E_1 = \{\alpha^\varepsilon(t) = i,\ \alpha^\varepsilon(s_{n-1}) = i_{n-1},\ \ldots, \alpha^\varepsilon(s_1) = i_1\} \text{ and}$$

$$E_2 = \{\alpha^\varepsilon(t+\varsigma) = j,\ \alpha^\varepsilon(t_1) = j_1,\ \ldots, \alpha^\varepsilon(t_l) = j_l\}.$$

Then in view of the Markovian property of $\alpha^\varepsilon(\cdot)$,

$$P(E_2|E_1) = P(E_2|\alpha^\varepsilon(t) = i)$$

$$= P(\alpha^\varepsilon(t+\varsigma) = j|\alpha^\varepsilon(t) = i)[p^\varepsilon_{j,j_1}(t_1, t+\varsigma)\cdots p^\varepsilon_{j_{l-1},j_l}(t_l, t_{l-1})].$$

Similarly, we have

$$P(E_2) = P(\alpha^\varepsilon(t+\varsigma) = j)[p^\varepsilon_{j,j_1}(t_1, t+\varsigma)\cdots p^\varepsilon_{j_{l-1},j_l}(t_l, t_{l-1})].$$

We first show that

$$\Big|P(E_2|E_1) - P(E_2)\Big| \le K \exp\left(-\frac{\kappa\varsigma}{\varepsilon}\right), \tag{5.37}$$

for some positive constants K and κ that are independent of $i, j \in \mathcal{M}$ and $t \in [0, T]$.

To verify (5.37), it suffices to show that for any $k \in \mathcal{M}$,

$$\Big|p^\varepsilon_{ij}(t+\varsigma, t) - p^\varepsilon_{kj}(t+\varsigma, t)\Big| \le K \exp\left(-\frac{2\kappa\varsigma}{\varepsilon}\right). \tag{5.38}$$

Since $P_0(\cdot)$ and $P_1(\cdot)$ have identical rows, the asymptotic expansion in (5.3) implies that $p^\varepsilon_{ij}(t+\zeta, t) - p^\varepsilon_{kj}(t+\zeta, t)$ is determined by $Q_0(\zeta/\varepsilon, t)$. By virtue of the asymptotic expansion (see Theorem 4.5 and Lemma 5.1), there exist a $K_1 > 0$ and a $\kappa_0 > 0$ such that

$$\left|Q_0\left(\frac{(t+\widetilde{t}) - t}{\varepsilon}, t\right)\right| \le K_1 \exp\left(-\frac{\kappa_0\widetilde{t}}{\varepsilon}\right), \text{ for all } \widetilde{t} \ge 0.$$

Choose $N > 0$ sufficiently large that $K_1 \exp(-\kappa_0 N) < 1$. Then for $\varepsilon > 0$ sufficiently small, there is a $0 < \rho < 1$ such that

$$\left| p_{ij}^{\varepsilon}(t + N\varepsilon, t) - p_{kj}^{\varepsilon}(t + N\varepsilon, t) \right| \leq \rho.$$

To proceed, subdivide $[t + N\varepsilon, t + \varsigma]$ into intervals of length $N\varepsilon$.

In view of the Chapman–Kolmogorov equation,

$$\begin{aligned}
&|p_{ij}^{\varepsilon}(t + 2N\varepsilon, t) - p_{kj}^{\varepsilon}(t + 2N\varepsilon, t)| \\
&\quad = \left| \sum_{l_0=1}^{m} [p_{il_0}^{\varepsilon}(t + N\varepsilon, t) - p_{kl_0}^{\varepsilon}(t + N\varepsilon, t)] p_{l_0 j}^{\varepsilon}(t + 2N\varepsilon, t + N\varepsilon) \right| \\
&\quad = \left| \sum_{l_0=1}^{m} [p_{il_0}^{\varepsilon}(t + N\varepsilon, t) - p_{kl_0}^{\varepsilon}(t + N\varepsilon, t)] \right. \\
&\qquad \left. \times [p_{l_0 j}^{\varepsilon}(t + 2N\varepsilon, t + N\varepsilon) - p_{l_1 j}^{\varepsilon}(t + 2N\varepsilon, t + N\varepsilon)] \right| \leq K\rho^2,
\end{aligned}$$

for any $l_1 \in \mathcal{M}$. Iterating on the inequality above, we arrive at

$$\left| p_{ij}^{\varepsilon}(t + k_0 N\varepsilon, t) - p_{kj}^{\varepsilon}(t + k_0 N\varepsilon, t) \right| \leq K\rho^{k_0}, \quad \text{for } k_0 \geq 1.$$

Choose $\kappa = -1/(2N) \log \rho$, and note that $\kappa > 0$. Then for any ς satisfying $k_0 N\varepsilon \leq \varsigma < (k_0 + 1)N\varepsilon$,

$$\left| p_{ij}^{\varepsilon}(t + \varsigma, t) - p_{kj}^{\varepsilon}(t + \varsigma, t) \right| \leq K \exp\left(-\frac{2\kappa\varsigma}{\varepsilon} \right).$$

Thus (5.37) holds. This implies that $\alpha^{\varepsilon}(\cdot)$ is a mixing process with exponential mixing rate. By virtue of Lemma A.16, (5.35) holds. □

Step 4. Prove that the sequence $n^{\varepsilon}(\cdot)$ is tight, and any weakly convergent subsequence of $\{n^{\varepsilon}(\cdot)\}$ has continuous paths with probability 1.

Lemma 5.16. *The following assertions hold:*

(a) $\{n^{\varepsilon}(t);\ t \in [0, T]\}$ *is tight in* $D([0, T]; \mathbb{R}^m)$, *where* $D([0, T]; \mathbb{R}^m)$ *denotes the space of functions that are defined on* $[0, T]$ *and that are right continuous with left limits.*

(b) *The limit* $n(\cdot)$ *of any weakly convergent subsequence of* $n^{\varepsilon}(\cdot)$ *has continuous sample paths with probability* 1.

Proof: For $i \in \mathcal{M}$, define

$$\widetilde{n}^{\varepsilon}(t, i) = \frac{1}{\sqrt{\varepsilon}} \int_0^t \left(I_{\{\alpha^{\varepsilon}(s) = i\}} - P(\alpha^{\varepsilon}(s) = i) \right) \beta_i(s) ds.$$

By virtue of Theorem 4.5,

$$\frac{1}{\sqrt{\varepsilon}}\int_0^t \left(P(\alpha^\varepsilon(s)=i)-\nu_i(s)\right)\beta_i(s)ds = O(\sqrt{\varepsilon}).$$

Thus $n^\varepsilon(t,i)=\widetilde{n}^\varepsilon(t,i)+O(\sqrt{\varepsilon})$, and as a result the tightness of $\{n^\varepsilon(\cdot)\}$ will follow from the tightness of $\{\widetilde{n}^\varepsilon(\cdot)\}$ (see Kushner [139, Lemma 5, p. 50]).

For the tightness of $\{\widetilde{n}^\varepsilon(\cdot)\}$, in view of Kushner [139, Theorem 5, p. 32], it suffices to show that

$$E|\widetilde{n}^\varepsilon(t+\varsigma)-\widetilde{n}^\varepsilon(t)|^4 \le K\varsigma^2. \tag{5.39}$$

To verify this assertion, it is enough to prove that for each $i\in\mathcal{M}$, $\widetilde{n}^\varepsilon(\cdot,i)$ satisfies the condition.

Fix $i\in\mathcal{M}$ and for any $0\le t\le T$, let

$$\theta(t)=\left(I_{\{\alpha^\varepsilon(t)=i\}}-P(\alpha^\varepsilon(t)=i)\right)\beta_i(t).$$

We have suppressed the i and ε dependence in $\theta(t)$ for ease of presentation. Let $D=\{(s_1,s_2,s_3,s_4): t\le s_i\le t+\varsigma, i=1,2,3,4\}$. It follows that

$$\begin{aligned} &E|\widetilde{n}^\varepsilon(t+\varsigma,i)-\widetilde{n}^\varepsilon(t,i)|^4\\ &\quad\le \frac{1}{\varepsilon^2}\int_D |E\theta(s_1)\theta(s_2)\theta(s_3)\theta(s_4)|ds_1ds_2ds_3ds_4. \end{aligned} \tag{5.40}$$

Let (i_1,i_2,i_3,i_4) denote a permutation of $(1,2,3,4)$ and

$$D_{i_1i_2i_3i_4}=\{(s_1,s_2,s_3,s_4):\ t\le s_{i_1}\le s_{i_2}\le s_{i_3}\le s_{i_4}\le t+\varsigma\}.$$

Then it is easy to see that $D=\cup D_{i_1i_2i_3i_4}$. This and (5.40) leads to

$$\begin{aligned} &E|\widetilde{n}^\varepsilon(t+\varsigma,i)-\widetilde{n}^\varepsilon(t,i)|^4\\ &\quad\le \frac{K}{\varepsilon^2}\int_{D_0} |E\theta(s_1)\theta(s_2)\theta(s_3)\theta(s_4)|ds_1ds_2ds_3ds_4, \end{aligned}$$

where $D_0=D_{1234}$.

Note that

$$\begin{aligned} &|E\theta(s_1)\theta(s_2)\theta(s_3)\theta(s_4)|\\ &\quad\le |E\theta(s_1)\theta(s_2)\theta(s_3)\theta(s_4)-E\theta(s_1)\theta(s_2)E\theta(s_3)\theta(s_4)|\\ &\qquad+|E\theta(s_1)\theta(s_2)||E\theta(s_3)\theta(s_4)|. \end{aligned} \tag{5.41}$$

By virtue of (5.36) and $E\theta(t) = 0$, $t \geq 0$,

$$\begin{aligned} |E\theta(s_1)\theta(s_2)| &= |E\theta(s_1)\theta(s_2) - E\theta(s_1)E\theta(s_2)| \\ &\leq K \exp\left(-\frac{\kappa(s_2 - s_1)}{\varepsilon}\right). \end{aligned}$$

Similarly, we have

$$\begin{aligned} |E\theta(s_3)\theta(s_4)| &= |E\theta(s_3)\theta(s_4) - E\theta(s_3)E\theta(s_4)| \\ &\leq K \exp\left(-\frac{\kappa(s_4 - s_3)}{\varepsilon}\right). \end{aligned}$$

Therefore, it follows that

$$\frac{K}{\varepsilon^2} \int_{D_0} |E\theta(s_1)\theta(s_2)| \cdot |E\theta(s_3)\theta(s_4)| ds_1 ds_2 ds_3 ds_4 \leq K\varsigma^2. \tag{5.42}$$

The elementary inequality $(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$ for nonnegative numbers a and b yields that

$$\begin{aligned} &|E\theta(s_1)\theta(s_2)\theta(s_3)\theta(s_4) - E\theta(s_1)\theta(s_2)E\theta(s_3)\theta(s_4)| \\ &= \left(|E\theta(s_1)\theta(s_2)\theta(s_3)\theta(s_4) - E\theta(s_1)\theta(s_2)E\theta(s_3)\theta(s_4)|^{\frac{1}{2}}\right)^2 \\ &\leq |E\theta(s_1)\theta(s_2)\theta(s_3)\theta(s_4) - E\theta(s_1)\theta(s_2)E\theta(s_3)\theta(s_4))|^{\frac{1}{2}} \\ &\quad \times \left(|E\theta(s_1)\theta(s_2)\theta(s_3)\theta(s_4)|^{\frac{1}{2}} + |E\theta(s_1)\theta(s_2)E\theta(s_3)\theta(s_4)|^{\frac{1}{2}}\right). \end{aligned}$$

In view of (5.36), we obtain

$$\begin{aligned} &|E\theta(s_1)\theta(s_2)\theta(s_3)\theta(s_4) - E\theta(s_1)\theta(s_2)E\theta(s_3)\theta(s_4))|^{\frac{1}{2}} \\ &\qquad \leq K \exp\left(-\frac{\kappa(s_3 - s_2)}{2\varepsilon}\right). \end{aligned}$$

Similarly, by virtue of (5.35) and the boundedness of $\theta(s)$,

$$\begin{aligned} &|E\theta(s_1)\theta(s_2)\theta(s_3)\theta(s_4)|^{\frac{1}{2}} \\ &= |E\theta(s_1)\theta(s_2)\theta(s_3)(E(\theta(s_4)|\alpha^\varepsilon(s) : \ s \leq s_3) - E\theta(s_4))|^{\frac{1}{2}} \\ &\leq K \exp\left(-\frac{\kappa(s_4 - s_3)}{2\varepsilon}\right), \end{aligned}$$

and

$$
\begin{aligned}
&|E\theta(s_1)\theta(s_2)E\theta(s_3)\theta(s_4)|^{\frac{1}{2}} \\
&\quad = |(E\theta(s_1)\theta(s_2) - E\theta(s_1)E\theta(s_2))(E\theta(s_3)\theta(s_4) - E\theta(s_3)E\theta(s_4))|^{\frac{1}{2}} \\
&\quad \le K \exp\left(-\frac{\kappa(s_2 - s_1)}{2\varepsilon}\right) \exp\left(-\frac{\kappa(s_4 - s_3)}{2\varepsilon}\right).
\end{aligned}
$$

By virtue of the estimates above, we arrive at

$$
\begin{aligned}
\frac{K}{\varepsilon^2} \int_{D_0} &|E\theta(s_1)\theta(s_2)\theta(s_3)\theta(s_4) \\
&-E\theta(s_1)\theta(s_2)E\theta(s_3)\theta(s_4)|ds_1 ds_2 ds_3 ds_4 \le K\varsigma^2.
\end{aligned} \tag{5.43}
$$

The estimate (5.39) then follows from (5.42) and (5.43), and so does the desired tightness of $\{n^\varepsilon(\cdot)\}$.

Since $\{n^\varepsilon(\cdot)\}$ is tight, by Prohorov's theorem, we extract a convergent subsequence, and for notational simplicity, we still denote the sequence by $\{n^\varepsilon(\cdot)\}$ whose limit is $n(\cdot)$. By virtue of Kushner [139, Theorem 5, p. 32] or Ethier and Kurtz [59, Proposition 10.3, p. 149], $n(\cdot)$ has continuous paths with probability 1. □

Remark 5.17. Step 4 implies that both $n^\varepsilon(\cdot)$ and $n(\cdot)$ have continuous sample paths with probability 1. It follows, in view of Prohorov's theorem (see Billingsley [13]), that $n^\varepsilon(\cdot)$ is tight in $C([0,T];\mathbb{R}^m)$.

Step 5. Show that the finite-dimensional distributions of $n^\varepsilon(\cdot)$ converge to that of a Gaussian process with independent increments.

This part of the proof is similar to Khasminskii [112] (see also Friedlin and Wentzel [67, pp. 224]). Use ι to denote the imaginary number $\iota^2 = -1$. To prove the convergence of the finite-dimensional distributions, we use the characteristic function $E\exp(\iota\langle z, n^\varepsilon(t)\rangle)$, where $z \in \mathbb{R}^m$ and $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^m$. Owing to the mixing property and repeated applications of Remark 5.15, for arbitrary positive real numbers s_l and t_l satisfying

$$
0 \le s_0 \le t_0 \le s_1 \le t_1 \le s_2 \le \cdots \le s_n \le t_n,
$$

we have

$$
\begin{aligned}
\Bigg| E\exp\Bigg(\iota \sum_{l=0}^{n} \big\langle z_l, (n^\varepsilon(t_l) - n^\varepsilon(s_l))\big\rangle\Bigg) \\
- \prod_{l=0}^{n} E\exp\Big(\iota\big\langle z_l, (n^\varepsilon(t_l) - n^\varepsilon(s_l))\big\rangle\Big)\Bigg| \to 0
\end{aligned}
$$

as $\varepsilon \to 0$, for $z_l \in \mathbb{R}^m$. This, in turn, implies that the limit process $n(\cdot)$ has independent increments. Moreover, in view of Lemma 5.16, the limit process has continuous path with probability 1. In accordance with a result in Skorohod [197, p. 7], if a process with independent increments has continuous paths w.p.1, then it must necessarily be a Gaussian process. This implies that the limits of the finite-dimensional distribution of $n(\cdot)$ are Gaussian.

Consequently, $n(\cdot)$ is a process having Gaussian finite-dimensional distributions, with mean zero and covariance $\int_0^t A(s)ds$ given by Lemma 5.13. Moreover, the limit does not depend on the chosen subsequence. Thus $n^\varepsilon(\cdot)$ converges weakly to the Gaussian process $n(\cdot)$. This completes the proof of the theorem. □

To illustrate, we give an example in which the covariance function of the limit process can be calculated explicitly.

Example 5.18. Let $\alpha^\varepsilon(t) \in \mathcal{M} = \{1, 2\}$ be a two-state Markov chain with a generator

$$Q(t) = \begin{pmatrix} -\mu_1(t) & \mu_1(t) \\ \mu_2(t) & -\mu_2(t) \end{pmatrix}$$

where $\mu_1(t) \geq 0$, $\mu_2(t) \geq 0$, and $\mu_1(t) + \mu_2(t) > 0$ for each $t \in [0, T]$. Moreover, $\mu_1(\cdot)$ and $\mu_2(\cdot)$ are twice continuously differentiable with Lipschitz continuous second derivatives. It is easy to see that assumptions (A5.1) and (A5.2) are satisfied. Therefore the desired asymptotic normality follows.

In this example,

$$\nu(t) = (\nu_1(t), \nu_2(t)) = \left(\frac{\mu_2(t)}{\mu_1(t) + \mu_2(t)}, \frac{\mu_1(t)}{\mu_1(t) + \mu_2(t)} \right).$$

Moreover,

$$Q_0(s, t_0) = -\frac{\exp(-(\mu_1(t_0) + \mu_2(t_0))s)}{\mu_1(t_0) + \mu_2(t_0)} Q(t_0).$$

Thus,

$$A(t) = \frac{2\mu_1(t)\mu_2(t)}{(\mu_1(t) + \mu_2(t))^3} \begin{pmatrix} (\beta_1(t))^2 & -\beta_1(t)\beta_2(t) \\ -\beta_1(t)\beta_2(t) & (\beta_2(t))^2 \end{pmatrix}.$$

5.2.5 Extensions

In this section, we generalize our results in the previous sections including asymptotic expansions, asymptotic normality, and exponential bounds, to the Markov chain $\alpha^\varepsilon(\cdot)$ with generator given by $Q^\varepsilon(t) = Q(t)/\varepsilon + \widehat{Q}(t)$ with

weakly irreducible generator $Q(t)$. Recall that the vector of probabilities $p^\varepsilon(t) = (P(\alpha^\varepsilon(t) = 1), \ldots, P(\alpha^\varepsilon(t) = m))$ satisfies the differential equation

$$\frac{dp^\varepsilon(t)}{dt} = p^\varepsilon(t)Q^\varepsilon(t),\ p^\varepsilon(t) \in \mathbb{R}^m,$$
$$p^\varepsilon(0) = p^0 \text{ with } p_i^0 \geq 0 \text{ for } i \in \mathcal{M} \text{ and } \sum_{i=1}^m p_i^0 = 1,$$

To proceed, the following conditions are needed.

(**A5.3**) Both $Q(t)$ and $\widehat{Q}(t)$ are generators. For each $t \in [0, T]$, $Q(t)$ is weakly irreducible.

(**A5.4**) For some positive integer n_0, $Q(\cdot)$ is $(n_0 + 1)$-times continuously differentiable on $[0, T]$ and $(d^{n_0+1}/dt^{n_0+1})Q(\cdot)$ is Lipschitz. Moreover, $\widehat{Q}(\cdot)$ is n_0-times continuously differentiable on $[0, T]$ and $(d^{n_0}/dt^{n_0})\widehat{Q}(\cdot)$ is Lipschitz.

Similarly to Section 4.2 for $k = 1, \ldots, n_0 + 1$, the outer expansions lead to equations

$$\begin{aligned}
&\varepsilon^0 :\ \varphi_0(t)Q(t) = 0,\\
&\varepsilon^1 :\ \varphi_1(t)Q(t) + \varphi_0(t)\widehat{Q}(t) = \frac{d\varphi_0(t)}{dt},\\
&\qquad \ldots\\
&\varepsilon^k :\ \varphi_k(t)Q(t) + \varphi_{k-1}(t)\widehat{Q}(t) = \frac{d\varphi_{k-1}(t)}{dt},
\end{aligned} \tag{5.44}$$

with constraints

$$\sum_{i=1}^m \varphi_{0,i}(t) = 1$$

and

$$\sum_{i=1}^m \varphi_{k,i}(t) = 0,\ \text{for } k \geq 1.$$

The initial-layer correction terms are

$$\begin{aligned}
\varepsilon^0: &\ \frac{d\psi_0(\tau)}{d\tau} = \psi_0(\tau)Q(0), \\
\varepsilon^1: &\ \frac{d\psi_1(\tau)}{d\tau} = \psi_1(\tau)Q(0) + \psi_0(\tau)\left(\tau\frac{dQ(0)}{dt} + \widehat{Q}(0)\right), \\
&\ \cdots \\
\varepsilon^k: &\ \frac{d\psi_k(\tau)}{d\tau} = \psi_k(\tau)Q(0) + r_k(\tau),
\end{aligned} \tag{5.45}$$

where

$$r_k(\tau) = \sum_{i=1}^{k} \psi_{k-i}(\tau)\left(\frac{\tau^i}{i!}\frac{d^iQ(0)}{dt^i} + \frac{\tau^{i-1}}{(i-1)!}\frac{d^{i-1}\widehat{Q}(0)}{dt^{i-1}}\right),$$

with initial conditions

$$\psi_0(0) = p^0 - \varphi_0(0), \text{ and}$$

$$\psi_k(0) = -\varphi_k(0) \text{ for } k \geq 1.$$

Theorem 5.19. *Suppose that* (A5.3) *and* (A5.4) *are satisfied. Then*

(a) $\varphi_i(\cdot)$ *is* $(n_0 + 1 - i)$*-times continuously differentiable on* $[0, T]$,

(b) *for each* i, *there is a* $\widehat{\kappa} > 0$ *such that*

$$\left|\psi_i\left(\frac{t}{\varepsilon}\right)\right| \leq K\exp\left(-\frac{\widehat{\kappa}t}{\varepsilon}\right), \text{ and}$$

(c) *the approximation error satisfies*

$$\sup_{t\in[0,T]}\left|p^\varepsilon(t) - \sum_{i=0}^{n_0}\varepsilon^i\varphi_i(t) - \sum_{i=0}^{n_0}\varepsilon^i\psi_i\left(\frac{t}{\varepsilon}\right)\right| \leq K\varepsilon^{n_0+1}. \tag{5.46}$$

The proof of this theorem is similar to those of Theorem 4.5, and is thus omitted. We also omit the proofs of the following two theorems because they are similar to that of Theorem 5.4 and Theorem 5.9, respectively.

Theorem 5.20. *Suppose* (A5.3) *and* (A5.4) *are satisfied with* $n_0 = 0$. *Then there exist positive constants* ε_0 *and* K *such that for* $0 < \varepsilon \leq \varepsilon_0$, $i \in \mathcal{M}$,

and for any deterministic process $\beta_i(\cdot)$ *satisfying* $|\beta_i(t)| \leq 1$ *for all* $t \geq 0$, *we have*

$$E \exp\left\{ \frac{\theta_T}{(T+1)^{\frac{3}{2}}} \sup_{0 \leq t \leq T} |n^\varepsilon(t)| \right\} \leq K,$$

where θ_T *and* $n^\varepsilon(\cdot)$ *are as defined previously.*

Corollary 5.21. *Consider* $Q^\varepsilon = Q/\varepsilon + \widehat{Q}$ *with constant generators* Q *and* $\widehat{Q}$ *such that* Q *is weakly irreducible. Then* (5.25) *and* (5.27) *hold with constants* K *and* K_j *independent of* T.

Theorem 5.22. *Suppose* (A5.3) *and* (A5.4) *are satisfied with* $n_0 = 1$. *Then for* $t \in [0, T]$, *the process* $n^\varepsilon(\cdot)$ *converges weakly to a Gaussian process* $n(\cdot)$ *such that*

$$En(t) = 0 \text{ and } E[n'(t)n(t)] = \int_0^t A(s)ds,$$

where $A(t) = (A_{ij}(t))$ *with*

$$A_{ij}(t) = \beta_i(t)\beta_j(t)\left[\nu_i(t) \int_0^\infty q_{0,ij}(r,t)dr + \nu_j(t) \int_0^\infty q_{0,ji}(r,t)dr\right],$$

and $Q_0(r,t) = (q_{0,ij}(r,t))$ *satisfying*

$$\frac{dQ_0(r,t)}{dr} = Q_0(r,t)Q(t), \ r \geq 0,$$

$$Q_0(0,t) = I - P_0(t),$$

with $P_0(t) = (\nu'(t), \ldots, \nu'(t))'$.

Remark 5.23. In view of Theorem 5.22, the asymptotic covariance is determined by the quasi-stationary distribution $\nu(t)$ and $Q_0(r,t)$. Both $\nu(t)$ and $Q_0(r,t)$ are determined by $Q(t)$, the dominating term in $Q^\varepsilon(t)$. In the asymptotic normality analysis, it is essential to have the irreducibility condition of $Q(t)$, whereas the role of $\widehat{Q}(t)$ is not as important. If $Q(t)$ is weakly irreducible, then there exists an $\varepsilon_0 > 0$ such that $Q^\varepsilon(t) = Q(t)/\varepsilon + \widehat{Q}(t)$ is weakly irreducible for $0 < \varepsilon \leq \varepsilon_0$, as shown in Sethi and Zhang [192, Lemma J.10].

By introducing another generator $\widehat{Q}(t)$, we are dealing with a singularly perturbed Markovian system with fast and slow motions. Nevertheless, the entire system under consideration is still weakly irreducible. This irreducibility allows us to extend our previous results with minor modifications.

Although most of the results in this section can be extended to the case with $Q^\varepsilon(t) = Q(t)/\varepsilon + \widehat{Q}(t)$, there are some exceptions. For example, Corollary 5.6 would not go through because even with constant matrix $\widehat{Q}(t) = \widehat{Q}$, $\varphi_1(t)$ in Lemma 5.1 does not equal 0 when $\widehat{Q} \neq 0$.

One may wonder what happens if $Q(t)$ in $Q^\varepsilon(t)$ is *not* weakly irreducible. In particular, one can consider the case in which $Q(t)$ consists of several blocks of irreducible submatrices. Related results of asymptotic normality and the exponential bounds are treated in subsequent sections.

5.3 Markov Chains with Weak and Strong Interactions

For brevity, unless otherwise noted, in the rest of the book, whenever the phrase "weak and strong interaction" is used, it refers to the case of two-time-scale Markov chains with all states being recurrent. Similar approaches can be used for the other cases as well. The remainder of the chapter concentrates on exploiting detailed structures of the weak and strong interactions. In addition, it deals with convergence of the probability distribution with merely measurable generators.

We continue our investigation of asymptotic properties of the Markov chain $\alpha^\varepsilon(\cdot)$ generated by $Q^\varepsilon(\cdot)$, with

$$Q^\varepsilon(t) = \frac{1}{\varepsilon}\widetilde{Q}(t) + \widehat{Q}(t), \text{ for } t \geq 0, \tag{5.47}$$

where $\widetilde{Q}(t) = \text{diag}(\widetilde{Q}^1(t), \ldots, \widetilde{Q}^l(t))$ is a block-diagonal matrix such that $\widehat{Q}(t)$ and $\widetilde{Q}^k(t)$, for $k = 1, \ldots, l$, are themselves generators. The state space of $\alpha^\varepsilon(\cdot)$ is given by

$$\mathcal{M} = \Big\{s_{11}, \ldots, s_{1m_1}, \ldots, s_{l1}, \ldots, s_{lm_l}\Big\}.$$

For each $k = 1, \ldots, l$, let $\mathcal{M}_k = \{s_{k1}, \ldots, s_{km_k}\}$, representing the group of states corresponding to $\widetilde{Q}^k(t)$.

The results in Section 5.3.1 reveal the structures of the Markov chains with weak and strong interactions based on the following observations. Intuitively, for small ε, the Markov chain $\alpha^\varepsilon(\cdot)$ jumps more frequently within the states in $\mathcal{M}_k$ and less frequently from $\mathcal{M}_k$ to $\mathcal{M}_j$ for $j \neq k$. Therefore, the states in $\mathcal{M}_k$ can be aggregated and represented by a single state k (one may view the state k as a super state). That is, one can approximate $\alpha^\varepsilon(\cdot)$ by an aggregated process, say, $\overline{\alpha}^\varepsilon(\cdot)$. Furthermore, by examining the tightness and finite-dimensional distribution of $\overline{\alpha}^\varepsilon(\cdot)$, it will be shown that $\overline{\alpha}^\varepsilon(\cdot)$ converges weakly to a Markov chain $\overline{\alpha}(\cdot)$ generated by

$$\overline{Q}(t) = \text{diag}(\nu^1(t), \ldots, \nu^l(t))\widehat{Q}(t)\text{diag}(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_l}). \tag{5.48}$$

Section 5.3.2 continues the investigation along the line of estimating the error bounds of the approximation. Our interest lies in finding how closely one can approximate an unscaled sequence of occupation measures. The study is through the examination of appropriate exponential-type bounds. To take a suitable scaled sequence, one first centers the sequence around the "mean," and then compares the actual sequence of occupation measures with this "mean." In contrast to the results of Section 5.2, in lieu of taking the difference of the occupation measure with that of a deterministic function, it is compared with a random process. One of the key points here is the utilization of solutions of linear time-varying stochastic differential equations, in which the stochastic integration is with respect to a square-integrable martingale.

In comparison with the central limit theorem obtained in Section 5.2, it is interesting to know whether these results still hold under the structure of weak and strong interactions. The answer to this question is in Section 5.3.3, which also contains further study on related scaled sequences of occupation measures. The approach is quite different from that of Section 5.2. We use the martingale formulation and apply the techniques of perturbed test functions. It is interesting to note that the limit process is a switching diffusion process, which does not have independent increments. When the generator is weakly irreducible as in Section 5.2, the motion of jumping around the grouped states disappears and the diffusion becomes the dominant force.

We have considered only Markov chains with smooth generators up to now. However, there are cases in certain applications in which the generators may be merely measurable. Section 5.4 takes care of the scenario in which the Markov chains are governed by generators that are only measurable. Formulation via weak derivatives is also discussed briefly. Finally the chapter is concluded with a few more remarks. Among other things, additional references are given.

5.3.1 Aggregation of Markov Chains

This section deals with an aggregation of $\alpha^\varepsilon(\cdot)$. The following assumptions will be needed:

(**A5.5**) For each $k = 1, \ldots, l$ and $t \in [0, T]$, $\widetilde{Q}^k(t)$ is weakly irreducible.

(**A5.6**) $\widetilde{Q}(\cdot)$ is differentiable on $[0, T]$ and its derivative is Lipschitz. Moreover, $\widehat{Q}(\cdot)$ is also Lipschitz.

The assumptions above guarantee the existence of an asymptotic expansion up to zeroth order. To prepare for the subsequent study, we first provide the following error estimate. Since only the zeroth-order expansion is needed here, the estimate is confined to such an approximation. Higher-order terms can be obtained in a similar way.

Lemma 5.24. *Assume* (A5.5) *and* (A5.6). *Let* $P^\varepsilon(t,t_0)$ *denote the transition probability of* $\alpha^\varepsilon(\cdot)$. *Then for some* $\kappa_0>0$,

$$P^\varepsilon(t,t_0)=P_0(t,t_0)+O\left(\varepsilon+\exp\left(-\frac{\kappa_0(t-t_0)}{\varepsilon}\right)\right),$$

where

$$\begin{aligned}P_0(t,t_0)&=\widetilde{\mathbb{1}}\Theta(t,t_0)\mathrm{diag}(\nu^1(t),\dots,\nu^l(t))\\&=\begin{pmatrix}\mathbb{1}_{m_1}\nu^1(t)\vartheta_{11}(t,t_0), & \dots & ,\mathbb{1}_{m_1}\nu^l(t)\vartheta_{1l}(t,t_0)\\ \vdots & \dots & \vdots\\ \mathbb{1}_{m_l}\nu^1(t)\vartheta_{l1}(t,t_0), & \dots & ,\mathbb{1}_{m_l}\nu^l(t)\vartheta_{ll}(t,t_0)\end{pmatrix},\end{aligned}\tag{5.49}$$

where $\nu^k(t)$ *is the quasi-stationary distribution of* $\widetilde{Q}^k(t)$, *and* $\Theta(t,t_0)=(\vartheta_{ij}(t,t_0))\in\mathbb{R}^{l\times l}$ *is the solution to the following initial value problem:*

$$\begin{aligned}&\frac{d\Theta(t,t_0)}{dt}=\Theta(t,t_0)\overline{Q}(t),\\&\Theta(t_0,t_0)=I.\end{aligned}\tag{5.50}$$

Proof: The proof is similar to those of Lemma 5.1 and Theorem 4.29, except that the notation is more involved. □

Define an aggregated process of $\alpha^\varepsilon(\cdot)$ on $[0,T]$ by

$$\overline{\alpha}^\varepsilon(t)=k \text{ if } \alpha^\varepsilon(t)\in\mathcal{M}_k.\tag{5.51}$$

The idea to follow is to treat a related Markov chain having only l states. The transitions among its states correspond to the jumps from one group $\mathcal{M}_k$ to another $\mathcal{M}_j$, $j\neq k$, in the original Markov chain.

Theorem 5.25. *Assume* (A5.5) *and* (A5.6). *Then, for any* $i=1,\dots,l$, $j=1,\dots,m_i$, *and bounded and measurable deterministic function* $\beta_{ij}(\cdot)$,

$$E\left(\int_0^T\left(I_{\{\alpha^\varepsilon(t)=s_{ij}\}}-\nu_j^i(t)I_{\{\overline{\alpha}^\varepsilon(t)=i\}}\right)\beta_{ij}(t)dt\right)^2=O(\varepsilon).$$

Proof: For any i,j and $0\le t\le T$, let

$$\eta^\varepsilon(t)=E\left(\int_0^t\left(I_{\{\alpha^\varepsilon(r)=s_{ij}\}}-\nu_j^i(r)I_{\{\overline{\alpha}^\varepsilon(r)=i\}}\right)\beta_{ij}(r)dr\right)^2.\tag{5.52}$$

We have suppressed the i, j dependence of $\eta^\varepsilon(\cdot)$ for notational simplicity. Loosely speaking, the argument used below is a Liapunov stability one, and $\eta^\varepsilon(\cdot)$ can be viewed as a Liapunov function. By differentiating $\eta^\varepsilon(\cdot)$, we have

$$\begin{aligned}\frac{d\eta^\varepsilon(t)}{dt} = 2E\Bigg[&\left(\int_0^t \left(I_{\{\alpha^\varepsilon(r)=s_{ij}\}} - \nu_j^i(r) I_{\{\overline{\alpha}^\varepsilon(r)=i\}}\right)\beta_{ij}(r)dr\right)\\ &\times \left(I_{\{\alpha^\varepsilon(t)=s_{ij}\}} - \nu_j^i(t) I_{\{\overline{\alpha}^\varepsilon(t)=i\}}\right)\beta_{ij}(t)\Bigg].\end{aligned}$$

The definition of $\overline{\alpha}^\varepsilon(\cdot)$ yields that $\{\overline{\alpha}^\varepsilon(t) = i\} = \{\alpha^\varepsilon(t) \in \mathcal{M}_i\}$. Thus,

$$\frac{d\eta^\varepsilon(t)}{dt} = 2\int_0^t \Phi^\varepsilon(t,r)\beta_{ij}(t)\beta_{ij}(r)dr,$$

where $\Phi^\varepsilon(t,r) = \Phi_1^\varepsilon(t,r) + \Phi_2^\varepsilon(t,r)$ with

$$\begin{aligned}\Phi_1^\varepsilon(t,r) = &\ P(\alpha^\varepsilon(t) = s_{ij}, \alpha^\varepsilon(r) = s_{ij})\\ &-\nu_j^i(t)P(\alpha^\varepsilon(t) \in \mathcal{M}_i, \alpha^\varepsilon(r) = s_{ij}),\end{aligned} \tag{5.53}$$

and

$$\begin{aligned}\Phi_2^\varepsilon(t,r) = &-\nu_j^i(r)P(\alpha^\varepsilon(t) = s_{ij}, \alpha^\varepsilon(r) \in \mathcal{M}_i)\\ &+\nu_j^i(r)\nu_j^i(t)P(\alpha^\varepsilon(t) \in \mathcal{M}_i, \alpha^\varepsilon(r) \in \mathcal{M}_i).\end{aligned} \tag{5.54}$$

Note that the Markov property of $\alpha^\varepsilon(\cdot)$ implies that for $0 \le r \le t$,

$$\begin{aligned}&P(\alpha^\varepsilon(t) = s_{ij}, \alpha^\varepsilon(r) = s_{ij})\\ &\quad = P(\alpha^\varepsilon(t) = s_{ij} | \alpha^\varepsilon(r) = s_{ij})P(\alpha^\varepsilon(r) = s_{ij}).\end{aligned}$$

In view of the asymptotic expansion, we have

$$\begin{aligned}&P(\alpha^\varepsilon(t) = s_{ij} | \alpha^\varepsilon(r) = s_{ij})\\ &\quad = \nu_j^i(t)\vartheta_{ii}(t,r) + O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-r)}{\varepsilon}\right)\right).\end{aligned} \tag{5.55}$$

It follows that

$$\begin{aligned} &P(\alpha^\varepsilon(t) \in \mathcal{M}_i | \alpha^\varepsilon(r) = s_{ij}) \\ &\quad = \sum_{k=1}^{m_i} \nu_k^i(t) \vartheta_{ii}(t,r) + O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-r)}{\varepsilon}\right)\right) \\ &\quad = \vartheta_{ii}(t,r) + O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-r)}{\varepsilon}\right)\right). \end{aligned} \tag{5.56}$$

Combining (5.55) and (5.56) leads to

$$\Phi_1^\varepsilon(t,r) = O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-r)}{\varepsilon}\right)\right).$$

Similarly, we can show that

$$\Phi_2^\varepsilon(t,r) = O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-r)}{\varepsilon}\right)\right),$$

by noting that

$$\begin{aligned} \Phi_2^\varepsilon(t,r) = &-\nu_j^i(r) \sum_{k=1}^{m_i} P(\alpha^\varepsilon(t) = s_{ij}, \alpha^\varepsilon(r) = s_{ik}) \\ &+\nu_j^i(r) \sum_{k=1}^{m_i} \nu_j^i(t) P(\alpha^\varepsilon(t) \in \mathcal{M}_i, \alpha^\varepsilon(r) = s_{ik}) \end{aligned}$$

and

$$\begin{aligned} &P(\alpha^\varepsilon(t) = s_{ij} | \alpha^\varepsilon(r) = s_{ik}) \\ &\quad = \nu_j^i(t) \vartheta_{ii}(t,r) + O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-r)}{\varepsilon}\right)\right), \end{aligned}$$

for any $k = 1, \ldots, m_i$. Therefore,

$$\frac{d\eta^\varepsilon(t)}{dt} = 2 \int_0^t O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-r)}{\varepsilon}\right)\right) dr = O(\varepsilon). \tag{5.57}$$

This together with $\eta^\varepsilon(0) = 0$ implies that $\eta^\varepsilon(t) = O(\varepsilon)$. □

Theorem 5.25 indicates that $\nu^k(t)$ together with $\overline{\alpha}^\varepsilon(\cdot)$ approximates well the Markov chain $\alpha^\varepsilon(\cdot)$ in an appropriate sense. Nevertheless, in general, $\{\alpha^\varepsilon(\cdot)\}$ is not tight. The following example provides a simple illustration.

Example 5.26. Let $\alpha^\varepsilon(\cdot) \in \{1,2\}$ denote a Markov chain generated by

$$\frac{1}{\varepsilon}\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix},$$

for some $\lambda, \mu > 0$. Then $\alpha^\varepsilon(\cdot)$ is *not* tight.

Proof: If $\alpha^\varepsilon(\cdot)$ is tight, then there exists a sequence $\varepsilon_k \to 0$ such that $\alpha^{\varepsilon_k}(\cdot)$ converges weakly to a stochastic process $\alpha(\cdot) \in D([0,T];\mathcal{M})$. In view of the Skorohod representation (without changing notation for simplicity), Theorem A.11, we may assume $\alpha^{\varepsilon_k}(\cdot) \to \alpha(\cdot)$ w.p.1. It follows from Lemma A.41 that

$$E\left|\int_0^t \alpha^{\varepsilon_k}(s)ds - \int_0^t \alpha(s)ds\right|^2 \to 0,$$

for all $t \in [0,T]$. Moreover, similarly as in Theorem 5.25, we obtain

$$E\left|\int_0^t \alpha^{\varepsilon_k}(s)ds - \int_0^t (\nu_1 + 2\nu_2)ds\right|^2 \to 0,$$

where (ν_1, ν_2) is the stationary distribution of $\alpha^\varepsilon(\cdot)$ and $\nu_1 + 2\nu_2$ is the mean with respect to the stationary distribution. As a consequence, it follows that $\alpha(t) = \nu_1 + 2\nu_2$ for all $t \in [0,T]$ w.p.1. Let

$$\delta_0 = \min\{|1 - (\nu_1 + 2\nu_2)|, |2 - (\nu_1 + 2\nu_2)|\} > 0.$$

Then for $t \in [0,T]$,

$$|\alpha^\varepsilon(t) - (\nu_1 + 2\nu_2)| \geq \delta_0.$$

Hence, under the Skorohod topology

$$d(\alpha^{\varepsilon_k}(\cdot), \nu_1 + 2\nu_2) \geq \delta_0.$$

This contradicts the fact that $\alpha^{\varepsilon_k}(\cdot) \to \alpha(\cdot) = \nu_1 + 2\nu_2$ w.p.1. Therefore, $\alpha^\varepsilon(\cdot)$ cannot be tight. □

Although $\alpha^\varepsilon(\cdot)$ is not tight because it fluctuates in $\mathcal{M}_k$ very rapidly for small ε, its aggregation $\overline{\alpha}^\varepsilon(\cdot)$ is tight, and converges weakly to $\overline{\alpha}(t)$, $t \geq 0$, a Markov chain generated by $\overline{Q}(t)$, $t \geq 0$, where $\overline{Q}(t)$ is defined in (5.48). The next theorem shows that $\overline{\alpha}^\varepsilon(\cdot)$ can be further approximated by $\overline{\alpha}(\cdot)$.

Theorem 5.27. *Assume* (A5.5) *and* (A5.6). *Then* $\overline{\alpha}^\varepsilon(\cdot)$ *converges weakly to* $\overline{\alpha}(\cdot)$ *in* $D([0,T];\overline{\mathcal{M}})$, *as* $\varepsilon \to 0$.

Proof: The proof is divided into two steps. First, we show that $\overline{\alpha}^\varepsilon(\cdot)$ defined in (5.51) is tight in $D([0,T];\overline{\mathcal{M}})$. The definition of $\overline{\alpha}^\varepsilon(\cdot)$ implies that

$$\{\overline{\alpha}^\varepsilon(t) = i\} = \{\alpha^\varepsilon(t) \in \mathcal{M}_i\} = \{\alpha^\varepsilon(t) = s_{ij} \text{ for some } j = 1, \ldots, m_i\}.$$

Consider the conditional expectation

$$\begin{aligned}
&E\left[(\overline{\alpha}^{\varepsilon}(t+s)-\overline{\alpha}^{\varepsilon}(s))^2\Big|\alpha^{\varepsilon}(s)=s_{ij}\right]\\
&\quad=E\left[(\overline{\alpha}^{\varepsilon}(t+s)-i)^2\Big|\alpha^{\varepsilon}(s)=s_{ij}\right]\\
&\quad=\sum_{k=1}^{l}E\left[(\overline{\alpha}^{\varepsilon}(t+s)-i)^2\,I_{\{\overline{\alpha}^{\varepsilon}(t+s)=k\}}\Big|\alpha^{\varepsilon}(s)=s_{ij}\right]\\
&\quad=\sum_{k=1}^{l}(k-i)^2P(\overline{\alpha}^{\varepsilon}(t+s)=k|\alpha^{\varepsilon}(s)=s_{ij})\\
&\quad\le l^2\sum_{k\neq i}P(\overline{\alpha}^{\varepsilon}(t+s)=k|\alpha^{\varepsilon}(s)=s_{ij}).
\end{aligned}$$

Since $\{\overline{\alpha}^{\varepsilon}(t+s)=k\}=\{\alpha^{\varepsilon}(t+s)\in\mathcal{M}_k\}$, it follows that

$$\begin{aligned}
&P(\overline{\alpha}^{\varepsilon}(t+s)=k|\alpha^{\varepsilon}(s)=s_{ij})\\
&\quad=\sum_{k_1=1}^{m_k}P(\alpha^{\varepsilon}(t+s)=s_{kk_1}|\alpha^{\varepsilon}(s)=s_{ij})\\
&\quad=\sum_{k_1=1}^{m_k}\nu_{k_1}^{k}(t+s)\vartheta_{ik}(t+s,s)+O\left(\varepsilon+\exp\left(-\frac{\kappa_0 t}{\varepsilon}\right)\right)\\
&\quad=\vartheta_{ik}(t+s,s)+O\left(\varepsilon+\exp\left(-\frac{\kappa_0 t}{\varepsilon}\right)\right).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&E\left[(\overline{\alpha}^{\varepsilon}(t+s)-\overline{\alpha}^{\varepsilon}(s))^2\Big|\alpha^{\varepsilon}(s)=s_{ij}\right]\\
&\qquad\le l^2\sum_{k\neq i}\vartheta_{ik}(t+s,s)+O\left(\varepsilon+\exp\left(-\frac{\kappa_0 t}{\varepsilon}\right)\right).
\end{aligned}$$

Note that $\lim_{t\to 0}\vartheta_{ik}(t+s,s)=0$ for $i\neq k$.

$$\lim_{t\to 0}\left(\lim_{\varepsilon\to 0}E\left((\overline{\alpha}^{\varepsilon}(t+s)-\overline{\alpha}^{\varepsilon}(s))^2|\alpha^{\varepsilon}(s)=s_{ij}\right)\right)=0.$$

Thus, the Markov property of $\alpha^{\varepsilon}(\cdot)$ implies

$$\lim_{t\to 0}\left(\lim_{\varepsilon\to 0}E\left((\overline{\alpha}^{\varepsilon}(t+s)-\overline{\alpha}^{\varepsilon}(s))^2|\alpha^{\varepsilon}(r):\ r\le s\right)\right)=0. \tag{5.58}$$

Recall that $\overline{\alpha}^{\varepsilon}(\cdot)$ is bounded. The tightness of $\overline{\alpha}^{\varepsilon}(\cdot)$ follows from Kurtz' tightness criterion (see Lemma A.17).

To complete the proof, it remains to show that the finite-dimensional distributions of $\overline{\alpha}^\varepsilon(\cdot)$ converge to that of $\overline{\alpha}(\cdot)$. In fact, for any

$$0 \le t_1 < t_2 < \cdots < t_n \le T \text{ and } i_1, i_2, \ldots, i_n \in \overline{\mathcal{M}} = \{1, \ldots, l\},$$

we have

$$\begin{aligned}
&P(\overline{\alpha}^\varepsilon(t_n) = i_n, \ldots, \overline{\alpha}^\varepsilon(t_1) = i_1) \\
&= P(\alpha^\varepsilon(t_n) \in \mathcal{M}_{i_n}, \ldots, \alpha^\varepsilon(t_1) \in \mathcal{M}_{i_1}) \\
&= \sum_{j_1,\ldots,j_n} P(\alpha^\varepsilon(t_n) = s_{i_n j_n}, \ldots, \alpha^\varepsilon(t_1) = s_{i_1 j_1}) \\
&= \sum_{j_1,\ldots,j_n} P(\alpha^\varepsilon(t_n) = s_{i_n j_n} | \alpha^\varepsilon(t_{n-1}) = s_{i_{n-1} j_{n-1}}) \\
&\qquad \times \cdots \times P(\alpha^\varepsilon(t_2) = s_{i_2 j_2} | \alpha^\varepsilon(t_1) = s_{i_1 j_1}) P(\alpha^\varepsilon(t_1) = s_{i_1 j_1}).
\end{aligned}$$

In view of Lemma 5.24, for each k, we have

$$P(\alpha^\varepsilon(t_k) = s_{i_k j_k} | \alpha^\varepsilon(t_{k-1}) = s_{i_{k-1} j_{k-1}}) \to \nu_{j_k}^{i_k}(t_k) \vartheta_{i_{k-1} i_k}(t_k, t_{k-1}).$$

Moreover, note that

$$\sum_{j_k=1}^{m_{i_k}} \nu_{j_k}^{i_k}(t_k) = 1.$$

It follows that

$$\begin{aligned}
&\sum_{j_1,\ldots,j_n} P(\alpha^\varepsilon(t_n) = s_{i_n j_n} | \alpha^\varepsilon(t_{n-1}) = s_{i_{n-1} j_{n-1}}) \\
&\qquad \times \cdots \times P(\alpha^\varepsilon(t_2) = s_{i_2 j_2} | \alpha^\varepsilon(t_1) = s_{i_1 j_1}) P(\alpha^\varepsilon(t_1) = s_{i_1 j_1}) \\
&\to \sum_{j_1,\ldots,j_n} \nu_{j_n}^{i_n}(t_n) \vartheta_{i_{n-1} i_n}(t_n, t_{n-1}) \cdots \nu_{j_2}^{i_2}(t_2) \vartheta_{i_1 i_2}(t_2, t_1) \nu_{j_1}^{i_1}(t_1) \widetilde{\vartheta}_{i_1}(t_1) \\
&= \vartheta_{i_{n-1} i_n}(t_n, t_{n-1}) \cdots \vartheta_{i_1 i_2}(t_2, t_1) \widetilde{\vartheta}_{i_1}(t_1) \\
&= P(\overline{\alpha}(t_n) = i_n, \ldots, \overline{\alpha}(t_1) = i_1),
\end{aligned}$$

where $\sum_{j_1,\ldots,j_n} = \sum_{j_1=1}^{m_{i_1}} \cdots \sum_{j_n=1}^{m_{i_n}}$ and $\widetilde{\vartheta}_{i_1}(t_1)$ denotes the initial distribution (also known as absolute probability in the literature of Markov chains). Thus, $\overline{\alpha}^\varepsilon(\cdot) \to \overline{\alpha}(\cdot)$ in distribution. □

This theorem implies that $\overline{\alpha}^\varepsilon(\cdot)$ converges to a Markov chain, although $\overline{\alpha}^\varepsilon(\cdot)$ itself is not a Markov chain in general. If, however, the generator $Q^\varepsilon(t)$

has some specific structure, then $\overline{\alpha}^\varepsilon(\cdot)$ is a Markov chain. The following example demonstrates this point.

Example 5.28. Let $\widetilde{Q}(t) = (\widetilde{q}_{ij}(t))$ and $\overline{Q}(t) = (\overline{q}_{ij}(t))$ denote generators with the corresponding state spaces $\{a_1, \ldots, a_{m_0}\}$ and $\{1, \ldots, l\}$, respectively. Consider

$$Q^\varepsilon(t) = \frac{1}{\varepsilon}\begin{pmatrix} \widetilde{Q}(t) & & \\ & \ddots & \\ & & \widetilde{Q}(t) \end{pmatrix} + \begin{pmatrix} \overline{q}_{11}(t)I_{m_0} & \cdots & \overline{q}_{1l}(t)I_{m_0} \\ \vdots & \vdots & \vdots \\ \overline{q}_{l1}(t)I_{m_0} & \cdots & \overline{q}_{ll}(t)I_{m_0} \end{pmatrix}, \quad (5.59)$$

where I_{m_0} is the $m_0 \times m_0$ identity matrix. In this case

$$m_1 = m_2 = \cdots = m_l = m_0.$$

Then $\overline{\alpha}^\varepsilon(\cdot)$ is a Markov chain generated by $\overline{Q}(t)$. In fact, let

$$\chi^\varepsilon(t) = \Big(I_{\{\alpha^\varepsilon(t)=s_{11}\}}, \ldots, I_{\{\alpha^\varepsilon(t)=s_{1m_0}\}}, \ldots, I_{\{\alpha^\varepsilon(t)=s_{l1}\}}, \ldots, I_{\{\alpha^\varepsilon(t)=s_{lm_0}\}}\Big).$$

Note that $s_{ij} = (i, a_j)$ for $j = 1, \ldots, m_0$ and $i = 1, \ldots, l$. In view of Lemma 2.4, we obtain that

$$\chi^\varepsilon(t) - \int_0^t \chi^\varepsilon(s)Q^\varepsilon(s)ds \quad (5.60)$$

is a martingale. Postmultiplying (multiplying from the right) (5.60) by

$$\widetilde{\mathbb{1}} = \operatorname{diag}(\mathbb{1}_{m_0}, \ldots, \mathbb{1}_{m_0})$$

and noting that $\{\overline{\alpha}^\varepsilon(t) = i\} = \{\alpha^\varepsilon(t) \in \mathcal{M}_i\}$ and

$$\chi^\varepsilon(t)\widetilde{\mathbb{1}} = (I_{\{\overline{\alpha}^\varepsilon(t)=1\}}, \ldots, I_{\{\overline{\alpha}^\varepsilon(t)=l\}}),$$

we obtain that

$$\big(I_{\{\overline{\alpha}^\varepsilon(t)=1\}}, \ldots, I_{\{\overline{\alpha}^\varepsilon(t)=l\}}\big) - \int_0^t \chi^\varepsilon(s)Q^\varepsilon(s)ds\widetilde{\mathbb{1}}$$

is still a martingale. In view of the special structure of $Q^\varepsilon(t)$ in (5.59),

$$\widetilde{Q}(t)\mathbb{1}_{m_0} = 0, \quad Q^\varepsilon(t)\widetilde{\mathbb{1}} = \widehat{Q}(t)\widetilde{\mathbb{1}},$$

and

$$\chi^\varepsilon(s)\widehat{Q}(s)\widetilde{\mathbb{1}} = \big(I_{\{\overline{\alpha}^\varepsilon(s)=1\}}, \ldots, I_{\{\overline{\alpha}^\varepsilon(s)=l\}}\big)\,\overline{Q}(s).$$

Therefore, (5.60) implies that

$$\big(I_{\{\overline{\alpha}^\varepsilon(t)=1\}}, \ldots, I_{\{\overline{\alpha}^\varepsilon(t)=l\}}\big) - \int_0^t \big(I_{\{\overline{\alpha}^\varepsilon(s)=1\}}, \ldots, I_{\{\overline{\alpha}^\varepsilon(s)=l\}}\big)\,\overline{Q}(s)ds$$

is a martingale. This implies, in view of Lemma 2.4, that $\overline{\alpha}^\varepsilon(\cdot)$ is a Markov chain generated by $\overline{Q}(t)$, $t \geq 0$.

5.3.2 Exponential Bounds

For each $i = 1, \dots, l$, $j = 1, \dots, m_i$, $\alpha \in \mathcal{M}$, and $t \geq 0$, let $\beta_{ij}(t)$ be a bounded, Borel measurable, deterministic function and let

$$W_{ij}(t, \alpha) = \left(I_{\{\alpha = s_{ij}\}} - \nu_j^i(t) I_{\{\alpha \in \mathcal{M}_i\}}\right) \beta_{ij}(t). \tag{5.61}$$

Consider normalized occupation measures

$$n^\varepsilon(t) = \left(n_{11}^\varepsilon(t), \dots, n_{1m_1}^\varepsilon(t), \dots, n_{l1}^\varepsilon(t), \dots, n_{lm_l}^\varepsilon(t)\right),$$

where

$$n_{ij}^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t W_{ij}(s, \alpha^\varepsilon(s)) ds.$$

In this section, we establish the exponential error bound for $n^\varepsilon(\cdot)$, a sequence of suitably scaled occupation measures for the singularly perturbed Markov chains with weak and strong interactions.

In view of Theorem 4.29, there exists $\kappa_0 > 0$ such that

$$\left|P^\varepsilon(t, s) - P_0(t, s)\right| = O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-s)}{\varepsilon}\right)\right). \tag{5.62}$$

Similar to Section 5.3.2, for fixed but otherwise arbitrary $T > 0$, let

$$K_T = \max\left\{1, \sup_{0 \leq s \leq t \leq T} \left(\frac{|P^\varepsilon(t, s) - P_0(t, s)|}{\varepsilon + \exp(-\kappa_0(t-s)/\varepsilon)}\right)\right\}. \tag{5.63}$$

We may write (5.62) in terms of K_T and $O_1(\cdot)$ as follows:

$$\left|P^\varepsilon(t, s) - P_0(t, s)\right| = K_T O_1\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-s)}{\varepsilon}\right)\right), \tag{5.64}$$

where $|O_1(y)|/|y| \leq 1$. The notation of K_T and $O_1(\cdot)$ above emphasizes the separation of the dependence of the constant and a "norm 1" function. Essentially, K_T serves as a magnitude of the bound indicating the size of the bounding region, and the rest is absorbed into the function $O_1(\cdot)$.

Theorem 5.29. *Assume* (A5.5) *and* (A5.6). *Then there exist* $\varepsilon_0 > 0$ *and* $K > 0$ *such that for* $0 < \varepsilon \leq \varepsilon_0$, $T \geq 0$, *and for any bounded, Borel measurable, and deterministic process* $\beta_{ij}(\cdot)$,

$$E \exp\left(\frac{\theta_T}{(T+1)^3} \sup_{0 \leq t \leq T} |n^\varepsilon(t)|\right) \leq K, \tag{5.65}$$

where θ_T *is any constant satisfying*

$$0 \leq \theta_T \leq \frac{\min\{1, \kappa_0\}}{K_T |\beta|_T (1 + |\widehat{Q}|_T)}, \tag{5.66}$$

and where $|\cdot|_T$ *denotes the matrix norm as defined in* (5.12)*, that is,*

$$|\beta|_T = \max_{i,j} \sup_{0\le t\le T} |\beta_{ij}(t)|,$$

similarly for $|\widehat{Q}|_T$.

Remark 5.30. This theorem is a natural extension to Theorem 5.4. Owing to the existence of the weak and strong interactions, slightly stronger conditions on K_T and θ_T are made in (5.63) and (5.66). Also the exponential constant in (5.65) is changed to $(T+1)^3$.

Proof of Theorem 5.29: Here the proof is again along the lines of Theorem 5.4. Since Steps 2-5 in the proof are similar to those of Theorem 5.4, we will only give the proof for Step 1.

Let $\chi^\varepsilon(\cdot)$ denote the vector of indicators corresponding to $\alpha^\varepsilon(\cdot)$, that is,

$$\chi^\varepsilon(t) = \left(I_{\{\alpha^\varepsilon(t)=s_{11}\}}, \ldots, I_{\{\alpha^\varepsilon(t)=s_{1m_1}\}}, \ldots, I_{\{\alpha^\varepsilon(t)=s_{l1}\}}, \ldots, I_{\{\alpha^\varepsilon(t)=s_{lm_l}\}}\right).$$

Then $w^\varepsilon(\cdot)$ defined by

$$w^\varepsilon(t) = \chi^\varepsilon(t) - \chi^\varepsilon(0) - \int_0^t \chi^\varepsilon(s)Q^\varepsilon(s)ds \tag{5.67}$$

is an $\mathbb{R}^m$-valued martingale. In fact, $w^\varepsilon(\cdot)$ is square integrable on $[0,T]$. It then follows from a well-known result (see Elliott [55] or Kunita and Watanabe [134]) that a stochastic integral with respect to $w^\varepsilon(t)$ can be defined. In view of the defining equation (5.67), the linear stochastic differential equation

$$d\chi^\varepsilon(t) = \chi^\varepsilon(t)Q^\varepsilon(t)dt + dw^\varepsilon(t) \tag{5.68}$$

makes sense. Recall that $P^\varepsilon(t,s)$ is the principal matrix solution of the matrix differential equation

$$\frac{dy(t)}{dt} = y(t)Q^\varepsilon(t). \tag{5.69}$$

The solution of this stochastic differential equation is

$$\begin{aligned}
\chi^\varepsilon(t) &= \chi^\varepsilon(0)P^\varepsilon(t,0) + \int_0^t (dw^\varepsilon(s))P^\varepsilon(t,s) \\
&= \chi^\varepsilon(0)\left(P^\varepsilon(t,0) - P_0(t,0)\right) \\
&\quad + \int_0^t (dw^\varepsilon(s))\left(P^\varepsilon(t,s) - P_0(t,s)\right) \\
&\quad + \chi^\varepsilon(0)P_0(t,0) + \int_0^t (dw^\varepsilon(s))P_0(t,s).
\end{aligned} \tag{5.70}$$

Use $\vartheta_{ij}(t,s)$ defined in Lemma 5.24 and write $\Theta(t,s) = (\vartheta_{ij}(t,s))$. Then it is easy to check that

$$P_0(t,s) = \widetilde{\mathbb{1}}\Theta(t,s)\text{diag}(\nu^1(t),\dots,\nu^l(t)). \tag{5.71}$$

Set

$$\overline{\chi}^\varepsilon(t) = \left(\nu^1(t)I_{\{\overline{\alpha}^\varepsilon(t)=1\}},\dots,\nu^l(t)I_{\{\overline{\alpha}^\varepsilon(t)=l\}}\right) \in \mathbb{R}^m$$

and

$$\widetilde{\chi}^\varepsilon(t) = \left(I_{\{\overline{\alpha}^\varepsilon(t)=1\}},\dots,I_{\{\overline{\alpha}^\varepsilon(t)=l\}}\right) \in \mathbb{R}^l.$$

Then it follows that

$$\begin{aligned} &\widetilde{\chi}^\varepsilon(t) = \chi^\varepsilon(t)\widetilde{\mathbb{1}} \quad \text{and} \\ &\overline{\chi}^\varepsilon(t) = \widetilde{\chi}^\varepsilon(t)\text{diag}(\nu^1(t),\dots,\nu^l(t)). \end{aligned} \tag{5.72}$$

Moreover, postmultiplying both sides of (5.67) by $\widetilde{\mathbb{1}}$ yields that

$$\chi^\varepsilon(t)\widetilde{\mathbb{1}} - \chi^\varepsilon(0)\widetilde{\mathbb{1}} - \int_0^t \chi^\varepsilon(s)Q^\varepsilon(s)\widetilde{\mathbb{1}}ds = w^\varepsilon(t)\widetilde{\mathbb{1}}. \tag{5.73}$$

Here $w^\varepsilon(\cdot)\widetilde{\mathbb{1}}$ is also a square-integrable martingale. Note that $\widetilde{Q}(s)\widetilde{\mathbb{1}} = 0$ and hence

$$\begin{aligned} &Q^\varepsilon(s)\widetilde{\mathbb{1}} = \widehat{Q}(s)\widetilde{\mathbb{1}} \quad \text{and} \\ &\overline{\chi}^\varepsilon(s)\widehat{Q}(s)\widetilde{\mathbb{1}} = \widetilde{\chi}^\varepsilon(s)\text{diag}(\nu^1(s),\dots,\nu^l(s))\widehat{Q}(s)\widetilde{\mathbb{1}} = \widetilde{\chi}^\varepsilon(s)\overline{Q}(s). \end{aligned}$$

We obtain from (5.73) that

$$\widetilde{\chi}^\varepsilon(t) - \widetilde{\chi}^\varepsilon(0) - \int_0^t \left((\chi^\varepsilon(s) - \overline{\chi}^\varepsilon(s))\widehat{Q}(s)\widetilde{\mathbb{1}} + \widetilde{\chi}^\varepsilon(s)\overline{Q}(s)\right) ds = w^\varepsilon(t)\widetilde{\mathbb{1}}.$$

Since $\Theta(t,s)$ is the principal matrix solution to

$$\frac{d\Theta(t,s)}{dt} = \Theta(t,s)\overline{Q}(t), \text{ with } \Theta(s,s) = I,$$

similar to (5.68), solving the stochastic differential equation for $\widetilde{\chi}^\varepsilon(\cdot)$ leads to the equation:

$$\begin{aligned} \widetilde{\chi}^\varepsilon(t) = \widetilde{\chi}^\varepsilon(0)\Theta(t,0) &+ \int_0^t (dw^\varepsilon(s)\widetilde{\mathbb{1}})\Theta(t,s) \\ &+ \int_0^t (\chi^\varepsilon(s) - \overline{\chi}^\varepsilon(s))\widehat{Q}(s)\widetilde{\mathbb{1}}\Theta(t,s)ds. \end{aligned} \tag{5.74}$$

Let us now return to the last two terms in (5.70) and use (5.71), (5.72), and (5.74) to obtain

$$\begin{aligned}
&\chi^\varepsilon(0)P_0(t,0)+\int_0^t(dw^\varepsilon(s))P_0(t,s)\\
&\quad=\left(\chi^\varepsilon(0)\widetilde{\mathbb{1}}\Theta(t,0)+\int_0^t(dw^\varepsilon(s))\widetilde{\mathbb{1}}\Theta(t,s)\right)\mathrm{diag}(\nu^1(t),\dots,\nu^l(t))\\
&\quad=\left(\widetilde{\chi}^\varepsilon(0)\Theta(t,0)+\int_0^t(dw^\varepsilon(s)\widetilde{\mathbb{1}})\Theta(t,s)\right)\mathrm{diag}(\nu^1(t),\dots,\nu^l(t))\\
&\quad=\left(\widetilde{\chi}^\varepsilon(t)-\int_0^t(\chi^\varepsilon(s)-\overline{\chi}^\varepsilon(s))\widehat{Q}(s)\widetilde{\mathbb{1}}\Theta(t,s)ds\right)\mathrm{diag}(\nu^1(t),\dots,\nu^l(t))\\
&\quad=\overline{\chi}^\varepsilon(t)-\int_0^t(\chi^\varepsilon(s)-\overline{\chi}^\varepsilon(s))\widehat{Q}(s)\widetilde{\mathbb{1}}\Theta(t,s)\mathrm{diag}(\nu^1(t),\dots,\nu^l(t))ds\\
&\quad=\overline{\chi}^\varepsilon(t)-\int_0^t(\chi^\varepsilon(s)-\overline{\chi}^\varepsilon(s))\widehat{Q}(s)P_0(t,s)ds.
\end{aligned}$$

Combining this with (5.70), we have

$$(\chi^\varepsilon(t)-\overline{\chi}^\varepsilon(t))+\int_0^t(\chi^\varepsilon(s)-\overline{\chi}^\varepsilon(s))\widehat{Q}(s)P_0(t,s)ds=\eta^\varepsilon(t),\tag{5.75}$$

where

$$\eta^\varepsilon(t)=\chi^\varepsilon(0)\left(P^\varepsilon(t,0)-P_0(t,0)\right)+\int_0^t(dw^\varepsilon(s))\left(P^\varepsilon(t,s)-P_0(t,s)\right).$$

Note that the matrix $P^\varepsilon(t,s)$ is invertible but $P_0(t,s)$ is not. The idea is to approximate the noninvertible matrix $P_0(t,s)$ by the invertible $P^\varepsilon(t,s)$. Let

$$\eta_1^\varepsilon(t)=\int_0^t(\chi^\varepsilon(s)-\overline{\chi}^\varepsilon(s))\widehat{Q}(s)\left(P_0(t,s)-P^\varepsilon(t,s)\right)ds\tag{5.76}$$

and

$$\phi^\varepsilon(t)=(\chi^\varepsilon(t)-\overline{\chi}^\varepsilon(t))-(\eta^\varepsilon(t)-\eta_1^\varepsilon(t)).$$

Then $\phi^\varepsilon(0)=0$ and $\phi^\varepsilon(t)$ satisfies the following equation:

$$\phi^\varepsilon(t)+\int_0^t\phi^\varepsilon(s)\widehat{Q}(s)P^\varepsilon(t,s)ds+\int_0^t(\eta^\varepsilon(s)-\eta_1^\varepsilon(s))\widehat{Q}(s)P^\varepsilon(t,s)ds=0.$$

The properties of the principal matrix solution imply that

$$P^\varepsilon(t,s)=P^\varepsilon(0,s)P^\varepsilon(t,0).$$

Set

$$\check{Q}^\varepsilon(t) = P^\varepsilon(t,0)\widehat{Q}(t)P^\varepsilon(0,t),$$

$$\psi^\varepsilon(t) = \phi^\varepsilon(t)P^\varepsilon(0,t), \quad \text{and}$$

$$\eta_2^\varepsilon(t) = (\eta^\varepsilon(t) - \eta_1^\varepsilon(t))\widehat{Q}(t)P^\varepsilon(0,t).$$

Owing to the properties of the principal matrix solution, for any $t \in [0,T]$, we have

$$P^\varepsilon(0,t)P^\varepsilon(t,0) = P^\varepsilon(t,t) = I, \tag{5.77}$$

$\psi^\varepsilon(0) = 0$ and $\psi^\varepsilon(t)$ satisfies the equation

$$\psi^\varepsilon(t) + \int_0^t \psi^\varepsilon(s)\check{Q}^\varepsilon(s)ds + \int_0^t \eta_2^\varepsilon(s)ds = 0.$$

The solution to this equation is given by

$$\psi^\varepsilon(t) = -\int_0^t \eta_2^\varepsilon(s)\check{\Phi}^\varepsilon(t,s)ds, \tag{5.78}$$

where $\check{\Phi}^\varepsilon(t,s)$ is the principal matrix solution to

$$\frac{d\check{\Phi}^\varepsilon(t,s)}{dt} = -\check{\Phi}^\varepsilon(t,s)\check{Q}^\varepsilon(t), \quad \text{with } \check{\Phi}^\varepsilon(s,s) = I.$$

Postmultiplying both sides of (5.78) by $P^\varepsilon(t,0)$ yields

$$\begin{aligned}\phi^\varepsilon(t) &= \psi^\varepsilon(t)P^\varepsilon(t,0)\\ &= -\int_0^t \eta_2^\varepsilon(s)\check{\Phi}^\varepsilon(t,s)P^\varepsilon(t,0)ds\\ &= -\int_0^t (\eta^\varepsilon(s) - \eta_1^\varepsilon(s))\widehat{Q}(s)\check{\Psi}^\varepsilon(t,s)ds,\end{aligned}$$

where

$$\check{\Psi}^\varepsilon(t,s) = P^\varepsilon(0,s)\check{\Phi}^\varepsilon(t,s)P^\varepsilon(t,0).$$

Thus it follows that

$$\chi^\varepsilon(t) - \overline{\chi}^\varepsilon(t) = \eta^\varepsilon(t) - \eta_1^\varepsilon(t) - \int_0^t (\eta^\varepsilon(s) - \eta_1^\varepsilon(s))\widehat{Q}(s)\check{\Psi}^\varepsilon(t,s)ds. \tag{5.79}$$

Again using (5.77), we have

$$\begin{aligned}
&\frac{d}{dt}\left(\check{\Phi}^{\varepsilon}(t,0)P^{\varepsilon}(t,0)\right)\\
&\quad=\left(\frac{d\check{\Phi}^{\varepsilon}(t,0)}{dt}\right)P^{\varepsilon}(t,0)+\check{\Phi}^{\varepsilon}(t,0)\left(\frac{dP^{\varepsilon}(t,0)}{dt}\right)\\
&\quad=-\check{\Phi}^{\varepsilon}(t,0)\check{Q}^{\varepsilon}(t)P^{\varepsilon}(t,0)+\check{\Phi}^{\varepsilon}(t,0)P^{\varepsilon}(t,0)Q^{\varepsilon}(t)\\
&\quad=-\check{\Phi}^{\varepsilon}(t,0)P^{\varepsilon}(t,0)\widehat{Q}(t)P^{\varepsilon}(0,t)P^{\varepsilon}(t,0)+\check{\Phi}^{\varepsilon}(t,0)P^{\varepsilon}(t,0)Q^{\varepsilon}(t)\\
&\quad=-\check{\Phi}^{\varepsilon}(t,0)P^{\varepsilon}(t,0)\widehat{Q}(t)+\check{\Phi}^{\varepsilon}(t,0)P^{\varepsilon}(t,0)Q^{\varepsilon}(t)\\
&\quad=\check{\Phi}^{\varepsilon}(t,0)P^{\varepsilon}(t,0)\left(-\widehat{Q}(t)+Q^{\varepsilon}(t)\right)\\
&\quad=\check{\Phi}^{\varepsilon}(t,0)P^{\varepsilon}(t,0)\left(\frac{1}{\varepsilon}\widetilde{Q}(t)\right).
\end{aligned}$$

This implies that $\check{\Psi}^{\varepsilon}(t,s)$ is the principal matrix solution to the differential equation

$$\frac{d\check{\Psi}^{\varepsilon}(t,s)}{dt}=\check{\Psi}^{\varepsilon}(t,s)\left(\frac{1}{\varepsilon}\widetilde{Q}(t)\right),\quad\text{with }\check{\Psi}^{\varepsilon}(s,s)=I. \tag{5.80}$$

Therefore, all entries of $\check{\Psi}^{\varepsilon}(t,s)$ are bounded below from 0 and bounded above by 1, and these bounds are uniform in $0\le s\le t\le T$. Thus, $|\check{\Psi}^{\varepsilon}(t,s)|_T\le 1$.

Multiplying both sides of (5.79) by the $m\times m$ matrix

$$\beta(t):=\operatorname{diag}(\beta_{11}(t),\dots,\beta_{1m_1}(t),\dots,\beta_{l1}(t),\dots,\beta_{lm_l}(t))$$

from the right and integrating over the interval $[0,\varsigma]$, for each $\varsigma\in[0,T]$, we have

$$\begin{aligned}
\int_0^{\varsigma}(\chi^{\varepsilon}(t)-\overline{\chi}^{\varepsilon}(t))\beta(t)dt\;=&\int_0^{\varsigma}\eta^{\varepsilon}(t)\beta(t)dt-\int_0^{\varsigma}\eta_1^{\varepsilon}(t)\beta(t)dt\\
&-\int_0^{\varsigma}\int_0^{t}(\eta^{\varepsilon}(s)-\eta_1^{\varepsilon}(s))\widehat{Q}(s)\check{\Psi}^{\varepsilon}(t,s)ds\beta(t)dt.
\end{aligned}$$

By changing the order of integration, we write the last term in the above expression as

$$\begin{aligned}
&\int_0^{\varsigma}\int_0^{t}(\eta^{\varepsilon}(s)-\eta_1^{\varepsilon}(s))\widehat{Q}(s)\check{\Psi}^{\varepsilon}(t,s)ds\beta(t)dt\\
&\qquad=\int_0^{\varsigma}(\eta^{\varepsilon}(s)-\eta_1^{\varepsilon}(s))\left(\int_s^{\varsigma}\widehat{Q}(s)\check{\Psi}^{\varepsilon}(t,s)\beta(t)dt\right)ds.
\end{aligned}$$

Therefore, it follows that

$$\int_0^\varsigma (\chi^\varepsilon(t) - \overline{\chi}^\varepsilon(t))\beta(t)dt = \int_0^\varsigma \eta^\varepsilon(t)\widetilde{\beta}(t)dt - \int_0^\varsigma \eta_1^\varepsilon(t)\widetilde{\beta}(t)dt, \tag{5.81}$$

where

$$\widetilde{\beta}(t) = \beta(t) + \int_t^\varsigma \widehat{Q}(t)\check{\Psi}^\varepsilon(r,t)\beta(r)dr.$$

Moreover, in view of the fact that $|\check{\Psi}^\varepsilon(t,s)|_T \le 1$, it is easy to see that

$$|\widetilde{\beta}|_T \le (1+T)|\beta|_T(1+|\widehat{Q}|_T). \tag{5.82}$$

Note that $n^\varepsilon(\cdot)$ can be written in terms of $\chi^\varepsilon(\cdot)$ and $\overline{\chi}^\varepsilon(\cdot)$ as

$$n^\varepsilon(\varsigma) = \frac{1}{\sqrt{\varepsilon}}\int_0^\varsigma (\chi^\varepsilon(t) - \overline{\chi}^\varepsilon(t))\beta(t)dt.$$

By virtue of (5.81), it follows that

$$|n^\varepsilon(\varsigma)| \le \frac{1}{\sqrt{\varepsilon}}\left|\int_0^\varsigma \eta^\varepsilon(t)\widetilde{\beta}(t)dt\right| + \frac{1}{\sqrt{\varepsilon}}\left|\int_0^\varsigma \eta_1^\varepsilon(t)\widetilde{\beta}(t)dt\right|.$$

Note that in view of the definition of $\eta_1^\varepsilon(\cdot)$ in (5.76),

$$|\eta_1^\varepsilon(t)| = \int_0^t O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-s)}{\varepsilon}\right)\right)ds = O(\varepsilon(t+1)).$$

Thus, in view of (5.82),

$$\begin{aligned}\sup_{0\le\varsigma\le T}\left|\int_0^\varsigma \eta_1^\varepsilon(t)\widetilde{\beta}(t)dt\right| &= |\widetilde{\beta}|_T \sup_{0\le\varsigma\le T}\int_0^\varsigma O(\varepsilon(t+1))dt\\ &= |\widetilde{\beta}|_T \sup_{0\le\varsigma\le T} O(\varepsilon(\varsigma^2+\varsigma))\\ &= |\widetilde{\beta}|_T(T^2+T)O(\varepsilon)\\ &\le (1+T)^3|\beta|_T(1+|\widehat{Q}|_T)O(\varepsilon).\end{aligned} \tag{5.83}$$

Thus, in view of (5.63) and (5.66), for some $\varepsilon_0 > 0$, and all $0 < \varepsilon \le \varepsilon_0$,

$$\begin{aligned}&\exp\left(\frac{\theta_T}{(T+1)^3}\sup_{0\le\varsigma\le T}\left|\int_0^\varsigma \frac{\eta_1^\varepsilon(t)\widetilde{\beta}(t)}{\sqrt{\varepsilon}}dt\right|\right)\\ &\quad\le \exp\left(\frac{O(\sqrt{\varepsilon})\min\{1,\kappa_0\}}{K_T}\right)\\ &\quad\le \exp\left(O(\sqrt{\varepsilon_0})\min\{1,\kappa_0\}\right) \le K.\end{aligned} \tag{5.84}$$

Moreover, using (5.64), as in the proof of Theorem 5.4, we obtain that

$$E\exp\left(\frac{\theta_T}{(T+1)^{\frac{3}{2}}}\sup_{0\le\varsigma\le T}\left|\int_0^{\varsigma}\left(\frac{\eta^{\varepsilon}(t)\widetilde{\beta}(t)}{\sqrt{\varepsilon}}\right)dt\right|\right)\le K, \tag{5.85}$$

for

$$0\le\theta_T\le\frac{\min\{1,\kappa_0\}}{K_T|\beta|_T(1+|\widehat{Q}|_T)}.$$

Finally, combine (5.81), (5.83), and (5.85) to obtain

$$E\exp\left(\frac{\theta_T}{(T+1)^3}\sup_{0\le t\le T}|n^{\varepsilon}(t)|\right)\le K.$$

This completes the proof. □

Remark 5.31. It is easily seen that the error bound so obtained has a form similar to that of the martingale inequality. If $n^{\varepsilon}(\cdot)$ were a martingale, the inequality would be obtained much more easily since $\exp(\cdot)$ is a convex function. As in Section 5.2, the error bound is still a measure of "goodness" of approximation. However, one cannot compare the unscaled occupation measures with a deterministic function. A sensible alternative is to use an approximation by the aggregated process that is no longer deterministic. The exponential bounds obtained tell us exactly how closely one can carry out the approximation. It should be particularly useful for many applications in stochastic control problems with Markovian jump disturbances under discounted cost criteria.

The next two corollaries show that the error bound can be improved under additional conditions by having smaller exponential constants, e.g., $(T+1)^{3/2}$ or $(T+1)^{5/2}$ instead of $(T+1)^3$.

Corollary 5.32. *Assume that the conditions of Theorem 5.29 hold. Let* $\widetilde{Q}(t)=(\widetilde{q}_{ij}(t))$ *and* $\overline{Q}(t)=(\overline{q}_{ij}(t))$ *denote generators with the corresponding state spaces* $\{a_1,\ldots,a_{m_0}\}$ *and* $\{1,\ldots,l\}$*, respectively. Consider*

$$Q^{\varepsilon}(t)=\frac{1}{\varepsilon}\begin{pmatrix}\widetilde{Q}(t) & & \\ & \ddots & \\ & & \widetilde{Q}(t)\end{pmatrix}+\begin{pmatrix}\overline{q}_{11}(t)I_{m_0} & \cdots & \overline{q}_{1l}(t)I_{m_0}\\ \vdots & \cdots & \vdots\\ \overline{q}_{l1}(t)I_{m_0} & \cdots & \overline{q}_{ll}(t)I_{m_0}\end{pmatrix},$$

where I_{m_0} *is the* $m_0\times m_0$ *identity matrix. Then there exist positive constants* ε_0 *and* K *such that for* $0<\varepsilon\le\varepsilon_0$*, and* $T\ge 0$*,*

$$E\exp\left(\frac{\theta_T}{(T+1)^{\frac{3}{2}}}\sup_{0\le t\le T}|n^{\varepsilon}(t)|\right)\le K.$$

Proof: Under the special structure of the generator Q^ε, it is easy to see that

$$\widehat{Q}(s)\widetilde{\mathbb{1}} = \widetilde{\mathbb{1}}\overline{Q}(s),$$

where $\widetilde{\mathbb{1}}$ now takes the form

$$\widetilde{\mathbb{1}} = \operatorname{diag}(\mathbb{1}_{m_0}, \ldots, \mathbb{1}_{m_0}).$$

Note that under current conditions on the fast-changing part of the generator $\widetilde{Q}(t)$,

$$\nu^1(t) = \nu^2(t) = \cdots = \nu^l(t) \text{ and } \operatorname{diag}(\nu^1(t), \ldots, \nu^l(t))\widetilde{\mathbb{1}} = I_l,$$

where I_l denotes the l-dimensional identity matrix. This together with (5.72) implies that

$$(\chi^\varepsilon(s) - \overline{\chi}^\varepsilon(s))\widehat{Q}(s)\widetilde{\mathbb{1}} = 0.$$

It follows from (5.71) that

$$\int_0^t (\chi^\varepsilon(s) - \overline{\chi}^\varepsilon(s))\widehat{Q}(s)P_0(t,s)ds = 0.$$

Then (5.75) becomes

$$\chi^\varepsilon(t) - \overline{\chi}^\varepsilon(t) = \eta^\varepsilon(t).$$

The rest of the proof follows exactly that of Theorem 5.29. □

Corollary 5.33. *Assume the conditions of Theorem 5.29. Suppose $\widetilde{Q}(t) = \widetilde{Q}$ and $\widehat{Q}(t) = \widehat{Q}$ for some constant matrices $\widetilde{Q}$ and $\widehat{Q}$. Then there exist positive constants ε_0 and K such that for $0 < \varepsilon \le \varepsilon_0$, and $T \ge 0$,*

$$E\exp\left(\frac{\theta_T}{(T+1)^{\frac{5}{2}}}\sup_{0\le t\le T}|n^\varepsilon(t)|\right) \le K.$$

Remark 5.34. Note that in view of Corollary 4.31, one can show under the condition $\widetilde{Q}(t) = \widetilde{Q}$ and $\widehat{Q}(t) = \widehat{Q}$ that there exists a constant K such that

$$P^\varepsilon(t,s) - P_0(t,s) = K(T+1)O_1\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-s)}{\varepsilon}\right)\right).$$

In this case, θ_T can be taken as

$$0 \le \theta_T \le \frac{\min\{1, \kappa_0\}}{K(T+1)|\beta|_T(1+|\widehat{Q}|_T)}.$$

That is, compared with the general result, the constant K_T can be further specified as $K_T = K(T+1)$.

Proof of Corollary 5.33: Note that when the generators are time independent, the quasi-stationary distribution $\nu^i(t)$ is also independent of time and is denoted by ν^i. In this case, the argument from (5.75) to (5.80) can be replaced by the following. Let

$$\check{Q}_0 = \widehat{Q}\widetilde{\mathbb{1}}\text{diag}(\nu^1, \ldots, \nu^l).$$

Then it can be shown that

$$\widehat{Q}\widetilde{\mathbb{1}}\left(\overline{Q}\right)^k \text{diag}(\nu^1, \ldots, \nu^l) = (\check{Q}_0)^{k+1}, \text{ for } k \geq 0.$$

This implies that

$$\begin{aligned}\widehat{Q}P_0(t,s) &= \widehat{Q}\widetilde{\mathbb{1}}\exp\left(\overline{Q}(t-s)\right)\text{diag}(\nu^1, \ldots, \nu^l)\\ &= \check{Q}_0 \exp(\check{Q}_0(t-s)).\end{aligned}$$

Let $\phi^\varepsilon(t) = (\chi^\varepsilon(t) - \overline{\chi}^\varepsilon(t)) - \eta^\varepsilon(t)$. Then $\phi^\varepsilon(\cdot)$ satisfies the equation

$$\phi^\varepsilon(t) + \int_0^t (\phi^\varepsilon(s) + \eta^\varepsilon(s))\check{Q}_0 \exp(\check{Q}_0(t-s))ds = 0.$$

Solving for $\phi^\varepsilon(\cdot)$, we obtain

$$\phi^\varepsilon(t) = -\int_0^t \eta^\varepsilon(s)\check{Q}_0 ds.$$

Writing $\chi^\varepsilon(t) - \overline{\chi}^\varepsilon(t)$ in terms of $\phi^\varepsilon(t)$ and $\eta^\varepsilon(t)$ yields,

$$\chi^\varepsilon(t) - \overline{\chi}^\varepsilon(t) = \eta^\varepsilon(t) - \int_0^t \eta^\varepsilon(s)\check{Q}_0 ds.$$

The rest of the proof follows that of Theorem 5.29. □

Similar to Section 5.2, we derive estimates that are analogous to Corollary 5.7 and Corollary 5.8. The details are omitted, however.

5.3.3 Asymptotic Distributions

In Section 5.2, we obtained a central limit theorem for a class of Markov chains generated by $Q^\varepsilon(t) = Q(t)/\varepsilon + \widehat{Q}(t)$ with a weakly irreducible $Q(t)$. In this case for sufficiently small $\varepsilon > 0$, $Q^\varepsilon(t)$ is weakly irreducible. What, if anything, can be said about the weak and strong interaction models, when $\widehat{Q}(t)$ is not weakly irreducible? Is there a central limit theorem for the corresponding occupation measure when one has a singularly perturbed Markov chain with weak and strong interactions? This section deals with

such an issue; our interest lies in the asymptotic distribution as $\varepsilon \to 0$. It is shown that the asymptotic distribution of the corresponding occupation measure can be obtained. However, the limit distribution is no longer Gaussian, but a Gaussian mixture, and the proof is quite different from that of the irreducible case in Section 5.2.

For each $i = 1, \ldots, l$, $j = 1, \ldots, m_i$, $\alpha \in \mathcal{M}$, and $t \geq 0$, let $\beta_{ij}(t)$ be a bounded Borel measurable deterministic function. Use $W_{ij}(t, \alpha)$ defined in (5.61) and the normalized occupation measure

$$n^\varepsilon(t) = \left(n^\varepsilon_{11}(t), \ldots, n^\varepsilon_{1m_1}(t), \ldots, n^\varepsilon_{l1}(t), \ldots, n^\varepsilon_{lm_l}(t)\right),$$

with

$$n^\varepsilon_{ij}(t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t W_{ij}(s, \alpha^\varepsilon(s))ds.$$

We will show in this section that $n^\varepsilon(\cdot)$ converges weakly to a switching diffusion modulated by $\overline{\alpha}(\cdot)$. The procedure is as follows:

(a) Show that $(n^\varepsilon(\cdot), \overline{\alpha}^\varepsilon(\cdot))$ is tight;

(b) verify that the limit of a subsequence of $(n^\varepsilon(\cdot), \overline{\alpha}^\varepsilon(\cdot))$ is a solution to a martingale problem that has a unique solution;

(c) characterize the solution of the associated martingale problem;

(d) construct a switching diffusion that is also a solution to the martingale problem and therefore the limit of $(n^\varepsilon(\cdot), \overline{\alpha}^\varepsilon(\cdot))$.

To accomplish our goal, these steps are realized by proving a series of lemmas. Recall that $\mathcal{F}^\varepsilon_t = \sigma\{\alpha^\varepsilon(s) : 0 \leq s \leq t\}$ denotes the filtration generated by $\alpha^\varepsilon(\cdot)$. The lemma below is on the order estimates of the conditional moments, and is useful for getting the tightness result in what follows.

Lemma 5.35. *Assume* (A5.5) *and* (A5.6). *Then for all* $0 \leq s \leq t \leq T$ *and* ε *small enough, the following hold:*

$$\text{(a)} \quad \sup_{s \leq t \leq T} E[n^\varepsilon(t) - n^\varepsilon(s)|\mathcal{F}^\varepsilon_s] = O(\sqrt{\varepsilon});$$

$$\text{(b)} \quad \sup_{\varepsilon} E\left[\, |n^\varepsilon(t) - n^\varepsilon(s)|^2 | \mathcal{F}^\varepsilon_s\right] = O(t - s).$$

Proof: First, note that for any fixed i, j,

$$E[(n^\varepsilon_{ij}(t) - n^\varepsilon_{ij}(s))|\mathcal{F}^\varepsilon_s] = \frac{1}{\sqrt{\varepsilon}} \int_s^t E[W_{ij}(r, \alpha^\varepsilon(r))|\mathcal{F}^\varepsilon_s]dr.$$

Moreover, in view of the definition of $W_{ij}(t,\alpha)$ and the Markov property, we have, for $0 \le s \le r$,

$$\begin{aligned}
&E[W_{ij}(r,\alpha^\varepsilon(r))|\mathcal{F}_s^\varepsilon] \\
&\quad = E\left[\left(I_{\{\alpha^\varepsilon(r)=s_{ij}\}} - \nu_j^i(r) I_{\{\alpha^\varepsilon(r)\in\mathcal{M}_i\}}\right)|\mathcal{F}_s^\varepsilon\right]\beta_{ij}(r) \\
&\quad = \left(P(\alpha^\varepsilon(r)=s_{ij}|\mathcal{F}_s^\varepsilon) - \nu_j^i(r)P(\alpha^\varepsilon(r)\in\mathcal{M}_i|\mathcal{F}_s^\varepsilon)\right)\beta_{ij}(r) \\
&\quad = \left(P(\alpha^\varepsilon(r)=s_{ij}|\alpha^\varepsilon(s)) - \nu_j^i(r)P(\alpha^\varepsilon(r)\in\mathcal{M}_i|\alpha^\varepsilon(s))\right)\beta_{ij}(r).
\end{aligned}$$

In view of Lemma 5.24, in particular, similar to (5.55) and (5.56), for all $i_0 = 1,\dots,l$ and $j_0 = 1,\dots,m_{i_0}$,

$$\begin{aligned}
&P(\alpha^\varepsilon(r)=s_{ij}|\alpha^\varepsilon(s)=s_{i_0j_0}) - \nu_j^i(r)P(\alpha^\varepsilon(r)\in\mathcal{M}_i|\alpha^\varepsilon(s)=s_{i_0j_0}) \\
&\qquad = O\left(\varepsilon + \exp\left(-\frac{\kappa_0(r-s)}{\varepsilon}\right)\right).
\end{aligned}$$

Thus owing to Lemma A.42, we have

$$\begin{aligned}
&\left(P(\alpha^\varepsilon(r)=s_{ij}|\alpha^\varepsilon(s)) - \nu_j^i(r)P(\alpha^\varepsilon(r)\in\mathcal{M}_i|\alpha^\varepsilon(s))\right)\beta_{ij}(r) \\
&\quad = \sum_{i_0=1}^{l}\sum_{j_0=1}^{m_{i_0}} I_{\{\alpha^\varepsilon(s)=s_{i_0j_0}\}}\Big(P(\alpha^\varepsilon(r)=s_{ij}|\alpha^\varepsilon(s)=s_{i_0j_0}) \\
&\qquad\quad -\nu_j^i(r)P(\alpha^\varepsilon(r)\in\mathcal{M}_i|\alpha^\varepsilon(s)=s_{i_0j_0})\Big)\beta_{ij}(r) \\
&\quad = O\left(\varepsilon + \exp\left(-\frac{\kappa_0(r-s)}{\varepsilon}\right)\right).
\end{aligned}$$

Note also that

$$\frac{1}{\sqrt{\varepsilon}}\int_s^t O\left(\varepsilon + \exp\left(-\frac{\kappa_0(r-s)}{\varepsilon}\right)\right)dr = O(\sqrt{\varepsilon}).$$

This implies (a).

To verify (b), fix and suppress i,j and define

$$\eta^\varepsilon(t) = E\left[\left(\int_s^t W_{ij}(r,\alpha^\varepsilon(r))dr\right)^2\Big|\mathcal{F}_s^\varepsilon\right].$$

Then by the definition of $n_{ij}(\cdot)$,

$$E\left[\left(n_{ij}^\varepsilon(t) - n_{ij}^\varepsilon(s)\right)^2\Big|\mathcal{F}_s^\varepsilon\right] = \frac{\eta^\varepsilon(t)}{\varepsilon}. \tag{5.86}$$

In accordance with the definition of $\overline{\alpha}^\varepsilon(\cdot)$, $\overline{\alpha}^\varepsilon(t) = i$ iff $\alpha^\varepsilon(t) \in \mathcal{M}_i$. In what follows, we use $\alpha^\varepsilon(t) \in \mathcal{M}_i$ and $\overline{\alpha}^\varepsilon(t) = i$ interchangeably. Set

$$\Psi_1^\varepsilon(t,r) = I_{\{\alpha^\varepsilon(r)=s_{ij}\}} I_{\{\alpha^\varepsilon(t)=s_{ij}\}} - \nu_j^i(t) I_{\{\alpha^\varepsilon(r)=s_{ij}\}} I_{\{\overline{\alpha}^\varepsilon(t)=i\}},$$

$$\Psi_2^\varepsilon(t,r) = -\nu_j^i(r) I_{\{\overline{\alpha}^\varepsilon(r)=i\}} I_{\{\alpha^\varepsilon(t)=s_{ij}\}} + \nu_j^i(r)\nu_j^i(t) I_{\{\overline{\alpha}^\varepsilon(r)=i\}} I_{\{\overline{\alpha}^\varepsilon(t)=i\}}.$$

Then as in the proof of Theorem 5.25,

$$\frac{d\eta^\varepsilon(t)}{dt} = 2\int_s^t E\left[\Psi_1^\varepsilon(t,r) + \Psi_2^\varepsilon(t,r)|\mathcal{F}_s^\varepsilon\right]\beta_{ij}(r)\beta_{ij}(t)dr.$$

Using Lemma 5.24, we obtain

$$E[\Psi_1^\varepsilon(t,r)|\alpha^\varepsilon(s) = s_{i_0j_0}] = O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-r)}{\varepsilon}\right)\right),$$
$$E[\Psi_2^\varepsilon(t,r)|\alpha^\varepsilon(s) = s_{i_0j_0}] = O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-r)}{\varepsilon}\right)\right),$$

for all $i_0 = 1, \ldots, l$ and $j_0 = 1, \ldots, m_{i_0}$. Then from Lemma A.42, we obtain

$$E[\Psi_1^\varepsilon(t,r)|\mathcal{F}_s^\varepsilon] = O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-r)}{\varepsilon}\right)\right),$$
$$E[\Psi_2^\varepsilon(t,r)|\mathcal{F}_s^\varepsilon] = O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t-r)}{\varepsilon}\right)\right).$$

As a consequence, we have

$$\frac{d\eta^\varepsilon(t)}{dt} = O(\varepsilon).$$

Integrating both sides over $[s,t]$ and recalling $\eta^\varepsilon(s) = 0$ yields

$$\frac{\eta^\varepsilon(t)}{\varepsilon} = O(t-s).$$

This completes the proof of the lemma. □

The next lemma is concerned with the tightness of $\{(n^\varepsilon(\cdot), \overline{\alpha}^\varepsilon(\cdot))\}$.

Lemma 5.36. *Assume* (A5.5) *and* (A5.6). *Then* $\{(n^\varepsilon(\cdot), \overline{\alpha}^\varepsilon(\cdot))\}$ *is tight in* $D([0,T]; \mathbb{R}^m \times \overline{\mathcal{M}})$.

Proof: The proof uses Lemma A.17. We first verify that the condition given in Remark A.18 holds. To this end, note that $0 \le \overline{\alpha}^\varepsilon(t) \le l$ for all $t \in [0,T]$. Moreover, by virtue of Theorem 5.25, for each $\delta > 0$ and each rational $t \ge 0$,

$$\begin{aligned}\inf_\varepsilon P\left(|n^\varepsilon(t)| \le K_{t,\delta}\right) &= \inf_\varepsilon [1 - P(|n^\varepsilon(t)| \ge K_{t,\delta})] \\ &\ge \inf_\varepsilon \left(1 - \frac{E|n^\varepsilon(t)|^2}{K_{t,\delta}^2}\right) \\ &\ge 1 - \frac{Kt}{K_{t,\delta}^2},\end{aligned}$$

where the last inequality is due to Theorem 5.25. Thus if we choose $K_{t,\delta} > \sqrt{KT/\delta}$, (A.6) will follow.

It follows from Lemma 5.35 and (5.58) that for all $t \in [0,T]$,

$$\begin{aligned}&\lim_{\Delta\to 0}\left\{\limsup_{\varepsilon\to 0}\left(\sup_{0\le s\le\Delta} E\left\{E\left[\,|n_{ij}^\varepsilon(t+s) - n_{ij}^\varepsilon(t)|^2|\mathcal{F}_t^\varepsilon\right]\right\}\right)\right\} = 0, \\ &\lim_{\Delta\to 0}\left\{\limsup_{\varepsilon\to 0}\left(\sup_{0\le s\le\Delta} E\left\{E\left[\,|\overline{\alpha}^\varepsilon(t+s) - \overline{\alpha}^\varepsilon(t)|^2|\mathcal{F}_t^\varepsilon\right]\right\}\right)\right\} = 0.\end{aligned} \tag{5.87}$$

Using (5.86) and (5.87), Theorem A.17 yields the desired result. □

The tightness of $(n^\varepsilon(\cdot), \overline{\alpha}^\varepsilon(\cdot))$ and Prohorov's theorem allow one to extract convergent subsequences. We next show that the limit of such a subsequence is uniquely determined in distribution. An equivalent statement is that the associated martingale problem has a unique solution. The following lemma is a generalization of Theorem 5.25 and is needed for proving such a uniqueness property.

Lemma 5.37. *Let $\xi(t,x)$ be a real-valued function that is Lipschitz in $(t,x) \in \mathbb{R}^{m+1}$. Then*

$$\sup_{0\le\varsigma\le T} E\left|\int_0^\varsigma W_{ij}(s,\alpha^\varepsilon(s))\xi(s,n^\varepsilon(s))ds\right|^2 \to 0,$$

where $W_{ij}(t,\alpha) = (I_{\{\alpha=s_{ij}\}} - \nu_j^i(t)I_{\{\alpha\in\mathcal{M}_i\}})\beta_{ij}(t)$ as defined in (5.61).

Remark 5.38. This lemma indicates that the weighted occupation measure (with weighting function $\xi(t,\alpha^\varepsilon(t))$) defined above goes to zero in mean square uniformly in $t \in [0,\varsigma]$. If $\xi(\cdot)$ were a bounded and measurable deterministic function not depending on $\alpha^\varepsilon(\cdot)$ or $n^\varepsilon(\cdot)$, this assertion would follow from Theorem 5.25 easily. In the current situation, it is a function of $n^\varepsilon(\cdot)$ and therefore a function of $\alpha^\varepsilon(\cdot)$, which results in much

of the difficulty. Intuitively, if we can "separate" the functions $W_{ij}(\cdot)$ and $\xi(\cdot)$ in the sense treating $\xi(\cdot)$ as deterministic, then Theorem 5.25 can be applied to obtain the desired limit. To do so, subdivide the interval $[0, \varsigma]$ into small intervals so that on each of the small intervals, the two functions can be separated. To be more specific, on each partitioned interval, use a piecewise-constant function to approximate $\xi(\cdot)$, and show that the error goes to zero. In this process, the Lipschitz condition of $\xi(t, x)$ plays a crucial role.

Proof of Lemma 5.37: For $0 < \delta < 1$ and $0 < \varsigma \leq T$, let $N = [\varsigma/\varepsilon^{1-\delta}]$. Use a partition of $[0, \varsigma]$ given by

$$[t_0, t_1] \cup [t_1, t_2) \cup \cdots \cup [t_N, t_{N+1}]$$

of $[0, \varsigma]$, where $t_k = \varepsilon^{1-\delta} k$ for $k = 0, 1, \ldots, N$ and $t_{N+1} = \varsigma$. Consider a piecewise-constant function

$$\widetilde{\xi}(t) = \begin{cases} \xi(0, n^\varepsilon(0)), & \text{if } 0 \leq t < t_2, \\ \xi(t_{k-1}, n^\varepsilon(t_{k-1})), & \text{if } t_k \leq t < t_{k+1},\ k = 2, \ldots N, \\ \xi(t_{N-1}, n^\varepsilon(t_{N-1})), & \text{if } t = t_{N+1}. \end{cases}$$

Let $W_{ij}^\varepsilon(t) = W_{ij}(t, \alpha^\varepsilon(t))$. Then

$$\begin{aligned} &E\left|\int_0^\varsigma W_{ij}^\varepsilon(t)\xi(t, n^\varepsilon(t))dt\right|^2 \\ &\quad \leq 2E\left|\int_0^\varsigma W_{ij}^\varepsilon(t)|\xi(t, n^\varepsilon(t)) - \widetilde{\xi}(t)|dt\right|^2 + 2E\left|\int_0^\varsigma W_{ij}^\varepsilon(t)\widetilde{\xi}(t)dt\right|^2. \end{aligned} \tag{5.88}$$

We now estimate the first term on the second line above. In view of the Cauchy inequality and the boundedness of $W_{ij}^\varepsilon(t)$, it follows, for $0 \leq \varsigma \leq T$, that

$$\begin{aligned} E\left|\int_0^\varsigma W_{ij}^\varepsilon(t)|\xi(t, n^\varepsilon(t)) - \widetilde{\xi}(t)|dt\right|^2 &\leq TE\int_0^\varsigma (\xi(t, n^\varepsilon(t)) - \widetilde{\xi}(t))^2 dt \\ &= T\int_0^\varsigma E(\xi(t, n^\varepsilon(t)) - \widetilde{\xi}(t))^2 dt. \end{aligned}$$

Note that Theorem 5.25 implies

$$E|n^\varepsilon(t)|^2 \leq K,$$

for a positive constant K and for all $t \in [0, T]$. Therefore, in view of the Lipschitz condition of $\xi(\cdot)$, we have

$$E|\xi(t, n^\varepsilon(t))| \leq K(1 + E|n^\varepsilon(t)|) \leq K(1 + (E|n^\varepsilon(t)|^2)^{\frac{1}{2}}) = O(1).$$

Noting that $t_2 = 2\varepsilon^{1-\delta} = O(\varepsilon^{1-\delta})$, it follows that

$$\begin{aligned}\int_0^{\varsigma} & E(\xi(t, n^{\varepsilon}(t)) - \widetilde{\xi}(t))^2 dt \\ & = \sum_{k=2}^{N} \int_{t_k}^{t_{k+1}} E(\xi(t, n^{\varepsilon}(t)) - \widetilde{\xi}(t))^2 dt + O(\varepsilon^{1-\delta}).\end{aligned}$$

Using the definition of $\widetilde{\xi}(t)$, the Lipschitz property of $\xi(t,x)$ in (t,x), the choice of the partition of $[0,\varsigma]$, and Lemma 5.35, we have

$$\begin{aligned}\sum_{k=2}^{N} & \int_{t_k}^{t_{k+1}} E(\xi(t, n^{\varepsilon}(t)) - \widetilde{\xi}(t))^2 dt \\ & = \sum_{k=2}^{N} \int_{t_k}^{t_{k+1}} E(\xi(t, n^{\varepsilon}(t)) - \xi(t_{k-1}, n^{\varepsilon}(t_{k-1})))^2 dt \\ & \le 2 \sum_{k=2}^{N} \int_{t_k}^{t_{k+1}} K\left((t - t_{k-1})^2 + E|n^{\varepsilon}(t) - n^{\varepsilon}(t_{k-1}))|^2\right) dt \\ & \le 2 \sum_{k=2}^{N} \int_{t_k}^{t_{k+1}} K\left((t - t_{k-1})^2 + O(t - t_{k-1})\right) dt \\ & = 2 \sum_{k=2}^{N} \int_{t_k}^{t_{k+1}} O(\varepsilon^{1-\delta}) dt = O(\varepsilon^{1-\delta}).\end{aligned}$$

Let us estimate the second term on the second line in (5.88). Set

$$\widetilde{\eta}^{\varepsilon}(t) = E\left(\int_0^t W_{ij}^{\varepsilon}(s)\widetilde{\xi}(s)ds\right)^2.$$

Then the derivative of $\widetilde{\eta}^{\varepsilon}(t)$ is given by

$$\begin{aligned}\frac{d\widetilde{\eta}^{\varepsilon}(t)}{dt} & = 2E \int_0^t W_{ij}^{\varepsilon}(s)\widetilde{\xi}(s)W_{ij}^{\varepsilon}(t)\widetilde{\xi}(t)ds \\ & = 2 \int_0^t E\left(W_{ij}^{\varepsilon}(s)\widetilde{\xi}(s)W_{ij}^{\varepsilon}(t)\widetilde{\xi}(t)\right) ds.\end{aligned}$$

For $0 \le t \le t_2$, in view of the Lipschitz property and Theorem 5.25, we obtain

$$\begin{aligned}\int_0^{t_2} E\left(W_{ij}^{\varepsilon}(s)\widetilde{\xi}(s)W_{ij}^{\varepsilon}(t)\widetilde{\xi}(t)\right) ds & \le \int_0^{t_2} E\left(|\widetilde{\xi}(s)| \cdot |\widetilde{\xi}(t)|\right) ds \\ & \le \int_0^{t_2} (E|\widetilde{\xi}(s)|^2)^{\frac{1}{2}} (E|\widetilde{\xi}(t)|^2)^{\frac{1}{2}} ds \\ & = O(t_2) = O(\varepsilon^{1-\delta}).\end{aligned}$$

If $t_k \le t < t_{k+1}$, for $k = 2, \ldots, N$, then using the same argument gives us

$$\int_{k-1}^{t} E\left(W_{ij}^{\varepsilon}(s)\widetilde{\xi}(s)W_{ij}^{\varepsilon}(t)\widetilde{\xi}(t)\right) ds$$
$$= O(t - t_{k-1}) = O(t_{k+1} - t_{k-1}) = O(\varepsilon^{1-\delta})$$

and

$$\frac{d\widetilde{\eta}^{\varepsilon}(t)}{dt} = 2\int_{0}^{t_{k-1}} E\left(W_{ij}^{\varepsilon}(s)\widetilde{\xi}(s)W_{ij}^{\varepsilon}(t)\widetilde{\xi}(t)\right) ds + O(\varepsilon^{1-\delta}).$$

Recall that $\mathcal{F}_t^{\varepsilon} = \sigma\{\alpha^{\varepsilon}(s) : \ 0 \le s \le t\}$. For $s \le t_{k-1} < t_k \le t < t_{k+1}$,

$$\begin{aligned} &E\left(W_{ij}^{\varepsilon}(s)\widetilde{\xi}(s)W_{ij}^{\varepsilon}(t)\widetilde{\xi}(t)\right) \\ &\quad = E\left(W_{ij}^{\varepsilon}(s)\widetilde{\xi}(s)E[W_{ij}^{\varepsilon}(t)\widetilde{\xi}(t)|\mathcal{F}_{t_{k-1}}]\right). \end{aligned} \tag{5.89}$$

Moreover, in view of the definition of $\widetilde{\xi}(\cdot)$ and the proof of Lemma 5.35, we have for some $\kappa_0 > 0$,

$$\begin{aligned} E[W_{ij}^{\varepsilon}(t)\widetilde{\xi}(t)|\mathcal{F}_{t_{k-1}}] &= \widetilde{\xi}(t)E[W_{ij}^{\varepsilon}(t)|\mathcal{F}_{t_{k-1}}] \\ &= \widetilde{\xi}(t)O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t - t_{k-1})}{\varepsilon}\right)\right) \\ &= \widetilde{\xi}(t)O\left(\varepsilon + \exp\left(-\frac{\kappa_0(t_k - t_{k-1})}{\varepsilon}\right)\right) \\ &= \widetilde{\xi}(t)O\left(\varepsilon + \exp\left(-\frac{\kappa_0}{\varepsilon^{\delta}}\right)\right) = \widetilde{\xi}(t)O(\varepsilon). \end{aligned}$$

Combine this with (5.89) to obtain

$$E\left(W_{ij}^{\varepsilon}(s)\widetilde{\xi}(s)W_{ij}^{\varepsilon}(t)\widetilde{\xi}(t)\right) = O(\varepsilon)E|\widetilde{\xi}(s)\widetilde{\xi}(t)| = O(\varepsilon).$$

Therefore,

$$\frac{d\widetilde{\eta}^{\varepsilon}(t)}{dt} = O(\varepsilon^{1-\delta})$$

uniformly on $[0, T]$, which implies, together with $\widetilde{\eta}^{\varepsilon}(0) = 0$, that

$$\sup_{0\le\varsigma\le T} \widetilde{\eta}^{\varepsilon}(\varsigma) = \sup_{0\le\varsigma\le T} \int_0^{\varsigma} \left(\frac{d\widetilde{\eta}^{\varepsilon}(t)}{dt}\right) dt = O(\varepsilon^{1-\delta}).$$

This completes the proof. □

To characterize the limit of $(n^{\varepsilon}(\cdot), \overline{\alpha}^{\varepsilon}(\cdot))$, consider the martingale problem associated with $(n^{\varepsilon}(\cdot), \overline{\alpha}^{\varepsilon}(\cdot))$. Note that

$$\frac{dn^{\varepsilon}(t)}{dt} = \frac{1}{\sqrt{\varepsilon}} W(t, \alpha^{\varepsilon}(t)) \text{ and } n^{\varepsilon}(0) = 0,$$

where

$$W(t,\alpha)=(W_{11}(t,\alpha),\dots,W_{1m_1}(t,\alpha),\dots,W_{l1}(t,\alpha),\dots,W_{lm_l}(t,\alpha))\,.$$

Let $\mathcal{G}^\varepsilon(t)$ be the operator

$$\mathcal{G}^\varepsilon(t)f(t,x,\alpha)=\frac{\partial}{\partial t}f(t,x,\alpha)+\frac{1}{\sqrt{\varepsilon}}\langle W(t,\alpha),\nabla_x f(t,x,\alpha)\rangle$$
$$+Q^\varepsilon(t)f(t,x,\cdot)(\alpha),$$

for all $f(\cdot,\cdot,\alpha)\in C^{1,1}$, where ∇_x denotes the gradient with respect to x and $\langle\cdot,\cdot\rangle$ denotes the usual inner product in Euclidean space. It is well known that (see Davis [41, Chapter 2])

$$f(t,n^\varepsilon(t),\alpha^\varepsilon(t))-\int_0^t\mathcal{G}^\varepsilon(s)f(s,n^\varepsilon(s),\alpha^\varepsilon(s))ds \tag{5.90}$$

is a martingale.

We use the perturbed test function method (see Ethier and Kurtz [59] and Kushner [139]) to study the limit as $\varepsilon\to 0$. To begin with, we define a functional space on $\mathbb{R}^m\times\overline{\mathcal{M}}$

$$C_L^2=\Big\{f^0(x,i):\ \text{with bounded derivatives up to the}$$
$$\text{second order such that the second derivative is Lipschitz}\Big\}. \tag{5.91}$$

For any real-valued function $f^0(\cdot,i)\in C_L^2$, define

$$\overline{f}(x,\alpha)=\sum_{i=1}^l f^0(x,i)I_{\{\alpha\in\mathcal{M}_i\}}=\begin{cases}f^0(x,1), & \text{if } \alpha\in\mathcal{M}_1,\\ \vdots & \vdots\\ f^0(x,l), & \text{if } \alpha\in\mathcal{M}_l,\end{cases}$$

and consider the function

$$f(t,x,\alpha)=\overline{f}(x,\alpha)+\sqrt{\varepsilon}h(t,x,\alpha), \tag{5.92}$$

where $h(t,x,\alpha)$ is to be specified later. The main idea is that by appropriate choice of $h(\cdot)$, the perturbation is small and results in the desired cancelation in the calculation.

In view of the block-diagonal structure of $\widetilde{Q}(t)$ and the definition of $\overline{f}(x,\alpha)$, it is easy to see that

$$\widetilde{Q}(t)\overline{f}(x,\cdot)(\alpha)=0.$$

Applying the operator $\mathcal{G}^\varepsilon(t)$ to the function $f(\cdot)$ defined in (5.92) yields that

$$\begin{aligned}
&\overline{f}(n^\varepsilon(t),\alpha^\varepsilon(t)) + \sqrt{\varepsilon}h(t,n^\varepsilon(t),\alpha^\varepsilon(t)) \\
&\quad - \int_0^t \Big\{ \frac{1}{\sqrt{\varepsilon}} \big\langle W(s,\alpha^\varepsilon(s)), \nabla_x \overline{f}(n^\varepsilon(s),\alpha^\varepsilon(s)) + \sqrt{\varepsilon}\nabla_x h(s,n^\varepsilon(s),\alpha^\varepsilon(s)) \big\rangle \\
&\qquad + \sqrt{\varepsilon}\frac{\partial}{\partial s}h(s,n^\varepsilon(s),\alpha^\varepsilon(s)) + \frac{1}{\sqrt{\varepsilon}}\widetilde{Q}(s)h(s,n^\varepsilon(s),\cdot)(\alpha^\varepsilon(s)) \\
&\qquad + \widehat{Q}(s)(\overline{f}(n^\varepsilon(s),\cdot) + \sqrt{\varepsilon}h(s,n^\varepsilon(s),\cdot)(\alpha^\varepsilon(s)) \Big\} ds
\end{aligned}$$

defines a martingale.

The basic premise of the perturbed test function method is to choose the function $h(\cdot)$ that cancels the "bad" terms of order $1/\sqrt{\varepsilon}$:

$$\widetilde{Q}(s)h(s,x,\cdot)(\alpha) = -\big\langle W(s,\alpha), \nabla_x \overline{f}(x,\alpha) \big\rangle. \tag{5.93}$$

Note that as mentioned previously, $\widetilde{Q}(t)$ has rank $m-l$. Thus the dimension of the null space is l; that is, $N(\widetilde{Q}(t)) = l$. A crucial observation is that in view of the Fredholm alternative (see Lemma A.37 and Corollary A.38), a solution of (5.93) exists iff the matrix $(\langle W(s,s_{ij}), \nabla_x \overline{f}(x,s_{ij})\rangle)$ is orthogonal to $\widetilde{\mathbb{1}}_{m_1},\ldots,\widetilde{\mathbb{1}}_{m_l}$, the span of $N(\widetilde{Q}(t))$ (see Remark 4.23 for the notation). Moreover, since $f^0(\cdot,i)$ is C_L^2, $h(\cdot)$ can be chosen to satisfy the following properties assuming $\beta_{ij}(\cdot)$ to be Lipschitz on $[0,T]$:

(1) $h(t,x,\alpha)$ is uniformly Lipschitz in t;

(2) $|h(t,x,\alpha)|$ and $|\nabla_x h(t,x,\alpha)|$ are bounded;

(3) $\nabla_x h(t,x,\alpha)$ is Lipschitz in (t,x).

Such an $h(\cdot)$ leads to

$$\begin{aligned}
&\overline{f}(n^\varepsilon(t),\alpha^\varepsilon(t)) + \sqrt{\varepsilon}h(t,n^\varepsilon(t),\alpha^\varepsilon(t)) \\
&\quad - \int_0^t \Big\{ \big\langle W(s,\alpha^\varepsilon(s)), \nabla_x h(s,n^\varepsilon(s),\alpha^\varepsilon(s)) \big\rangle \\
&\qquad + \sqrt{\varepsilon}\Big(\frac{\partial}{\partial s}h(s,n^\varepsilon(s),\alpha^\varepsilon(s)) \Big) + \widehat{Q}(s)\overline{f}(n^\varepsilon(s),\cdot)(\alpha^\varepsilon(s)) \\
&\qquad + \sqrt{\varepsilon}\widehat{Q}(s)h(s,n^\varepsilon(s),\cdot)(\alpha^\varepsilon(s)) \Big\} ds
\end{aligned} \tag{5.94}$$

being a martingale. For each s, x, α, define

$$g(s,x,\alpha) = \big\langle W(s,\alpha), \nabla_x h(s,x,\alpha)\big\rangle. \tag{5.95}$$

With $f^0 \in C_L^2$, it is easy to see that $g(s,x,\alpha)$ is Lipschitz in (s,x). This function will be used in defining the operator for the limit problem later.

Remark 5.39. Note that the choice of $h(\cdot)$ in (5.93) is not unique. If $h_1(\cdot)$ and $h_2(\cdot)$ are both solutions to (5.93), then the irreducibility of $\widetilde{Q}^i(s)$ implies that, for each $i = 1, \ldots, l$,

$$\begin{pmatrix} h_1(s,x,s_{i1}) \\ \vdots \\ h_1(s,x,s_{im_i}) \end{pmatrix} - \begin{pmatrix} h_2(s,x,s_{i1}) \\ \vdots \\ h_2(s,x,s_{im_i}) \end{pmatrix} = h^0(s,x,i)\mathbb{1}_{m_i}$$

for some scalar functions $h^0(s,x,i)$. Although the choice of h is not unique, the resulting function $g(s,x,\alpha)$ is well defined. As in Remark 4.23, the consistency condition or solvability condition due to Fredholm alternative is in force. Therefore, if h_1 and h_2 are both solutions to (5.93), then

$$\big\langle W(s,\alpha), \nabla_x h_1(s,x,\alpha)\big\rangle = \big\langle W(s,\alpha), \nabla_x h_2(s,x,\alpha)\big\rangle,$$

for $\alpha \in \mathcal{M}_i$ and $i = 1, \ldots, l$.

Using $g(s,x,\alpha)$ defined above, we obtain

$$\begin{aligned}
&\int_0^t \big\langle W(s,\alpha^\varepsilon(s)), \nabla_x h(s,n^\varepsilon(s),\alpha^\varepsilon(s))\big\rangle ds \\
&\quad = \int_0^t g(s,n^\varepsilon(s),\alpha^\varepsilon(s))ds \\
&\quad = \int_0^t \sum_{i=1}^{l}\sum_{j=1}^{m_i} I_{\{\alpha^\varepsilon(s)=s_{ij}\}} g(s,n^\varepsilon(s),s_{ij})ds \\
&\quad = \int_0^t \sum_{i=1}^{l}\sum_{j=1}^{m_i} (I_{\{\alpha^\varepsilon(s)=s_{ij}\}} - \nu_j^i(s) I_{\{\overline{\alpha}^\varepsilon(s)=i\}}) g(s,n^\varepsilon(s),s_{ij})ds \\
&\qquad + \int_0^t \sum_{i=1}^{l}\sum_{j=1}^{m_i} I_{\{\overline{\alpha}^\varepsilon(s)=i\}} \nu_j^i(s) g(s,n^\varepsilon(s),s_{ij})ds.
\end{aligned}$$

In view of Lemma 5.37, the term in the fourth line above goes to zero in mean square uniformly in $t \in [0,T]$. Let

$$\overline{g}(s,x,i) = \sum_{j=1}^{m_i} \nu_j^i(s) g(s,x,s_{ij}).$$

Then it follows that

$$\begin{aligned}
&\int_0^t \sum_{i=1}^{l} \sum_{j=1}^{m_i} I_{\{\overline{\alpha}^\varepsilon(s)=i\}} \nu_j^i(s) g(s, n^\varepsilon(s), s_{ij}) ds \\
&\quad = \int_0^t \sum_{i=1}^{l} I_{\{\overline{\alpha}^\varepsilon(s)=i\}} \overline{g}(s, n^\varepsilon(s), i) ds \\
&\quad = \int_0^t \overline{g}(s, n^\varepsilon(s), \overline{\alpha}^\varepsilon(s)) ds.
\end{aligned}$$

Therefore, as $\varepsilon \to 0$, we have

$$\begin{aligned}
E\bigg| \int_0^t \big\langle W(s, \alpha^\varepsilon(s)), \nabla_x h(s, n^\varepsilon(s), \alpha^\varepsilon(s)) \big\rangle ds \\
- \int_0^t \overline{g}(s, n^\varepsilon(s), \overline{\alpha}^\varepsilon(s)) ds \bigg|^2 \to 0
\end{aligned} \tag{5.96}$$

uniformly in $t \in [0, T]$.

Furthermore, we have

$$\begin{aligned}
&\int_0^t \widehat{Q}(s) \overline{f}(n^\varepsilon(s), \cdot)(\alpha^\varepsilon(s)) ds \\
&\quad = \int_0^t \sum_{i=1}^{l} \sum_{j=1}^{m_i} I_{\{\alpha^\varepsilon(s)=s_{ij}\}} \widehat{Q}(s) \overline{f}(n^\varepsilon(s), \cdot)(s_{ij}) ds \\
&\quad = \int_0^t \sum_{i=1}^{l} \sum_{j=1}^{m_i} (I_{\{\alpha^\varepsilon(s)=s_{ij}\}} - \nu_j^i(s) I_{\{\overline{\alpha}^\varepsilon(s)=i\}}) \widehat{Q}(s) \overline{f}(n^\varepsilon(s), \cdot)(s_{ij}) ds \\
&\qquad + \int_0^t \sum_{i=1}^{l} \sum_{j=1}^{m_i} \nu_j^i(s) I_{\{\overline{\alpha}^\varepsilon(s)=i\}} \widehat{Q}(s) \overline{f}(n^\varepsilon(s), \cdot)(s_{ij}) ds.
\end{aligned}$$

Again, Lemma 5.37 implies that the third line above goes to 0 in mean square uniformly in $t \in [0, T]$. The last term above equals

$$\int_0^t \overline{Q}(s) f^0(n^\varepsilon(s), \cdot)(\overline{\alpha}^\varepsilon(s)) ds,$$

where $\overline{Q}(s) = \operatorname{diag}(\nu^1(t), \ldots, \nu^l(t)) \widehat{Q}(s) \widetilde{\mathbb{1}}$. It follows that as $\varepsilon \to 0$,

$$\begin{aligned}
E\bigg| \int_0^t \widehat{Q}(s) \overline{f}(n^\varepsilon(s), \cdot)(\alpha^\varepsilon(s)) ds \\
- \int_0^t \overline{Q}(s) f^0(n^\varepsilon(s), \cdot)(\overline{\alpha}^\varepsilon(s)) ds \bigg| \to 0
\end{aligned} \tag{5.97}$$

uniformly in $t \in [0, T]$.

We next examine the function $\overline{g}(s,x,i)$ closely. Using the block-diagonal structure of $\widetilde{Q}(s)$, we can write (5.93) in terms of each block $\widetilde{Q}^j(s)$. For $j=1,\ldots,l$,

$$\widetilde{Q}^j(s)\begin{pmatrix} h(s,x,s_{j1}) \\ \vdots \\ h(s,x,s_{jm_j}) \end{pmatrix} = -\begin{pmatrix} \langle W(s,s_{j1}), \nabla_x f^0(x,j)\rangle \\ \vdots \\ \langle W(s,s_{jm_j}), \nabla_x f^0(x,j)\rangle \end{pmatrix}. \tag{5.98}$$

Note that $\widetilde{Q}^j(s)$ is weakly irreducible so $\text{rank}(\widetilde{Q}^j(s)) = m_j - 1$. As in Remark 4.9, equation (5.98) has a solution since it is consistent and the solvability condition in the sense of Fredholm alternative is satisfied. We can solve (5.98) using exactly the same technique as in Section 4.2 for obtaining the $\varphi_i(t)$, that is, replacing one of the rows of the augmented matrix in (5.98) by $(1,1,\ldots,1,0)$, which represents the equation $\sum_{k=1}^{m_j} h(s,x,s_{jk}) = 0$. The coefficient matrix of the resulting equation then has full rank; one readily obtains a solution. Equivalently, the solution may be written as

$$\begin{pmatrix} h(s,x,s_{j1}) \\ \vdots \\ h(s,x,s_{jm_j}) \end{pmatrix} = - \left[\begin{pmatrix} \widetilde{Q}^j(s) \\ \mathbb{1}'_{m_j} \end{pmatrix}' \begin{pmatrix} \widetilde{Q}^j(s) \\ \mathbb{1}'_{m_j} \end{pmatrix}\right]^{-1} \begin{pmatrix} \widetilde{Q}^j(s) \\ \mathbb{1}'_{m_j} \end{pmatrix}' \begin{pmatrix} \langle W(s,s_{j1}), \nabla_x f^0(x,j)\rangle \\ \vdots \\ \langle W(s,s_{jm_j}), \nabla_x f^0(x,j)\rangle \\ 0 \end{pmatrix}.$$

Note that

$$I_{\{\alpha=s_{jk}\}} - \nu_k^j(t) I_{\{\alpha\in\mathcal{M}_j\}} = 0 \text{ if } \alpha \notin \mathcal{M}_j.$$

Recall the notation for the partitioned vector $x=(x^1,\ldots,x^l)$ where x^j is an m_j-dimensional vector and $x^j=(x_1^j,\ldots,x_{m_j}^j)$. For the partial derivatives, use the notation

$$\partial_{j,k} = \frac{\partial}{\partial x_k^j} \text{ and } \partial^2_{j,j_1j_2} = \frac{\partial^2}{\partial x_{j_1}^j \partial x_{j_2}^j}.$$

Then $h(s,x,s_{jk})$ is a functional of $\partial_{j,j1}f^0(x,j)$,...$\partial_{j,m_j}f^0(x,j)$. It follows that $g(s,x,s_{jk})$ is a functional of $\partial^2_{j,j_1j_2}f^0(x,j)$, for $j_1,j_2=1,\ldots,m_j$, and so is $\overline{g}(s,x,j)$. Write

$$\overline{g}(s,x,j) = \frac{1}{2}\sum_{j_1,j_2=1}^{m_j} a_{j_1j_2}(s,j)\partial^2_{j,j_1j_2}f^0(x,j), \tag{5.99}$$

for some continuous functions $a_{j_1j_2}(s,j)$.

Lemma 5.40. *Assume* (A5.5) *and* (A5.6). *Suppose* $(n^\varepsilon(\cdot), \overline{\alpha}^\varepsilon(\cdot))$ *converges weakly to* $(n(\cdot), \overline{\alpha}(\cdot))$. *Then for* $f^0(\cdot, i) \in C_L^2$,

$$f^0(n(t), \overline{\alpha}(t)) - \int_0^t \left(\overline{g}(s, n(s), \overline{\alpha}(s)) + \overline{Q}(s) f^0(n(s), \cdot)(\overline{\alpha}(s))\right) ds$$

is a martingale.

Proof: Define

$$\begin{aligned} H^\varepsilon(t) &= \overline{f}(n^\varepsilon(t), \alpha^\varepsilon(t)) + \sqrt{\varepsilon} h(t, n^\varepsilon(t), \alpha^\varepsilon(t)) \\ &\quad - \int_0^t \Big\{ \left\langle W(s, \alpha^\varepsilon(s)), \nabla_x h(s, n^\varepsilon(s), \alpha^\varepsilon(s)) \right\rangle \\ &\qquad + \sqrt{\varepsilon} \frac{\partial}{\partial s} h(s, n^\varepsilon(s), \alpha^\varepsilon(s)) + \widehat{Q}(s) \overline{f}(n^\varepsilon(s), \cdot)(\alpha^\varepsilon(s)) \\ &\qquad + \sqrt{\varepsilon} \widehat{Q}(s) h(s, n^\varepsilon(s), \cdot)(\alpha^\varepsilon(s)) \Big\} ds. \end{aligned}$$

The martingale property implies that

$$E\left[(H^\varepsilon(t) - H^\varepsilon(s)) z_1(n^\varepsilon(t_1), \overline{\alpha}^\varepsilon(t_1)) \cdots z_k(n^\varepsilon(t_k), \overline{\alpha}^\varepsilon(t_k))\right] = 0,$$

for any $0 \le t_1 \le \cdots \le t_k \le s \le t$ and any bounded and continuous functions $z_1(\cdot), \ldots, z_k(\cdot)$.

In view of the choice of $h(\cdot)$, it follows that all the three terms

$$\begin{aligned} &\sqrt{\varepsilon} h(t, n^\varepsilon(t), \alpha^\varepsilon(t)), \\ &\sqrt{\varepsilon} \left(\frac{\partial}{\partial t} h(t, n^\varepsilon(t), \alpha^\varepsilon(t)) \right), \text{ and} \\ &\sqrt{\varepsilon} \widehat{Q}(t) h(t, n^\varepsilon(t), \cdot)(\alpha^\varepsilon(t)) \end{aligned}$$

converge to 0 in mean square. Recall (5.96), (5.97), and

$$\overline{f}(n^\varepsilon(t), \alpha^\varepsilon(t)) = f^0(n^\varepsilon(t), \overline{\alpha}^\varepsilon(t)).$$

Denote the weak limit of $H^\varepsilon(\cdot)$ by $\overline{H}(\cdot)$. We have

$$E\left[\left(\overline{H}(t) - \overline{H}(s)\right) z_1(n(t_1), \overline{\alpha}(t_1)) \cdots z_k(n(t_k), \overline{\alpha}(t_k))\right] = 0,$$

where $\overline{H}(\cdot)$ is given by

$$\begin{aligned} \overline{H}(t) &= f^0(n(t), \overline{\alpha}(t)) \\ &\quad - \int_0^t \left(\overline{g}(r, n(r), \overline{\alpha}(r)) + \overline{Q}(r) f^0(n(r), \cdot)(\overline{\alpha}(r))\right) dr. \end{aligned}$$

Thus $(n(\cdot), \overline{\alpha}(\cdot))$ is a solution to the martingale problem. □

Lemma 5.41. *Let $\mathcal{L}$ denote the operator given by*

$$\mathcal{L}f^0(x,j) = \frac{1}{2}\sum_{j_1,j_2=1}^{m_j} a_{j_1j_2}(s,j)\partial^2_{j,j_1j_2}f^0(x,j) + \overline{Q}(s)f^0(x,\cdot)(j).$$

Then the martingale problem with operator $\mathcal{L}$ has a unique solution.

Proof: In view of Lemma A.14, we need only verify the uniqueness in distribution of $(n(t),\overline{\alpha}(t))$ for each $t\in[0,T]$. Let

$$f(x,j) = \exp\left(\iota\{\langle\theta,x\rangle + \theta_0 j\}\right),$$

where $\theta\in\mathbb{R}^m$, $\theta_0\in\mathbb{R}$, $j\in\mathcal{M}$, and ι is the pure imaginary number with $\iota^2=-1$.

For fixed j_0,k_0, let $F_{j_0k_0}(x,j) = I_{\{j=j_0\}}f(x,k_0)$. Then

$$F_{j_0k_0}(n(t),\overline{\alpha}(t)) = I_{\{\overline{\alpha}(t)=j_0\}}f(n(t),k_0).$$

Moreover, note that

$$\begin{aligned}
\overline{g}(s,n(s),\overline{\alpha}(s)) &= \sum_{j=1}^{l} I_{\{\overline{\alpha}(s)=j\}}\overline{g}(s,n(s),j)\\
&= \frac{1}{2}\sum_{j=1}^{l} I_{\{\overline{\alpha}(s)=j\}}\sum_{j_1,j_2=1}^{m_j} a_{j_1j_2}(s,j)\partial^2_{j,j_1j_2}F_{j_0k_0}(n(s),j)\\
&= \frac{1}{2}I_{\{\overline{\alpha}(s)=j_0\}}\sum_{j_1,j_2=1}^{m_{j_0}} a_{j_1j_2}(s,j_0)\partial^2_{j_0,j_1j_2}f(n(s),k_0)\\
&= \frac{1}{2}\sum_{j_1,j_2=1}^{m_j} a_{j_1j_2}(s,j_0)(-\theta_{j_0j_1}\theta_{j_0j_2})(I_{\{\overline{\alpha}(s)=j_0\}}f(n(s),k_0)).
\end{aligned} \tag{5.100}$$

Furthermore, we have

$$\begin{aligned}
&\overline{Q}(s)F_{j_0k_0}(n(s),\cdot)(\overline{\alpha}(s))\\
&\quad= \sum_{j=1}^{l} I_{\{\overline{\alpha}(s)=j\}}\overline{Q}(s)F_{j_0k_0}(n(s),\cdot)(j)\\
&\quad= \sum_{j=1}^{l} I_{\{\overline{\alpha}(s)=j\}}\sum_{k=1}^{l}\overline{q}_{jk}(s)F_{j_0k_0}(n(s),k)\\
&\quad= \sum_{j=1}^{l} I_{\{\overline{\alpha}(s)=j\}}\sum_{k=1}^{l}\overline{q}_{jk}(s)I_{\{k=j_0\}}f(n(s),k_0)\\
&\quad= \sum_{j=1}^{l} I_{\{\overline{\alpha}(s)=j\}}\overline{q}_{jj_0}(s)f(n(s),k_0)\\
&\quad= \sum_{j=1}^{l} \overline{q}_{jj_0}(s)I_{\{\overline{\alpha}(s)=j\}}f(n(s),k_0).
\end{aligned} \tag{5.101}$$

Let

$$\phi_{jk}(t) = E\left(I_{\{\overline{\alpha}(t)=j\}} f(n(t),k)\right), \quad \text{for} \quad j,k = 1,\dots,l.$$

Then in view of (5.100) and (5.101),

$$\begin{aligned}\phi_{j_0k_0}(t) - \phi_{j_0k_0}(0) - \int_0^t \Bigg\{ &\sum_{j_1,j_2=1}^{m_{j_0}} a_{j_1j_2}(s,j_0)(-\theta_{j_0j_1}\theta_{j_0j_2})\phi_{j_0k_0}(s) \\ &+ \sum_{j=1}^{l} \overline{q}_{jj_0}(s)\phi_{jk_0}(s)\Bigg\} ds = 0.\end{aligned} \tag{5.102}$$

Let

$$\phi(t) = (\phi_{11}(t),\dots,\phi_{1m_1}(t),\dots,\phi_{l1}(t),\dots,\phi_{lm_l}(t)).$$

Rewrite (5.102) in terms of $\phi(\cdot)$ as

$$\phi(t) = \phi(0) + \int_0^t \phi(s)B(s)ds,$$

where $\phi(0) = (\phi_{jk}(0))$ with $\phi_{jk}(0) = EI_{\{\overline{\alpha}(0)=j\}} f(0,k)$, and $B(t)$ is a matrix-valued function whose entries are defined by the integrand of (5.102). The equation for $\phi(t)$ is a linear ordinary differential equation. It is well known that such a differential equation has a unique solution. Hence, $\phi(t)$ is uniquely determined. In particular,

$$\begin{aligned}&E\exp\left(\iota\{\langle\theta,n(t)\rangle + \theta_0\overline{\alpha}(t)\}\right) \\ &\quad = \sum_{j=1}^{l} E\left(I_{\{\overline{\alpha}(t)=j\}} \exp\left(\iota\{\langle\theta,n(t)\rangle + j\theta_0\}\right)\right)\end{aligned}$$

is uniquely determined for all θ, θ_0, so is the distribution of $(n(t),\overline{\alpha}(t))$ by virtue of the uniqueness theorem and the inversion formula of the characteristic function (see Chow and Teicher [30]). □

The tightness of $(n^\varepsilon(\cdot),\overline{\alpha}^\varepsilon(\cdot))$ together with Lemma 5.40 and Lemma 5.41 implies that $(n^\varepsilon(\cdot),\overline{\alpha}^\varepsilon(\cdot))$ converges weakly to $(n(\cdot),\overline{\alpha}(\cdot))$. We will show that $n(\cdot)$ is a switching diffusion, i.e., a diffusion process modulated by a Markov process such that the covariance of the diffusion depends on the Markov jump process. Precisely, owing to the presence of the jump Markov chains, the limit process does not possess the independent increment property shared by many processes. A moment of reflection reveals that, necessarily, the coefficients in $\overline{g}(s,x,i)$ must consist of a symmetric nonnegative definite matrix serving as a covariance matrix. The following lemma verifies this assertion.

Lemma 5.42. *For $s \in [0,T]$ and $j = 1, \ldots, l$, the matrix*

$$A(s,j) = (a_{j_1 j_2}(s,j))$$

is symmetric and nonnegative definite.

Proof: Let $\eta^j = (\eta_{j1}, \ldots, \eta_{jm_j})'$ and $x^j = (x_{j1}, \ldots, x_{jm_j})'$. Define

$$f_j(x) = \frac{1}{2}\left(\langle \eta^j, x^j \rangle\right)^2.$$

Then the corresponding $\overline{g}(\cdot)$ defined in (5.99) has the following form:

$$\overline{g}(s,x,j) = \frac{1}{2}\eta^{j,\prime} A(s,j)\eta^j.$$

Moreover, let $f_j(x,k) = f_j(x)$, independent of k. Then for all $k = 1, \ldots, l$,

$$\overline{Q}(s) f_j(n^\varepsilon(s), \cdot)(k) = 0.$$

To verify the nonnegativity of $A(s,j)$, it suffices to show that

$$\int_s^t \eta^{j,\prime} A(r,j)\eta^j dr \geq 0,$$

for all $0 \leq s \leq t \leq T$. Recall that $f_j(x)$ is a quadratic function. In view of (5.94) and the proof of Lemma 5.40, it then follows that

$$\frac{1}{2}\int_s^t \eta^{j,\prime} A(r,j)\eta^j dr = \lim_{\varepsilon \to 0} \left(E f_j(n^\varepsilon(t)) - E f_j(n^\varepsilon(s))\right).$$

We are in a position to show that the limit is nonnegative. Let

$$n^{\varepsilon,j}(t) = (n^\varepsilon_{j1}(t), \ldots, n^\varepsilon_{jm_j}(t)).$$

Then

$$\begin{aligned} &E\left(f_j(n^\varepsilon(t)) - f_j(n^\varepsilon(s))\right) \\ &\qquad = \frac{1}{2} E\left(\left\langle \eta^j, n^{\varepsilon,j}(t)\right\rangle^2 - \left\langle \eta^j, n^{\varepsilon,j}(s)\right\rangle^2\right). \end{aligned}$$

For $t \geq s \geq 0$, using

$$\left\langle \eta^j, n^{\varepsilon,j}(t)\right\rangle = \left\langle \eta^j, n^{\varepsilon,j}(s)\right\rangle + \left\langle \eta^j, n^{\varepsilon,j}(t) - n^{\varepsilon,j}(s)\right\rangle,$$

we have

$$\begin{aligned} &E\left(\left\langle \eta^j, n^{\varepsilon,j}(t)\right\rangle^2 - \left\langle \eta^j, n^{\varepsilon,j}(s)\right\rangle^2\right) \\ &\quad = E\left(2\left\langle \eta^j, n^{\varepsilon,j}(s)\right\rangle\left\langle \eta^j, n^{\varepsilon,j}(t) - n^{\varepsilon,j}(s)\right\rangle + \left\langle \eta^j, n^{\varepsilon,j}(t) - n^{\varepsilon,j}(s)\right\rangle^2\right) \\ &\quad \geq 2E\left(\left\langle \eta^j, n^{\varepsilon,j}(s)\right\rangle\left\langle \eta^j, n^{\varepsilon,j}(t) - n^{\varepsilon,j}(s)\right\rangle\right) \\ &\quad = 2E\left(\left\langle \eta^j, n^{\varepsilon,j}(s)\right\rangle E\left[\left\langle \eta^j, n^{\varepsilon,j}(t) - n^{\varepsilon,j}(s)\right\rangle \middle| \mathcal{F}^\varepsilon_s\right]\right). \end{aligned}$$

We next show that the last term goes to 0 as $\varepsilon \to 0$. In fact, in view of (a) in Lemma 5.35, it follows that

$$E[n^{\varepsilon,j}(t) - n^{\varepsilon,j}(s)|\mathcal{F}_s^\varepsilon] = O(\sqrt{\varepsilon}),$$

and hence

$$E\left[\left\langle \eta^j, n^{\varepsilon,j}(t) - n^{\varepsilon,j}(s)\right\rangle \middle| \mathcal{F}_s^\varepsilon\right]$$
$$= \left\langle \eta^j, E[(n^{\varepsilon,j}(t) - n^{\varepsilon,j}(s))|\mathcal{F}_s^\varepsilon]\right\rangle = O(\sqrt{\varepsilon}).$$

Using (b) in Lemma 5.35, we derive the following inequalities

$$E\left\langle \eta^j, n^{\varepsilon,j}(s)\right\rangle^2 \le |\eta^j|^2 E|n^{\varepsilon,j}(s)|^2 \le |\eta^j|^2 O(s).$$

The Cauchy–Schwarz inequality then leads to

$$\left|E\left(\left\langle \eta^j, n^{\varepsilon,j}(s)\right\rangle E\left[\left\langle \eta^j, n^{\varepsilon,j}(t) - n^{\varepsilon,j}(s)\right\rangle \middle| \mathcal{F}_s^\varepsilon\right]\right)\right|$$
$$\le \left(E\left\langle \eta^j, n^{\varepsilon,j}(s)\right\rangle^2\right)^{\frac{1}{2}} \left(E\left[\left\langle \eta^j, n^{\varepsilon,j}(t) - n^{\varepsilon,j}(s)\right\rangle \middle| \mathcal{F}_s^\varepsilon\right]^2\right)^{\frac{1}{2}}$$
$$= \left(E\left\langle \eta^j, n^{\varepsilon,j}(s)\right\rangle^2\right)^{\frac{1}{2}} O(\sqrt{\varepsilon}) \to 0, \quad \text{as } \varepsilon \to 0.$$

As a result for some $K > 0$, we have

$$E\left(\left\langle \eta^j, n^{\varepsilon,j}(s)\right\rangle E\left[\left\langle \eta^j, n^{\varepsilon,j}(t) - n^{\varepsilon,j}(s)\right\rangle \middle| \mathcal{F}_s^\varepsilon\right]\right) \ge -K|\eta^j|s\sqrt{\varepsilon} \to 0,$$

as $\varepsilon \to 0$. The nonnegativity of $A(s,j)$ follows.

To show that $A(s,j)$ is symmetric, consider

$$f_{j,j_1j_2}(x) = x_{jj_1}x_{jj_2} \text{ for } j_1, j_2 = 1, \ldots, m_j.$$

Then, we have

$$\frac{1}{2}\int_0^t a_{j_1j_2}(s,j)ds = \lim_{\varepsilon\to 0} E(n_{j_1}^{\varepsilon,j}(t)n_{j_2}^{\varepsilon,j}(t)) = \frac{1}{2}\int_0^t a_{j_2j_1}(s,j)ds, \quad (5.103)$$

for all $t \in [0,T]$. Thus, $A(s,j)$ is symmetric. □

Next, we derive an explicit representation of the nonnegative definite matrix $A(s,j)$ similar to that of Theorem 5.9. Recall that given a function $f^0(\cdot)$, one can find $h(\cdot)$ as in (5.93). Using this $h(\cdot)$, one defines $f(\cdot)$ as in

(5.95) which leads to $\overline{g}(\cdot)$ given in (5.99). In view of the result in Theorem 5.9 for a single block of the irreducible matrix $\widetilde{Q}^j(t)$ together with the computations of $\overline{g}(s,x,j)$, it follows that $A(s,j) = 2A^0(s,j)$, where

$$A^0(t,j) = \beta^j_{\text{diag}}(t)\Bigg(\nu^j_{\text{diag}}(t)\int_0^\infty Q_0(r,t,j)dr + \Bigg(\int_0^\infty Q_0(r,t,j)dr\Bigg)\nu^j_{\text{diag}}(t)\Bigg)\beta^j_{\text{diag}}(t),$$

with

$$\beta^j_{\text{diag}}(t) = \text{diag}(\beta_{j1}(t),\ldots,\beta_{jm_j}(t)),$$

$$\nu^j_{\text{diag}}(t) = \text{diag}(\nu^j_1(t),\ldots,\nu^j_{m_j}(t)),$$

and

$$Q_0(r,t,j) = \left[I - \begin{pmatrix}\nu^j(t)\\ \vdots \\ \nu^j(t)\end{pmatrix}\right]\exp\left(\widetilde{Q}^j(t)r\right).$$

Applying Lemma 5.42 to the case of $\widetilde{Q}(s)$ a single block irreducible matrix $\widetilde{Q}^j(s)$, it follows that $A^0(s,j)$ is symmetric and nonnegative definite. Hence, standard results in linear algebra yield that there exists an $m_j \times m_j$ matrix $\sigma^0(s,j)$ such that

$$\sigma^0(s,j)\sigma^{0,\prime}(s,j) = A^0(s,j). \tag{5.104}$$

Note that the definition of $\overline{g}(s,x,j)$ is independent of $\widehat{Q}(t)$, so for determining $A^0(s,j)$, we may consider $\widehat{Q}(t) = 0$. Note also that

$$\widetilde{Q}(t) = \text{diag}(\widetilde{Q}^1(t),0,\ldots,0) + \cdots + \text{diag}(0,\ldots,0,\widetilde{Q}^l(t)).$$

The foregoing statements suggest that in view of (5.104), the desired covariance matrix is given by

$$\sigma(s,j) = \begin{pmatrix} 0_{m_1\times m_1} & & & & & \\ & 0_{m_2\times m_2} & & & & \\ & & \ddots & & & \\ & & & \sigma^0(s,j) & & \\ & & & & \ddots & \\ & & & & & 0_{m_l\times m_l}\end{pmatrix} \tag{5.105}$$

$$= \text{diag}(0_{m_1\times m_1}, 0_{m_2\times m_2},\ldots,\sigma^0(s,j)\ldots,0_{m_l\times m_l}),$$

where $0_{m_k\times m_k}$ is the $m_k \times m_k$ zero matrix. That is, it is a matrix with the jth block-diagonal submatrix equal to $\sigma^0(s,j)$ and the rest of its elements equal to zero.

Theorem 5.43. *Assume that* (A5.5) *holds. Suppose* $\widetilde{Q}(\cdot)$ *is twice differentiable with Lipschitz continuous second derivative and* $\widehat{Q}(\cdot)$ *is differentiable with Lipschitz continuous derivative. Let* $\beta_{ij}(\cdot)$ *be bounded and Lipschitz continuous deterministic functions. Then* $n^\varepsilon(\cdot)$ *converges weakly to a switching diffusion* $n(\cdot)$*, where*

$$n(t) = \Big(\int_0^t \sigma(s, \overline{\alpha}(s)) dw(s) \Big)'$$

and $w(\cdot)$ *is a standard* m*-dimensional Brownian motion.*

Proof: Let

$$\widetilde{n}(t) = \Big(\int_0^t \sigma(s, \overline{\alpha}(s)) dw(s) \Big)'$$

and $\overline{\alpha}(\cdot)$ be a Markov chain generated by $\overline{Q}(t)$. Then for all $f^0(\cdot, i) \in C_L^2$,

$$f^0(\widetilde{n}(t), \overline{\alpha}(t)) - \int_0^t \big(\overline{g}(s, \widetilde{n}(s), \overline{\alpha}(s)) + \overline{Q}(s) f^0(\widetilde{n}(s), \cdot)(\overline{\alpha}(s)) \big)\, ds$$

is a martingale. This and the uniqueness of the martingale problem in Lemma 5.41 yields that $(\widetilde{n}(\cdot), \overline{\alpha}(\cdot))$ has the same probability distribution as $(n(\cdot), \overline{\alpha}(\cdot))$. This proves the theorem. □

Remark 5.44. Note that the Lipschitz condition on $\beta_{ij}(\cdot)$ is not required in analyzing the asymptotic normality in Section 5.3.3. It is needed in this section because the perturbed test function method typically requires smoothness conditions of the associated processes.

It appears that the conditions in (A5.5) and (A5.6) together with the Lipschitz property of $\beta_{ij}(\cdot)$ are sufficient for the convergence of $n^\varepsilon(\cdot)$ to a switching diffusion $n(\cdot)$. The additional assumptions on further derivatives of $\widetilde{Q}(\cdot)$ and $\widehat{Q}(\cdot)$ are needed for computing the covariance of the limit process $n(\cdot)$.

Remark 5.45. If $\overline{\alpha}(\cdot)$ were a deterministic function, $n(\cdot)$ above would be a diffusion process in the usual sense. However since the limit $\overline{\alpha}(\cdot)$ is a Markov chain, the diffusion process is modulated by this jump process; the resulting distribution has the features of the "continuous" diffusion process and the "discrete" Markov chain limit.

In this section, we use the perturbed test function method, which is quite different from the approach of Section 5.2. The method used in that section, which might be called a direct approach, is interesting in its own right and makes a close connection between asymptotic expansion and asymptotic normality. It is effective whenever it can be applied. One of the main ingredients is that the direct approach makes use of the mixing properties of the scaled occupation measures heavily. In fact, using asymptotic expansion, it was shown that the scaled sequence of occupation measures is a mixing

process with exponential mixing rate. For the weak and strong interaction cases presented, the mixing condition, and even approximate mixing conditions, no longer hold. To illustrate, consider Example 4.20 with constant jump rates and calculate

$$E[n^{\varepsilon,\prime}(s)(n^{\varepsilon}(t) - n^{\varepsilon}(s))].$$

By virtue of the proof of Theorem 5.25, a straightforward but tedious calculation shows that

$$E\left[n^{\varepsilon,\prime}(s)(n^{\varepsilon}(t) - n^{\varepsilon}(s))\right] \not\to 0 \quad \text{as } \varepsilon \to 0$$

for the weak and strong interaction models because $E[n^{\varepsilon,\prime}(s)(n^{\varepsilon}(t)-n^{\varepsilon}(s))]$ depends on $P_1(t,s)$, generally a nonzero function. A direct consequence is that the limit process does *not* have independent increments in general. It is thus difficult to characterize the limit process via the direct approach. The perturbed test function method, on the other hand, can be considered as a combined approach. It uses enlarged or augmented states by treating the scaled occupation measure $n^{\varepsilon}(\cdot)$ and the Markov chain $\alpha^{\varepsilon}(\cdot)$ together. That is, one considers a new state variable with two components (x, α). This allows us to bypass the verification of mixing-like properties such that the limit process is characterized by means of solutions of appropriate martingale problems via perturbed test functions, which underlies the rationale and essence of the approach. As a consequence, the limit process is characterized via the limit of the underlying sequence of operators.

Note that if $\widetilde{Q}(t)$ itself is weakly irreducible (i.e., $\widetilde{Q}(t)$ consists of only one block), then the covariance matrix is given by (5.30). In this case, since there is only one group of recurrent states, the jump behavior due to the limit process $\overline{\alpha}(\cdot)$ will disappear. Moreover, owing to the fast transition rate $\widetilde{Q}(t)/\varepsilon$, the singularly perturbed Markov chain rapidly reaches its quasi-stationary regime. As a result, the jump behavior does not appear in the asymptotic distribution, and the diffusion becomes the dominant factor. Although the method employed in this chapter is different from that of Section 5.2, the result coincides with that of Section 5.2 under irreducibility. We state this in the following corollary.

Corollary 5.46. *Assume that the conditions of Theorem 5.43 are fulfilled with $l = 1$ (i.e., $\widetilde{Q}(t)$ has only one block). Then $n^{\varepsilon}(\cdot)$ converges weakly to the diffusion process*

$$n(t) = \Big(\int_0^t \sigma(s) dw(s) \Big)',$$

where $w(\cdot)$ is an m-dimensional standard Brownian motion with covariance

$$A(t) = \sigma(t)\sigma'(t)$$

given by (5.30).

To further illustrate, consider the following example. This problem is concerned with a singularly perturbed Markov chain with four states divided into two groups. It has been used in modeling production planning problems with failure-prone machines. As was mentioned, from a modeling point of view, it may be used to depict the situation that two machines operate in tandem, in which the operating conditions (the machine capacity) of one of the machines change much faster than the other; see also related discussions in Chapters 7 and 8.

Example 5.47. Let $\alpha^\varepsilon(\cdot)$ be a Markov chain generated by

$$Q^\varepsilon(t) = \frac{1}{\varepsilon}\begin{pmatrix} -\lambda_1(t) & \lambda_1(t) & 0 & 0 \\ \mu_1(t) & -\mu_1(t) & 0 & 0 \\ 0 & 0 & -\lambda_1(t) & \lambda_1(t) \\ 0 & 0 & \mu_1(t) & -\mu_1(t) \end{pmatrix}$$

$$+ \begin{pmatrix} -\lambda_2(t) & 0 & \lambda_2(t) & 0 \\ 0 & -\lambda_2(t) & 0 & \lambda_2(t) \\ \mu_2(t) & 0 & -\mu_2(t) & 0 \\ 0 & \mu_2(t) & 0 & -\mu_2(t) \end{pmatrix}.$$

Then

$$\overline{Q}(t) = \begin{pmatrix} -\lambda_2(t) & \lambda_2(t) \\ \mu_2(t) & -\mu_2(t) \end{pmatrix}.$$

Let $\overline{\alpha}(\cdot)$ be a Markov chain generated by $\overline{Q}(t)$, $t \geq 0$. In this example,

$$\sigma^0(s,1) = 2\left(\frac{\lambda_1(s)\mu_1(s)}{(\lambda_1(s)+\mu_1(s))^3}\right)^{\frac{1}{2}} \begin{pmatrix} \beta_{11}(s) & 0 \\ -\beta_{12}(s) & 0 \end{pmatrix},$$

$$\sigma^0(s,2) = 2\left(\frac{\lambda_1(s)\mu_1(s)}{(\lambda_1(s)+\mu_1(s))^3}\right)^{\frac{1}{2}} \begin{pmatrix} \beta_{21}(s) & 0 \\ -\beta_{22}(s) & 0 \end{pmatrix},$$

$$\sigma(s,1) = 2\left(\frac{\lambda_1(s)\mu_1(s)}{(\lambda_1(s)+\mu_1(s))^3}\right)^{\frac{1}{2}} \begin{pmatrix} \beta_{11}(s) & 0 & 0 & 0 \\ -\beta_{12}(s) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\sigma(s,2) = 2\left(\frac{\lambda_1(s)\mu_1(s)}{(\lambda_1(s)+\mu_1(s))^3}\right)^{\frac{1}{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_{21}(s) & 0 \\ 0 & 0 & -\beta_{22}(s) & 0 \end{pmatrix}.$$

The limit of $n^\varepsilon(\cdot)$ is given by

$$n(t) = \Big(\int_0^t \sigma(s, \overline{\alpha}(s)) dw(s) \Big)',$$

where $w(\cdot)$ is a standard Brownian motion taking values in $\mathbb{R}^4$.

5.4 Measurable Generators

In Section 4.2, we considered the asymptotic expansions of probability distributions. A natural requirement of such expansions is that the generator $Q^\varepsilon(t)$ be smooth enough to establish the desired error bounds. It would be interesting to consider the case in which the generator $Q^\varepsilon(t)$, $t \geq 0$, is merely measurable. The method used in this section is very useful in some manufacturing problems; see Sethi and Zhang [192]. Moreover, the results are used in Section 8.6 to deal with a control problem under relaxed control formulation. Given only the measurability of $Q^\varepsilon(t)$, there seems to be little hope to obtain an asymptotic expansion. Instead of constructing an asymptotic series of the corresponding probability distribution, we consider the convergence of $P(\alpha^\varepsilon(t) = s_{ij})$ under the framework of convergence of

$$\int_0^T P(\alpha^\varepsilon(t) = s_{ij}) f(t) dt \quad \text{for} \quad f(\cdot) \in L^2[0, T]; \mathbb{R}).$$

Since the phrase "weak convergence" is reserved throughout the book for the convergence of probability measures, to avoid confusion, we refer to the convergence above as convergence in the weak sense on $L^2([0, T]; \mathbb{R})$ or convergence under the weak topology of $L^2([0, T]; \mathbb{R})$.

Case I: Weakly Irreducible $\widetilde{Q}(t)$

Let $\alpha^\varepsilon(\cdot) \in \mathcal{M} = \{1, \ldots, m\}$ denote the Markov chain generated by

$$Q^\varepsilon(t) = \frac{1}{\varepsilon} \widetilde{Q}(t) + \widehat{Q}(t),$$

where both $\widetilde{Q}(t)$ and $\widehat{Q}(t)$ are generators.

We assume the following conditions in this subsection.

(**A5.7**) $\widetilde{Q}(t)$ and $\widehat{Q}(t)$ are bounded and Borel measurable. Moreover, $\widetilde{Q}(t)$ is weakly irreducible.

Remark 5.48. In fact, both the boundedness and the Borel measurability in (A5.7) are redundant. Recall that our definition of generators (see Definition 2.2) uses the q-Property, which includes both the Borel measurability and the boundedness. Thus, (A5.7) requires only weak irreducibility.

Nevertheless, we retain both boundedness and measurability for those who read only this section. Similar comments apply to assumption (A5.8) in what follows.

Define the probability distribution vector

$$p^\varepsilon(t) = (P(\alpha^\varepsilon(t) = 1), \ldots, P(\alpha^\varepsilon(t) = m))$$

and the transition matrix

$$P^\varepsilon(t,s) = (p_{ij}^\varepsilon(t,s)) = (P(\alpha^\varepsilon(t) = j | \alpha^\varepsilon(s) = i)).$$

Then using the martingale property in Lemma 2.4, we have

$$p^\varepsilon(t) = p^\varepsilon(s) + \int_s^t p^\varepsilon(r) Q^\varepsilon(r) dr \tag{5.106}$$

and

$$P^\varepsilon(t,s) = I + \int_s^t P^\varepsilon(r,s) Q^\varepsilon(r) dr. \tag{5.107}$$

The next two lemmas are concerned with the asymptotic properties of $p^\varepsilon(t)$ and $P^\varepsilon(t,s)$.

Lemma 5.49. *Assume* (A5.7). *Then for each i, j, and $T > 0$, $P(\alpha^\varepsilon(t) = i)$ and $P(\alpha^\varepsilon(t) = i | \alpha^\varepsilon(s) = j)$ both converge weakly to $\nu_i(t)$ on $L^2([0,T];\mathbb{R})$ and $L^2([s,T];\mathbb{R})$, respectively, that is, as $\varepsilon \to 0$,*

$$\int_0^T [P(\alpha^\varepsilon(t) = i) - \nu_i(t)] f(t) dt \to 0 \tag{5.108}$$

and

$$\int_s^T [P(\alpha^\varepsilon(t) = i | \alpha^\varepsilon(s) = j) - \nu_i(t)] f(t) dt \to 0, \tag{5.109}$$

for all $f(\cdot) \in L^2([0,T];\mathbb{R})$ and $L^2([s,T];\mathbb{R})$, respectively.

Proof: We only verify (5.108); the proof of (5.109) is similar. Recall that

$$p^\varepsilon(t) = (p_1^\varepsilon(t), \ldots, p_m^\varepsilon(t)) = (P(\alpha^\varepsilon(t) = 1), \ldots, P(\alpha^\varepsilon(t) = m)).$$

Since $p^\varepsilon(\cdot) \in L^2([0,T];\mathbb{R}^m)$ (space of square-integrable functions on $[0,T]$ taking values in $\mathbb{R}^m$), for each subsequence of $\varepsilon \to 0$ there exists (see Lemma A.36) a further subsequence of $\varepsilon \to 0$ (still denoted by ε for simplicity), and for such ε, the corresponding $\{p^\varepsilon(\cdot)\}$ converges (in the weak sense on $L^2([0,T];\mathbb{R}^m)$) to some $p(\cdot) = (p_1(\cdot), \ldots, p_m(\cdot)) \in L^2([0,T];\mathbb{R}^m)$, that is,

$$\int_0^T p^\varepsilon(r)(f_1(r), \ldots, f_m(r))' dr \to \int_0^T p(r)(f_1(r), \ldots, f_m(r))' dr,$$

for any $(f_1(\cdot), \ldots, f_m(\cdot))' \in L^2([0,T];\mathbb{R}^m)$. Moreover,

$$0 \le p_i(t) \le 1 \quad \text{and} \quad p_1(t) + \cdots + p_m(t) = 1 \tag{5.110}$$

almost everywhere. Since $\widetilde{Q}(\cdot) \in L^2([0,T];\mathbb{R}^{m\times m})$, we have for $0 \le s \le t \le T$,

$$\int_s^t p^\varepsilon(r)\widetilde{Q}(r)dr \to \int_s^t p(r)\widetilde{Q}(r)dr.$$

Thus, using (5.106) we obtain

$$\begin{aligned}\int_s^t p(r)\widetilde{Q}(r)dr &= \lim_{\varepsilon\to 0}\int_s^t p^\varepsilon(r)\widetilde{Q}(r)dr \\ &= \lim_{\varepsilon\to 0}\left(\varepsilon(p^\varepsilon(t) - p^\varepsilon(s)) - \varepsilon\int_s^t p^\varepsilon(r)\widehat{Q}(r)dr\right) = 0.\end{aligned}$$

Since s and t are arbitrary, it follows immediately that

$$p(t)\widetilde{Q}(t) = 0 \text{ a.e. in } t.$$

By virtue of (5.110), the irreducibility of $\widetilde{Q}(t)$ implies $p(t) = \nu(t)$ almost everywhere. Thus the limit is independent of the chosen subsequence. Therefore, $p^\varepsilon(\cdot) \to \nu(\cdot)$ in the weak sense on $L^2([0,T];\mathbb{R}^m)$. □

Theorem 5.50. *Assume* (A5.7). *Then for any bounded deterministic function* $\beta_i(\cdot)$ *and for each* $i \in \mathcal{M}$ *and* $t \ge 0$,

$$E\left|\int_0^t (I_{\{\alpha^\varepsilon(s)=i\}} - \nu_i(s))\beta_i(s)ds\right|^2 \to 0 \text{ as } \varepsilon \to 0. \tag{5.111}$$

Proof: Let

$$\eta(t) = E\left|\int_0^t (I_{\{\alpha^\varepsilon(s)=i\}} - \nu_i(s))\beta_i(s)ds\right|^2.$$

Then as in the proof of Theorem 5.25, we can show that

$$\eta(t) = 2(\eta_1(t) + \eta_2(t)),$$

where

$$\begin{aligned}\eta_1(t) &= \int_0^t\int_0^s (-\nu_i(r))[P(\alpha^\varepsilon(s) = i) - \nu_i(s)]\beta_i(s)\beta_i(r)drds, \\ \eta_2(t) &= \int_0^t\int_0^s P(\alpha^\varepsilon(r) = i)[P(\alpha^\varepsilon(s) = i|\alpha^\varepsilon(r) = i) - \nu_i(s)] \\ &\qquad \times\beta_i(s)\beta_i(r)drds.\end{aligned}$$

By virtue of Lemma 5.49, $P(\alpha^\varepsilon(s) = i) \to \nu_i(s)$ in the weak sense on $L^2([0,T];\mathbb{R})$ and therefore as $\varepsilon \to 0$,

$$\eta_1(t) = \int_0^t [P(\alpha^\varepsilon(s) = i) - \nu_i(s)]\beta_i(s) \left(\int_0^s (-\nu_i(r))\beta_i(r)dr \right) ds \to 0.$$

Similarly, in view of the convergence of

$$P(\alpha^\varepsilon(s) = i | \alpha^\varepsilon(r) = i) \to \nu_i(s)$$

under the weak topology of $L^2([r,t];\mathbb{R})$, we have

$$\eta_2(t) = \int_0^t \left[\int_r^t [P(\alpha^\varepsilon(s) = i | \alpha^\varepsilon(r) = i) - \nu_i(s)]\beta_i(s)ds \right] \times P(\alpha^\varepsilon(r) = i)\beta_i(r)dr \to 0.$$

This concludes the proof of the theorem. □

Case II: $\widetilde{Q}(t) = \text{diag}(\widetilde{Q}^1(t), \ldots, \widetilde{Q}^l(t))$

This subsection extends the preceding result to the cases in which $\widetilde{Q}(t)$ is a block-diagonal matrix with irreducible blocks. We make the following assumptions:

(**A5.8**) $\widehat{Q}(t)$ and $\widetilde{Q}^i(t)$, for $i = 1, \ldots, l$, are bounded and Borel measurable. Moreover, $\widetilde{Q}^i(t)$, $i = 1, \ldots, l$, are weakly irreducible.

Lemma 5.51. *Assume* (A5.8). *Then the following assertions hold:*

(a) *For each $i = 1, \ldots, l$ and $j = 1, \ldots, m_i$, $P(\alpha^\varepsilon(t) = s_{ij})$ converges in the weak sense to $\nu_j^i(t)\vartheta^i(t)$ on $L^2([0,T];\mathbb{R})$, that is,*

$$\int_0^T [P(\alpha^\varepsilon(t) = s_{ij}) - \nu_j^i(t)\vartheta^i(t)]f(t)dt \to 0, \tag{5.112}$$

for all $f(\cdot) \in L^2([0,T];\mathbb{R})$, where

$$(\vartheta^1(t), \ldots, \vartheta^l(t)) = p_0\widetilde{\mathbb{1}} + \int_0^t (\vartheta^1(s), \ldots, \vartheta^l(s))\overline{Q}(s)ds.$$

(b) *For each i, j, i_1, j_1, $P(\alpha^\varepsilon(t) = s_{ij} | \alpha^\varepsilon(s) = s_{i_1 j_1})$ converges in the weak sense to $\nu_j^i(t)\vartheta_{ii}(t,s)$ on $L^2([s,T];\mathbb{R})$, that is,*

$$\int_s^T [P(\alpha^\varepsilon(t) = s_{ij} | \alpha^\varepsilon(s) = s_{i_1 j_1}) - \nu_j^i(t)\vartheta_{ii}(t,s)]f(t)dt \to 0, \tag{5.113}$$

for all $f(\cdot) \in L^2([s,T];\mathbb{R})$, where $\vartheta_{ij}(t,s)$ is defined in Lemma 5.24 (see (5.50)).

Proof: We only derive (5.112); the proof of (5.113) is similar. Let

$$p^{\varepsilon}(t) = \left(p_{11}^{\varepsilon}(t), \ldots, p_{1m_1}^{\varepsilon}(t), \ldots, p_{l1}^{\varepsilon}(t), \ldots, p_{lm_l}^{\varepsilon}(t)\right)$$

where $p_{ij}^{\varepsilon}(t) = P(\alpha^{\varepsilon}(t) = s_{ij})$. Since $p^{\varepsilon}(\cdot) \in L^2([0,T];\mathbb{R}^m)$, there exists (see Lemma A.36) a subsequence of $\varepsilon \to 0$ (still denoted by ε for simplicity), such that corresponding to this ε, $p^{\varepsilon}(t)$ converges to some $p(\cdot) \in L^2([0,T];\mathbb{R}^m)$ under the weak topology. Let

$$p(t) = (p_{11}(t), \ldots, p_{1m_1}(t), \ldots, p_{l1}(t), \ldots, p_{lm_l}(t)).$$

Then $0 \le p_{ij}(t) \le 1$ and $\sum_{i,j} p_{ij}(t) = 1$ almost everywhere. Similarly as in the proof of Lemma 5.49, for $0 \le t \le T$,

$$p(t)\widetilde{Q}(t) = 0 \text{ a.e. in } t.$$

The irreducibility of $\widetilde{Q}^k(t)$, $k = 1, \ldots, l$, implies that

$$p(t) = (\vartheta^1(t), \ldots, \vartheta^l(t))\mathrm{diag}(\nu^1(t), \ldots, \nu^l(t)), \tag{5.114}$$

for some functions $\vartheta^1(t), \ldots, \vartheta^l(t)$.

In view of (5.106), we have

$$p^{\varepsilon}(t)\widetilde{\mathbb{1}} = p_0\widetilde{\mathbb{1}} + \int_0^t p^{\varepsilon}(s)\left(\frac{1}{\varepsilon}\widetilde{Q}(s) + \widehat{Q}(s)\right)\widetilde{\mathbb{1}}ds.$$

Since $\widetilde{Q}(s)\widetilde{\mathbb{1}} = 0$, it follows that

$$p^{\varepsilon}(t)\widetilde{\mathbb{1}} = p_0\widetilde{\mathbb{1}} + \int_0^t p^{\varepsilon}(s)\widehat{Q}(s)\widetilde{\mathbb{1}}ds.$$

Owing to the convergence of $p^{\varepsilon}(t) \to p(t)$ under the weak topology of $L^2([0,T];\mathbb{R}^m)$, we have

$$p(t)\widetilde{\mathbb{1}} = p_0\widetilde{\mathbb{1}} + \int_0^t p(s)\widehat{Q}(s)\widetilde{\mathbb{1}}ds.$$

Using (5.114) and noting that

$$\mathrm{diag}(\nu^1(t), \ldots, \nu^l(t))\widetilde{\mathbb{1}} = I,$$

we have

$$(\vartheta^1(t), \ldots, \vartheta^l(t)) = p_0\widetilde{\mathbb{1}} + \int_0^t (\vartheta^1(s), \ldots, \vartheta^l(s))\overline{Q}(s)ds.$$

The uniqueness of the solution then yields the lemma. □

Theorem 5.52. *Assume* (A5.8). *Then for any* $i = 1, \ldots, l$, $j = 1, \ldots, m_i$, *and bounded deterministic function* $\beta_{ij}(t)$, $t \geq 0$,

$$E\left(\int_0^T \left(I_{\{\alpha^\varepsilon(t)=s_{ij}\}} - \nu_j^i(t) I_{\{\overline{\alpha}^\varepsilon(t)=i\}}\right)\beta_{ij}(t)dt\right)^2 \to 0, \text{ as } \varepsilon \to 0.$$

Proof: Let $\eta(t)$ be defined as in (5.52). Then we can show similarly as in the proof of Theorem 5.25 that

$$\eta(T) = 2\int_0^T \int_0^t \Phi^\varepsilon(t,r)\beta_{ij}(t)\beta_{ij}(r)drdt,$$

where $\Phi^\varepsilon(t,r) = \Phi_1^\varepsilon(t,r) + \Phi_2^\varepsilon(t,r)$ with $\Phi_1^\varepsilon(t,r)$ and $\Phi_2^\varepsilon(t,r)$ defined by (5.53) and (5.54), respectively.

Note that by changing the order of integration,

$$\begin{aligned}
&\int_0^T \int_0^t \Phi_1^\varepsilon(t,r)\beta_{ij}(t)\beta_{ij}(r)drdt \\
&\quad = \int_0^T P(\alpha^\varepsilon(r) = s_{ij})\beta_{ij}(r)\Big\{\int_r^T [P(\alpha^\varepsilon(t) = s_{ij}|\alpha^\varepsilon(r) = s_{ij}) \\
&\qquad\qquad -\nu_j^i(t)P(\alpha^\varepsilon(t) \in \mathcal{M}_i|\alpha^\varepsilon(r) = s_{ij})]\beta_{ij}(t)dt\Big\}dr.
\end{aligned}$$

Since the $\beta_{ij}(\cdot)$ are bounded uniformly on $[0,T]$, $\beta_{ij}(\cdot) \in L^2([0,T];\mathbb{R})$. As a result, Lemma 5.51 implies that

$$\begin{aligned}
&\int_r^T [P(\alpha^\varepsilon(t) = s_{ij}|\alpha^\varepsilon(r) = s_{ij}) \\
&\qquad -\nu_j^i(t)P(\alpha^\varepsilon(t) \in \mathcal{M}_i|\alpha^\varepsilon(r) = s_{ij})]\beta_{ij}(t)dt \to 0.
\end{aligned}$$

Hence as $\varepsilon \to 0$,

$$\int_0^T \int_0^t \Phi_1^\varepsilon(t,r)\beta_{ij}(t)\beta_{ij}(r)drdt \to 0.$$

Similarly,

$$\int_0^T \int_0^t \Phi_2^\varepsilon(t,r)\beta_{ij}(t)\beta_{ij}(r)drdt \to 0, \text{ as } \varepsilon \to 0.$$

The proof is complete. □

Theorem 5.53. *Assume* (A5.8). *Then* $\overline{\alpha}^\varepsilon(\cdot)$ *converges weakly to* $\overline{\alpha}(\cdot)$ *on* $D([0,T];\overline{\mathcal{M}})$, *as* $\varepsilon \to 0$.

Proof: Recall that $\chi^\varepsilon(t)$ denotes the vector of indicator functions

$$\left(I_{\{\alpha^\varepsilon(t)=s_{11}\}},\ldots,I_{\{\alpha^\varepsilon(t)=s_{1m_1}\}},\ldots,I_{\{\alpha^\varepsilon(t)=s_{l1}\}},\ldots,I_{\{\alpha^\varepsilon(t)=s_{lm_l}\}}\right),$$

and let

$$\overline{\chi}^\varepsilon(t)=(\overline{\chi}_1^\varepsilon(t),\ldots,\overline{\chi}_l^\varepsilon(t))=\chi^\varepsilon(t)\widetilde{\mathbb{1}}.$$

Then $\overline{\chi}_i^\varepsilon(t)=I_{\{\overline{\alpha}^\varepsilon(t)=i\}}$ for $i=1,\ldots,l$.

We show that $\overline{\chi}^\varepsilon(\cdot)$ is tight in $D^l[0,T]$ first. Let $\mathcal{F}_t^\varepsilon=\sigma\{\alpha^\varepsilon(r):\ r\le t\}$. Then in view of the martingale property associated with $\alpha^\varepsilon(\cdot)$, we have, for $0\le s\le t$,

$$E\left[\chi^\varepsilon(t)-\chi^\varepsilon(s)-\int_s^t\chi^\varepsilon(r)Q^\varepsilon(r)dr\Big|\mathcal{F}_s^\varepsilon\right]=0.$$

Right multiplying both sides of the equation by $\widetilde{\mathbb{1}}$ and noting that $\widetilde{Q}(r)\widetilde{\mathbb{1}}=0$, we obtain

$$E\left[\overline{\chi}^\varepsilon(t)-\overline{\chi}^\varepsilon(s)-\int_s^t\chi^\varepsilon(r)\widehat{Q}(r)\widetilde{\mathbb{1}}dr\Big|\mathcal{F}_s^\varepsilon\right]=0. \tag{5.115}$$

Note that

$$\left|\int_s^t\chi^\varepsilon(r)\widehat{Q}(r)\widetilde{\mathbb{1}}dr\right|=O(t-s).$$

It follows from (5.115) that

$$E\left[I_{\{\overline{\alpha}^\varepsilon(t)=i\}}|\mathcal{F}_s^\varepsilon\right]=I_{\{\overline{\alpha}^\varepsilon(s)=i\}}+O(t-s). \tag{5.116}$$

Note also that $(I_A)^2=I_A$ for any set A. We have, in view of (5.116),

$$\begin{aligned}
&E\left[\left(I_{\{\overline{\alpha}^\varepsilon(t)=i\}}-I_{\{\overline{\alpha}^\varepsilon(s)=i\}}\right)^2\Big|\mathcal{F}_s^\varepsilon\right]\\
&\quad=E\left[I_{\{\overline{\alpha}^\varepsilon(t)=i\}}-2I_{\{\overline{\alpha}^\varepsilon(t)=i\}}I_{\{\overline{\alpha}^\varepsilon(s)=i\}}+I_{\{\overline{\alpha}^\varepsilon(s)=i\}}\Big|\mathcal{F}_s^\varepsilon\right]\\
&\quad=E\left[I_{\{\overline{\alpha}^\varepsilon(t)=i\}}\Big|\mathcal{F}_s^\varepsilon\right]-2E\left[I_{\{\overline{\alpha}^\varepsilon(t)=i\}}\Big|\mathcal{F}_s^\varepsilon\right]I_{\{\overline{\alpha}^\varepsilon(s)=i\}}+I_{\{\overline{\alpha}^\varepsilon(s)=i\}}\\
&\quad=I_{\{\overline{\alpha}^\varepsilon(s)=i\}}+O(t-s)\\
&\qquad\qquad-2\left(I_{\{\overline{\alpha}^\varepsilon(s)=i\}}+O(t-s)\right)I_{\{\overline{\alpha}^\varepsilon(s)=i\}}+I_{\{\overline{\alpha}^\varepsilon(s)=i\}}\\
&\quad=O(t-s),
\end{aligned}$$

for each $i=1,\ldots,l$. Hence,

$$\lim_{t\to s}\lim_{\varepsilon\to0}E\left\{E\left[\left(I_{\{\overline{\alpha}^\varepsilon(t)=i\}}-I_{\{\overline{\alpha}^\varepsilon(s)=i\}}\right)^2\Big|\mathcal{F}_s^\varepsilon\right]\right\}=0.$$

Therefore, by Lemma A.17, $\overline{\chi}^{\varepsilon}(\cdot)$ is tight.

The tightness of $\overline{\chi}^{\varepsilon}(\cdot)$ implies that for any sequence $\varepsilon_k \to 0$, there exists a subsequence of $\{\varepsilon_k\}$ (still denoted by $\{\varepsilon_k\}$) such that $\overline{\chi}^{\varepsilon_k}(\cdot)$ converges weakly. We next show that the limit of such a subsequence is uniquely determined by $\overline{Q}(\cdot) := \text{diag}(\nu^1(\cdot), \ldots, \nu^l(\cdot))\widehat{Q}(\cdot)\widetilde{\mathbb{1}}$.

Note that

$$\int_s^t \chi^{\varepsilon}(r)\widehat{Q}(r)\widetilde{\mathbb{1}}dr = \int_s^t \overline{\chi}^{\varepsilon}(r)\overline{Q}(r)dr + \int_s^t \left(\chi^{\varepsilon}(r) - \overline{\chi}^{\varepsilon}(r)\text{diag}(\nu^1(r), \ldots, \nu^l(r))\right)\widehat{Q}(r)\widetilde{\mathbb{1}}dr.$$

In view of Theorem 5.52, we have, as $\varepsilon \to 0$,

$$E\left|\int_s^t \left[\chi^{\varepsilon}(r) - \overline{\chi}^{\varepsilon}(r)\text{diag}(\nu^1(r), \ldots, \nu^l(r))\right]\widehat{Q}(r)\widetilde{\mathbb{1}}dr\right| \to 0. \tag{5.117}$$

Now by virtue of (5.115),

$$E\left[\left(\overline{\chi}^{\varepsilon}(t) - \overline{\chi}^{\varepsilon}(s) - \int_s^t \chi^{\varepsilon}(r)\widehat{Q}(r)\widetilde{\mathbb{1}}dr\right) z_1(\overline{\chi}^{\varepsilon}(t_1)) \cdots z_j(\overline{\chi}^{\varepsilon}(t_j))\right] = 0,$$

for $0 \le t_1 \le \cdots \le t_j \le s \le t$ and bounded and continuous functions $z_1(\cdot), \ldots, z_j(\cdot)$.

Let $\overline{\chi}(\cdot)$ denote the limit in distribution of $\overline{\chi}^{\varepsilon_k}(\cdot)$. Then in view of (5.117) and the continuity of $\int_s^t \eta(r)\overline{Q}(r)dr$ with respect to $\eta(\cdot)$ (see Lemma A.40), we have $\overline{\chi}^{\varepsilon}(\cdot) \to \overline{\chi}(\cdot)$ as $\varepsilon_k \to 0$, and $\overline{\chi}(\cdot)$ satisfies

$$E\left[\left(\overline{\chi}(t) - \overline{\chi}(s) - \int_s^t \overline{\chi}(r)\overline{Q}(r)dr\right) z_1(\overline{\chi}(t_1)) \cdots z_j(\overline{\chi}(t_j))\right] = 0.$$

It is easy to see that $\overline{\chi}(\cdot) = (\overline{\chi}_1(\cdot), \ldots, \overline{\chi}_l(\cdot))$ is an l-valued measurable process having sample paths in $D^l[0,T]$ and satisfying $\overline{\chi}_i(t) = 0$ or 1 and $\overline{\chi}_1(\cdot) + \cdots + \overline{\chi}_l(\cdot) = 1$ w.p.1. Let

$$\overline{\alpha}(t) = \sum_{i=1}^{l} i I_{\{\overline{\chi}_i(t)=1\}},$$

or in an expanded form,

$$\overline{\alpha}(t) = \begin{cases} 1, & \text{if } \overline{\chi}_1(t) = 1, \\ 2, & \text{if } \overline{\chi}_1(t) = 0,\ \overline{\chi}_2(t) = 1, \\ \vdots & \vdots \\ l, & \text{if } \overline{\chi}_i(t) = 0, \text{ for } i \le l-1,\ \overline{\chi}_l(t) = 1. \end{cases}$$

Then $\overline{\alpha}(\cdot)$ is a process with sample paths in $D([0,T];\overline{\mathcal{M}})$ and

$$\overline{\chi}(t) = (I_{\{\overline{\alpha}(t)=1\}}, \ldots, I_{\{\overline{\alpha}(t)=l\}}) \text{ w.p.1.}$$

Therefore, $\overline{\alpha}(\cdot)$ is a Markov chain generated by $\overline{Q}(\cdot)$. As a result, its distribution is uniquely determined by $\overline{Q}(\cdot)$. It follows that $\overline{\alpha}^{\varepsilon}(\cdot)$ converges weakly to $\overline{\alpha}(\cdot)$. □

Remark 5.54. Note that Theorem 5.53 gives the same result as Theorem 5.27 under weaker conditions. The proofs are quite different. The proof of Theorem 5.53 is based on martingale properties associated with the Markov chain, whereas the proof of Theorem 5.27 follows the traditional approach, i.e., after the tightness is verified, the convergence of finite-dimensional distributions is proved.

Remark 5.55. In view of the development in Chapter 4, apart from the smoothness conditions, one of the main ingredients is the use of the Fredholm alternative. One hopes that this will carry over (under suitable conditions) to the measurable generators. A possible approach is the utilization of the formulation of weak derivatives initiated in the study of partial differential equations (see Hutson and Pym [90]).

Following the tactics of the weak sense formulation, for some $T < \infty$ and for given $g(\cdot) \in L^2([0,T];\mathbb{R})$, a function $f(\cdot) \in L^2([0,T];\mathbb{R})$ is a weak solution of $(d/dt)f = g$ if

$$\int_0^T f(t)\left(\frac{d\phi(t)}{dt}\right)dt = \int_0^T g(t)\phi(t)dt$$

for any C^∞-functions on $[0,T]$ vanishing on the boundary together with their derivatives (denoted by $\phi \in C_0^\infty([0,T];\mathbb{R})$). Write the weak solution as $(d/dt)f \overset{\text{W}}{=} g$.

Recall that L^2_{loc} is the set of functions that lie in $L^2(S;\mathbb{R})$ for every closed and bounded set $S \subset (0,T)$. A function $f(\cdot) \in L^2_{\text{loc}}$ has a jth-order weak derivative if there is a function $g(\cdot) \in L^2_{\text{loc}}$ such that

$$\int_0^T g(t)\phi(t)dt = (-1)^j \int_0^T f(t)\frac{d^j\phi(t)}{dt^j}dt$$

for all $\phi \in C_0^\infty([0,T];\mathbb{R})$. The function $g(\cdot)$ above is called the jth-order weak derivative of $f(\cdot)$, and is denoted by $D^j f = g$.

To proceed, define the space of functions H^n as

$$H^n = \{f \text{ on } [0,T];\ \text{ for } 0 \le j \le n,\ D^j f \text{ exist and are in } L^2([0,T];\mathbb{R})\}.$$

Equip H^n with an inner product and a norm as

$$(f,g)_n = \sum_{j\le n} \int_0^T D^j f D^j g dt,$$
$$|f|_n^2 = (f,f)_n = \sum_{j\le n} \int_0^T |D^j f|^2 dt.$$

One can then work under such a framework and proceed to obtain the asymptotic expansion of the probability distribution. It seems that the conditions required are not much different from those in the case of smooth generators; we will not pursue this issue further.

5.5 Remarks on Inclusion of Transient and Absorbing States

So far, the development in this chapter has focused on Markov chains with only recurrent states (either a single weakly irreducible class or a number of weakly irreducible classes). This section extends the results obtained to the case that a transient class or a group of absorbing states is included.

5.5.1 Inclusion of Transient States

Consider the Markov chain $\alpha^\varepsilon(\cdot) \in \mathcal{M}$, where its generator is still given by (5.47) and the state space of $\alpha^\varepsilon(t)$ is given by

$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots \cup \mathcal{M}_l \cup \mathcal{M}_*, \tag{5.118}$$

with $\mathcal{M}_i = \{s_{i1}, \ldots, s_{im_i}\}$ and $\mathcal{M}_* = \{s_{*1}, \ldots, s_{*m_*}\}$. In what follows, we present results concerning the asymptotic distributions of scaled occupation measures and properties of measurable generators. While main assumptions and results are provided, the full proofs are omitted. The interested reader can derive the results using the ideas presented in the previous sections.

To proceed, assume that $\widetilde{Q}(t)$ is a generator of a Markov chain satisfying

$$\widetilde{Q}(t) = \begin{pmatrix} \widetilde{Q}^1(t) & & & \\ & \ddots & & \\ & & \widetilde{Q}^l(t) & \\ \widetilde{Q}_*^1(t) & \cdots & \widetilde{Q}_*^l(t) & \widetilde{Q}_*(t) \end{pmatrix} \tag{5.119}$$

such that for each $t \in [0,T]$ and each $i = 1, \ldots, l$, $\widetilde{Q}^i(t)$ is a generator with dimension $m_i \times m_i$, $\widetilde{Q}_*(t)$ is an $m_* \times m_*$ matrix, $\widetilde{Q}_*^i(t) \in \mathbb{R}^{m_* \times m_i}$, and $m_1 + m_2 + \cdots + m_l + m_* = m$. We impose the following conditions.

(**A5.9**) For all $t \in [0,T]$, and $i = 1,\ldots,l$, $\widetilde{Q}^i(t)$ are weakly irreducible, and $\widetilde{Q}_*(t)$ is Hurwitz (i.e., all of its eigenvalues have negative real parts). Moreover, $\widetilde{Q}(\cdot)$ is differentiable on $[0,T]$ and its derivative is Lipschitz; $\widehat{Q}(\cdot)$ is Lipschitz continuous on $[0,T]$.

Use the partition

$$\widehat{Q}(t) = \begin{pmatrix} \widehat{Q}^{11}(t) & \widehat{Q}^{12}(t) \\ \widehat{Q}^{21}(t) & \widehat{Q}^{22}(t) \end{pmatrix}$$

where

$$\widehat{Q}^{11}(t) \in \mathbb{R}^{(m-m_*)\times(m-m_*)},\ \widehat{Q}^{12}(t) \in \mathbb{R}^{(m-m_*)\times m_*},$$

$$\widehat{Q}^{21}(t) \in \mathbb{R}^{m_*\times(m-m_*)},\ \text{ and }\ \widehat{Q}^{22}(t) \in \mathbb{R}^{m_*\times m_*},$$

and write

$$\begin{aligned} \overline{Q}_*(t) &= \operatorname{diag}(\nu^1(t),\ldots,\nu^l(t))(\widehat{Q}^{11}(t)\widetilde{\mathbb{1}} + \widehat{Q}^{12}(t)(a_{m_1}(t),\ldots,a_{m_l}(t))) \\ \overline{Q}(t) &= \operatorname{diag}(\overline{Q}_*(t), 0_{m_*\times m_*}), \end{aligned} \tag{5.120}$$

where

$$\widetilde{\mathbb{1}} = \operatorname{diag}(\mathbb{1}_{m_1},\ldots,\mathbb{1}_{m_l}),\ \mathbb{1}_{m_j} = (1,\ldots,1)' \in \mathbb{R}^{m_j\times 1},$$

and

$$a_{m_i}(t) = -\widetilde{Q}_*^{-1}(t)\widetilde{Q}_*^i(t)\mathbb{1}_{m_i},\ \text{for } i = 1,\ldots,l. \tag{5.121}$$

In what follows, if $a_{m_i}(t)$ is time independent, we will simply write it as a_{m_i}. The requirement on $\widetilde{Q}_*(t)$ in (A5.9) implies that the corresponding states are transient. The Hurwitzian property also has the following interesting implication: For each $t \in [0,T]$, and each $i = 1,\ldots,l$, $a_{m_i}(t) = (a_{m_i,1}(t),\ldots,a_{m_i,m_*}(t))' \in \mathbb{R}^{m_*\times 1}$. Then

$$a_{m_i,j}(t) \geq 0\ \text{ and }\ \sum_{i=1}^{l} a_{m_i,j}(t) = 1 \tag{5.122}$$

for each $j = 1,\ldots,m_*$. That is, for each $t \in [0,T]$ and each $j = 1,\ldots,l$, $(a_{m_1,j}(t),\ldots,a_{m_l,j}(t))$ can be considered a probability row vector. To see this, note that

$$\int_0^\infty \exp(\widetilde{Q}_*(t)s)ds = -\widetilde{Q}_*^{-1}(t),$$

which has nonnegative components. It follows from the definition that $a_{m_i}(t) \geq 0$. Furthermore,

$$\sum_{i=1}^{l} a_{m_i}(t) = -\widetilde{Q}_*^{-1}(t)\sum_{i=1}^{l}\widetilde{Q}_*^i(t)\mathbb{1}_{m_i} = (-\widetilde{Q}_*^{-1}(t))(-\widetilde{Q}_*(t))\mathbb{1}_{m_*} = \mathbb{1}_{m_*}.$$

Thus (5.122) follows. Similar to the development in the section for the case of weak and strong interactions, we can derive the following results.

Theorem 5.56. *Define*

$$\chi_{ij}^{\varepsilon}(t) = \begin{cases} \displaystyle\int_0^t \left(I_{\{\alpha^{\varepsilon}(s)=s_{ij}\}} - \nu_j^i(s) I_{\{\alpha^{\varepsilon}(s)\in\mathcal{M}_i\}}\right) ds, & \text{for } i = 1,\dots,l, \\ \displaystyle\int_0^t I_{\{\alpha^{\varepsilon}(s)=s_{*j}\}} ds, & \text{for } i = *, \end{cases} \tag{5.123}$$

and assume (A5.9). *Then for each* $j = 1,\dots,m_i$,

$$\sup_{t\in[0,T]} E|\chi_{ij}^{\varepsilon}(t)|^2 = \begin{cases} O(\varepsilon), & \text{for } i = 1,\dots,l, \\ O(\varepsilon^2), & \text{for } i = *. \end{cases} \tag{5.124}$$

Next, for each fixed $t \in [0,T]$, let ξ be a random variable uniformly distributed on $[0,1]$ that is independent of $\alpha^{\varepsilon}(\cdot)$. For each $j = 1,\dots,m_*$, define an integer-valued random variable $\xi_j(t)$ by

$$\begin{aligned} \xi_j(t) \ &= I_{\{0\le\xi\le a_{m_1,j}(t)\}} + 2I_{\{a_{m_1,j}(t)<\xi\le a_{m_1,j}(t)+a_{m_2,j}(t)\}} \\ &\quad + \cdots + lI_{\{a_{m_1,j}(t)+\cdots+a_{m_{l-1},j}(t)<\xi\le 1\}}. \end{aligned}$$

Now redefine the aggregated process $\overline{\alpha}^{\varepsilon}(\cdot)$ by

$$\overline{\alpha}^{\varepsilon}(t) = \begin{cases} i, & \text{if } \alpha^{\varepsilon}(t) \in \mathcal{M}_i, \\ \xi_j(t), & \text{if } \alpha^{\varepsilon}(t) = s_{*j}(t). \end{cases} \tag{5.125}$$

Note that the state space of $\overline{\alpha}^{\varepsilon}(t)$ is $\overline{\mathcal{M}} = \{1,\dots,l\}$, and that $\overline{\alpha}^{\varepsilon}(\cdot) \in D([0,T];\overline{\mathcal{M}})$. Similar to the weak and strong interaction case, but with more effort, we can obtain the following result.

Theorem 5.57. *Under conditions* (A5.9), $\overline{\alpha}^{\varepsilon}(\cdot)$ *converges weakly to* $\overline{\alpha}(\cdot)$, *a Markov chain generated by* $\overline{Q}_*(\cdot)$ *given by* (5.120).

Next, for $t \ge 0$, and $\alpha \in \mathcal{M}$, let $\beta_{ij}(t)$ be bounded Borel measurable deterministic functions, and let

$$W_{ij}(t,\alpha) = \begin{cases} (I_{\{\alpha=s_{ij}\}} - \nu_j^i(t) I_{\{\alpha\in\mathcal{M}_i\}})\beta_{ij}(t), & \text{if } i = 1,\dots,l,\ j = 1,\dots,m_i, \\ I_{\{\alpha=s_{*j}\}}\beta_{ij}(t), & \text{if } i = *,\ j = 1,\dots,m_*. \end{cases} \tag{5.126}$$

Consider the normalized occupation measure

$$n^\varepsilon(t) = (n^\varepsilon_{11}(t), \dots, n^\varepsilon_{1m_1}(t), \dots, n^\varepsilon_{*1}(t), \dots, n^\varepsilon_{*m_*}(t)),$$

where

$$n^\varepsilon_{ij}(t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t W_{ij}(s, \alpha^\varepsilon(s))ds.$$

We can then proceed to obtain the asymptotic distribution.

Theorem 5.58. *Assume* (A5.9), *and suppose* $\widetilde{Q}(\cdot)$ *is twice differentiable with Lipschitz continuous second derivative. Moreover,* $\widehat{Q}(\cdot)$ *is differentiable with Lipschitz continuous derivative. Let* $\beta_{ij}(\cdot)$ *(for* $i = 1, \dots, l$, $j = 1, \dots, m_i$*) be bounded and Lipschitz continuous deterministic functions. Then* $n^\varepsilon(\cdot)$ *converges weakly to a switching diffusion* $n(\cdot)$*, where*

$$n(t) = \Big(\int_0^t \sigma(s, \overline{\alpha}(s))dw(s) \Big)', \tag{5.127}$$

where $\sigma(s, i)$ *is similar to* (5.105) *with the following modifications:*

$$\sigma(s, i) = \text{diag}(0_{m_1 \times m_1}, \dots, \sigma^0(s, i), \dots, 0_{m_l \times m_l}, 0_{m_* \times m_*}) \tag{5.128}$$

and $w(\cdot)$ *is a standard* m*-dimensional Brownian motion.*

Finally, we confirm that the case of the generators being merely measurable can be treated as well. We state this as the following theorem.

Theorem 5.59. *Assume the generator is given by* (5.47) *with* $\widetilde{Q}(\cdot)$ *given by* (5.120) *such that* $\widetilde{Q}$ *and* $\widehat{Q}$ *are measurable and bounded and that* $\widetilde{Q}^i(t)$ *is weakly irreducible for each* $i = 1, \dots, l$*. Then the following assertions hold:*

- *For any* $i = 1, \dots, l$, $j = 1, \dots, m_i$, *and bounded deterministic function* $\beta_{ij}(t)$, $t \geq 0$,

$$E\left(\int_0^T \left(I_{\{\alpha^\varepsilon(t) = s_{ij}\}} - \nu^i_j(t) I_{\{\overline{\alpha}^\varepsilon(t) = i\}} \right) \beta_{ij}(t)dt \right)^2 \to 0, \text{ as } \varepsilon \to 0.$$

- $\overline{\alpha}^\varepsilon(\cdot)$ *converges weakly to* $\overline{\alpha}(\cdot)$*, a Markov chain generated by* $\overline{Q}_*(\cdot)$.

5.5.2 Inclusion of Absorbing States

Consider the Markov chain $\alpha^\varepsilon(\cdot) \in \mathcal{M}$, where the generator of $\alpha^\varepsilon(\cdot)$ is still given by (5.47) with

$$\widetilde{Q}(t) = \text{diag}(\widetilde{Q}^1(t), \dots, \widetilde{Q}^l(t), 0_{m_a \times m_a}), \tag{5.129}$$

where $0_{m_a \times m_a}$ is the $m_a \times m_a$ zero matrix, the state space of $\alpha^\varepsilon(t)$ is given by

$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots \cup \mathcal{M}_l \cup \mathcal{M}_a, \tag{5.130}$$

with $\mathcal{M}_i = \{s_{i1}, \dots, s_{im_i}\}$ and $\mathcal{M}_a = \{s_{a1}, \dots, s_{am_a}\}$, and $m_1 + m_2 + \cdots + m_l + m_a = m$. Assume the following conditions.

(**A5.10**) For all $t \in [0,T]$ and $i = 1,\dots,l$, $\widetilde{Q}^i(t)$ is weakly irreducible. Furthermore, $\widetilde{Q}(\cdot)$ is differentiable on $[0,T]$ and its derivative is Lipschitz. Moreover, $\widehat{Q}(\cdot)$ is Lipschitz continuous on $[0,T]$.

Define

$$\begin{aligned}&\widetilde{\mathbb{1}} = \text{diag}(\mathbb{1}_{m_1},\dots,\mathbb{1}_{m_l}) \quad \text{and} \quad \widetilde{\mathbb{1}}_a = \text{diag}(\widetilde{\mathbb{1}}, I_{m_a})\\ &\overline{Q}(t) = \text{diag}(\nu^1(t),\nu^2(t),\dots,\nu^l(t),I_{m_a})\widehat{Q}(t)\widetilde{\mathbb{1}}_a.\end{aligned} \tag{5.131}$$

Assume that the conditions in (A5.10) are satisfied. Then we can prove the following:

(a) As $\varepsilon \to 0$,

$$p^\varepsilon(t) = (\vartheta(t),\vartheta^a(t))\text{diag}(\nu^1(t),\dots,\nu^l(t),I_{m_a}) + O\left(\varepsilon + \exp(-\kappa_0 t/\varepsilon)\right),$$

where

$$\vartheta(t) = (\vartheta^1(t),\dots,\vartheta^l(t))) \in \mathbb{R}^{1\times l} \quad \text{and}$$

$$\vartheta^a(t) = (\vartheta^a_1(t),\dots,\vartheta^a_{m_a}(t)) \in \mathbb{R}^{1\times m_a},$$

satisfying

$$\frac{d(\vartheta(t),\vartheta^a(t))}{dt} = (\vartheta(t),\vartheta^a(t))\overline{Q}(t), \quad (\vartheta(0),\vartheta^a(0)) = p^\varepsilon(0)\widetilde{\mathbb{1}}_a$$

where $\overline{Q}(t)$ is given in (5.131) and $p^\varepsilon(0) = (p^{\varepsilon,1}(0),\dots,p^{\varepsilon,l}(0),p^{\varepsilon,a}(0))$ with $p^{\varepsilon,i}(0) \in \mathbb{R}^{1\times m_i}$ and $p^{\varepsilon,a}(0) \in \mathbb{R}^{1\times m_a}$.

(b) For the transition probability $P^\varepsilon(t,t_0)$, we have

$$P^\varepsilon(t,t_0) = P^0(t,t_0) + O\left(\varepsilon + \exp(-\kappa_0(t-t_0)/\varepsilon)\right), \tag{5.132}$$

for some $\kappa_0 > 0$, where

$$P^0(t,t_0) = \widetilde{\mathbb{1}}_a\Theta(t,t_0)\text{diag}(\nu^1(t),\dots,\nu^l(t),I_{m_a}),$$

and

$$\frac{d\Theta(t,t_0)}{dt} = \Theta(t,t_0)\overline{Q}(t), \quad \Theta(t_0,t_0) = I.$$

To proceed, we aggregate the states in $\mathcal{M}_i$ for $i = 1,\dots,l$ as one state leading to the definition of the following process:

$$\overline{\alpha}^\varepsilon(t) = \begin{cases} i, & \text{if } \alpha^\varepsilon(t) \in \mathcal{M}_i,\\ \alpha^\varepsilon(t), & \text{if } \alpha^\varepsilon(t) \in \mathcal{M}_a. \end{cases} \tag{5.133}$$

For each $j = 1, \ldots, m_i$, we also define a sequence of centered occupation measures by

$$\chi_{ij}^{\varepsilon}(t) = \begin{cases} \displaystyle\int_0^t \left(I_{\{\alpha^{\varepsilon}(s)=s_{ij}\}} - \nu_j^i(s) I_{\{\overline{\alpha}^{\varepsilon}(s)=i\}}\right) ds, & \text{for } i = 1, \ldots, l, \\ \displaystyle\int_0^t (I_{\{\alpha^{\varepsilon}(s)=s_{aj}\}} - \vartheta_j^a(s)) ds. \end{cases} \tag{5.134}$$

For $t \geq 0$ and $\alpha \in \mathcal{M}$, let

$$W_{ij}(t,\alpha) = \begin{cases} I_{\{\alpha=s_{ij}\}} - \nu_j^i(t) I_{\{\alpha\in\mathcal{M}_i\}}, & \text{for } i = 1, \ldots, l,\ j = 1, \ldots, m_i, \\ I_{\{\alpha=s_{aj}\}} - \vartheta_j^a(t), & \text{for } j = 1, \ldots, m_a. \end{cases} \tag{5.135}$$

Consider the normalized occupation measure

$$n^{\varepsilon}(t) = (n_{11}^{\varepsilon}(t), \ldots, n_{1m_1}^{\varepsilon}(t), \ldots, n_{l1}^{\varepsilon}(t), \ldots, n_{lm_l}^{\varepsilon}(t), n_{a1}^{\varepsilon}(t), \ldots, n_{am_a}^{\varepsilon}(t)),$$

where

$$n_{ij}^{\varepsilon}(t) = \begin{cases} \dfrac{1}{\sqrt{\varepsilon}} \displaystyle\int_0^t W_{ij}(s, \alpha^{\varepsilon}(s)) \beta_{ij}(s) ds,\ i = 1, \ldots, l,\ j = 1, \ldots, m_i, \\ \displaystyle\int_0^t W_{aj}(s, \alpha^{\varepsilon}(s)) \beta_{aj}(s) ds,\ j = 1, \ldots, m_a. \end{cases}$$

Note that

$$\frac{dn^{\varepsilon}(t)}{dt} = \begin{cases} \dfrac{1}{\sqrt{\varepsilon}} W^r(t, \alpha^{\varepsilon}(t)), & \text{for } \alpha^{\varepsilon}(t) \in \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l, \\ W^a(t, \alpha^{\varepsilon}(t)), & \text{for } \alpha^{\varepsilon}(t) \in \mathcal{M}_a, \end{cases}$$

$$n^{\varepsilon}(0) = 0,$$

where

$$W^r(t,\alpha) = (W_{11}(t,\alpha), \ldots, W_{1m_1}(t,\alpha), \ldots, W_{l1}(t,\alpha), \ldots, W_{lm_l}(t,\alpha)),$$

$$W^a(t,\alpha) = (W_{a1}(t,\alpha), \ldots, W_{am_a}(t,\alpha)), \text{ and}$$

$$W(t,\alpha) = (W^r(t,\alpha), W^a(t,\alpha)).$$

$$\langle W^a(u,\alpha)), \nabla_x^a f^0(x,\alpha)\rangle = \sum_{j=1}^{m_a} b_j(u,\alpha) \frac{\partial}{\partial_{a,j}} f^0(x,\alpha).$$

We can obtain the following results.

Theorem 5.60. *Assume* (A5.10). *Then the following assertions hold.*

(a) *For all* $i = 1, \ldots, l$ *and* $j = 1, \ldots, m_i$, *corresponding to the recurrent states,* $\sup_{t\in[0,T]} E|O_{ij}^{\varepsilon}(t)|^2 = O(\varepsilon)$.

(b) $\overline{\alpha}^{\varepsilon}(\cdot)$ *converges weakly to* $\overline{\alpha}(\cdot)$, *a Markov chain generated by* $\overline{Q}(\cdot)$.

(c) *Define the generator* $\mathcal{L}$ *by*

$$\begin{aligned}\mathcal{L}f^0(x,\alpha) &= \frac{1}{2}\sum_{j_1,j_2=1}^{m_\alpha} a_{j_1j_2}(s,\alpha)\partial^2_{\alpha,j_1j_2}f^0(x,\alpha)\\ &\quad+\sum_{j=1}^{m_a} b_j(s,\alpha)\partial_{a,j}f^0(x,\alpha)+\overline{Q}(s)f^0(x,\cdot)(\alpha).\end{aligned}$$

Then the sequence $Y^{\varepsilon}(\cdot) = (n^{\varepsilon}(\cdot), \overline{\alpha}^{\varepsilon}(\cdot))$ *converges weakly to* $\overline{Y}(\cdot) = (n(\cdot), \overline{\alpha}(\cdot))$ *that is a solution of the martingale problem with operator* $\mathcal{L}$.

Next, assume that $\widetilde{Q}(\cdot)$ *and* $\widehat{Q}(\cdot)$ *are bounded and measurable and* $\widetilde{Q}^i(t)$ *for each* $i = 1, \ldots, l$ *is weakly irreducible. Then*

$$p^{\varepsilon}(t) = (p^{\varepsilon}_{11}(t), \ldots, p^{\varepsilon}_{1m1}(t), \ldots, p^{\varepsilon}_{l1}(t), \ldots, p^{\varepsilon}_{lm_l}(t), p^{\varepsilon}_{a1}(t), \ldots, p^{\varepsilon}_{am_a}(t))$$

converges in the weak topology of $L^2([0,T];\mathbb{R}^m)$ *(with* $m = \sum_{i=1}^{l} m_i + m_a$*) to*

$$p(t) = (\vartheta_1(t)\nu^1(t), \ldots, \nu^1(t)\vartheta_l(t), p^{0,a}),$$

where $p^{0,a}$ *is the subvector in the initial data* p^0 *corresponding the absorbing state.*

Note that in deriving the asymptotic distribution of the scaled occupation measures, we need to compute the asymptotic covariance of the limit process. That is, we need to evaluate the limit of

$$\begin{aligned}E\int_0^t \begin{pmatrix}\frac{1}{\sqrt{\varepsilon}}(W^r(s,\alpha^{\varepsilon}(s)))'\\ (W^a(s,\alpha^{\varepsilon}(s)))'\end{pmatrix}\left(\frac{1}{\sqrt{\varepsilon}}W^r(s,\alpha^{\varepsilon}(s)),\quad W^a(s,\alpha^{\varepsilon}(s))\right)ds\\ \stackrel{\text{def}}{=}\begin{pmatrix}W^{rr}_{\varepsilon}(t) & W^{ra}_{\varepsilon}(t)\\ W^{ar}_{\varepsilon}(t) & W^{aa}_{\varepsilon}(t)\end{pmatrix},\end{aligned} \tag{5.136}$$

where

$$\begin{aligned}W^{rr}_{\varepsilon}(t) &= \frac{1}{\varepsilon}\int_0^t E(W^r(s,\alpha^{\varepsilon}(s)))'W^r(s,\alpha^{\varepsilon}(s))ds\\ W^{ra}_{\varepsilon}(t) &= \frac{1}{\sqrt{\varepsilon}}\int_0^t E(W^r(s,\alpha^{\varepsilon}(s)))'W^a(s,\alpha^{\varepsilon}(s))ds\\ W^{ar}_{\varepsilon}(t) &= \frac{1}{\sqrt{\varepsilon}}\int_0^t E(W^a(s,\alpha^{\varepsilon}(s)))'W^r(s,\alpha^{\varepsilon}(s))ds\\ W^{aa}_{\varepsilon}(t) &= \int_0^t E(W^a(s,\alpha^{\varepsilon}(s)))'W^a(s,\alpha^{\varepsilon}(s))ds.\end{aligned}$$

It can be shown that

$$W_\varepsilon^{rr}(t) \to \overline{W}^r(t),\ W_\varepsilon^{ra}(t) \to 0,\ W_\varepsilon^{ar}(t) \to 0,\ \text{ and } W_\varepsilon^{aa}(t) \to \overline{W}^a(t),$$

as $\varepsilon \to 0$, where for $i = 1, \ldots, l$, $\overline{W}^r(t) = \overline{W}^r(t, i) = \int_0^t \widehat{W}^r(s, i)ds$ with

$$\widehat{W}^r(s, i) = \text{diag}(0_{m_1 \times m_1}, \ldots, \sigma(s, i), \ldots, 0_{m_l \times m_l}) \tag{5.137}$$

with $\sigma(s, i)$ the $m_i \times m_i$ matrix such that $\sigma(s, i)\sigma'(s, i) = A(s, i)$ for $i = 1, \ldots, l$, and

$$\overline{W}^a(t) = (\overline{W}^a_{jk}(t)) \text{ with } \overline{W}^a_{jk}(t) = \int_0^t \left(\delta_{jk}\vartheta^a_j(s) - \vartheta^a_j(s)\vartheta^a_k(s)\right) ds, \tag{5.138}$$

where $\delta_{jk} = 1$ if $j = k$, $\delta_{jk} = 0$ if $j \neq k$. The detailed proof of Theorem 5.60 can be found in Yin, Zhang, and Badowski [241].

5.6 Remarks on a Stability Problem

So far, our study has been devoted to systems with two time scales in a finite interval. In many problems arising in networked control systems, stability is often a main concern. A related problem along this line is in Badowski and Yin [5].

It is interesting to note that intuitive ideas sometimes are not necessarily true for systems with switching, for example, if one put together two stable systems by using, for instance, Markovian switching. Our intuition may lead to the conclusion that the combined systems should also be stable. Nevertheless, this is, in fact, not true. Such an idea was illustrated in Wang, Khargonekar, and Beydoun [212] for deterministically switched systems; see also Chapter 1 of this book concerning this matter.

As a variation of the system in [212], we consider the following example. Suppose that $\alpha^\varepsilon(\cdot)$ is a continuous-time Markov chain with state space $\mathcal{M} = \{1, 2\}$ and generator $Q^\varepsilon = Q/\varepsilon$, where $Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. Consider a controlled system

$$\dot{x} = A(\alpha^\varepsilon(t))x + B(\alpha^\varepsilon(t))u(t),$$

with state feedback $u(t) = K(\alpha^\varepsilon(t))x(t)$. Then we obtain the equivalent representation

$$\dot{x} = [A(\alpha^\varepsilon(t)) - B(\alpha^\varepsilon(t))K(\alpha^\varepsilon(t))]x. \tag{5.139}$$

Suppose that

$$G(1) = A(1) - B(1)K(1) = \begin{pmatrix} -100 & 20 \\ 200 & -100 \end{pmatrix},$$

$$G(2) = A(2) - B(2)K(2) = \begin{pmatrix} -100 & 200 \\ 20 & -100 \end{pmatrix}.$$

Note that both matrices are Hurwitz (i.e., their eigenvalues have negative real parts). A question of interest is this: Is system (5.139) stable? The key to understanding the system is to examine

$$\dot{x}^\varepsilon(t) = G(\alpha^\varepsilon(t))x^\varepsilon(t), \tag{5.140}$$

where both $G(1)$ and $G(2)$ are stable matrices.

Since Q is irreducible, the stationary distribution associated with Q is given by $(1/2, 1/2)$. As a result, as $\varepsilon \to 0$, using our weak convergence result, $x^\varepsilon(\cdot)$ converges weakly to $x(\cdot)$, which is a solution of the system

$$\begin{aligned} &\dot{x}(t) = \overline{G}x(t), \quad \text{where} \\ &\overline{G} = \frac{1}{2}(G(1) + G(2)) = \begin{pmatrix} -100 & 110 \\ 110 & -100 \end{pmatrix}. \end{aligned} \tag{5.141}$$

In addition, for any $T < \infty$, using the large deviations result obtained in He, Yin, and Zhang [84], we can show that for any $\delta > 0$, there is a $c_1 > 0$ such that

$$P(\rho_{0,T}(x^\varepsilon(t), x(t)) \geq \delta) \leq \exp(-c_1/\varepsilon), \tag{5.142}$$

where $\rho_{0,T}(x, y) = \sup_{0 \leq t \leq T} |x(t) - y(t)|$.

Note that $\overline{G}$ is an unstable matrix with eigenvalues -210 and 10. Thus for (5.141), the critical point $(0, 0)'$ is a saddle point. But why should the stability of the averaged system dominate that of the original system? To see this, from a result of differential equations, there is a nonsingular matrix H such that $H\overline{G}H^{-1} = \Lambda = \text{diag}(-210, 10)$. Clearly, the stability of (5.141) is equivalent to that of

$$\dot{y}(t) = \Lambda y(t) = H \sum_{i=1}^{2} \nu_i G(i) H^{-1} y(t) = \text{diag}(-210, 10)y(t), \tag{5.143}$$

where $y = Hx = (y_1, y_2)'$. The stability of (5.141) is equivalent to that of (5.143), which is completely decoupled and $y_1(t) = \exp(-210t)y_1(0) \to 0$

and $y_2(t) = \exp(10t)y_2(0) \to \infty$. To see how the original system (5.140) behaves, we apply the same transformation to get

$$\dot{y}^\varepsilon(t) = H \sum_{i=1}^{2} I_{\{\alpha^\varepsilon(t)=i\}} G(i) H^{-1} y^\varepsilon(t). \tag{5.144}$$

For the transformed system (5.143), by choosing $V(y) = y_2^2/2$, we obtain $\dot{V}(y(t)) = 10y_2^2 > 0$ for all $y_2 \neq 0$. Define $L^\varepsilon z(t) = \lim_{\delta \downarrow 0} E_t^\varepsilon [z(t+\delta) - z(t)]/\delta$ for a real-valued function $z(t)$ that is continuously differentiable, where E_t^ε denotes the conditioning on the $\mathcal{F}_t^\varepsilon = \sigma\{\alpha^\varepsilon(s) : s \leq t\}$. With $V(y) = y_2^2/2$, we have

$$L^\varepsilon V(y^\varepsilon(t)) = 10(y_2^\varepsilon(t))^2 + V_y'(y^\varepsilon(t)) H \sum_{i=1}^{2} [I_{\{\alpha^\varepsilon(t)=i\}} - \nu_i] G(i) H^{-1} y^\varepsilon(t),$$

where $V_y'(y) = (0, y_2) \in \mathbb{R}^{1\times 2}$. Using perturbed Liapunov function techniques as done in Badowski and Yin [5], define a perturbation

$$V_2^\varepsilon(y,t) = E_t^\varepsilon \int_t^\infty e^{t-s} V_y'(y) H \sum_{i=1}^{2} [I_{\{\alpha^\varepsilon(s)=i\}} - \nu_i] G(i) H^{-1} y.$$

It can be shown that $V_2^\varepsilon(y,t) = O(\varepsilon)V(y)$. In addition,

$$L^\varepsilon V_2^\varepsilon(y^\varepsilon(t),t) = -V_y'(y^\varepsilon(t)) H \sum_{i=1}^{2} [I_{\{\alpha^\varepsilon(t)=i\}} - \nu_i] G(i) H^{-1} y^\varepsilon(t)$$

$$+ O(\varepsilon) V(y^\varepsilon(t)).$$

Define

$$V^\varepsilon(y,t) = V(y) + V_2^\varepsilon(y,t).$$

Evaluate $L^\varepsilon V^\varepsilon(y^\varepsilon(t),t)$. Upon cancelation, for sufficiently small ε, we can make

$$O(\varepsilon)V(y^\varepsilon(t)) \geq -(y_2^\varepsilon(t))^2.$$

It then follows that

$$\begin{aligned} L^\varepsilon V^\varepsilon(y^\varepsilon(t),t) &= 10(y_2^\varepsilon(t))^2 + O(\varepsilon)V(y^\varepsilon(t)) \\ &\geq 9(y_2^\varepsilon(t))^2. \end{aligned}$$

Taking expectation of the left- and right-hand sides above leads to

$$\frac{d}{dt} E|y_2^\varepsilon(t)|^2 \geq 9E|y_2^\varepsilon(t)|^2,$$

which in turn yields that

$$E(y_2^\varepsilon(t))^2 \geq E(y_2^\varepsilon(0))^2 \exp(9t) \to \infty \quad \text{as} \quad t \to \infty.$$

Similar to the previous development, choose $V(y) = y_1^2/2$, define

$$V_1^\varepsilon(y,t) = E_t^\varepsilon \int_t^\infty e^{t-s} V_y'(y) H \sum_{i=1}^{2} [I_{\{\alpha^\varepsilon(s)=i\}} - \nu_i] G(i) H^{-1} y,$$

and redefine

$$V^\varepsilon(y,t) = V(y) + V_1^\varepsilon(y,t).$$

Using the upper bound $O(\varepsilon)V(y^\varepsilon(t)) \leq (y_1^\varepsilon(t))^2$ this time and calculating $L^\varepsilon V^\varepsilon(y^\varepsilon(t),t)$, we obtain

$$\frac{d}{dt} E|y_1^\varepsilon(t)|^2 \leq -209 E|y_1^\varepsilon(t)|^2,$$

which in turn yields that

$$E(y_1^\varepsilon(t))^2 \leq E(y_1^\varepsilon(0))^2 \exp(-209t) \to 0 \quad \text{as} \quad t \to \infty.$$

This yields that (5.144) and hence (5.140) are unstable in probability (see Yin and Zhu [244, p. 220] for a definition). In fact, it can be seen that the trivial solution of the original system is also a saddle.

In the same spirit of the last example, consider a system given by

$$\begin{aligned} &\dot{x}^\varepsilon(t) = G(\alpha^\varepsilon(t)) x^\varepsilon(t), \quad \alpha^\varepsilon(t) \sim Q/\varepsilon, \quad \text{where} \\ &G(1) = \begin{pmatrix} -\frac{7}{3} & -1 \\ 0 & 1 \end{pmatrix}, \; G(2) = \begin{pmatrix} 1 & 0 \\ -1 & -\frac{7}{3} \end{pmatrix}, \end{aligned} \tag{5.145}$$

where Q is as in the last example. Then it can be shown that $x^\varepsilon(\cdot)$ converges weakly to $x(\cdot)$ that is a solution of the following system

$$\begin{aligned} \dot{x}(t) &= \overline{G} x(t), \\ \overline{G} &= \begin{pmatrix} -\frac{4}{3} & -1 \\ -1 & -\frac{4}{3} \end{pmatrix}. \end{aligned} \tag{5.146}$$

Neither $G(1)$ nor $G(2)$ is a stable matrix, but the system (5.146) is a stable one. The stability analysis is again carried out using perturbed Liapunov function methods. Here exactly the same kind of argument as in [5] can be applied. Using the techniques of perturbed Liapunov functions, we can

show that the stability of the averaged system "implies" that of the original system.

These two examples illustrate that one can combine two stable systems using Markovian switching to produce an unstable limit system. Likewise, one can combine two unstable systems to produce a limit stable systems. More importantly, using our weak convergence result of this chapter and the large deviations results in He, Yin, and Zhang [84], combined with the perturbed Liapunov function argument, we can give the reason why such a thing can happen.

5.7 Notes

This chapter concerns sequences of functional occupation measures. It includes convergence of an unscaled sequence (in probability) and central-limit-type results for suitably scaled sequences. For a general introduction to central limit theorems, we refer to the book by Chow and Teicher [30] and the references therein. In the stationary case, that is, $Q(t) = Q$, a constant matrix, the central limit theorem may be obtained as in Friedlin and Wentzell [67]. Some results of central limit type for discrete Markov chains are in Dobrushin [50] (see also the work of Linnik on time-inhomogeneous Markov chains [147]). Work in the context of random evolution, with primary concern the central limit theorem involving a singularly perturbed Markov chain, is in Pinsky [176]; see also Kurtz [135, 137] for related discussions and the martingale problem formulation. Exponential bounds for Markov processes are quite useful in analyzing the behavior of the underlying stochastic processes. Some results in connection with diffusions can be found in Kallianpur [102]. Corollary 5.8 can be viewed as a large deviations result. For extensive treatment of large deviations, see Varadhan [207].

The central theme here is limit results of unscaled as well as scaled sequences of occupation measures, which include the law of large numbers for an unscaled sequence, exponential upper bounds, and asymptotic distribution of a suitably scaled sequence of occupation times. Results in Section 5.2 are based on the paper of Zhang and Yin [252]; however, a somewhat different approach to the central limit theorem was used in [252]. Some of the results in Section 5.3 are based on Zhang and Yin [253]. The result on exponential error bound in Section 5.3 is a natural extension for the irreducible generators. Such result holds uniformly in $t \in [0, T]$ for fixed but otherwise arbitrary $T > 0$. The main motivation for treating T as a parameter stems from various control and optimization problems with discounted cost over the infinite horizon. In such a situation, the magnitude of the bound counts. Thus detailed information on the bounding constant is helpful for dealing with the near optimality of the underlying problem. Section 5.3 also presents a characterization of the limit process using martingale problem formulations. Much of the foundation of this useful approach is

in the work of Stroock and Varadhan [203]. Using perturbed operators to study limit behavior may be traced back to Kurtz [135]. The general idea of perturbed test functions was used in Blankenship and Papanicolaou [16], and Papanicolaou, Stroock, and Varadhan [168]. It was further developed and extended by Kushner [139] for various stochastic systems, and singularly perturbed systems in Kushner [140]; see also Kushner and Yin [145] for related stochastic approximation problems, and Ethier and Kurtz [59] and Kurtz [137] for related work in stochastic processes. The results of this section have benefited from the discussion with Thomas Kurtz, who suggested treating the pair of processes $(n^\varepsilon(\cdot), \alpha^\varepsilon(\cdot))$ together, which led to the current version. Earlier treatment of a pair of processes may be found in the work of Kesten and Papanicolaou [110] for stochastic acceleration.

The results on asymptotic properties for the inclusion of transient states can be found in Yin, Zhang, and Badowski [239]; the results for the case of generators being measurable can be found in the work of Yin, Zhang, and Badowski [240]; the results on asymptotic properties of occupation measures with absorbing states can be found in Yin, Zhang, and Badowski [241].

6

Asymptotic Expansions of Solutions for Backward Equations

6.1 Introduction

In Chapter 4, we focused on obtaining an approximation to the solutions of Kolmogorov forward equations; the emphasis was on the associated probability distribution vectors. However, in many applications, instead of treating the forward equations, we need to deal with the backward equations. In this chapter, we take a dual point of view by examining the associated backward equations.

An important question to answer is this: Is it possible to construct asymptotic expansions? In this chapter, we answer this question using analytic techniques. We aim to obtain asymptotic expansions of solutions of backward equations. Using matched asymptotic expansions, we construct approximations to the solutions of backward equations, show that the asymptotic expansions are valid, and obtain uniform asymptotic error bounds.

The rest of this chapter is arranged as follows. The precise formulation is given next. Then Section 6.3 proceeds with the constructions of the formal asymptotic expansions. Section 6.4 validates the asymptotic expansions by providing uniform error bounds. In this chapter, we consider finite-state Markov chains either including only recurrent states, or containing transient states in addition to recurrent states. Section 6.5 takes up the issue of inclusion of transient states. Section 6.6 provides additional remarks. Finally, the chapter is concluded with the end-of-chapter notes.

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6.2 Problem Formulation

We first present a result needed in the subsequent development, which is the key for the construction of asymptotic expansions. Then we give the formulation of the problem we wish to study.

6.2.1 A Preliminary Lemma

We use v' to denote the transpose of $v \in \mathbb{R}^{i_1 \times i_2}$ with i_1 and $i_2 \geq 1$. For subsequent use, we state a lemma. Part (i) of the lemma is more or less a Fredholm alternative. Part (ii) is based on the observation that any solution of an inhomogeneous algebraic system is the sum of a general solution of the corresponding homogeneous equation and a particular solution of the inhomogeneous equation. A key point is this: Owing to the Markovian structure, among the infinitely many particular solutions, there is only one that satisfies the orthogonality condition (i.e., it is orthogonal to the stationary distribution of the Markov chain); see Yin [223] for further details.

Lemma 6.1. *Suppose that a constant matrix $Q \in \mathbb{R}^{m \times m}$ is a generator of a continuous-time Markov chain and that Q is weakly irreducible.*

(i) *Then for any $b \in \mathbb{R}^m$, the equation*

$$Q\zeta = b \tag{6.1}$$

has a solution if and only if $\nu b = 0$, where $\nu = (\nu_1, \ldots, \nu_m)$ is the quasi-stationary distribution associated with Q. Moreover, suppose that $\widetilde{\zeta}_1$ and $\widetilde{\zeta}_2$ are two solutions of (6.1). Then $\widetilde{\zeta}_1 - \widetilde{\zeta}_2 = c_0 \mathbb{1}_m$ for some $c_0 \in \mathbb{R}$.

(ii) *Any solution of (6.1) can be written as $\zeta = c_0 \mathbb{1}_m + \xi$, where $c_0 \in \mathbb{R}$ is an arbitrary constant and ξ is the unique solution of (6.1) satisfying $\nu\xi = 0$.*

Proof: The proof of part (i) is standard in linear algebra. We proceed to prove part (ii). By part (i) of Lemma 6.1, since (6.1) has a solution, $\nu b = 0$. Consequently, the solution of (6.1) consists of two parts, an arbitrary solution of the homogeneous equation and a particular solution of (6.1).

Since Q is a generator, the null space of Q is spanned by $\mathbb{1}_m$. Thus, the solution of the homogeneous equation can be written as $c_0 \mathbb{1}_m$ for an arbitrary constant $c_0 \in \mathbb{R}$. It remains only to show that there is a unique solution ξ of (6.1) satisfying $\nu\xi = 0$.

Set $b = (b_1, \ldots, b_m)' \in \mathbb{R}^m$ and $\xi = (\xi_1, \ldots, \xi_m)' \in \mathbb{R}^m$. Consider the following system:

$$\begin{cases} Q\xi = b, \\ \nu\xi = 0. \end{cases} \tag{6.2}$$

There are $m+1$ equations and m unknowns. Recall that Q is weakly irreducible. The null space of Q is one-dimensional and the dimension of range(Q) is $(m-1)$, so the rank of Q is also $m-1$. Thus we can replace one of the first m equations in (6.1) by the last equation, and the resulting system of equations is uniquely solvable. A shortcut is to define an augmented matrix $\check{Q} = \begin{pmatrix} Q \\ \nu \end{pmatrix} \in \mathbb{R}^{(m+1)\times m}$ and a new vector $B = \begin{pmatrix} b \\ 0 \end{pmatrix} \in \mathbb{R}^{m+1}$. Then (6.2) can be written as

$$\check{Q}\xi = B. \tag{6.3}$$

Note that $\check{Q}'\check{Q}$ has full rank m due to the weak irreducibility of Q. Thus the solution of (6.3) can be represented by $\xi = (\check{Q}'\check{Q})^{-1}\check{Q}'B$. □

6.2.2 Formulation

To reduce complexity, we focus on a finite-state Markov chain with a large state space $\mathcal{M}$. We deal with time-inhomogeneous Markov chains. The generator of the Markov chain is time-dependent. Let us introduce a small parameter $\varepsilon > 0$, and suppose that $\alpha^\varepsilon(\cdot)$ is a continuous-time Markov chain with state space $\mathcal{M}$ and is time-inhomogeneous with generator

$$Q^\varepsilon(t) = \frac{\widetilde{Q}(t)}{\varepsilon} + \widehat{Q}(t), \tag{6.4}$$

where both $\widetilde{Q}(t)$ and $\widehat{Q}(t)$ are generators of some continuous-time Markov chains (the same model as (4.39)).

Note that $\widetilde{Q}(t)/\varepsilon$ represents the fast-changing part, whereas $\widehat{Q}(t)$ delineates the slowly varying part. For the fast-changing part, we consider two cases. Note that in a finite-state Markov chain, there is at least one recurrent state, and not all states can be transient; either all states are recurrent, or in addition to recurrent states, there is a group of transient states. In the first case, the states belong to l weakly irreducible classes (all states are "recurrent"):

$$\widetilde{Q}(t) = \text{diag}(\widetilde{Q}^1(t), \ldots, \widetilde{Q}^l(t)), \tag{6.5}$$

whereas in the second case, the transient states are included,

$$\widetilde{Q}(t) = \begin{pmatrix} \widetilde{Q}^1(t) & & & \\ & \ddots & & \\ & & \widetilde{Q}^l(t) & \\ \widetilde{Q}^1_*(t) & \ldots & \widetilde{Q}^l_*(t) & \widetilde{Q}_*(t) \end{pmatrix}. \tag{6.6}$$

In both (6.5) and (6.6), we assume that $\widetilde{Q}^i(t)$ is weakly irreducible for each $i = 1, \ldots, l$. In (6.6), we also assume that $\widetilde{Q}^i_*(t)$, $i = 1, \ldots, l$, are of proper dimensions and $\widetilde{Q}_*(t)$ is Hurwitz. If $\widehat{Q}(t)$ were missing or had the same kind of structure as that of $\widetilde{Q}(t)$, the resulting chain would be completely decomposable into l "weakly irreducible" classes. Nevertheless, the presence of the matrix $\widehat{Q}(t)$ makes the corresponding Markov chain only nearly decomposable. In view of the form (6.5) and (6.6), the decomposition of the state space is carried out in accordance with the structure of the generator $\widetilde{Q}(t)$. That is, we write the space $\mathcal{M}$ as

$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots \cup \mathcal{M}_l \tag{6.7}$$

for a chain with $\widetilde{Q}(t)$ given by (6.5), and write the state space as

$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots \cup \mathcal{M}_l \cup \mathcal{M}_* \tag{6.8}$$

for a chain with $\widetilde{Q}(t)$ given by (6.6), where for $i = 1, \ldots, l$,

$$\mathcal{M}_i = \{s_{i1}, \ldots, s_{im_i}\}, \quad \mathcal{M}_* = \{s_{*,1}, \ldots, s_{*,m_*}\}. \tag{6.9}$$

In this chapter, we focus on the system of backward equations (see Chiang [27, p. 402])

$$\begin{aligned} &\frac{d}{dt}u^\varepsilon(t) = -Q^\varepsilon(t)u^\varepsilon(t), \\ &u^\varepsilon(T) = u_0, \end{aligned} \tag{6.10}$$

for some $0 < T < \infty$, where $u^\varepsilon(t) \in \mathbb{R}^{m\times 1}$. Equation (6.10) is known as the backward Kolomogrov equation. In the next two sections, we focus on the case that the generator is of the form (6.5). Then in Section 6.5, we treat the case of $\widetilde{Q}(t)$ given by (6.6). Note that in the above, if $u_0 = (u_0^1, \ldots, u_0^m)'$ satisfies $u_0^i \geq 0$ and $\sum_{i\in\mathcal{M}} u_0^i = 1$, then $u^\varepsilon(t)$ represents a probability vector; otherwise in general $u^\varepsilon(\cdot)$ is not a probability vector.

6.3 Construction of Asymptotic Expansions

This section constructs the formal asymptotic expansions of the solution $u^\varepsilon(\cdot)$ for the case that all states are recurrent using (6.7) with $\sum_{k=1}^l m_k = m$. To carry out the needed analysis, we use the following conditions. The first one concerns the weak irreducibility of $\widetilde{Q}^i(t)$ for each t, and the second one deals with the smoothness of the generators. It follows from (A6.2) that the $(n+1)$st derivatives of $\widetilde{Q}(\cdot)$ and $\widehat{Q}(\cdot)$ are Lipschitz continuous. Slightly weaker conditions are possible, but for most applications, the smoothness condition given here poses little restriction.

(**A6.1**) For each $i = 1, \ldots, l$ and each $t \in [0, T]$, $\widetilde{Q}^i(t)$ is weakly irreducible, and its quasi-stationary distribution is denoted by $\nu^i(t)$.

(**A6.2**) For some positive integer n, $\widetilde{Q}(\cdot)$ and $\widehat{Q}(\cdot)$ are $(n+2)$-times continuously differentiable.

Following the methods of matched asymptotic expansions, we aim to approximate $u^\varepsilon(t)$, the solution of the Cauchy problem (6.10), by

$$\Phi_n^\varepsilon(t) + \Psi_n^\varepsilon\left(\frac{T-t}{\varepsilon}\right),$$

where

$$\begin{aligned} \Phi_n^\varepsilon(t) &= \sum_{i=0}^{n} \varepsilon^i \varphi_i(t), \\ \Psi_n^\varepsilon(\tau) &= \sum_{i=0}^{n} \varepsilon^i \psi_i(\tau), \end{aligned} \tag{6.11}$$

and $\tau = (T-t)/\varepsilon$. Here τ is known as a stretched variable that magnifies the details of the solution near the terminal time T. Using the terminology of singular perturbations, the $\varphi_i(t)$ are the outer expansion terms, and the $\psi_i(\tau)$ are the so-called terminal-layer (or boundary-layer) correction terms. Similar to the approximation to the solutions of forward equations in Chapter 4, the idea is that away from a terminal layer of thickness $O(\varepsilon)$, the solution $\Phi_n^\varepsilon(t)$ is a good approximation to the solution of (6.10), but this solution generally fails to satisfy the terminal condition. Thus to get a good approximation with reasonable accuracy, one has to add a correction term to take care of the approximation in a neighborhood of T.

In constructing the asymptotic expansions, to get the desired error estimates, we need to compute a few more terms. Thus, we need to compute $\Phi_i^\varepsilon(t)$ for $i \le n+1$. Substituting $\Phi_i^\varepsilon(t)$ for $i = 0, \ldots, n+1$ into (6.10) and equating coefficients of like powers of ε^i, we obtain:

$$\begin{aligned} &\widetilde{Q}(t)\varphi_0(t) = 0, \\ &\widetilde{Q}(t)\varphi_1(t) = -\frac{d}{dt}\varphi_0(t) - \widehat{Q}(t)\varphi_0(t), \\ &\cdots \cdots \\ &\widetilde{Q}(t)\varphi_{i+1}(t) = -\frac{d}{dt}\varphi_i(t) - \widehat{Q}(t)\varphi_i(t), \quad i = 1, 2, \ldots, n. \end{aligned} \tag{6.12}$$

Likewise, substituting $\Psi_i(\tau)$ for $i \le n+1$ into (6.10), we obtain

$$\frac{d}{d\tau}\left(\sum_{i=0}^{k} \varepsilon^i \psi_i(\tau)\right) = \sum_{i=0}^{k} \varepsilon^i \psi_i(\tau)\left(-\widetilde{Q}(T-\varepsilon\tau) - \varepsilon\widehat{Q}(T-\varepsilon\tau)\right). \tag{6.13}$$

It is easily seen that the differential equations for $\psi_i(\tau)$ involve a term with time-varying coefficient matrix $\widetilde{Q}(T-\varepsilon\tau)$, which is complicated to deal with. To overcome the difficulty, we use Taylor expansions of $\widetilde{Q}(\cdot)$ and $\widehat{Q}(\cdot)$ about the terminal time T to obtain

$$\begin{aligned}\widetilde{Q}(T-\varepsilon\tau) &= \sum_{j=0}^{i}\frac{\widetilde{Q}^{(j)}(T)}{j!}(-\varepsilon\tau)^j + \widetilde{R}_i(\varepsilon\tau)\\ \varepsilon\widehat{Q}(T-\varepsilon\tau) &= \sum_{j=0}^{i-1}\frac{\widehat{Q}^{(j)}(T)}{j!}\varepsilon(-\varepsilon\tau)^j + \varepsilon\widehat{R}_{i-1}(\varepsilon\tau),\end{aligned}$$

where $Q^{(j)}(T) = (d^j/dt^j)Q(t)|_{t=T}$, $\widetilde{R}_i(t) = O(t^{i+1})$, and $\widehat{R}_{i-1}(t) = O(t^i)$. Equating coefficients of like powers of ε^i in (6.13) for $i \le n+1$ and using the Taylor expansions above, we obtain

$$\begin{aligned}&\frac{d\psi_0(\tau)}{d\tau} = \widetilde{Q}(T)\psi_0(\tau),\\ &\frac{d\psi_1(\tau)}{d\tau} = \widetilde{Q}(T)\psi_1(\tau) + \left(-\tau\widetilde{Q}'(T) + \widehat{Q}(T)\right)\psi_0(\tau),\\ &\quad\cdots\ \cdots\\ &\frac{d\psi_i(\tau)}{d\tau} = \widetilde{Q}(T)\psi_i(\tau) + r_i,\\ &r_i = \sum_{j=0}^{i-1}\left((-\tau)^{i-j}\frac{\widetilde{Q}^{(i-j)}(T)}{(i-j)!} + (-\tau)^{i-j-1}\frac{\widehat{Q}^{(i-j-1)}(T)}{(i-j-1)!}\right)\psi_j(\tau),\\ &\qquad\qquad i = 2,\dots,n+1.\end{aligned} \tag{6.14}$$

To satisfy the terminal conditions, we demand that

$$\varphi_0(T) + \psi_0(0) = u_0 \text{ and } \varphi_i(T) + \psi_i(0) = 0, \text{ for } i = 1,\dots,n+1. \tag{6.15}$$

Solving (6.14) together with terminal conditions (6.15), we obtain

$$\begin{aligned}&\psi_0(\tau) = \exp(\widetilde{Q}(T)\tau)(u_0 - \varphi_0(T)),\\ &\psi_i(\tau) = \exp(\widetilde{Q}(T)\tau)(-\varphi_i(T)) + \int_0^\tau \exp(\widetilde{Q}(T)(\tau-s))r_i(s)ds, \text{ for } i > 0.\end{aligned} \tag{6.16}$$

Set

$$\widetilde{\mathbb{1}} = \operatorname{diag}(\mathbb{1}_{m_1},\dots,\mathbb{1}_{m_l})$$

and

$$\overline{P} = \widetilde{\mathbb{1}}\nu(T) = \operatorname{diag}(\mathbb{1}_{m_1}\nu^1(T),\dots,\mathbb{1}_{m_l}\nu^l(T)). \tag{6.17}$$

Before proceeding further, we present a lemma.

Lemma 6.2. *There exist positive constants κ and C such that*

$$|\exp(\widetilde{Q}(T)\tau) - \overline{P}| \le C\exp(-\kappa\tau), \quad \textit{for} \quad \tau > 0.$$

Proof: Note that $\exp(\widetilde{Q}^k(T)t)$ is a solution of the initial value problem

$$\frac{dP^k(t)}{dt} = \widetilde{Q}^k(T)P^k(t), \ P^k(0) = I_{m_k},$$

where I_{m_k} is the $m_k \times m_k$ identity matrix. Moreover, in view of Lemma A.2, there are C_k and $\kappa_k > 0$ for $k = 1, \ldots, l$ such that

$$|\exp(\widetilde{Q}^k(T)\tau) - \mathbb{1}_{m_k}\nu^k(T)| \le C_k\exp(-\kappa_k\tau).$$

We thus have

$$\begin{aligned}
&|\exp(\widetilde{Q}(T)\tau) - \overline{P}| \\
&\quad = \left|\mathrm{diag}\Big(\exp(\widetilde{Q}^1(T)\tau) - \mathbb{1}_{m_1}\nu^1(T), \ldots, \exp(\widetilde{Q}^l(T)\tau) - \mathbb{1}_{m_l}\nu^1(T)\Big)\right| \\
&\quad \le C\exp(-\kappa\tau),
\end{aligned}$$

where

$$\kappa = \min(\kappa_1, \ldots, \kappa_l) \ \text{ and } \ C = \max(C_1, \ldots, C_l).$$

The desired estimate thus follows. □

6.3.1 Leading Term $\varphi_0(t)$ and Zero-Order Terminal-Layer Term $\psi_0(\tau)$

We may write $\varphi_0(t)$ in a partitioned form as

$$\varphi_0(t) = ((\varphi_0^1(t))', \ldots, (\varphi_0^l(t))')',$$

where $\varphi_0^i(t) \in \mathbb{R}^{m_i \times 1}$. Since $\widetilde{Q}(t)\varphi_0(t) = 0$, $\widetilde{Q}^i(t)\varphi_0^i(t) = 0$ for each $i = 1, \ldots, l$. Then $\varphi_0^i(t)$ is in the null space of $\widetilde{Q}^i(t)$ spanned by $\mathbb{1}_{m_i}$. We can thus write $\varphi_0^i(t)$ as $\varphi_0^i(t) = \beta_0^i(t)\mathbb{1}_{m_i}$ for a smooth function $\beta_0^i(t) \in \mathbb{R}$ to be determined later. As a result, $\varphi_0(t)$ must be of the form

$$\varphi_0(t) = \widetilde{\mathbb{1}}\beta_0(t) \tag{6.18}$$

with

$$\begin{aligned}
&\beta_0(t) = (\beta_0^1(t), \ldots, \beta_0^l(t))' \in \mathbb{R}^l, \\
&\widetilde{\mathbb{1}} = \mathrm{diag}(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_l}).
\end{aligned} \tag{6.19}$$

For a suitable function $f(t)$, set $\dot{f}(t) = df(t)/dt$. Then

$$\dot{\varphi}_0(t) = \widetilde{\mathbb{1}}\dot{\beta}_0(t).$$

So

$$\widetilde{Q}(t)\varphi_1(t) = -\widetilde{\mathbb{1}}\dot{\beta}_0(t) - \widehat{Q}(t)\widetilde{\mathbb{1}}\beta_0(t). \tag{6.20}$$

Define

$$\nu(t) = \text{diag}(\nu^1(t), \ldots, \nu^l(t)). \tag{6.21}$$

Then $\nu(t)\widetilde{Q}(t) = 0$ and $\nu(t)\widetilde{\mathbb{1}} = I_l \in \mathbb{R}^{l\times l}$, the $l \times l$ identity matrix. Using (6.21) and multiplying both sides of equation (6.20) from the left by $\nu(t)$, we obtain

$$\dot{\beta}_0(t) = -\overline{Q}(t)\beta_0(t), \tag{6.22}$$

where

$$\overline{Q}(t) = \nu(t)\widehat{Q}(t)\widetilde{\mathbb{1}}. \tag{6.23}$$

Note that $\overline{Q}(t)$ is the average of $\widehat{Q}(t)$ with respect to the quasi-stationary measures associated with the $\widetilde{Q}^i(t)$. In view of (6.16),

$$\psi_0(\tau) = \exp(\widetilde{Q}(T)\tau)(u_0 - \varphi_0(T)). \tag{6.24}$$

We demand that $\psi_0(\tau) \to 0$ as $\tau \to \infty$. Letting $\tau \to \infty$ in (6.24) and noting that $\exp(\widetilde{Q}(T)\tau) \to \overline{P}$ with $\overline{P}$ given in (6.17), we obtain

$$\overline{P}\psi_0(0) = 0. \tag{6.25}$$

Using (6.21) and multiplying both sides of the above equation from the left by $\nu(T)$, we obtain the following equivalent equation

$$\nu(T)\psi_0(0) = 0. \tag{6.26}$$

In addition,

$$\begin{aligned} \nu(T)\psi_0(0) &= \nu(T)(u_0 - \varphi_0(T)) \\ &= \nu(T)u_0 - \beta_0(T). \end{aligned} \tag{6.27}$$

Thus,

$$\beta_0(T) = \nu(T)u_0. \tag{6.28}$$

As a result, $\beta_0(t)$ can be uniquely determined from the differential equation (6.22) and the terminal condition (6.28).

6.3.2 Higher-Order Terms

For future use, we state a lemma. The proof uses elementary row operations of matrices.

Lemma 6.3. *Define* $Q_v(t) = \begin{pmatrix} \widetilde{Q}(t) \\ \nu(t) \end{pmatrix}$. *Under condition* (A6.1), *for each* $t \in [0,T]$,

$$\operatorname{rank}(Q_v(t)) = m \quad \textit{and} \quad \operatorname{rank}(Q_v'(t)Q_v(t)) = m.$$

Proof: Interchanging rows, we can show that

$$Q_v(t) \to Q_c(t) = \operatorname{diag}\left(\begin{pmatrix} \widetilde{Q}^1(t) \\ \nu^1(t) \end{pmatrix}, \begin{pmatrix} \widetilde{Q}^2(t) \\ \nu^2(t) \end{pmatrix}, \ldots, \begin{pmatrix} \widetilde{Q}^l(t) \\ \nu^l(t) \end{pmatrix} \right).$$

Since elementary row operations do not change the rank of a matrix, $\operatorname{rank}(Q_v(t)) = \operatorname{rank}(Q_c(t))$. Note that for each $i = 1, \ldots, l$, the matrix $\begin{pmatrix} \widetilde{Q}^i(t) \\ \nu^i(t) \end{pmatrix}$ has rank m_i owing to the weak irreducibility of $\widetilde{Q}^i(t)$. This yields that

$$\operatorname{rank}(Q_c(t)) = \sum_{i=1}^{l} \operatorname{rank} \begin{pmatrix} \widetilde{Q}^i(t) \\ \nu^i(t) \end{pmatrix} = \sum_{i=1}^{l} m_i = m.$$

The lemma is thus proved. □

To proceed, for $i > 0$, we construct $\varphi_i(t)$ and $\psi_i(\tau)$ by induction. Suppose that the terms $\varphi_j(t)$ and $\psi_j(\tau)$ for $j < i$ have been constructed such that for each $j < i$, $\psi_j(\tau)$ decays exponentially fast and $\varphi_j(t)$ is smooth. We next construct $\varphi_i(t)$ and $\psi_i(\tau)$.

Using (6.12), we have

$$\widetilde{Q}(t)\varphi_i(t) = -\dot{\varphi}_{i-1}(t) - \widehat{Q}(t)\varphi_{i-1}(t) \stackrel{\text{def}}{=} \widetilde{b}_{i-1}(t). \tag{6.29}$$

The right-hand side above, namely $\widetilde{b}_{i-1}$ is a known function since $\varphi_{i-1}(t)$ has been constructed. Using the partitioned form of the vector

$$\varphi_i(t) = ((\varphi_i^1(t))', \ldots, (\varphi_i^l(t))')',$$

by part (ii) of Lemma 6.1, the ith block of the solution of (6.29) is given by

$$\varphi_i(t) = \widetilde{\mathbb{1}}\beta_i(t) + \widehat{\varphi}_i(t), \tag{6.30}$$

where $\widetilde{\mathbb{1}}\beta_i(t)$ is an arbitrary solution to the homogeneous equation and $\widehat{\varphi}_i(t)$ the unique solution of the inhomogeneous equation satisfying $\nu^k(t)\widetilde{b}_{i-1}^k = 0$, where $\widetilde{b}_{i-1}^k(t)$ denotes the kth partitioned vector of $\widetilde{b}_{i-1}(t)$.

Define $Q_v(t)$ as in Lemma 6.3 and $\widehat{b}_{i-1} = \begin{pmatrix} \widetilde{b}_{i-1} \\ 0 \end{pmatrix}$. Then

$$Q_v(t)\widehat{\varphi}_i(t) = \begin{pmatrix} \widetilde{Q}(t)\widehat{\varphi}_i(t) \\ \nu(t)\widehat{\varphi}_i(t) \end{pmatrix} = \widehat{b}_i.$$

Lemma 6.3 together with Lemma 6.1 implies

$$\widehat{\varphi}_i(t) = (Q_v'(t)Q_v(t))^{-1}Q_v'(t)\overline{b}_{i-1}. \tag{6.31}$$

Using (6.12), we obtain

$$\widetilde{Q}(t)\varphi_{i+1}(t) = -\dot{\varphi}_i(t) - \widehat{Q}(t)\varphi_i(t).$$

Multiplying both sides by $\nu(t)$ from the left, noting (6.30) and the fact $\nu(t)\widehat{\varphi}_i(t) = 0$, we deduce

$$\dot{\beta}_i(t) = -\overline{Q}(t)\beta_i(t) - \nu(T)\dot{\widehat{\varphi}}_i(t), \tag{6.32}$$

where $\overline{Q}(t)$ is given in (6.23). Equation (6.32) is uniquely solvable if the terminal condition is specified. We need to use the terminal-layer term to determine the terminal condition. In view of (6.16),

$$\psi_i(\tau) = \exp(\widetilde{Q}(T)\tau)\psi_i(0) + \int_0^\tau \exp(\widetilde{Q}(T)(\tau - s))r_i(s)ds. \tag{6.33}$$

We demand that $\psi_i(\tau) \to 0$ as $\tau \to \infty$. Letting $\tau \to \infty$ in (6.33) and noting that $\exp(\widetilde{Q}(T)\tau) \to \overline{P}$ with $\overline{P}$ given in (6.17), we obtain

$$\overline{P}\psi_i(0) + \int_0^\infty \overline{P}r_i(s)ds = 0. \tag{6.34}$$

Using (6.21) and multiplying both sides of the above equation from the left by $\nu(T)$, we obtain the following equivalent equation

$$\nu(T)\psi_i(0) + \int_0^\infty \nu(T)r_i(s)ds = 0. \tag{6.35}$$

We have

$$\begin{aligned} &\nu(T)\psi_i(0) + \int_0^\infty \nu(T)r_i(s)ds \\ &= -\nu(T)\varphi_i(T) + \int_0^\infty \nu(T)r_i(s)ds \\ &= -(\beta_i(T) + \nu(T)\widehat{\varphi}_i(T)) + \int_0^\infty \nu(T)r_i(s)ds \\ &= -\beta_i(T) + \int_0^\infty \nu(T)r_i(s)ds. \end{aligned} \tag{6.36}$$

Note that the integral involving $r_i(s)$ is well defined since $|r_i(s)| \leq Ce^{-\kappa s}$ a consequence of $r_i(\cdot)$ being a linear combination of $\psi_j(\cdot)$ for $j \leq i-1$, and the $\psi_j(\cdot)$ decay exponentially fast by the induction hypothesis. By virtue of (6.35) and (6.36),

$$\beta_i(T) = \int_0^\infty \nu(T) r_i(s) ds. \tag{6.37}$$

Moreover, when $\beta_i(T)$ satisfies (6.37), $\psi_i(\tau) \to 0$ as $\tau \to \infty$ as desired. The terminal condition for $\beta_i(t)$ has thus been found. Then

$$\varphi_i(t) = \widetilde{\mathbb{1}}\beta_i(t) + \widehat{\varphi}_i(t) = \widetilde{\mathbb{1}}\beta_i(t) + (Q_v(t)'Q_v(t))^{-1}Q_v(t)'\widehat{b}_{i-1}, \tag{6.38}$$

with $\beta_i(t)$ determined uniquely by the differential equation (6.32) and the terminal condition (6.37).

Proposition 6.4. *For each $i = 0, \ldots, n+1$, $\varphi_i(\cdot) \in C^{n+2-i}([0,T])$.*

Proof: We prove this by induction. First note that (6.22) implies that $\beta_0(\cdot) \in C^{n+1}([0,T])$. Therefore, $\varphi_0(\cdot) \in C^{n+2}([0,T])$. Assume that $\varphi_j(\cdot) \in C^{n+2-j}([0,T])$ for any $j < i$. Then $\widehat{b}_{i-1}(\cdot) \in C^{n+2-i}([0,T])$. Set $Q_1(t) = (\widetilde{Q}(t)\ \widetilde{\mathbb{1}})$. Then

$$\nu(t)Q_1(t) = (0_{l\times m}\ I_l).$$

Moreover, using the weak irreducibility of $\widetilde{Q}^k(t)$ for $k = 1, \ldots, l$, similarly to Lemma 6.3, we can show that $\operatorname{rank}(Q_1(t)Q_1'(t)) = m$. As a result, $\nu(t) = (0_{l\times m}\ I_l)Q_1'(t)(Q_1(t)Q_1'(t))^{-1}$. Thus $\nu(t) \in C^{n+2}([0,T])$. As a result, (6.38) implies $\varphi_i(\cdot) \in C^{n+2-i}([0,T])$. □

Proposition 6.5. *For $0 \leq i \leq n+1$, there exist constants C and $0 < \kappa_i < \kappa$ such that*

$$|\psi_i(\tau)| \leq C\exp(-\kappa_i\tau) \quad \textit{for} \quad \tau \geq 0.$$

Proof: We prove this by induction. First, by Lemma 6.2 and (6.25),

$$\begin{aligned} |\psi_0(\tau)| &= \left|\overline{P}\psi_0(0) + (\exp(\widetilde{Q}(T)\tau) - \overline{P})\psi_0(0)\right| \\ &= |\exp(\widetilde{Q}(T)\tau) - \overline{P}||\psi_0(0)| \\ &\leq C\exp(-\kappa\tau) \quad \text{for some} \quad \kappa > 0. \end{aligned}$$

Assume that for all $j < i$,

$$|\psi_j(\tau)| \leq C\exp(-\kappa_j\tau) \quad \text{for some} \quad 0 < \kappa_j < \kappa.$$

Then for some $0 < \widetilde{\kappa} \leq \kappa_{i-1}$,

$$|r_i(s)| \leq C \exp(-\widetilde{\kappa} s) \quad \text{for } s \geq 0.$$

We proceed to show that $\psi_i(\tau)$ also decays exponentially fast. In fact,

$$\begin{aligned} |\psi_i(\tau)| &= \Big| (\exp(\widetilde{Q}(T)\tau) - \overline{P})\psi_i(0) \\ &\quad + \int_0^\tau (\exp(\widetilde{Q}(T)(\tau - s)) - \overline{P}) r_i(s) ds + \int_\tau^\infty -\overline{P} r_i(s) ds \Big| \\ &\leq C \exp(-\kappa\tau) + C \int_0^\tau \exp(-\kappa(\tau - s)) \exp(-\widetilde{\kappa} s) ds \\ &\quad + C \int_\tau^\infty \exp(-\widetilde{\kappa} s) ds \end{aligned}$$

Thus $|\psi_i(\tau)| \leq C \exp(-\kappa_i \tau)$ for some $0 < \kappa_i < \widetilde{\kappa} < \kappa$ as desired. □

6.4 Error Estimates

Consider

$$\mathcal{L}^\varepsilon f = \varepsilon \frac{df}{dt} + (\widetilde{Q}(t) + \varepsilon \widehat{Q}(t)) f. \tag{6.39}$$

Lemma 6.6. *Suppose that $\xi^\varepsilon(t)$ is a solution of $\mathcal{L}^\varepsilon \xi^\varepsilon(t) = 0$ with $\xi^\varepsilon(T) = 0$ such that $|\mathcal{L}^\varepsilon \xi^\varepsilon(t)| \leq C\varepsilon^{k+1}$ for a positive integer $1 \leq k \leq n$. Then $\sup_{t \in [0,T]} |\xi^\varepsilon(t)| = O(\varepsilon^k)$.*

Proof: Consider the solution of the following Cauchy problem for $s \geq t$,

$$\begin{cases} \dfrac{\partial \Sigma^\varepsilon(s,t)}{\partial t} = -Q^\varepsilon(t) \Sigma^\varepsilon(s,t) \\ \Sigma^\varepsilon(s,s) = I. \end{cases} \tag{6.40}$$

Because the equation is linear, there is a unique solution. Owing to the well-known setup for Markov chains and the associated backward differential equations (see, e.g., [27, pp. 398-403]), $\Sigma^\varepsilon(s,t)$ is the solution of the backward equations together with the terminal condition being an identity matrix, so it is a transition matrix. As a result, $\Sigma^\varepsilon(s,t)$ is bounded uniformly.

To proceed, let us examine the following terminal value problem

$$\begin{cases} \mathcal{L}^\varepsilon \xi^\varepsilon(t) = \zeta^\varepsilon(t), \\ |\zeta^\varepsilon(t)| = O(\varepsilon^{k+1}), \\ \xi^\varepsilon(T) = 0. \end{cases}$$

The solution is given by

$$\xi^\varepsilon(t) = -\frac{1}{\varepsilon}\int_t^T \Sigma^\varepsilon(s,t)\zeta^\varepsilon(s)ds,$$

where $\Sigma^\varepsilon(s,t)$ is the principal matrix solution (i.e., the solution of (6.40)) that is bounded. Thus

$$|\xi^\varepsilon(t)| \le \frac{C}{\varepsilon}\int_t^T \varepsilon^{k+1}ds \le C\varepsilon^k.$$

Furthermore, taking $\sup_{t\in[0,T]}$, the lemma is proved. □

Proposition 6.7. *Let $u^\varepsilon(t)$ be the solution of (6.10) and let $\Phi_n^\varepsilon(t)$ and $\Psi_n^\varepsilon(\tau)$ be given by (6.11), as constructed in the last section. Under the conditions* (A6.1) *and* (A6.2)*, there exists a $C>0$ such that*

$$\sup_{t\in[0,T]}\left|u^\varepsilon(t) - \Phi_n^\varepsilon(t) - \Psi_n^\varepsilon\left(\frac{T-t}{\varepsilon}\right)\right| = C\varepsilon^{n+1}.$$

Proof: Put

$$e^{\varepsilon,k}(t) = u^\varepsilon(t) - \Phi_k^\varepsilon(t) - \Psi_k^\varepsilon\left(\frac{T-t}{\varepsilon}\right),\ k \le n+1.$$

Since $u^\varepsilon(t)$ is a solution of (6.10), $\mathcal{L}^\varepsilon u^\varepsilon(t) = 0$. For $k \le n+1$,

$$\mathcal{L}^\varepsilon e_k^\varepsilon(t) = -\mathcal{L}^\varepsilon\Phi_k^\varepsilon(t) - \mathcal{L}^\varepsilon\Psi_k^\varepsilon\left(\frac{T-t}{\varepsilon}\right).$$

Moreover, recall that $\widetilde{Q}(t)\varphi_0(t) = 0$. It follows that

$$\begin{aligned}
\mathcal{L}^\varepsilon\Phi_k^\varepsilon(t) &= \sum_{i=0}^k \varepsilon^{i+1}\dot{\varphi}_i(t) + \sum_{i=0}^k \varepsilon^i\widetilde{Q}(t)\varphi_i(t) + \sum_{i=0}^k \varepsilon^{i+1}\widehat{Q}(t)\varphi_i(t)\\
&= \sum_{i=0}^k \varepsilon^{i+1}(-\widetilde{Q}(t)\varphi_{i+1}(t) - \widehat{Q}(t)\varphi_i(t)) + \sum_{i=0}^k \varepsilon^i\widetilde{Q}(t)\varphi_i(t)\\
&\quad + \sum_{i=0}^k \varepsilon^{i+1}\widehat{Q}(t)\varphi_i(t)\\
&= -\varepsilon^{k+1}\widetilde{Q}(t)\varphi_{k+1}(t) + \widetilde{Q}(t)\varphi_0(t)\\
&= -\varepsilon^{k+1}\widetilde{Q}(t)\varphi_{k+1}(t).
\end{aligned}$$

The smoothness of $\varphi_i(t)$ then yields

$$|\mathcal{L}^\varepsilon\Phi_k^\varepsilon(t)| \le C\varepsilon^{k+1},\ \forall t\in[0,T].$$

Furthermore, the bound holds uniformly in $t \in [0,T]$. Now using the definition $\tau = (T-t)/\varepsilon$,

$$\varepsilon \frac{d}{dt}\psi_k\Big(\frac{T-t}{\varepsilon}\Big) = -\frac{d}{d\tau}\psi_k(\tau),$$

which yields

$$\begin{aligned}
&\mathcal{L}^\varepsilon \Psi_k^\varepsilon(\tau)\\
&\quad = \sum_{i=0}^{k} -\varepsilon^i \frac{d}{d\tau}\psi_i(\tau) + \sum_{i=0}^{k} \varepsilon^i \widetilde{Q}(t)\psi_i(\tau) + \sum_{i=0}^{k} \varepsilon^{i+1}\widehat{Q}(t)\psi_i(t)\\
&\quad = \sum_{i=0}^{k} -\varepsilon^i(\widetilde{Q}(T)\psi_i(\tau) + r_i(\tau)) + \sum_{i=0}^{k} \varepsilon^i \widetilde{Q}(t)\psi_i(\tau) + \sum_{i=0}^{k} \varepsilon^{i+1}\widehat{Q}(t)\psi_i(\tau),
\end{aligned}$$

where

$$\begin{aligned}
&\sum_{i=1}^{k} \varepsilon^i r_i(\tau)\\
&\quad = \sum_{j=0}^{k-1}\sum_{i=j+1}^{k} \Big(\varepsilon^i(-\tau)^{i-j}\frac{\widetilde{Q}^{(i-j)}(T)}{(i-j)!} + \varepsilon^i(-\tau)^{i-j-1}\frac{\widehat{Q}^{(i-j-1)}(T)}{(i-j-1)!}\Big)\psi_j(\tau)\\
&\quad = \sum_{j=0}^{k-1}\sum_{i=j+1}^{k} \Big(\varepsilon^j(t-T)^{i-j}\frac{\widetilde{Q}^{(i-j)}(T)}{(i-j)!}\\
&\qquad\qquad + \varepsilon^{j+1}(t-T)^{i-j-1}\frac{\widehat{Q}^{(i-j-1)}(T)}{(i-j-1)!}\Big)\psi_j(\tau)\\
&\quad = \sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1} \Big(\varepsilon^i(t-T)^{j+1}\frac{\widetilde{Q}^{(j+1)}(T)}{(j+1)!} + \varepsilon^{i+1}(t-T)^j\frac{\widehat{Q}^{(j)}(T)}{j!}\Big)\psi_i(\tau).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathcal{L}^\varepsilon \Psi_k^\varepsilon(\tau)\\
&\quad = \sum_{i=0}^{k} -\varepsilon^i(\widetilde{Q}(T)\psi_i(\tau) + r_i(\tau)) + \sum_{i=0}^{k} \varepsilon^i \widetilde{Q}(t)\psi_i(\tau) + \sum_{i=0}^{k} \varepsilon^{i+1}\widehat{Q}(t)\psi_i(\tau)\\
&\quad = \varepsilon^k\Big(\widetilde{Q}(t) - \widetilde{Q}(T)\Big)\psi_k(\tau) + \sum_{i=0}^{k-1}\varepsilon^i\Big(\widetilde{Q}(t) - \widetilde{Q}(T)\\
&\qquad - \sum_{j=0}^{k-i-1}(t-T)^{j+1}\frac{\widetilde{Q}^{(j+1)}(T)}{(j+1)!}\Big)\psi_i(\tau)\\
&\qquad + \varepsilon^{k+1}\widehat{Q}(t)\psi_k(\tau) + \sum_{i=0}^{k-1}\varepsilon^{i+1}\left(\widehat{Q}(t) - \sum_{j=0}^{k-i-1}(t-T)^j\frac{\widehat{Q}^{(j)}(T)}{j!}\right)\psi_i(\tau).
\end{aligned}$$

Moreover, the Taylor expansion implies that

$$\left|\widetilde{Q}(t)-\widetilde{Q}(T)\right| \le C|T-t| = C\varepsilon\tau,$$

$$\left|\widetilde{Q}(t)-\widetilde{Q}(T)-\sum_{j=0}^{k-i-1}(t-T)^{j+1}\frac{\widetilde{Q}^{(j+1)}(T)}{(j+1)!}\right| \le C(T-t)^{k-i},$$

$$= C\varepsilon^{k-i}\tau^{k-i}$$

$$\left|\widehat{Q}(t)-\sum_{j=0}^{k-i}(t-T)^{j}\frac{\widehat{Q}^{(j)}(T)}{j!}\right| \le C(T-t)^{k-i} = C\varepsilon^{k-i}\tau^{k-i}.$$

As a result,

$$\begin{aligned}|\mathcal{L}^{\varepsilon}\Psi_k^{\varepsilon}(\tau)| \ &\le C\varepsilon^{k+1}\tau\exp(-\kappa_k\tau)\\ &\quad+\sum_{i=0}^{k-1}C\varepsilon^{k+1}\tau^{k-i}\exp(-\kappa_i\tau)\\ &\quad+C\varepsilon^{k+1}+\sum_{i=0}^{k-1}C\varepsilon^{k+1}\tau^{k-i}\exp(-\kappa_i\tau)\\ &\le C\varepsilon^{k+1}.\end{aligned}$$

Piecing this together with the estimates on $\mathcal{L}^{\varepsilon}\Phi_k^{\varepsilon}(t)$, we have shown that $|\mathcal{L}^{\varepsilon}e_k^{\varepsilon}(t)| \le C\varepsilon^{k+1}$. Note the terminal condition $e_k^{\varepsilon}(T)=0$. Thus Lemma 6.6 implies $\sup_{t\in[0,T]}|e_k^{\varepsilon}(t)| = O(\varepsilon^k)$. Taking $k=n+1$, we obtain

$$\sup_{t\in[0,T]}|e_{n+1}^{\varepsilon}(t)| = O(\varepsilon^{n+1}).$$

Finally, note that

$$e_{n+1}^{\varepsilon}(t) = e_n^{\varepsilon}(t)+\varepsilon^{n+1}\varphi_{n+1}(t)+\varepsilon^{n+1}\psi_{n+1}(\tau). \tag{6.41}$$

The continuity of $\varphi_{n+1}(\cdot)$ and the exponential decay properties of $\psi_{n+1}(\tau)$ yield that

$$\sup_{t\in[0,T]}|\varepsilon^{n+1}\varphi_{n+1}(t)+\varepsilon^{n+1}\psi_{n+1}((T-t)/\varepsilon)| = O(\varepsilon^{n+1}).$$

Substituting this into (6.41), we obtain

$$\sup_{t\in[0,T]}|e_n^{\varepsilon}(t)| = O(\varepsilon^{n+1}).$$

The proposition is proved. □

Next, we summarize what has been obtained so far in the following theorem. It provides a detailed construction of the asymptotic series as well as the error bounds.

Theorem 6.8. *Assume* (A6.1) *and* (A6.2). *Then asymptotic expansions* $\Phi_n^\varepsilon(t) + \Psi^\varepsilon((T-t)/\varepsilon)$ *can be constructed as follows:*

- $\varphi_0(t)$ *is obtained from* (6.18), (6.22), *and* (6.28), *and* $\psi_0((T-t)/\varepsilon)$ *is obtained from the first equation in* (6.16);
- $\varphi_k(t)$ *is obtained from* (6.32), (6.37), *and* (6.38), *and* $\psi_k((T-t)/\varepsilon)$ *is obtained from the second equation in* (6.16);
- $\varphi_k(\cdot) \in C^{n+2-k}([0,T])$;
- $\psi_k((T-t)/\varepsilon)$ *decays exponentially fast in that*

$$|\psi_k((T-t)/\varepsilon)| \le C\exp(-\kappa_0(T-t)/\varepsilon);$$

- *the following error bound holds:*

$$\sup_{t\in[0,T]} |u^\varepsilon(t) - \Phi^\varepsilon(t) - \Psi^\varepsilon((T-t)/\varepsilon)| = O(\varepsilon^{n+1}).$$

6.5 Asymptotic Expansions Including Transient States

This section concerns the asymptotic expansions of solutions of (6.10) when the generator is given by (6.6). We only outline the differences and state the main results. We describe in some detail how to get the terms $\varphi_0(t)$ and $\psi_0(\tau)$, and then present the results for higher-order terms. In addition to (A6.1) and (A6.2), we assume (A6.3).

(**A6.3**) For each $t \in [0,T]$, $\widetilde{Q}_*(t)$ is Hurwitz (i.e., all of its eigenvalues have negative real parts).

We construct the formal expansions as in Section 6.3, and obtain in (6.12) the outer expansion terms. Then we take Taylor expansions about T for both $\widetilde{Q}(t)$ and $\widehat{Q}(t)$ as in Section 6.3, which lead to (6.14). To proceed, we denote $\varphi_k(t)$ and $\psi_k(t)$ in a partitioned form by

$$\begin{aligned}\varphi_k(t) &= ((\varphi_k^1(t))', \ldots, (\varphi_k^l(t))', (\varphi_k^*(t))')',\\ \psi_k(\tau) &= ((\psi_k^1(\tau))', \ldots, (\varphi_k^l(\tau))', (\varphi_k^*(\tau))')',\end{aligned} \tag{6.42}$$

whenever it is convenient. In the above, $\varphi_k^i(t), \psi_k^i(\tau) \in \mathbb{R}^{m_i \times 1}$ with $i = 1, \dots, l, *$. Then in the zeroth-order expansion, we have

$$\begin{aligned} &\widetilde{Q}^i(t)\varphi_0^i(t) = 0, \quad i = 1, \dots, l, \\ &\sum_{i=1}^{l} \widetilde{Q}_*^i(t)\varphi_0^i(t) + \widetilde{Q}_*(t)\varphi_0^*(t) = 0. \end{aligned} \tag{6.43}$$

As in the previous section,

$$\varphi_0^i(t) = \beta_0^i(t)\mathbb{1}_{m_i}, \ i = 1, \dots, l, \tag{6.44}$$

with $\beta_0^i(t)$ to be determined and $\beta_0(t) = (\beta_0^1(t), \dots, \beta_0^l(t))' \in \mathbb{R}^{l \times 1}$. Since $\widetilde{Q}_*(t)$ is nonsingular owing to (A6.3), it is readily seen that

$$\varphi_0^*(t) = -\sum_{i=1}^{l} \beta_0^i(t)\widetilde{Q}_*^{-1}(t)\widetilde{Q}_0^i(t)\mathbb{1}_{m_i} = \sum_{i=1}^{l} \beta_0^i(t)a_i(t), \tag{6.45}$$

where

$$a_i(t) = -\widetilde{Q}_*^{-1}(t)\widetilde{Q}_*^i(t)\mathbb{1}_{m_i} \in \mathbb{R}^{m_* \times m_i}, \quad \text{for} \quad i = 1, \dots, l. \tag{6.46}$$

Thus,

$$\varphi_0(t) = \left(\beta_0^1(t)\mathbb{1}_{m_1}, \dots, \beta_0^l(t)\mathbb{1}_{m_l}, \sum_{i=1}^{l} \beta_0^i(t)a_i(t) \right). \tag{6.47}$$

Denote $\widetilde{m}_* = m_1 + \cdots + m_l + m_*$, and

$$\widetilde{\mathbb{1}}_*(t) = \begin{pmatrix} \mathbb{1}_{m_1} & & & \\ & \ddots & & \\ & & & \mathbb{1}_{m_l} \\ a_1(t) & \dots & a_l(t) & \end{pmatrix}, \tag{6.48}$$

$$a(t) = (a_1(t), \dots, a_l(t)).$$

Note that

$$\sum_{i=1}^{l} \widetilde{Q}_*^i(t)\mathbb{1}_{m_i} + \sum_{i=1}^{l} \widetilde{Q}_*(t)a_i = 0.$$

Then $\widetilde{Q}(t)\widetilde{\mathbb{1}}_*(t) = 0$ for each $t \in [0,T]$. Define also

$$\nu_*(t) = (\nu(t)\vdots 0_{l\times m_*}) = \begin{pmatrix} \nu^1(t) & & & 0_{1\times m_*} \\ & \ddots & & \vdots \\ & & \nu^l(t) & 0_{1\times m_*} \end{pmatrix} \in \mathbb{R}^{l\times \widetilde{m}_*},$$

$$\overline{P}_* = \widetilde{\mathbb{1}}_*(T)\nu_*(T) = \begin{pmatrix} \mathbb{1}_{m_1}\nu^1(T) & & & 0_{m_1\times m_*} \\ & \ddots & & \vdots \\ & & \mathbb{1}_{m_l}\nu^l(T) & 0_{m_l\times m_*} \\ a_1(T)\nu^1(T) & \dots & a_l(T)\nu^l(T) & 0_{m_*\times m_*} \end{pmatrix}, \tag{6.49}$$

where $\nu(t) = \operatorname{diag}(\nu^1(t),\dots,\nu^l(t))$. Define

$$\overline{Q}_*(t) = \nu_*(t)\widehat{Q}(t)\widetilde{\mathbb{1}}_*(t) \in \mathbb{R}^{l\times l}. \tag{6.50}$$

Following similar arguments to those of Section 6.3, we obtain

$$\widetilde{Q}(t)\varphi_1(t) = -\widetilde{\mathbb{1}}_*(t)\dot{\beta}_0(t) - \widehat{Q}(t)\widetilde{\mathbb{1}}_*(t)\beta_0(t).$$

Then $\nu_*(t)\widetilde{Q}(t) = 0$ and $\nu_*(t)\widetilde{\mathbb{1}}_*(t) = I_l$ lead to

$$\dot{\beta}_0(t) = -\overline{Q}_*(t)\beta_0(t), \tag{6.51}$$

where $\beta_0(T)$ is yet to be chosen.

Lemma 6.9. *Consider the system of equations*

$$\frac{d}{d\tau}y(\tau) = \widetilde{Q}(T)y(\tau), \quad y(0) = y_0$$

where $y(\tau) = (y^1(\tau),\dots,y^l(\tau),y^*(\tau))' \in \mathbb{R}^{\widetilde{m}_*\times 1}$ *satisfy* $\nu^i(T)y_0^i = 0$ *for each* $i = 1,\dots,l$. *Then there exist positive constants* κ *and* C *such that*

$$|y^i(\tau)| \le C\exp(-\kappa_i\tau),\ i = 1,\dots,l,$$

$$|y^*(\tau)| \le C\exp(-\kappa\tau) \quad \textit{for} \quad \tau > 0.$$

Proof: Using the partitioned vector notation, we obtain

$$\frac{d}{d\tau}y^i(\tau) = \widetilde{Q}^i(T)y^i(\tau),\ i = 1,\dots,l,$$
$$\frac{d}{d\tau}y^*(\tau) = \widetilde{Q}_*(T)y^*(\tau) + \sum_{i=1}^{l}\widetilde{Q}_*^i(T)y^i(\tau).$$

As a consequence of the above orthogonality condition,

$$\overline{P}_* y_0 = \widetilde{\mathbb{1}}_*(T)\nu_*(T)y_0 = \begin{pmatrix} \mathbb{1}_{m_1}\nu^1(T)y_0^1 \\ \cdots \\ \mathbb{1}_{m_l}\nu^l(T)y_0^l \\ \sum_{i=1}^{l} a_i(T)\nu^i(T)y_0^i + 0_{m_*\times m_*}y_0^* \end{pmatrix} = 0,$$

since $\nu^i(T)$ is orthogonal to y_0^i. Thus

$$\begin{aligned} |y^i(\tau)| &= |\exp(\widetilde{Q}^i(T)\tau)y_0^i| \\ &= |[\exp(\widetilde{Q}^i(T)\tau) - \mathbb{1}_{m_i}\nu^i(T)]y_0^i + \mathbb{1}_{m_k}\nu^i(T)y_0^i| \\ &\le C\exp(-\kappa_i\tau) \quad \text{for some} \quad \kappa_i > 0. \end{aligned} \tag{6.52}$$

As for $y^*(\tau)$, we obtain

$$y^*(\tau) = \exp(\widetilde{Q}_*(T)\tau)y_0^* + \sum_{i=1}^{l}\int_0^\tau \exp(\widetilde{Q}_*(T)(\tau - s))\widetilde{Q}_*^i(T)y^i(s)ds.$$

Since $\widetilde{Q}_*(T)$ has all of its eigenvalues on the left-hand side of the complex plane, it follows that

$$|\exp(\widetilde{Q}_*(T)\tau)| \le C\exp(-\kappa_*\tau) \quad \text{for some} \quad \kappa_* > 0.$$

As a result,

$$\begin{aligned} &|y^*(\tau)| \\ &\le |\exp(\widetilde{Q}_*(T)\tau)||y_0^*| + \sum_{i=1}^{l}\int_0^\tau |\exp(\widetilde{Q}_*(T)(\tau - s))||\widetilde{Q}_*^i(T)||y^i(s)|ds \\ &\le C\exp(\kappa_*\tau) + C\sum_{i=1}^{l}\int_0^\tau \exp(-\kappa_*(\tau - s))\exp(-\kappa_i s)ds \\ &\le C\exp(-\kappa\tau) \quad \text{for some} \quad \kappa < \min\{\kappa_*, \kappa_i : i = 1, \ldots, l\}. \end{aligned}$$

Hence the lemma follows. □

Choosing $\psi_0(0) + \varphi_0(T) = u_0$, we obtain

$$\psi_0(\tau) = [\exp(\widetilde{Q}(T)\tau) - \overline{P}_*](u_0 - \varphi_0(T)) + \overline{P}_*(u_0 - \varphi_0(T)). \tag{6.53}$$

In view of (6.53), we choose $\psi_0(0)$ such that $\nu^i(T)\psi_0^i = 0$ or equivalently, $\overline{P}_*\psi_0(0) = 0$. Since

$$\overline{P}_*\psi_0(0) = \widetilde{\mathbb{1}}_*(T)\nu_*(T)(u_0 - \varphi_0(T)),$$

multiplying from the left by $\nu_*(T)$ yields that

$$\beta_0(T) = \nu_*(T)u_0. \tag{6.54}$$

Partition $\psi_0(\tau)$ so that

$$\psi(\tau) = ((\psi_0^1(\tau))', \ldots, (\psi_0^l(\tau))', (\psi_0^*(\tau))')'.$$

Then we have

$$\psi_0^i(\tau) = \exp(\widetilde{Q}^i(T)\tau)\psi_0^i(0), \quad i = 1, \ldots, l,$$

$$\psi_0^*(\tau) = \exp(\widetilde{Q}_*(T)\tau)\psi_0^*(0) + \sum_{i=1}^{l}\int_0^\tau \exp(\widetilde{Q}_*(T)(\tau - s))\widetilde{Q}_*(T)\psi_0^i(s)ds.$$

Then using Lemma 6.9, for some $\kappa > 0$,

$$|\psi_0^i(\tau)| \le C\exp(-\kappa\tau), \ i = 1, \ldots, l,$$

$$|\psi_0^*(\tau)| \le C\exp(-\kappa\tau).$$

Once the leading expansion terms are obtained, we proceed to obtain higher-order expansion terms. We summarize the result as follows.

Theorem 6.10. *Under conditions* (A6.1)–(A6.3)*, the asymptotic expansions* $\Phi_n^\varepsilon(t) + \Psi_n^\varepsilon(\tau)$ *(with* $\tau = (T-t)/\varepsilon$*) can be constructed as follows:*

- $\varphi_0(t)$ *is obtained from*

$$\varphi_0(t) = \widetilde{\mathbb{1}}(t)\beta_0(t)$$

$$\dot{\beta}_0(t) = -\overline{Q}(t)\beta_0(t), \ \beta_0(T) = \nu(T)u_0;$$

- $\psi_0(t)$ *is obtained from*

$$\psi_0(\tau) = \exp(\widetilde{Q}(T)\tau)(u_0 - \varphi_0(T));$$

- $\varphi_k(t)$ *is obtained from*

$$\varphi_k(t) = \widetilde{\mathbb{1}}(t)\beta_0(t) + (Q_v'(t)Q_v(t))^{-1}Q_v'(t)\widehat{b}_{k-1}$$

$$\dot{\beta}_k(t) = -\overline{Q}(t)\beta_0(t), \ \beta_k(T) = \int_0^\infty \nu(T)r_k(s)ds,$$

where

$$Q_v(t) = \begin{pmatrix} \widetilde{Q}(t) \\ \nu(t) \end{pmatrix}$$

$$\widehat{b}_{k-1} = \begin{pmatrix} \widetilde{b}_{k-1} \\ 0 \end{pmatrix}, \quad \textit{with } \widetilde{b}_{k-1} = -\dot{\varphi}_{k-1}(t) - \widehat{Q}(t)\varphi_{k-1}(t);$$

- $\psi_k(\tau)$ *is obtained from*

$$\psi_k(\tau) = \exp(\widetilde{Q}(T)\tau)(-\varphi_k(T)) + \int_0^\tau \exp(\widetilde{Q}(T)(\tau - s))r_k(s)ds,$$

where

$$r_k(\tau) = \sum_{j=0}^{k-1} \left((-\tau)^{k-j} \frac{\widetilde{Q}^{(k-j)}(T)}{(k-j)!} + (-\tau)^{k-j-1} \frac{\widehat{Q}^{(k-j-1)}(T)}{(k-j-1)!} \right) \psi_j(\tau);$$

- $\varphi_k(\cdot) \in C^{n+1-k}([0,T])$*;*
- $\psi_k(\tau)$ *decays exponentially fast, whereby* $|\psi_k(\tau)| \le C\exp(-\kappa_0\tau)$ *for some* $\kappa_0 > 0$*;*
- *the following error bound holds:*

$$\sup_{t\in[0,T]} |u^\varepsilon(t) - \Phi^\varepsilon(t) - \Psi^\varepsilon((T-t)/\varepsilon)| = O(\varepsilon^{n+1}).$$

6.6 Remarks

This section provides several further remarks. In the first part, it considers Markov chains that are weakly irreducible. The second part deals with fully degenerate switching diffusions with the diffusion coefficients being identically 0. Although many details are omitted, an interesting problem is outlined. Finally, a few further remarks are made at the end.

Example 6.11. We have derived asymptotic expansions of Markov chains with fast and slow motions. The asymptotic expansions obtained can be used in deriving so-called reduced systems. The rationale is that reduction of complexity will be achieved by appropriate aggregation. This is particularly pronounced when the dominating force is an irreducible generator. To illustrate, let the generator $Q^\varepsilon(t)$ be given by

$$Q^\varepsilon(t) = \frac{1}{\varepsilon}Q_0(t) + \widehat{Q}(t),$$

where $Q_0(t) \in \mathbb{R}^{m\times m}$ is weakly irreducible for each $t \in [0,T]$ and $\widehat{Q}(t) \in \mathbb{R}^{m\times m}$ is a generator of another Markov chain. Moreover, $Q_0(\cdot)$, and $\widehat{Q}(\cdot)$ satisfy (A6.2). Again, let $u^\varepsilon(t)$ be the solution of the corresponding systems of backward equations with $u^\varepsilon(T) = u_0$ such that u_0 is a probability vector (i.e., $u_0'\mathbb{1} = 1$). Denote the quasi-stationary distribution associated with $Q_0(t)$ by $\nu_0(t)$. Then our results in the last section indicates that

$$u^\varepsilon(t) = \nu_0'(t) + O(\varepsilon + \exp(-(\kappa_0(T-t))/\varepsilon).$$

6.6.1 Related Problems

In this section, we consider a variation of the systems of backward equations considered in the previous sections. The system is a hybrid system with both continuous dynamics and discrete events included, which is represented by a two-component process $(X(t), \alpha^\varepsilon(t))$. These two processes are intertwined. We have to treat the joint pair as an entity in lieu of treating them separately. Aiming at obtaining asymptotic expansions of solutions of certain systems of differential equations and with the main interest of getting uniform asymptotic expansions, we let the continuous component be in a compact set. For simplicity, we consider the compact set to be $[0,1]$. Extension to more general sets is possible; see Khasminskii and Yin [118] and also related work in Khasminskii and Yin [117]. Let us begin by examining an ordinary differential equation with random switching of the form

$$\dot{X}(t) = b(X(t), \alpha^\varepsilon(t)), \tag{6.55}$$

where $b(\cdot,\cdot) : [0,1]\times\mathcal{M} \mapsto \mathbb{R}$. Associated with (6.55), there is an operator defined by

$$\mathcal{L}^\varepsilon f(x,i) = b(x,i)\frac{\partial}{\partial x}f(x,i) + Q^\varepsilon(t)f(x,\cdot)(i), \tag{6.56}$$

for each $i \in \mathcal{M}$ and any $f(\cdot,i)$ that is continuously differentiable. The process associated with the operator can be thought of as a special case of switching diffusions with diffusion part identically 0, which is the fully degenerate case. Thus, we also have the associated system of backward equations

$$-\frac{\partial}{\partial t}u^\varepsilon(x,t,i) = \mathcal{L}^\varepsilon u^\varepsilon(x,t,i), \quad u^\varepsilon(x,T,i) = g(x,i). \tag{6.57}$$

To approximate the solution of (6.57), we can construct asymptotic expansions of the form

$$\sum_{k=0}^{n}\varepsilon^k\varphi_k(x,t) + \sum_{k=0}^{n}\varepsilon^k\psi_k(x,\tau), \ i \in \mathcal{M},$$

where

$$\varphi_k(x,t) = (\varphi_k(x,t,1),\ldots,\varphi_k(x,t,m))',$$

$$\psi_k(x,t) = (\psi_k(x,t,1),\ldots,\psi_k(x,t,m))',$$

$$\tau = \frac{T-t}{\varepsilon}.$$

The developments are similar to the previous case. For example, we obtain

$$\widetilde{Q}(t)\varphi_0(x,t,\cdot)(i) = 0, \quad i \in \mathcal{M}.$$

Thus, the leading term is given by $\varphi_0(x,t) = \widetilde{\mathbb{1}}\beta_0(x,t)$, where similarly to Section 6.3, $\beta_0(x,t)$ can be determined. In this process, we have to work with the terminal-layer terms $\psi_0(x,\tau)$ as before. Since the techniques are similar to the previous case, we omit the details.

It should be noted that the use of compact set for the values of $X(t)$ is crucial to get the desired uniform error bounds. A more general setup is to assume $b(\cdot,\cdot) : K_r \times \mathcal{M} \mapsto K_r$, where $K_r \subset \mathbb{R}^r$ is a compact subset of $\mathbb{R}^r$. We refer the reader to [118] for such a setup. Note that the error bound will be in the sense of using the sup-norm over $(x,t) \in [0,1] \times [0,T]$.

6.7 Notes

Because of modeling requirements and other practical considerations, numerous problems arising in the physical sciences, biological sciences, and engineering involve continuous-time Markov chains that are large and complex. To overcome the difficulty, one uses partition and aggregation for dynamic systems motivated by the work of Simon and Ando [196] (see also Courtois [35]) for nearly completely decomposable systems.

Complementing the results presented in Chapter 4, in which we used a singular perturbation approach to treat forward equations for two-time-scale Markov chains, this chapter demonstrated that such an approach can be employed to treat asymptotic expansions of backward equations for Markov chains with two time scales. The result of this chapter is based on Yin and Nguyen [226]. Our approach is constructive leading to practically useful approximation schemes. A worthwhile future research direction is to consider the case that the generator of the switching process depends not only on t but also on the continuous state x, that is, $Q(x,t)$. Such a case may also be handled by similar techniques as presented in this work.

Part III

Applications: MDPs, Near-optimal Controls, Numerical Methods, and LQG with Switching

7

Markov Decision Problems

7.1 Introduction

Continuing the central themes of this book, as an application of the asymptotic properties of two-time-scale Markov chains, this chapter focuses on a class of Markov decision processes (MDPs). Our main attention is on finite-state continuous-time Markov decision processes having weak and strong interactions.

Markov decision processes have received much attention lately owing to their ability to deal with a large class of problems under uncertainty. Many problems in applied mathematics such as inventory planning, resource allocation, queueing, and machine replacement, fit well in the framework of (or can be recast to) Markov decision processes. For a complete treatment of discrete-time MDP models, we refer the reader to the books of Derman [46], Ross [184], and White [218]. For a comprehensive study of continuous-time Markov decision processes as well as some recent progress, we refer the reader to Guo and Hernández-Lerma [78].

Many systems in real life are large and complex, so it is necessary to treat them in a hierarchical fashion. Using hierarchical control to deal with large-dimensional systems, a typical approach is to derive a limit control problem by replacing the fast-changing processes with their "average" in terms of the corresponding quasi-stationary distributions together with appropriate modification of the objective function. The limit control problem is much simpler to handle than the original one. Based on an optimal decision for the limit problem, one can make a decision for the original problem so as

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to achieve its asymptotic optimality. For related literature on hierarchical control in the context of manufacturing, we refer the reader to the books by Gershwin [71] and Sethi and Zhang [192].

A common practice in dealing with optimization of stochastic dynamical systems is the method of dynamic programming. To find the optimal decisions, one solves a set of associated dynamic programming (DP) equations. However, such an approach is only computationally feasible when the dimension of the underlying system is not very large. For large-scale systems, one has to resort to approximate optimal schemes. The hierarchical control approach provides a powerful tool for dealing with large and complex systems and for finding approximate optimal solutions. An approximate optimal solution is often more desirable than an exact optimal solution because it can usually be obtained by working with relatively simpler models with reduced dimensionality. This is especially the case when the effort of working with a simpler model is substantially less than that of a large-dimensional model (see Simon and Ando [196]).

In this chapter, we devote our attention to the Markov decision processes with weak and strong interactions such that the states of the process can be divided into several groups and that transitions among the states within each group occur much more frequently than the transitions among the states belonging to different groups. Both discounted cost and long-run average cost criteria are considered. By aggregating the states in each group as a single state, we derive a limit problem. Using an optimal solution of the limit problem, we then construct a solution for the original problem that is asymptotically optimal. Estimates of the error bounds for the constructed controls are also derived.

Concentrating on controlled Markov chains and considering a relative simpler model, we are able to deal with hierarchical controls avoiding the use of hard-to-verify conditions, e.g., the Lipschitz continuity of an optimal control for the limit problem (see Sethi and Zhang [192, Chapter 5] and Chapter 8 of this book for related problem); obtain much better convergence rates of value functions and error bounds of constructed controls than that of problems considered in Section 8.5; incorporate a long-run average cost criterion. The method used in dealing with long-run average cost models is quite different from that for models with discounted costs, however.

Our results provide a rigorous justification for hierarchical controls of complex systems, elicit insights of the design of hierarchies, and suggest nearly optimal procedures of hierarchical decision making in a general setting. By establishing a criterion for determining the quality of hierarchical solutions, the results provide guidance for a systematic approach for the design of hierarchical structures within an organization.

The plan of the chapter is as follows. We formulate the MDP model in Section 7.2, derive a limit problem in Section 7.3, give a method for constructing asymptotically optimal decisions of the original problem using

the optimal solution of the limit problem in Section 7.4, and obtain the estimate of the deviation of the constructed decisions from optimality in Section 7.5. Particular attention is paid to the rate of convergence of the value functions and error bounds of constructed controls. Sections 7.2–7.5 are mainly concerned with discounted costs, whereas long-run average costs are dealt with in Section 7.6, in which the main difficulty is to verify the irreducibility of the limit Markov decision process. Finally, some related computational methods are discussed in Section 7.7.

7.2 Problem Formulation

Consider a real-valued Markov decision process $x^\varepsilon(\cdot) = \{x^\varepsilon(t) : \ t \geq 0\}$ and a feedback control $u(\cdot) = \{u(t) = u(x^\varepsilon(t)) : \ t \geq 0\}$ such that $u(t) \in \Gamma$, $t \geq 0$, where Γ is a compact subset of an Euclidean space. Let $Q^\varepsilon(u(t)) = (q_{ij}^\varepsilon(u(t)))$, $t \geq 0$, denote the generator of $x^\varepsilon(\cdot)$ such that

$$Q^\varepsilon(u) = \frac{1}{\varepsilon}\widetilde{Q}(u) + \widehat{Q}(u), \tag{7.1}$$

where

$$\widetilde{Q}(u) = \text{diag}\left(\widetilde{Q}^1(u), \ldots, \widetilde{Q}^l(u)\right), \quad \widetilde{Q}^k(u) = (\tilde{q}_{ij}^k(u))_{m_k \times m_k}$$

with $\tilde{q}_{ij}^k(u) \geq 0$ for $j \neq i$ and $\sum_j \tilde{q}_{ij}^k(u) = 0$, $\widehat{Q}(u) = (\widehat{q}_{ij}(u))_{m\times m}$ with $m = m_1 + \cdots + m_l$, $\widehat{q}_{ij}(u) \geq 0$ for $j \neq i$ and $\sum_j \widehat{q}_{ij}(u) = 0$, and $\varepsilon > 0$ is a small parameter. For $k = 1, \ldots, l$, denote by $\mathcal{M}_k = \{s_{k1}, \ldots, s_{km_k}\}$ the substate space of $x^\varepsilon(\cdot)$ corresponding to $\widetilde{Q}^k(u)$. The entire state space of $x^\varepsilon(\cdot)$ is given by

$$\begin{aligned}\mathcal{M} &= \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l \\ &= \left\{s_{11}, \ldots, s_{1m_1}, s_{21}, \ldots, s_{2m_2}, \ldots, s_{l1}, \ldots, s_{lm_l}\right\}.\end{aligned}$$

Let $u = u(i)$ denote a function such that $u(i) \in \Gamma$ for all $i \in \mathcal{M}$. We call u an admissible control and use $\mathcal{A}_f$ to denote the collection of all such functions. For each $u = u(x) \in \mathcal{A}_f$, the Markov process generated by $Q^\varepsilon(u(x^\varepsilon(t)))$ can be constructed as in Section 2.4.

Consider the cost functional $J^\varepsilon(i, u)$ defined on $\mathcal{M} \times \mathcal{A}_f$ by

$$J^\varepsilon(i, u) = E\int_0^\infty e^{-\rho t} G(x^\varepsilon(t), u(x^\varepsilon(t)))dt,$$

where $i = x^\varepsilon(0)$ is the initial value of $x^\varepsilon(\cdot)$, $G(x, u)$ is the cost-to-go function, and $\rho > 0$ is the discount factor. Our objective is to find a function $u(\cdot) \in \mathcal{A}_f$ that minimizes $J^\varepsilon(i, u)$.

The original problem, termed as $\mathcal{P}^\varepsilon$, takes the form

$$\mathcal{P}^\varepsilon : \begin{cases} \text{minimize: } J^\varepsilon(i,u) = E\displaystyle\int_0^\infty e^{-\rho t} G(x^\varepsilon(t), u(x^\varepsilon(t)))dt, \\ \text{subject to: } x^\varepsilon(t) \sim Q^\varepsilon(u(x^\varepsilon(t))),\ t \geq 0, \\ \qquad\qquad\qquad\qquad x^\varepsilon(0) = i,\ u \in \mathcal{A}_f, \\ \text{value function: } v^\varepsilon(i) = \displaystyle\inf_{u\in\mathcal{A}_f} J^\varepsilon(i,u), \end{cases} \tag{7.2}$$

where $x^\varepsilon(t) \sim Q^\varepsilon(u(x^\varepsilon(t)))$ means that $x^\varepsilon(\cdot)$ is a Markov chain generated by $Q^\varepsilon(u(x^\varepsilon(t)))$, $t \geq 0$.

For each $i \in \mathcal{M}$, the associated DP equation is

$$\rho v^\varepsilon(i) = \min_{u\in\Gamma} \Big\{ G(i,u) + Q^\varepsilon(u) v^\varepsilon(\cdot)(i) \Big\}, \tag{7.3}$$

where

$$Q^\varepsilon(u) v^\varepsilon(\cdot)(i) = \sum_{j\neq i} q_{ij}^\varepsilon(u)(v^\varepsilon(j) - v^\varepsilon(i)).$$

As a continuous-time analog of Ross [184], it can be shown that $v^\varepsilon(i)$ is the unique solution to the above DP equation as in Theorem A.30 without using differential equations. Moreover, for each $i \in \mathcal{M}$, let $u_{*\varepsilon}(i)$ denote the minimizer of $G(i,u) + Q^\varepsilon(u)v^\varepsilon(\cdot)(i)$. Then following the proof of Theorem A.31, it can be shown that $u_{*\varepsilon} = u_{*\varepsilon}(x^\varepsilon(t))$ is optimal, that is, $J^\varepsilon(i, u_{*\varepsilon}) = v^\varepsilon(i)$. To further our understanding and to provide a better picture of the underlying problem, let us consider the following example.

Example 7.1. Consider a manufacturing system consisting of two failure-prone machines in a flowshop configuration. Each of the two machines has two states, up (denoted by 1) and down (denoted by 0). Then the system has four states, represented by $\{(1,1),(0,1),(1,0),(0,0)\}$. Let

$$s_{11} = (1,1),\ s_{12} = (0,1),\ s_{21} = (1,0),\ \text{and } s_{22} = (0,0).$$

The function $u(\cdot)$ is the rate of preventive maintenance used to reduce the failure rate of machines and to improve the productivity of the system. Our objective is to choose $u(\cdot)$ to keep the average machine capacity at a reasonable level and, in the meantime, to avoid excessive costly preventive maintenance.

Suppose that the state of the first machine is changing more rapidly than the second one. A typical way of modeling the machine state process is to

formulate the process as a Markov chain with the following generator

$$Q^{\varepsilon}(u) = \frac{1}{\varepsilon}\begin{pmatrix} -\lambda_1(u) & \lambda_1(u) & 0 & 0 \\ \mu_1(u) & -\mu_1(u) & 0 & 0 \\ 0 & 0 & -\lambda_1(u) & \lambda_1(u) \\ 0 & 0 & \mu_1(u) & -\mu_1(u) \end{pmatrix} + \begin{pmatrix} -\lambda_2(u) & 0 & \lambda_2(u) & 0 \\ 0 & -\lambda_2(u) & 0 & \lambda_2(u) \\ \mu_2(u) & 0 & -\mu_2(u) & 0 \\ 0 & \mu_2(u) & 0 & -\mu_2(u) \end{pmatrix}.$$

The breakdown and repair rates are $\lambda_1(u)/\varepsilon$ and $\mu_1(u)/\varepsilon$ for the first machine, and $\lambda_2(u)$ and $\mu_2(u)$ for the second machine, respectively. The quantity ε represents the frequency of transitions of $x^{\varepsilon}(\cdot)$ between s_{11} and s_{12} or s_{21} and s_{22}. For small ε, the transition of $x^{\varepsilon}(\cdot)$ within either $\{s_{11}, s_{12}\}$ or $\{s_{21}, s_{22}\}$ is much more frequent than that between the two groups $\{s_{11}, s_{12}\}$ and $\{s_{21}, s_{22}\}$. In this example,

$$\mathcal{M}_1 = \{s_{11}, s_{12}\},\ \mathcal{M}_2 = \{s_{21}, s_{22}\},$$

and

$$\widetilde{Q}(u) = \text{diag}\left(\widetilde{Q}^1(u), \widetilde{Q}^2(u)\right)$$

with

$$\widetilde{Q}^1(u) = \widetilde{Q}^2(u) = \begin{pmatrix} -\lambda_1(u) & \lambda_1(u) \\ \mu_1(u) & -\mu_1(u) \end{pmatrix} \quad \text{and}$$

$$\widehat{Q}(u) = \begin{pmatrix} -\lambda_2(u) & 0 & \lambda_2(u) & 0 \\ 0 & -\lambda_2(u) & 0 & \lambda_2(u) \\ \mu_2(u) & 0 & -\mu_2(u) & 0 \\ 0 & \mu_2(u) & 0 & -\mu_2(u) \end{pmatrix}.$$

The matrix $\widetilde{Q}(u)$ governs the fast transition of $x^{\varepsilon}(\cdot)$ within each group $\mathcal{M}_k$, $k = 1, 2$, and the matrix $\widehat{Q}(u)$ dictates the slow transition of $x^{\varepsilon}(\cdot)$ between $\mathcal{M}_1$ and $\mathcal{M}_2$. For small ε, the MDP $x^{\varepsilon}(\cdot)$ has strong interactions within each group $\mathcal{M}_k$, $k = 1, 2$, and weak interactions between the groups $\mathcal{M}_1$ and $\mathcal{M}_2$.

7.3 Limit Problem

This section is devoted to the derivation of the corresponding limit control problem. In view of Example 7.1, the Markov process $x^{\varepsilon}(\cdot)$ can be regarded

as a process with a fast-changing component and a slowly varying one. The fast-changing process is governed by $\widetilde{Q}(u)/\varepsilon$, while the slowly changing process is governed by $\widehat{Q}(u)$ and the quasi-stationary distributions of $\widetilde{Q}^k(u)$, $k = 1, \ldots, l$. As $\varepsilon \to 0$, the fast-changing process is averaged out. Consequently, the Markov process $x^\varepsilon(\cdot)$ converges to a process $\overline{x}(\cdot)$ in which the states within each group corresponding to $\widetilde{Q}^k(u)$ are aggregated into a single state.

To proceed, define the control set for the limit problem. For each $k = 1, \ldots, l$, let

$$\Gamma_k = \Big\{U^k := (u^{k1}, \ldots, u^{km_k}) : \text{ such that } u^{kj} \in \Gamma,\ j = 1, \ldots, m_k\Big\}.$$

The control set for the limit problem is defined as $\overline{\Gamma} = \Gamma_1 \times \cdots \times \Gamma_l$;

$$\overline{\Gamma} = \Big\{U = (U^1, \ldots, U^l) = (u^{11}, \ldots, u^{1m_1}, \ldots, u^{l1}, \ldots, u^{lm_l}) : \text{ such that } U^k \in \Gamma_k,\ k = 1, \ldots, l\Big\}.$$

Define matrices $\widetilde{Q}_0^k$, $\widehat{Q}_0$, and $\overline{Q}^\varepsilon$ as follows. For each $U = (U^1, \ldots, U^l) \in \overline{\Gamma}$, let

$$\widetilde{Q}_0^k(U^k) = (\widetilde{q}_{ij}^k(u^{ki})), \text{ for } k = 1, \ldots, l \text{ and } i = 1, \ldots, m_k,$$

and

$$\widehat{Q}_0(U) = (\widehat{q}_{ij}^0(U)),$$

where

$$\widehat{q}_{ij}^0(U) = \begin{cases} \widehat{q}_{ij}(u^{1i}), & \text{if } 1 \le i \le m_1, \\ \widehat{q}_{ij}(u^{2(i-m_1)}), & \text{if } m_1 < i \le m_1 + m_2, \\ \cdots & \cdots \\ \widehat{q}_{ij}(u^{l(i-m+m_l)}), & \text{if } m - m_l < i \le m, \end{cases}$$

with $m = m_1 + \cdots + m_l$, and

$$\overline{Q}^\varepsilon(U) = \frac{1}{\varepsilon}\text{diag}\left(\widetilde{Q}_0^1(U^1), \ldots, \widetilde{Q}_0^l(U^l)\right) + \widehat{Q}_0(U). \tag{7.4}$$

Writing it more compactly, for $k = 1, \ldots, l$,

$$\widehat{q}_{ij}^0(U) = \widehat{q}_{ij}(u^{ki}) \ \text{ for } \sum_{r=1}^{k-1} m_r < i \le \sum_{r=1}^{k} m_r.$$

This definition reveals the dependence of $\widehat{Q}_0(U)$ on the controls. Denoting $U \in \overline{\Gamma}$ as an m-vector $U = (u^1, \ldots, u^m)$, $\overline{Q}^\varepsilon(U)$ is obtained from $Q^\varepsilon(u)$ by replacing the control variable u in the ith row with u^i.

Remark 7.2. For each $u \in \mathcal{A}_f$, define

$$\widetilde{U} = (u^{11}, \ldots, u^{1m_1}, \ldots, u^{l1}, \ldots, u^{lm_l}), \quad \text{with } u^{ij} = u(s_{ij}).$$

Let $x^\varepsilon(\cdot)$ denote the Markov chain generated by $Q^\varepsilon(u(x^\varepsilon(t)))$, $t \geq 0$, and $\tilde{x}^\varepsilon(\cdot)$ denote the Markov chain generated by $\overline{Q}^\varepsilon(\widetilde{U})$. Since

$$Q^\varepsilon(u(i))f(\cdot)(i) = \overline{Q}^\varepsilon(\widetilde{U})f(\cdot)(i)$$

for any function f on $\mathcal{M}$, it is easy to see that $x^\varepsilon(\cdot)$ and $\tilde{x}^\varepsilon(\cdot)$ have the same probability distribution (see Section 2.4).

We make the following assumptions on the generator $Q^\varepsilon(u)$ and the cost-to-go function $G(x, u)$.

(**A7.1**) $Q^\varepsilon(u)$ is a continuous function of u. Moreover, for each $U^k \in \Gamma_k$ and $k = 1, \ldots, l$, $\widetilde{Q}_0^k(U^k)$ is weakly irreducible. Furthermore, there exists a $U_0 = (U_0^1, \ldots, U_0^l) \in \overline{\Gamma}$ such that $\widetilde{Q}_0^k(U_0^k)$ is irreducible, for $k = 1, \ldots, l$.

(**A7.2**) For each $x \in \mathcal{M}$, $G(x, u)$ is a continuous function on Γ.

For each $U^k \in \Gamma_k$, let $\nu^k(U^k)$ denote the quasi-stationary distribution of $\widetilde{Q}_0^k(U^k)$ for $k = 1, \ldots, l$. Define

$$\nu(U) = \operatorname{diag}(\nu^1(U^1), \ldots, \nu^l(U^l))$$

and recall that $\tilde{\mathbb{1}} = \operatorname{diag}(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_l})$ with $\mathbb{1}_{m_k} = (1, \ldots, 1)' \in \mathbb{R}^{m_k}$. Using $\nu(U)$ and $\tilde{\mathbb{1}}$, define another matrix $\overline{Q}(U)$ as a function of $U \in \overline{\Gamma}$

$$\overline{Q}(U) = \nu(U)\widehat{Q}_0(U)\tilde{\mathbb{1}}. \tag{7.5}$$

Note that the ith row of $\overline{Q}(U)$ depends only on U^i and that

$$\overline{Q}(U) = (\overline{q}_{ij}(U^i))_{l \times l}.$$

With a slight abuse of notation, write

$$\overline{Q}(U^k)f(\cdot)(k) = \sum_{k' \neq k} \overline{q}_{kk'}(U^k)(f(k') - f(k))$$

instead of $\overline{Q}(U)f(\cdot)(k)$, for a function $f(\cdot)$ defined on $\{1, \ldots, l\}$. Thus the process $\overline{x}(\cdot)$ generated by $\overline{Q}(U)$ can be viewed as a Markov chain generated by $\overline{Q}(U(\overline{x}(t)))$, $t \geq 0$.

We proceed to define the limit problem. For $k = 1, \ldots, l$ and $U^k \in \Gamma_k$, define the average of $G(x, u)$ with respect to the quasi-stationary distribution as

$$\overline{G}(k, U^k) = \sum_{j=1}^{m_k} \nu_j^k(U^k)G(s_{kj}, u^{kj}), \ k = 1, \ldots, l,$$

where $\nu^k(U^k) = (\nu_1^k(U^k), \ldots, \nu_{m_k}^k(U^k))$ is the quasi-stationary distribution of $\widetilde{Q}_0^k(U^k)$. Let $\mathcal{A}^0$ denote a class of functions $U = U(k) \in \Gamma_k$, $k = 1, \ldots, l$. For convenience, call $U = (U(1), \ldots, U(l)) \in \mathcal{A}^0$ an admissible control for the limit problem, termed as $\mathcal{P}^0$.

The limit problem $\mathcal{P}^0$ is

$$\mathcal{P}^0 : \begin{cases} \text{minimize: } J^0(k, U) = E \displaystyle\int_0^\infty e^{-\rho t} \overline{G}(\overline{x}(t), U(\overline{x}(t)))dt, \\ \text{subject to: } \overline{x}(t) \sim \overline{Q}(U(\overline{x}(t))), \quad \overline{x}(0) = k, \ U \in \mathcal{A}^0, \\ \text{value function: } v(k) = \displaystyle\inf_{U \in \mathcal{A}^0} J^0(k, U), \end{cases} \tag{7.6}$$

where $\overline{x}(t) \sim \overline{Q}(U(\overline{x}(t)))$ means that $\overline{x}(\cdot)$ is a Markov chain generated by $\overline{Q}(U(\overline{x}(t)))$, $t \geq 0$. As in the proof of Theorem A.30, it can be shown that $v(k)$ is the unique solution to the following DP equation

$$\rho v(k) = \min_{U^k \in \Gamma_k} \left\{ \overline{G}(k, U^k) + \overline{Q}(U^k) v(\cdot)(k) \right\}, \tag{7.7}$$

where $\overline{Q}(U^k)v(\cdot)(k) = \sum_{k' \neq k} \overline{q}_{kk'}(U^k)(v(k') - v(k))$. Moreover, let $U_* = (U_*^1, \ldots, U_*^l) \in \overline{\Gamma}$ denote a minimizer of the right-hand side of (7.7). Then following the proof of Theorem A.31, it can be shown that $U_* \in \mathcal{A}^0$ is optimal for $\mathcal{P}^0$.

Remark 7.3. Note that the number of the DP equations for $\mathcal{P}^\varepsilon$ is equal to $m = m_1 + \cdots + m_l$, while the number of that for $\mathcal{P}^0$ is only l. For each $k = 1, \ldots, l$, $m_k \geq 2$, so it follows that $m - l \geq l$. The difference between m and l could be very large for either large l or a large m_k for some k. As is well known (see Hillier and Lieberman [86]), the computation effort in solving the DP equations depends largely on the number of equations involved. Thus the effort in solving the DP equations for $\mathcal{P}^0$ is substantially less than that of $\mathcal{P}^\varepsilon$ if $m - l$ is large (i.e., $m \gg l$).

Example 7.4. In Example 7.1, with

$$\widetilde{Q}_0^1(U^1) = \begin{pmatrix} -\lambda_1(u^{11}) & \lambda_1(u^{11}) \\ \mu_1(u^{12}) & -\mu_1(u^{12}) \end{pmatrix} \text{ and}$$

$$\widetilde{Q}_0^2(U^2) = \begin{pmatrix} -\lambda_1(u^{21}) & \lambda_1(u^{21}) \\ \mu_1(u^{22}) & -\mu_1(u^{22}) \end{pmatrix},$$

the quasi-stationary distributions of $\widetilde{Q}_0^1(U^1)$ and $\widetilde{Q}_0^2(U^2)$ are given by

$$\nu^1(U^1) = \left(\frac{\mu_1(u^{12})}{\lambda_1(u^{11}) + \mu_1(u^{12})}, \frac{\lambda_1(u^{11})}{\lambda_1(u^{11}) + \mu_1(u^{12})} \right)$$

and

$$\nu^2(U^2) = \left(\frac{\mu_1(u^{22})}{\lambda_1(u^{21}) + \mu_1(u^{22})}, \frac{\lambda_1(u^{21})}{\lambda_1(u^{21}) + \mu_1(u^{22})} \right),$$

respectively. The generator $\overline{Q}(U)$ is given by

$$\begin{pmatrix} -\eta_1(U) & \eta_1(U) \\ \eta_2(U) & -\eta_2(U) \end{pmatrix},$$

where

$$\eta_1(U) = \frac{\mu_1(u^{12})\lambda_2(u^{11}) + \lambda_1(u^{11})\lambda_2(u^{12})}{\lambda_1(u^{11}) + \mu_1(u^{12})}$$

and

$$\eta_2(U) = \frac{\mu_1(u^{22})\mu_2(u^{21}) + \lambda_1(u^{21})\mu_2(u^{22})}{\lambda_1(u^{21}) + \mu_1(u^{22})}.$$

7.4 Asymptotic Optimality

This section is devoted to the convergence of the sequence of value functions v^ε to v and the construction of asymptotic optimal controls for $\mathcal{P}^\varepsilon$.

Lemma 7.5. *For $i \in \mathcal{M}_k$, $k = 1, \ldots, l$, if there exists a subsequence of $\varepsilon \to 0$ (still denoted by ε for simplicity) such that $v^\varepsilon(i) \to v(i)$, then the limit function $v(i)$ depends only on k (i.e., $v(i) = v(k)$).*

Proof: Let $x^\varepsilon(0) = i = s_{kj} \in \mathcal{M}_k$ for some $k = 1, \ldots, l$ and $j = 1, \ldots, m_k$. In view of (A7.1), there exists $U_0^k = (u_0^{k1}, \ldots, u_0^{km_k}) \in \Gamma_k$ such that $\widetilde{Q}_0^k(U_0^k)$ is irreducible. Let $U_0 = (U_0^1, \ldots, U_0^l)$. Then the DP equation in (7.3) implies

$$\rho v^\varepsilon(s_{kj}) \le G(s_{kj}, u_0^{kj}) + \overline{Q}^\varepsilon(U_0) v^\varepsilon(\cdot)(s_{kj}).$$

Multiplying both sides by ε and sending $\varepsilon \to 0$ lead to

$$\widetilde{Q}_0^k(U_0^k) \begin{pmatrix} v(s_{k1}) \\ \vdots \\ v(s_{km_k}) \end{pmatrix} \ge 0,$$

for $k = 1, \ldots, l$. Now, the irreducibility of $\widetilde{Q}_0^k(U_0^k)$ and Lemma A.39 imply

$$v(s_{k1}) = v(s_{k2}) = \cdots = v(s_{km_k}) \ \ (:= v(k)). \tag{7.8}$$

This proves the lemma. □

Theorem 7.6. *For each* $i \in \mathcal{M}_k$,

$$\lim_{\varepsilon \to 0} v^{\varepsilon}(i) = v(k).$$

Proof: Since $\mathcal{M}$ is a set containing finitely many elements, it is easy to show that $v^{\varepsilon}(i)$ is uniformly bounded. Thus, there exist a subsequence of $\varepsilon \to 0$ (denoted by ε) and (in view of Lemma 7.5) a $v(k)$, with $i\ (= s_{kj}) \in \mathcal{M}_k$ such that $v^{\varepsilon}(i) \to v(k)$. It will be shown that $v(k)$ is a solution to (7.7). Since the DP equation (7.7) has a unique solution, one concludes that $v^{\varepsilon}(i) \to v(k)$.

Since $v^{\varepsilon}(i)$ is a solution to (7.3), for each $j = 1, \ldots, m_k$, $k = 1, \ldots, l$, and $U^k \in \Gamma_k$,

$$\rho v^{\varepsilon}(s_{kj}) \le G(s_{kj}, u^{kj}) + Q^{\varepsilon}(u^{kj}) v^{\varepsilon}(\cdot)(s_{kj}).$$

Use $\nu^k(U^k) = (\nu_1^k(U^k), \ldots, \nu_{m_k}^k(U^k))$ to denote the quasi-stationary distribution of $\widetilde{Q}_0^k(U^k)$. It follows that

$$\begin{aligned}\rho \sum_{j=1}^{m_k} \nu_j^k(U^k) v^{\varepsilon}(s_{kj}) \le & \sum_{j=1}^{m_k} \nu_j^k(U^k) G(s_{kj}, u^{kj}) \\ & + \sum_{j=1}^{m_k} \nu_j^k(U^k) Q^{\varepsilon}(u^{kj}) v^{\varepsilon}(\cdot)(s_{kj}).\end{aligned}$$

Letting $\varepsilon \to 0$ and in view of the definition of $\overline{Q}(U)$ in (7.5) and Lemma 7.5,

$$\rho v(k) \le \min_{U^k \in \Gamma_k} \left\{ \overline{G}(k, U^k) + \overline{Q}(U^k) v(\cdot)(k) \right\}.$$

To derive the reverse inequality, let $U_{*\varepsilon} = (U_{*\varepsilon}^1, \ldots, U_{*\varepsilon}^l) \in \overline{\Gamma}$ denote the minimizer of the right-hand side of (7.3). Then

$$\rho v^{\varepsilon}(s_{kj}) = G(s_{kj}, u_{*\varepsilon}^{kj}) + Q^{\varepsilon}(u_{*\varepsilon}^{kj}) v^{\varepsilon}(\cdot)(s_{kj}),$$

for $j = 1, \ldots, m_k$ and $k = 1, \ldots, l$. Thus we have

$$\begin{aligned}\rho \sum_{j=1}^{m_k} \nu_j^k(U_{*\varepsilon}^k) v^{\varepsilon}(s_{kj}) = & \sum_{j=1}^{m_k} \nu_j^k(U_{*\varepsilon}^k) G(s_{kj}, u_{*\varepsilon}^{kj}) \\ & + \sum_{j=1}^{m_k} \nu_j^k(U_{*\varepsilon}^k) Q^{\varepsilon}(u_{*\varepsilon}^{kj}) v^{\varepsilon}(\cdot)(s_{kj}).\end{aligned}$$

Note that $\overline{\Gamma}$ is a bounded set. As $\varepsilon \to 0$, there exists a subsequence of ε (still denoted by ε for simplicity) such that $U_{*\varepsilon} \to \widetilde{U} \in \overline{\Gamma}$. Hence,

$$\begin{aligned}\rho v(k) = & \ \overline{G}(k, \widetilde{U}^k) + \overline{Q}(\widetilde{U}^k) v(\cdot)(k) \\ \ge & \min_{U^k \in \Gamma_k} \left\{ \overline{G}(k, U^k) + \overline{Q}(U^k) v(\cdot)(k) \right\}.\end{aligned}$$

This completes the proof. □

Next, our task is to construct asymptotic optimal control policies for $\mathcal{P}^\varepsilon$. Let us begin with the optimal control $U_* = (U_*^1, \ldots, U_*^l) \in \mathcal{A}^0$ for the limit problem $\mathcal{P}^0$, which is obtained by minimizing the right-hand side of (7.7), that is,

$$\begin{aligned} &\overline{G}(k, U_*^k) + \overline{Q}(U_*^k)v(\cdot)(k) \\ &\qquad = \min_{U^k \in \Gamma_k} \left\{ \overline{G}(k, U^k) + \overline{Q}(U^k)v(\cdot)(k) \right\}. \end{aligned} \tag{7.9}$$

Construct a control u_c as

$$u_c = u_c(x) = \sum_{k=1}^{l} \sum_{j=1}^{m_k} I_{\{x = s_{kj}\}} u_*^{kj}. \tag{7.10}$$

It is clear that $u_c \in \mathcal{A}_f$. We show next that u_c is nearly optimal. To verify this, the following lemma, based on the asymptotic expansion in Section 4.3, is needed.

Lemma 7.7. *Given $U \in \overline{\Gamma}$, let $x^\varepsilon(\cdot)$ denote the Markov chain generated by $\overline{Q}^\varepsilon(U)$ defined by (7.4). Then there exist positive constants K and $\kappa_0 > 0$ (both independent of ε and t) such that*

$$\left| P(x^\varepsilon(t) = s_{kj}) - \nu_j^k(U^k) f_k(t) \right| \le K \left(\varepsilon(t+1) + \exp\left(-\frac{\kappa_0 t}{\varepsilon} \right) \right), \tag{7.11}$$

where $f_k(t)$ satisfies

$$\frac{d}{dt}(f_1(t), \ldots, f_l(t)) = (f_1(t), \ldots, f_l(t))\overline{Q}(U), \tag{7.12}$$

with $(f_1(0), \ldots, f_l(0)) = (P(x^\varepsilon(0) \in \mathcal{M}_1), \ldots, P(x^\varepsilon(0) \in \mathcal{M}_l))$. Moreover, let $\overline{x}(t)$ denote the Markov chain generated by $\overline{Q}(U)$. Then

$$f_k(t) = P(\overline{x}(t) = k).$$

Proof: It follows from Corollary 4.31 by identifying $\widetilde{Q}^k$ and $\widehat{Q}$ with $\widetilde{Q}_0^k(U^k)$ and $\widehat{Q}_0(U)$, respectively. □

Theorem 7.8. *The control $u_c = u_c(x)$ constructed in (7.10) is asymptotically optimal in that*

$$\lim_{\varepsilon \to 0} |J^\varepsilon(i, u_c) - v^\varepsilon(i)| = 0.$$

Proof: In view of the convergence of $v^\varepsilon(i)$, it suffices to show that for $x^\varepsilon(0) = i \in \mathcal{M}_k$,

$$\lim_{\varepsilon \to 0} J^\varepsilon(i, u_c) = v(k).$$

Let $x^\varepsilon(\cdot)$ denote the Markov chain generated by $Q^\varepsilon(u_c(x^\varepsilon(t)))$, $t \geq 0$, with u_c given in (7.10). Then in view of Remark 7.2, $Q^\varepsilon(u_c(x^\varepsilon(t)))$ and $\overline{Q}^\varepsilon(U_*)$ generate Markov chains with identical probability distribution.

Using the definition of $u_c(x)$, we have

$$\begin{aligned} J^\varepsilon(i,u_c) &= E\int_0^\infty e^{-\rho t}G(x^\varepsilon(t),u_c(x^\varepsilon(t)))dt \\ &= E\sum_{k=1}^{l}\sum_{j=1}^{m_k}\int_0^\infty e^{-\rho t}G(s_{kj},u_*^{kj})I_{\{x^\varepsilon(t)=s_{kj}\}}dt \\ &= \sum_{k=1}^{l}\sum_{j=1}^{m_k}\int_0^\infty e^{-\rho t}G(s_{kj},u_*^{kj})P(x^\varepsilon(t)=s_{kj})dt. \end{aligned} \tag{7.13}$$

Let $\overline{x}(\cdot)$ denote a Markov chain generated by $\overline{Q}(U_*)$. Then in view of Lemma 7.7, the definition of $\overline{G}(k,U)$, and (7.13), we have

$$\begin{aligned} &|J^\varepsilon(i,u_c)-v(k)| = \left|J^\varepsilon(i,u_c)-J^0(k,U_*)\right| \\ &\leq \sum_{k=1}^{l}\sum_{j=1}^{m_k}\int_0^\infty e^{-\rho t}G(s_{kj},u_*^{kj}) \\ &\qquad \times\left|P(x^\varepsilon(t)=s_{kj})-\nu_j^k(U_*^k)P(\overline{x}(t)=k)\right|dt \\ &\leq \sum_{k=1}^{l}\sum_{j=1}^{m_k}\int_0^\infty e^{-\rho t}G(s_{kj},u_*^{kj}) \\ &\qquad \times K\left(\varepsilon(t+1)+\exp\left(-\frac{\kappa_0 t}{\varepsilon}\right)\right)dt = O(\varepsilon). \end{aligned} \tag{7.14}$$

This proves the theorem. □

Remark 7.9. The inequality in Lemma 7.7 is only valid for a discounted cost problem. It does not work for long-run average cost problems, since the upper bound in (7.11) depends on time t.

7.5 Convergence Rate and Error Bound

It is interesting from a computational point of view to estimate the convergence rate of v^ε to v and to obtain the error bound of the control $u_c = u_c(x)$ constructed in (7.10). The next theorem shows that such convergence rate and error bound are of the order ε.

Theorem 7.10. *Assume that the control set* Γ *contains finitely many elements. Then for all* $i \in \mathcal{M}_k$,

$$v^\varepsilon(i) - v(k) = O(\varepsilon)$$

and

$$J^\varepsilon(i, u_c) - v^\varepsilon(i) = O(\varepsilon).$$

Proof: In view of (7.14) and the triangle inequality,

$$|J^\varepsilon(i, u_c) - v^\varepsilon(i)| \le |J^\varepsilon(i, u_c) - v(k)| + |v^\varepsilon(i) - v(k)|.$$

It suffices to show $v^\varepsilon(i) - v(k) = O(\varepsilon)$. Note that the inequality (7.14) implies that for $i \in \mathcal{M}_k$

$$v^\varepsilon(i) - v(k) \le J^\varepsilon(i, u_c) - v(k) \le O(\varepsilon).$$

To derive the reverse inequality, let $u_{*\varepsilon}(x)$ denote an optimal control for $\mathcal{P}^\varepsilon$ and let

$$U_{*\varepsilon} = (U^1_{*\varepsilon}, \dots, U^l_{*\varepsilon}) \in \overline{\Gamma},$$

where

$$U^k_{*\varepsilon} = (u^{k1}, \dots, u^{km_k}) := (u_{*\varepsilon}(s_{k1}), \dots, u_{*\varepsilon}(s_{km_k})).$$

The control set Γ contains finitely many elements by the hypothesis, so does $\overline{\Gamma}$. Suppose $\overline{\Gamma} = \{\gamma_1, \dots, \gamma_L\}$ for some positive integer L. Define

$$\mathcal{E}_j = \{\varepsilon \in (0,1) : \ U_{*\varepsilon} = \gamma_j\}.$$

Then $\{\mathcal{E}_j\}$ consists of a class of disjoint sets such that $(0,1) = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_L$. Moreover, for each $j = 1, \dots, L$,

$$\lim_{\varepsilon \in \mathcal{E}_j, \varepsilon \to 0} U_{*\varepsilon} = \gamma_j.$$

For fixed j and $\varepsilon \in \mathcal{E}_j$, consider $x^\varepsilon(\cdot)$ generated by $\overline{Q}^\varepsilon(\gamma_j)$. Then in view of (7.14) and the optimality of $u_{*\varepsilon}$, we have

$$v^\varepsilon(i) = J^\varepsilon(i, u_{*\varepsilon}) = J^0(k, \gamma_j) + O(\varepsilon) \ge v(k) + O(\varepsilon).$$

Thus for $0 < \varepsilon < 1$,

$$v^\varepsilon(i) \ge v(k) + O(\varepsilon).$$

This completes the proof. □

Remark 7.11. The use of $O(\varepsilon)$ allows us to simplify the notation and to suppress various constants. We will also keep this practice in the following section. Working with a production-planning model of manufacturing system described by differential equations having random machine capacity, it

is shown in Sethi and Zhang [192, Chapter 5] (via an example) that the best possible convergence rate of the value function for the original problem to that of the limit problem is of the order $\sqrt{\varepsilon}$. In this section, we are able to obtain much better estimates, because in the current model the dynamics are driven by a Markov chain rather than a governing differential equation.

7.6 Long-Run Average Cost

Sections 7.2–7.5 concentrated on the MDP with a discounted cost criterion. This section is concerned with the corresponding MDP with a long-run average cost. Replace (A7.1) and (A7.2) with the following assumptions throughout this section.

(**A7.3**) For each $U \in \overline{\Gamma}$ and $k = 1, \ldots, l$, $\widetilde{Q}_0^k(U^k)$ is irreducible; for sufficiently small $\varepsilon > 0$, $\overline{Q}^\varepsilon(U)$ is irreducible, where $\overline{Q}^\varepsilon(U)$ is defined in (7.4).

(**A7.4**) Γ is a set containing finitely many elements.

Remark 7.12. Assumption (A7.3) is not restrictive. In Example 7.1, this assumption is satisfied when the jump rates $\lambda_1(u)$, $\mu_1(u)$, $\lambda_2(u)$, and $\mu_2(u)$ are strictly positive.

Consider the following problem

$$\mathcal{P}_{\rm av}^\varepsilon : \begin{cases} \text{minimize: } J^\varepsilon(u) = \limsup\limits_{T\to\infty} \dfrac{1}{T} E \displaystyle\int_0^T G(x^\varepsilon(t), u(x^\varepsilon(t)))dt, \\ \text{subject to: } x^\varepsilon(t) \sim Q^\varepsilon(u(t)),\ x^\varepsilon(0) = i \in \mathcal{M},\ u \in \mathcal{A}_f, \\ \text{value function: } \lambda^\varepsilon = \inf\limits_{u\in\mathcal{A}_f} J^\varepsilon(u). \end{cases}$$

Note that for any given $U \in \overline{\Gamma}$, $\overline{Q}^\varepsilon(U)$ is irreducible. Thus the corresponding Markov chain $x^\varepsilon(\cdot)$ has a stationary distribution. Consequently, the average cost function is independent of the initial condition $x^\varepsilon(0) = i$, so is the value function.

The DP equation for $\mathcal{P}_{\rm av}^\varepsilon$ is

$$\lambda^\varepsilon = \min_{u\in\Gamma} \Big\{ G(i,u) + Q^\varepsilon(u) h^\varepsilon(\cdot)(i) \Big\}, \tag{7.15}$$

where $h^\varepsilon(i)$ is a function to be determined later. The next theorem gives necessary and sufficient conditions for optimality. The proof of the theorem is standard and can be found in Ross [184].

Theorem 7.13. *The following assertions hold:*

(a) *For each fixed $\varepsilon > 0$, there exists a pair $(\lambda^\varepsilon, h^\varepsilon(i))$ that satisfies the DP equation (7.15).*

(b) *The DP equation (7.15) has a unique solution in the sense that if $(\tilde{\lambda}^\varepsilon, \tilde{h}^\varepsilon(i))$ is another solution to (7.15), $\tilde{\lambda}^\varepsilon = \lambda^\varepsilon$ and for some constant K_0, $\tilde{h}^\varepsilon(i) = h^\varepsilon(i) + K_0$, for $i \in \mathcal{M}$.*

(c) *Let $u_{*\varepsilon} = u_{*\varepsilon}(i) \in \Gamma$ denote a minimizer of the right-hand side of (7.15). Then $u_{*\varepsilon}(i) \in \mathcal{A}_f$ is optimal and*

$$J^\varepsilon(u_{*\varepsilon}) = \lambda^\varepsilon.$$

As an analog to the discounted cost case, the limit problem with the long-run average cost is given as follows:

$$\mathcal{P}^0_{\rm av}: \begin{cases} \text{minimize: } J^0(U) = \limsup\limits_{T\to\infty} \dfrac{1}{T} E \displaystyle\int_0^T \overline{G}(\overline{x}(t), U(\overline{x}(t)))dt, \\ \text{subject to: } \overline{x}(t) \sim \overline{Q}(U(\overline{x}(t))),\ \overline{x}(0) = k,\ U \in \mathcal{A}^0, \\ \text{value function: } \lambda^0 = \inf\limits_{U\in\mathcal{A}^0} J^0(U). \end{cases}$$

The next lemma is concerned with the irreducibility of $\overline{Q}(U)$, for any $U \in \overline{\Gamma}$, which is required to guarantee the existence of the corresponding stationary distribution.

Lemma 7.14. *The generator $\overline{Q}(U)$ is irreducible for each $U \in \overline{\Gamma}$.*

Proof: The following proof is along the line of the Gaussian elimination procedure in which elementary row operations do not alter the rank of a matrix. It proceeds in two steps. The first step derives the weak irreducibility of $\overline{Q}(U)$ and the second step shows that it is also (strongly) irreducible. For simplicity, the control variable U will be suppressed in the proof whenever no confusion arises.

Step 1. $\text{rank}(\overline{Q}) = l - 1$.

Write $\widehat{Q}_0 = (\widehat{Q}_{ij})$ as the blocks of submatrices such that $\widehat{Q}_{ij}$ has dimension $m_i \times m_j$. Then $\overline{Q} = (\overline{q}_{ij})_{l\times l}$ with $\overline{q}_{ij} = \nu^i \widehat{Q}_{ij} \mathbb{1}_{m_j}$, where $\nu^i = \nu^i(U^i)$, $\widehat{Q}_{ij} = \widehat{Q}_{ij}(U^i)$, and $\mathbb{1}_{m_j} = (1, \ldots, 1)'$. Since $\nu^i > 0$ and $\mathbb{1}_{m_j} > 0$, it follows that $\widehat{Q}_{ij} = 0$ if $\overline{q}_{ij} = 0$ for $i \neq j$. It is not difficult to see that the irreducibility of Q^ε implies $\overline{q}_{kk} < 0$ for $k = 1, \ldots, l$. Multiply the first row of $\overline{Q}$ by $-\overline{q}_{k1}/\overline{q}_{11}$ and add to the kth row, $k = 2, \ldots, l$, to make the first component of that row 0. Let $\overline{Q}^{(1)} = (\overline{q}^{(1)}_{ij})$ denote the

resulting matrix. Then it follows immediately that

$$\overline{q}_{1j}^{(1)} = \overline{q}_{1j},\ j = 1, \ldots, l,$$

$$\overline{q}_{k1}^{(1)} = 0,\ k = 2, \ldots, l,$$

$$\overline{q}_{kk}^{(1)} \leq 0,\ k = 2, \ldots, l,\ \text{and}\ \sum_{j=1}^{l} \overline{q}_{kj}^{(1)} = 0.$$

We claim that $\overline{q}_{kk}^{(1)} < 0$ for $k = 2, \ldots, l$. For $k = 2$, if $\overline{q}_{22}^{(1)} \not< 0$, then it must be equal to 0. Thus,

$$(\overline{q}_{23}, \ldots, \overline{q}_{2l}) + \left(-\frac{\overline{q}_{21}}{\overline{q}_{11}}\right)(\overline{q}_{13}, \ldots, \overline{q}_{1r}) = 0. \tag{7.16}$$

Recall that $\overline{q}_{22} \neq 0$. One must have $\overline{q}_{21} > 0$ since $\overline{q}_{21} = 0$ implies $\overline{q}_{22} = \overline{q}_{22}^{(1)} = 0$, which contradicts the fact that $\overline{q}_{kk} < 0$ for $k = 1, \ldots, l$. Thus, $-\overline{q}_{21}/\overline{q}_{11} > 0$. It follows that both vectors in (7.16) must be equal to 0, that is,

$$(\overline{q}_{23}, \ldots, \overline{q}_{2l}) = 0 \text{ and } (\overline{q}_{13}, \ldots, \overline{q}_{1r}) = 0.$$

Consequently, one must have $\widehat{Q}_{1k} = 0$ and $\widehat{Q}_{2k} = 0$ for $k = 3, \ldots, l$. This implies that $\overline{Q}^{\varepsilon}$ cannot be irreducible since a state in $(\mathcal{M}_1 \cup \mathcal{M}_2)^c$ (the complement of $\mathcal{M}_1 \cup \mathcal{M}_2$) is not accessible from a state in $\mathcal{M}_1 \cup \mathcal{M}_2$. The contradiction implies that $\overline{q}_{22}^{(1)} < 0$. Similarly, we can show $\overline{q}_{kk}^{(1)} < 0$ for $k = 3, \ldots, l$.

Repeat this procedure. Multiply the second row of $\overline{Q}^{(1)}$ by $-\overline{q}_{k2}^{(1)}/\overline{q}_{22}^{(1)}$ for $k = 3, \ldots, l$, and add to the kth row. Let $\overline{Q}^{(2)} = (\overline{q}_{ij}^{(2)})$ denote the resulting matrix. Then

$$\overline{q}_{ij}^{(2)} = \overline{q}_{ij}^{(1)},\ i = 1, 2,\ j = 1, \ldots, l,$$

$$\overline{q}_{ij}^{(2)} = 0,\ i = 3, \ldots, l,\ j = 1, 2,$$

$$\overline{q}_{kk}^{(2)} \leq 0,\ k = 3, \ldots, l,\ \text{and}\ \sum_{j=1}^{l} \overline{q}_{kj}^{(2)} = 0.$$

Similarly, we can show $\overline{q}_{kk}^{(2)} < 0$ for $k = 3, \ldots, l$.

Continue this procedure and transform $\overline{Q} \to \overline{Q}^{(1)} \to \cdots \to \overline{Q}^{(l-1)}$ with $\overline{Q}^{(l-1)} = (\overline{q}_{ij}^{(l-1)})$ such that

$$\begin{aligned}
&\overline{q}_{ij}^{(l-1)} = 0, i > j,\\
&\overline{q}_{kk}^{(l-1)} < 0,\ k = 1, \ldots, l-1,\\
&\sum_{j=1}^{l} \overline{q}_{ij}^{(l-1)} = 0, \text{ and } \overline{q}_{ll}^{(l-1)} = 0.
\end{aligned}$$

Note that the prescribed transformation does not change the rank of the original matrix. Thus,

$$\operatorname{rank}\Big(\overline{Q}\Big) = \operatorname{rank}\Big(\overline{Q}^{(1)}\Big) = \cdots = \operatorname{rank}\Big(\overline{Q}^{(l-1)}\Big) = l - 1.$$

By virtue of Lemma A.5, $\overline{Q}$ is weakly irreducible.

Step 2. $\overline{Q}$ is irreducible (i.e., $(\overline{\nu}_1, \ldots, \overline{\nu}_l) > 0$).

Suppose that this is not true. Without loss of generality, we may assume $\overline{\nu}_1 > 0$, …, $\overline{\nu}_{k_0} > 0$, and $\overline{\nu}_{k_0+1} = 0$, …, $\overline{\nu}_l = 0$, for some k_0. Note that $(\overline{\nu}_1, \ldots, \overline{\nu}_l)\overline{Q} = 0$ implies that $\overline{q}_{ij} = 0$ for $i = 1, \ldots, k_0$ and $j = k_0+1, \ldots, l$, which in turn implies that $\widehat{Q}_{ij} = 0$ for $i = 1, \ldots, k_0$ and $j = k_0 + 1, \ldots, l$. Again, $\overline{Q}^{\varepsilon}$ is not irreducible since the process $x^{\varepsilon}(\cdot)$ cannot jump from a state in $\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_{k_0}$ to a state in $\mathcal{M}_{k_0+1} \cup \cdots \cup \mathcal{M}_l$. The contradiction yields the irreducibility of $\overline{Q}$. □

To proceed, consider the DP equation for $\mathcal{P}_{\text{av}}^0$

$$\lambda^0 = \min_{U^k \in \Gamma_k} \Big\{ \overline{G}(k, U^k) + \overline{Q}(U^k) h^0(\cdot)(k) \Big\}, \tag{7.17}$$

for some function $h^0(k)$. Next we give the verification theorem on the limit problem. Again, the proof of the following theorem is standard and can be found in Ross [184].

Theorem 7.15. *The following assertions hold:*

(a) *There exists a pair* $(\lambda^0, h^0(k))$ *that satisfies the DP equation* (7.17).

(b) *Any two solutions of the DP equation* (7.17) *differ by a constant, that is, if* $(\lambda^0, h^0(k))$ *and* $(\tilde{\lambda}^0, \tilde{h}^0(k))$ *are solutions of* (7.17), *then* $\tilde{\lambda}^0 = \lambda^0$ *and for some constant* K_0, $\tilde{h}^0(k) = h^0(k) + K_0$, *for* $k = 1, \ldots, l$.

(c) *Let* $U_* \in \overline{\Gamma}$ *denote a minimizer of the right-hand side of* (7.17). *Then* $U_* \in \mathcal{A}^0$ *is optimal, and* $J^0(U_*) = \lambda^0$.

We next show that the quasi-stationary distribution corresponding to $\overline{Q}^{\varepsilon}(U)$ can be approximated by the quasi-stationary distributions corresponding to $\overline{Q}(U)$ and $\widetilde{Q}^k(U^k)$, $k = 1, \ldots, l$.

Lemma 7.16. *For any $U \in \overline{\Gamma}$, let $\nu_\varepsilon(U)$ denote the quasi-stationary distribution of $\overline{Q}^{\varepsilon}(U)$. Then*

$$\nu_\varepsilon(U) = \nu_0(U) + O(\varepsilon),$$

where

$$\nu_0(U) = (\nu^1(U^1)\overline{\nu}_1(U), \ldots, \nu^l(U^l)\overline{\nu}_l(U))$$

with $\nu^k(U^k)$ being the quasi-stationary distribution of $\widetilde{Q}^k(U^k)$, $k = 1, \ldots, l$, and $(\overline{\nu}_1(U), \ldots, \overline{\nu}_l(U))$ the quasi-stationary distribution of $\overline{Q}(U)$.

Proof: For simplicity, we suppress the dependence of U. Note that ν_ε is a bounded vector-valued function. It follows that for each sequence of $\varepsilon \to 0$, there exists a further subsequence of ε (still denoted by ε) such that $\nu_\varepsilon \to \nu_0 := (\nu_0^1, \ldots, \nu_0^l)$. Sending $\varepsilon \to 0$ in $\nu_\varepsilon \overline{Q}^{\varepsilon} = 0$, we have $\nu_0^k \widetilde{Q}^k = 0$ for $k = 1, \ldots, l$. In view of the irreducibility of $\widetilde{Q}^k$, $\nu_0^k = a_k \nu^k$ for some scalar $a_k \geq 0$ with $a_1 + \cdots + a_l = 1$. Note also that $\nu_\varepsilon \overline{Q}^{\varepsilon} \tilde{\mathbb{1}} = 0$, where $\tilde{\mathbb{1}} = \operatorname{diag}(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_l})$. Thus, $\nu_\varepsilon \widehat{Q} \tilde{\mathbb{1}} = 0$. This implies $\nu_0 \widehat{Q} \tilde{\mathbb{1}} = 0$, that is, $(a_1, \ldots, a_l)\overline{Q} = 0$. The irreducibility of $\overline{Q}$ implies that $(a_1, \ldots, a_l)$ is equal to the quasi-stationary distribution $(\overline{\nu}_1, \ldots, \overline{\nu}_l)$ of $\overline{Q}$.

Since $\overline{Q}^{\varepsilon}$ is irreducible, we can always write the solution ν_ε to

$$\begin{cases} \nu_\varepsilon \overline{Q}^{\varepsilon} = 0, \\ \displaystyle\sum_{i,j} \nu_\varepsilon^{ij} = 1, \end{cases}$$

as (see Remark 4.10)

$$\nu_\varepsilon = (1, 0, \ldots, 0)\big(\mathbb{1} \vdots\, \overline{Q}^{\varepsilon}\big)' \Big[\big(\mathbb{1} \vdots\, \overline{Q}^{\varepsilon}\big)\big(\mathbb{1} \vdots\, \overline{Q}^{\varepsilon}\big)'\Big]^{-1},$$

which is a rational function of ε, where ν_ε^{ij} denotes the jth component of ν_ε^i for each i. Therefore, the rate of convergence of $\nu_\varepsilon \to \nu_0$ must be at least linear, i.e., $\nu_\varepsilon = \nu_0 + O(\varepsilon)$. □

Theorem 7.17. *Let $U_* \in \mathcal{A}^0$ denote an optimal control for $\mathcal{P}^0_{\text{av}}$ and construct $u_c \in \mathcal{A}_f$ as in (7.10). Then u_c is asymptotically optimal with an ε-order error bound, that is,*

$$J^\varepsilon(u_c) - \lambda^\varepsilon = O(\varepsilon).$$

Proof: Let $x^\varepsilon(t)$ and $\overline{x}(t)$ denote the Markov chains generated by $\overline{Q}^\varepsilon(U_*)$ and $\overline{Q}(U_*)$, respectively. Recall the irreducibility of $\overline{Q}^\varepsilon(U_*)$ and $\overline{Q}(U_*)$. It follows that as $t \to \infty$

$$P(x^\varepsilon(t) = s_{ij}) \to \nu_\varepsilon^{ij}(U_*^i),\ j = 1, \ldots, m_i,\ i = 1, \ldots, l, \quad \text{and}$$

$$P(\overline{x}(t) = i) \to \overline{\nu}_i(U_*),\ i = 1, \ldots, l.$$

Therefore, we obtain

$$\begin{aligned} J^\varepsilon(u_c) = &\limsup_{T\to\infty} \frac{1}{T} E \int_0^T G(x^\varepsilon(t), u_c(t))dt \\ = &\limsup_{T\to\infty} \frac{1}{T} \int_0^T \sum_{i,j} G(s_{ij}, u_*^{ij}) P(x^\varepsilon(t) = s_{ij})dt \\ = &\sum_{i,j} G(s_{ij}, u_*^{ij}) \nu_\varepsilon^{ij}(U_*^i) \end{aligned} \tag{7.18}$$

and

$$\lambda^0 = J^0(U_*) = \sum_{i,j} G(s_{ij}, u_*^{ij}) \nu_j^i(U_*^i) \overline{\nu}_i(U_*). \tag{7.19}$$

In view of Lemma 7.16, we have

$$|\nu_\varepsilon^{ij}(U_*^i) - \nu_j^i(U_*^i)\overline{\nu}_i(U_*)| = O(\varepsilon).$$

It follows that

$$J^\varepsilon(u_c) - \lambda^0 = O(\varepsilon). \tag{7.20}$$

Note that (7.18)–(7.20) imply the inequality

$$\lambda^\varepsilon \le \lambda^0 + O(\varepsilon). \tag{7.21}$$

Let $u_{*\varepsilon} \in \mathcal{A}_f$ denote an optimal control for $\mathcal{P}^\varepsilon$ and let

$$U_{*\varepsilon} = (u_{*\varepsilon}(s_{11}), \ldots, u_{*\varepsilon}(s_{1m_1}), \ldots, u_{*\varepsilon}(s_{l1}), \ldots, u_{*\varepsilon}(s_{lm_l})).$$

Since Γ is a set containing finitely many elements, so is $\overline{\Gamma} = \{\gamma_1, \ldots, \gamma_L\}$. As in the proof of Theorem 7.10, we let $\mathcal{E}_{i_1} = \{\varepsilon \in (0,1) : U_{*\varepsilon} = \gamma_{i_1}\}$, for $i_1 = 1, \ldots, L$. Then for $\varepsilon \in \mathcal{E}_{i_1}$, $U_{*\varepsilon} = \gamma_{i_1} (:= U_{*i_1})$ (independent of ε) and

$$\begin{aligned} \lambda^\varepsilon = J^\varepsilon(u_c) = &\sum_{i,j} G(s_{ij}, u_{*i_1}^{ij}) \nu_\varepsilon^{ij}(U_{*i_1}^i) \\ = &\sum_{i,j} G(s_{ij}, u_{*i_1}^{ij}) \nu_j^i(U_{*i_1}^i) \overline{\nu}_i(U_{*i_1}) + O(\varepsilon) \\ = &\ J^0(U_{*i_1}) + O(\varepsilon) \ge \lambda^0 + O(\varepsilon). \end{aligned}$$

Therefore,

$$\lambda^\varepsilon \geq \lambda^0 + O(\varepsilon). \tag{7.22}$$

Combining (7.21) and (7.22), we have

$$\lambda^\varepsilon = \lambda^0 + O(\varepsilon). \tag{7.23}$$

Finally, in view of (7.20), we obtain

$$J^\varepsilon(u_c) - \lambda^\varepsilon = O(\varepsilon).$$

This completes the proof. □

Remark 7.18. Note that (7.21) and (7.22) mean that there are some positive constants $K_1 > 0$ and $K_2 > 0$ such that

$$\lambda^\varepsilon \leq \lambda^0 + K_1\varepsilon \quad \text{and} \quad \lambda^\varepsilon \geq \lambda^0 - K_2\varepsilon.$$

We can certainly select a single $K > 0$ such that both inequalities hold; see also Remark 7.11 for the use of $O(\varepsilon)$. If $\widetilde{Q}(u)$ is irreducible, we can show $J^\varepsilon(u_c) = \lambda^0 = \lambda^\varepsilon$ by using (7.18), (7.19), and the fact that $\nu_\varepsilon^{ij}(U_*^i)$ is independent of ε and j.

7.7 Computational Procedures

Owing to Theorems 7.8 and 7.17, to find a nearly optimal solution for $\mathcal{P}^\varepsilon$ and $\mathcal{P}_{\rm av}^\varepsilon$, one need only solve the limit problems $\mathcal{P}^0$ and $\mathcal{P}_{\rm av}^0$, respectively, which requires that the solution of the associated DP equations be found. Although an optimal control policy can be obtained as in (7.10), an analytic solution to the DP equations is usually not obtainable except in some simple cases. To apply our results to real-life problems, one has to resort to numerical methods. In this section, we discuss computational methods for solving the DP equations associated with continuous-time models.

Discounted Cost Problems

Let us begin with the problem having discounted costs. The DP equation (7.7) for $\mathcal{P}^0$ can be written as

$$\rho v(k) = \min_{U^k \in \Gamma_k} \left\{ \overline{G}(k, U^k) + \sum_{k' \neq k} \overline{q}_{kk'}(U^k)(v(k') - v(k)) \right\}.$$

We next show that this equation is equivalent to a DP equation of a discrete-time Markov decision problem.

In fact, for each $U \in \overline{\Gamma}$,

$$\rho v(k) \le \overline{G}(k, U^k) + \sum_{k' \ne k} \overline{q}_{kk'}(U^k)(v(k') - v(k)). \tag{7.24}$$

Note that

$$\sum_{k' \ne k} \overline{q}_{kk'}(U^k) v(k) = |\overline{q}_{kk}(U^k)| v(k).$$

Since $\rho > 0$, the inequality (7.24) is equivalent to

$$v(k) \le \frac{\overline{G}(k, U^k)}{\rho + |\overline{q}_{kk}(U^k)|} + \sum_{k' \ne k} \frac{\overline{q}_{kk'}(U^k)}{\rho + |\overline{q}_{kk}(U^k)|} v(k').$$

It follows that

$$v(k) \le \min_{U^k \in \Gamma_k} \left\{ \frac{\overline{G}(k, U^k)}{\rho + |\overline{q}_{kk}(U^k)|} + \sum_{k' \ne k} \frac{\overline{q}_{kk'}(U^k)}{\rho + |\overline{q}_{kk}(U^k)|} v(k') \right\}.$$

The equality holds if and only if U is equal to the minimizer U_* of the right-hand side of (7.7). Thus, the DP equation (7.7) is equivalent to

$$v(k) = \min_{U^k \in \Gamma_k} \left\{ \frac{\overline{G}(k, U^k)}{\rho + |\overline{q}_{kk}(U^k)|} + \sum_{k' \ne k} \frac{\overline{q}_{kk'}(U^k)}{\rho + |\overline{q}_{kk}(U^k)|} v(k') \right\}. \tag{7.25}$$

To show that (7.25) is equivalent to a DP equation of a discrete-time Markov decision problem, let

$$\widetilde{G}(k, U^k) = \frac{\overline{G}(k, U^k)}{\rho + |\overline{q}_{kk}(U^k)|},$$

$$\varpi = \max_{k=1,\dots,l, U \in \overline{\Gamma}} \frac{|\overline{q}_{kk}(U^k)|}{\rho + |\overline{q}_{kk}(U^k)|},$$

and

$$\tilde{p}_{kk'} = \frac{\overline{q}_{kk'}(U^k)}{\varpi(\rho + |\overline{q}_{kk}(U^k)|)} \text{ for } k' \ne k \text{ and } \tilde{p}_{kk} = 0.$$

Then $0 < \varpi < 1$, $\sum_{k'} \tilde{p}_{kk'}(U^k) = 1$. The corresponding discrete-time version of the DP equation is

$$v(k) = \min_{U^k \in \Gamma_k} \left\{ \widetilde{G}(k, U^k) + \varpi \sum_{k'} \tilde{p}_{kk'}(U^k) v(k') \right\}.$$

There are a number of methods available for solving discrete-time DP equations. For example, the methods of successive approximation, policy improvement, and linear programming can be used. We refer the reader to Ross [184] for discussions on these methods.

Long-Run Average Cost Problems

Similar to the case of discounted costs, we can write the discrete-time version of the DP equation corresponding to (7.17) for $\mathcal{P}^0_{\rm av}$ as

$$\tilde{\lambda}^0 + \tilde{h}^0(k) = \min_{U^k \in \Gamma_k} \left\{ \widetilde{G}(k, U^k) + \sum_{k'} \tilde{p}_{kk'}(U^k) \tilde{h}^0(k') \right\},$$

where

$$\tilde{\lambda}^0 = \max_{k=1,\ldots,l, U \in \overline{\Gamma}} \frac{\lambda^0}{|\overline{q}_{kk}(U^k)|},$$

$$\widetilde{G}(k, U^k) = \frac{\overline{G}(k, U^k)}{|\overline{q}_{kk}(U^k)|} + \tilde{\lambda}^0 - \frac{\lambda^0}{|\overline{q}_{kk}(U^k)|},$$

$$\tilde{h}^0(k) = h^0(k),$$

and

$$\tilde{p}_{kk'}(U^k) = \frac{\overline{q}_{kk'}(U^k)}{|\overline{q}_{kk}(U^k)|} \text{ for } k \neq k', \text{ and } \tilde{p}_{kk}(U^k) = 0.$$

One may also design algorithms for solving the discrete-time version DP equations with long-run average costs. A typical method uses the idea of linear programming (see Ross [184]); see also Kushner and Dupuis [141, Chapter 6] for more details on computational methods.

7.8 Notes

This chapter is based on the results obtained in Zhang [249]. For related results with discounted costs, we refer the reader to the papers of Delebecque and Quadrat [44] and Phillips and Kokotovic [175].

The hierarchical control approach in this chapter is concerned with reduction of dimensionality of a class of stochastic dynamic systems. In conjunction with solving DP equations, where the number of equations is the main factor that affects the computational effort (see Hillier and Lieberman [86]), our results pave the way to a substantial reduction of complexity. The recent book of Guo and Hernández-Lerma [78] collects a number of new results on continuous-time Markov decision processes, which includes the so-called advanced (bias, overtaking, sensitive discount, and Blackwell) criteria in addition to the usual discounted cost and average cost per unit time problems.

Note that the main results in this chapter are obtained under assumptions that the state space and the control space are finite. It would be interesting to extend these results to more general models without such finite-dimensionality restrictions (see Sethi and Zhang [193]). Recently,

Costa and Dufour formulated and treated piecewise deterministic Markov processes under singular perturbation [33] and singularly perturbed discounted Markov control processes in a general state space in [34]. From another angle, it is also interesting to consider problems with robust cost criteria such as risk-sensitive cost; see Zhang [248] for discounted cost, and Fleming and Zhang [66] for long-run average cost criteria in this connection.

8

Stochastic Control of Dynamical Systems

8.1 Introduction

While Chapter 7 deals with Markov decision processes, this chapter is concerned with stochastic dynamical systems with the state $x^\varepsilon(t) \in \mathbb{R}^n$ and the control $u(t) \in \Gamma \subset \mathbb{R}^{n_1}$ satisfying

$$\frac{dx^\varepsilon(t)}{dt} = f(x^\varepsilon(t), \alpha^\varepsilon(t), u(t)),\ x^\varepsilon(0) = x,\ t \geq 0, \tag{8.1}$$

where $\varepsilon > 0$ is a small parameter, and $\alpha^\varepsilon(t)$, $t \geq 0$, is a Markov chain defined on a probability space $(\Omega, \mathcal{F}, P)$ taking values in

$$\mathcal{M} = \Big\{s_{11}, \ldots, s_{1m_1}, \ldots, s_{l1}, \ldots, s_{lm_l}\Big\}.$$

Let $u(\cdot) = \{u(t) \in \Gamma :\ t \geq 0\}$. Assuming that $\alpha^\varepsilon(0) = \alpha$, $\rho > 0$ is the discount factor, and $G(x, \alpha, u)$ is the cost-to-go function. The problems of interest are: Subject to the constraint (8.1),

(1) find a control process $u(t)$ as a function of $x^\varepsilon(s), \alpha^\varepsilon(s)$, $s \leq t$ to minimize a discounted cost function

$$J^\varepsilon(x, \alpha, u(\cdot)) = E\int_0^\infty e^{-\rho t} G(x^\varepsilon(t), \alpha^\varepsilon(t), u(t))dt, \tag{8.2}$$

over the infinite horizon $[0, \infty)$, and

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(2) find a control process $u(t)$ as a function of $x^\varepsilon(s), \alpha^\varepsilon(s)$, $s \le t$ to minimize a finite horizon cost function

$$J^\varepsilon(x, \alpha, u(\cdot)) = E \int_0^T G(x^\varepsilon(t), \alpha^\varepsilon(t), u(t))dt \tag{8.3}$$

over $[0, T]$ for a given T satisfying $0 < T < \infty$.

As was demonstrated previously, singularly perturbed Markovian models typically arise from either systems displaying multiple-time scales such as Example 4.20, or from large-scale systems naturally leading to hierarchical decomposition as illustrated in Chapters 1 and 3. One introduces a small parameter $\varepsilon > 0$ to account for the time-scale separation and to achieve the goal of dimensionality reduction.

Consider the case that the process $\alpha^\varepsilon(t)$ has weak and strong interactions among different group of states as discussed in Chapters 4 and 5; assume the generator is given by $Q^\varepsilon(t) = \widetilde{Q}(t)/\varepsilon + \widehat{Q}(t)$, where $\widetilde{Q}(t) = \text{diag}(\widetilde{Q}^1(t), \ldots, \widetilde{Q}^l(t))$ is a block diagonal matrix such that $\widehat{Q}(t)$ and $\widetilde{Q}^k(t)$ for $k = 1, \ldots, l$, are themselves generators of appropriate dimensions. Moreover, for each $k = 1,\ldots,l$, the block $\widetilde{Q}^k(t)$ corresponds to $\mathcal{M}_k = \{s_{k1}, \ldots, s_{km_k}\}$. As in Chapter 7, the focus of this chapter is on $\widetilde{Q}(t)$ corresponding to a number of groups with recurrent states.

The control systems given above belong to the category of piecewise-deterministic processes (see Davis [41] for the terminology). They adequately describe interconnections and transformations that occur within the subsystems, and are suited for various applications in manufacturing, queueing networks, etc. The random process $\alpha^\varepsilon(t)$ and the weak and strong interactions among its states are difficult to deal with, however. One cannot obtain closed-form solutions except in some special cases. The computational problem is even more acute if the state space of $\alpha^\varepsilon(t)$ is large. It is thus vital to reduce the complexity of the underlying problem. Keeping these points in mind, our effort is devoted to obtaining asymptotic optimal and nearly optimal controls.

The rationale of our approach is that, when ε is small, one may ignore some of the details of the process $\alpha^\varepsilon(t)$ at each time t and obtain an averaged model in which the stochastic process $\alpha^\varepsilon(t)$ is aggregated so that the states in $\mathcal{M}_k$ can be replaced by a single state. The limit of the aggregated problem is much simpler than the original one and is easier to analyze.

In the first five sections, we consider the generator of $\alpha^\varepsilon(t)$ having the form

$$Q^\varepsilon = \frac{1}{\varepsilon}\widetilde{Q} + \widehat{Q}, \tag{8.4}$$

with time independent $\widetilde{Q}$ and $\widehat{Q}$. Our suggestion is to obtain the optimality and to derive the feedback control for the limit problem first, then, using this feedback control, to construct a control for the original problem, and to show that the control so constructed is asymptotically optimal. To use

the dynamic programming approach for the problems, and to study the asymptotic properties of the underlying system for sufficiently small ε, the viscosity solution method is employed to verify the convergence of the value function, which allows us to obtain the convergence in a general setting and helps us to characterize the structure of the limit system. Then in Section 8.6, we switch gears and present an alternative approach, namely, the weak convergence approach via the use of relaxed control representation. In lieu of the Hamilton-Jacobi-Bellman (HJB) equations, the investigation is on the corresponding probability measures.

The detailed arrangement of the rest of the chapter is as follows. Section 8.2 presents the formulation and the assumptions of the system under consideration. Section 8.3 discusses the dynamic properties of the original problem $\mathcal{P}^\varepsilon$. It is shown that the value function v^ε is continuous and is the only viscosity solution to the associated HJB equation. Then we introduce a limit control problem $\mathcal{P}^0$ and show that $\mathcal{P}^\varepsilon \to \mathcal{P}^0$ as $\varepsilon \to 0$ in terms of the convergence of their value functions. Section 8.4 constructs feedback controls for the original problem $\mathcal{P}^\varepsilon$ using the optimal control of $\mathcal{P}^0$; it is shown that the controls so constructed are asymptotically optimal. Section 8.5 gives an estimate of the convergence rate of the value function under assumptions such as linear system equations and irreducibility of $\widetilde{Q}$. The convergence rate is shown to be the order of $\sqrt{\varepsilon}$, which turns out to be the best rate possible. To demonstrate the versatility of the asymptotic properties, the weak convergence approach to the near optimality is given in Section 8.6, which enables us to obtain asymptotic optimality under milder conditions. Finally, Section 8.7 concludes the chapter with additional notes.

8.2 Problem Formulation

To begin, let us make the following assumptions.

(**A8.1**) There exist bounded functions $f_1(x,\alpha) \in \mathbb{R}^n$ and $f_2(x,\alpha) \in \mathbb{R}^{n\times n_1}$ on $\mathbb{R}^n \times \mathcal{M}$ such that

$$f(x,\alpha,u) = f_1(x,\alpha) + f_2(x,\alpha)u,$$

where $u \in \Gamma$, a convex and compact subset of $\mathbb{R}^{n_1}$. Moreover, for $i = 1, 2$, $f_i(x,\alpha)$ are Lipschitz in that

$$|f_i(x,\alpha) - f_i(y,\alpha)| \le K|x-y|, \text{ for all } \alpha \in \mathcal{M},$$

for a constant K.

(**A8.2**) For each $\alpha \in \mathcal{M}$, $G(x,\alpha,u)$ is jointly convex in (x,u) and locally Lipschitz in the sense that

$$|G(x,\alpha,u) - G(y,\alpha,u)| \le K(1 + |x|^\kappa + |y|^\kappa)|x-y|,$$

for some positive constants K and κ. Moreover,

$$0 \le G(x, \alpha, u) \le K(1 + |x|^{\kappa}),$$

that is, $G(x, \alpha, u)$ has at most polynomial growth rate in x.

(**A8.3**) The process $\alpha^{\varepsilon}(\cdot) = \{\alpha^{\varepsilon}(t),\ t \ge 0\}$ is a finite-state Markov chain on $(\Omega, \mathcal{F}, P)$ generated by Q^{ε} given in (8.4). Each block $\widetilde{Q}^{k}$ is irreducible.

Definition 8.1. A control $u(\cdot) = \{u(t) :\ t \ge 0\}$ is *admissible* if $u(t) \in \Gamma$ and is progressively measurable with respect to $\sigma\{\alpha^{\varepsilon}(s) :\ s \le t\}$. Use $\mathcal{A}^{\varepsilon}$ to denote the set of all admissible controls.

Definition 8.2. A Borel measurable function $u(x, \alpha)$ is an *admissible feedback* (or simply feedback) control if under $u(t) = u(x^{\varepsilon}(t), \alpha^{\varepsilon}(t))$ the system equation (8.1) has a unique solution $x^{\varepsilon}(t)$, $t \ge 0$, and $u(t)$, $t \ge 0$, is admissible.

Let $v^{\varepsilon}(x, \alpha)$ denote the corresponding value function of the control problem (8.1) and (8.2), i.e.,

$$v^{\varepsilon}(x, \alpha) = \inf_{u(\cdot) \in \mathcal{A}^{\varepsilon}} J^{\varepsilon}(x, \alpha, u(\cdot)).$$

In what follows, use $\mathcal{P}^{\varepsilon}$ to denote our original control problem, that is,

$$\mathcal{P}^{\varepsilon} : \begin{cases} \text{minimize: } J^{\varepsilon}(x, \alpha, u(\cdot)), \\ \text{subject to: } \dfrac{dx^{\varepsilon}(t)}{dt} = f(x^{\varepsilon}(t), \alpha^{\varepsilon}(t), u(t)), \\ \qquad x^{\varepsilon}(0) = x,\ \alpha^{\varepsilon}(0) = \alpha,\ u(\cdot) \in \mathcal{A}^{\varepsilon}, \\ \text{value function: } v^{\varepsilon}(x, \alpha) = \inf\limits_{u(\cdot) \in \mathcal{A}^{\varepsilon}} J^{\varepsilon}(x, \alpha, u(\cdot)). \end{cases} \tag{8.5}$$

Example 8.3. Consider the following failure-prone manufacturing system consisting of a single machine and producing one part type. Let $x^{\varepsilon}(t) \in \mathbb{R}^1$ be the surplus of finished goods, $c^{\varepsilon}(t) \in \{0, 1\}$ the production capacity rate, and $z^{\varepsilon}(t) \in \{z_1, z_2\}$ the part demand rate for $0 < z_1 < z_2 < 1$. Let $u(t) \in \mathbb{R}^1$ denote the control process so that $c^{\varepsilon}(t)u(t)$ represents the rate of production.

In this example, $c^{\varepsilon}(t) = 0$ means the machine is down and $c^{\varepsilon}(t) = 1$ means the machine is up with maximum capacity 1. Using $c^{\varepsilon}(t)$, the

production constraint is given as $0 \le u(t) \le 1$, $t \ge 0$. Let $\mathcal{M}_1 = \{(1, z_1), (0, z_1)\}$ and $\mathcal{M}_2 = \{(1, z_2), (0, z_2)\}$. Consider a Markov chain defined as

$$\alpha^\varepsilon(t) = (c^\varepsilon(t), z^\varepsilon(t)) \in \mathcal{M} = \Big\{(1, z_1), (0, z_1), (1, z_2), (0, z_2)\Big\},$$

generated by Q^ε.

The system equation is

$$\frac{dx^\varepsilon(t)}{dt} = \alpha^\varepsilon(t)(u(t), -1)' = c^\varepsilon(t)u(t) - z^\varepsilon(t),\ x^\varepsilon(0) = x.$$

Our objective is to choose a control $u(\cdot)$ to minimize the production and surplus costs

$$J^\varepsilon(x, \alpha, u(\cdot)) = E\int_0^\infty e^{-\rho t}\left(c^+(x^\varepsilon(t))^+ + c^-(x^\varepsilon(t))^-\right) dt,$$

where c^+ and c^- are positive constants, $x^+ = \max\{0, x\}$, and $x^- = \max\{0, -x\}$.

Let λ_c, μ_c, λ_z, and μ_z be positive real numbers. Suppose that the generator Q^ε has the form

$$Q^\varepsilon = \frac{1}{\varepsilon}\begin{pmatrix} -\lambda_c & \lambda_c & 0 & 0 \\ \mu_c & -\mu_c & 0 & 0 \\ 0 & 0 & -\lambda_c & \lambda_c \\ 0 & 0 & \mu_c & -\mu_c \end{pmatrix} + \begin{pmatrix} -\lambda_z & 0 & \lambda_z & 0 \\ 0 & -\lambda_z & 0 & \lambda_z \\ \mu_z & 0 & -\mu_z & 0 \\ 0 & \mu_z & 0 & -\mu_z \end{pmatrix}.$$

In this case, the rate of fluctuation of $c^\varepsilon(t)$ in $\alpha^\varepsilon(t)$ is faster than that of $z^\varepsilon(t)$.

Next rearrange the order of the states in $\mathcal{M}$ as

$$\mathcal{M} = \Big\{(1, z_1), (1, z_2), (0, z_1), (0, z_2)\Big\}$$

and consider

$$Q^\varepsilon = \frac{1}{\varepsilon}\begin{pmatrix} -\lambda_z & \lambda_z & 0 & 0 \\ \mu_z & -\mu_z & 0 & 0 \\ 0 & 0 & -\lambda_z & \lambda_z \\ 0 & 0 & \mu_z & -\mu_z \end{pmatrix} + \begin{pmatrix} -\lambda_c & 0 & \lambda_c & 0 \\ 0 & -\lambda_c & 0 & \lambda_c \\ \mu_c & 0 & -\mu_c & 0 \\ 0 & \mu_c & 0 & -\mu_c \end{pmatrix}.$$

In the second case, the demand $z^\varepsilon(t)$ is the fast-changing process. The results to follow provide guidance on how the asymptotically optimal feedback controls can be constructed for such systems.

8.3 Properties of the Value Functions

This section concerns properties of the value function $v^\varepsilon(x,\alpha)$. By applying the viscosity solution methods, we show that $v^\varepsilon(x,\alpha)$ converges to a function $v(x,k)$ whenever $\alpha \in \mathcal{M}_k$. It turns out that $v(x,k)$ is equal to the value function of a limit problem in which the stochastic process is replaced by the "limit" of its aggregated process.

Formally, the HJB equation of $\mathcal{P}^\varepsilon$ takes the following form

$$\rho v^\varepsilon(x,\alpha) = \min_{u\in\Gamma}\left\{ f(x,\alpha,u)\frac{\partial v^\varepsilon(x,\alpha)}{\partial x} + G(x,\alpha,u)\right\} + Q^\varepsilon v^\varepsilon(x,\cdot)(\alpha). \tag{8.6}$$

Remark 8.4. Note that the term $f(x,\alpha,u)(\partial v^\varepsilon(x,\alpha)/\partial x)$ in the HJB equation is meant to be the inner product of f and $(\partial v^\varepsilon/\partial x)$ and should have been written as $\langle f, (\partial v^\varepsilon/\partial x)\rangle$. Nevertheless, we write it as in (8.6) for the sake of notational simplicity. Similar notation will also be used in the subsequent development.

Let $m = m_1 + \cdots + m_l$ be the total number of states in $\mathcal{M}$. Denote

$$\Omega_H := \mathbb{R}^n \times \mathcal{M} \times \mathbb{R}^m \times \mathbb{R}^n,$$

and define a Hamiltonian H on Ω_H as

$$H(x,\alpha,v(x,\cdot),p) = \min_{u\in\Gamma}\left\{ f(x,\alpha,u)p + G(x,\alpha,u)\right\} + Q^\varepsilon v(x,\cdot)(\alpha) - \rho v(x,\alpha). \tag{8.7}$$

Then (8.6) can be written equivalently as

$$H\left(x,\alpha,v^\varepsilon(x,\cdot),\frac{\partial v^\varepsilon(x,\alpha)}{\partial x}\right) = 0. \tag{8.8}$$

In general, the partial derivatives of v^ε may not exist. To handle possible non-smoothness of v^ε, we use viscosity solutions developed by Crandall and Lions [39].

The proof of the next lemma is provided in Lemma A.28 for the convexity and Lipschitz property, and in Theorem A.30 for the uniqueness of the viscosity solution to (8.8).

Lemma 8.5. *Assume* (A8.1)–(A8.3).

(a) *There exists a constant K such that, for all x, $y \in \mathbb{R}^n$,*

$$|v^\varepsilon(x,\alpha) - v^\varepsilon(y,\alpha)| \le K(1+|x|^\kappa + |y|^\kappa)|x-y|.$$

Therefore the value function $v^\varepsilon(x,\alpha)$ is uniformly continuous in x.

(b) *$v^\varepsilon(x,\alpha)$ is the unique viscosity solution to the HJB equation* (8.8).

(c) *If $f(x,\alpha,u)$ is only a function of (α,u), then $v^\varepsilon(\cdot,\alpha)$ is convex.*

In view of the Lipschitz property of the value function in Lemma 8.5 and the Arzelà-Ascoli theorem, on any compact subset of $\mathbb{R}^n$, for each subsequence of ε, there exists a further subsequence of $\{\varepsilon \to 0\}$ (still denoted by ε) such that v^ε converges to a limit function v on that set. The next lemma shows that if this is so, the limit function depends only on k whenever $\alpha \in \mathcal{M}_k$ for $k = 1,\ldots,l$. That is, the aggregation of states in $\mathcal{M}_k$ gives the primary information, whereas the detailed variations within the group $\mathcal{M}_k$ are not as crucial, which reveals the hierarchical features of the decomposition/aggregation.

Lemma 8.6. *Assume* (A8.1)–(A8.3). *For each x, if for some subsequence of ε, $v^\varepsilon(x,\alpha) \to v(x,\alpha)$, for all $\alpha \in \mathcal{M}$, then $v(x,\alpha) = v(x,k)$ whenever $\alpha \in \mathcal{M}_k$ for some function $v(x,k)$ depending only on k.*

Proof: Let $\alpha = s_{i_0 j_0} \in \mathcal{M}_{i_0}$. Let τ^ε denote the first jump time of $\alpha^\varepsilon(t)$ for $t \geq 0$. Then $\tau^\varepsilon \to 0$ in probability as $\varepsilon \to 0$. Moreover, for any $u(\cdot) \in \mathcal{A}^\varepsilon$, the dynamic programming principle in Lemma A.29 yields

$$\begin{aligned} v^\varepsilon(x,\alpha) \leq\ & E\bigg(\int_0^{\tau^\varepsilon} e^{-\rho t} G(x^\varepsilon(t),\alpha^\varepsilon(t),u(t))dt \\ & + e^{-\rho\tau^\varepsilon} v^\varepsilon(x^\varepsilon(\tau^\varepsilon),\alpha^\varepsilon(\tau^\varepsilon))\bigg). \end{aligned} \tag{8.9}$$

In view of (A8.1) and (A8.2), we have

$$\begin{aligned} &|x^\varepsilon(t)| \leq K(1+t+|x|), \\ &|G(x^\varepsilon(t),\alpha^\varepsilon(t),u(t))| \leq K(1+t^\kappa+|x|^\kappa). \end{aligned} \tag{8.10}$$

It follows from the Lebesgue dominated convergence theorem that

$$\lim_{\varepsilon\to 0} E\int_0^{\tau^\varepsilon} e^{-\rho t} G(x^\varepsilon(t),\alpha^\varepsilon(t),u(t))dt = 0. \tag{8.11}$$

Moreover, the uniform Lipschitz property of $v^\varepsilon(x,\alpha)$ implies

$$\lim_{\varepsilon\to 0} E\,|v^\varepsilon(x^\varepsilon(\tau^\varepsilon),\alpha^\varepsilon(\tau^\varepsilon)) - v^\varepsilon(x,\alpha^\varepsilon(\tau^\varepsilon))| = 0. \tag{8.12}$$

Combine (8.9), (8.11), and (8.12) to obtain

$$v(x,\alpha) \leq \lim_{\varepsilon\to 0} E v^\varepsilon(x,\alpha^\varepsilon(\tau^\varepsilon)).$$

Note that

$$Ev^{\varepsilon}(x,\alpha^{\varepsilon}(\tau^{\varepsilon}))=\sum_{i=1}^{l}\sum_{j=1}^{m_i}v^{\varepsilon}(x,s_{ij})P(\alpha^{\varepsilon}(\tau^{\varepsilon})=s_{ij}).$$

Since $v^{\varepsilon}(x,\alpha)\to v(x,\alpha)$, we need to analyze the limit of $P(\alpha^{\varepsilon}(\tau^{\varepsilon})=s_{ij})$ as $\varepsilon\to 0$. To this end,

$$\begin{aligned}P(\alpha^{\varepsilon}(\tau^{\varepsilon})=s_{ij}) &= EI_{\{\alpha^{\varepsilon}(\tau^{\varepsilon})=s_{ij}\}}\\ &= E\left(E[I_{\{\alpha^{\varepsilon}(\tau^{\varepsilon})=s_{ij}\}}|\tau^{\varepsilon}]\right)\\ &= E\left(P(\alpha^{\varepsilon}(\tau^{\varepsilon})=s_{ij}|\tau^{\varepsilon})\right).\end{aligned}$$

Recall that $\widetilde{Q}^k=(\tilde{q}^k_{ij})$. If $s_{ij}\notin\mathcal{M}_{i_0}$, then the construction of Markov chain in Section 2.4 implies that

$$P(\alpha^{\varepsilon}(\tau^{\varepsilon})=s_{ij}|\tau^{\varepsilon})=O(\varepsilon)\to 0.$$

If $s_{ij}\in\mathcal{M}_{i_0}$, then $i=i_0$, and

$$P(\alpha^{\varepsilon}(\tau^{\varepsilon})=s_{i_0j}|\tau^{\varepsilon})=-\frac{\varepsilon^{-1}\tilde{q}^{i_0}_{j_0j}+\hat{q}_{s_{i_0j_0}s_{i_0j}}}{\varepsilon^{-1}\tilde{q}^{i_0}_{j_0j_0}+\hat{q}_{s_{i_0j_0}s_{i_0j_0}}}\to-\frac{\tilde{q}^{i_0}_{j_0j}}{\tilde{q}^{i_0}_{j_0j_0}}.$$

Hence,

$$v(x,s_{i_0j_0})\le\sum_{j\neq i_0}v(x,s_{i_0j})\left(-\frac{\tilde{q}^{i_0}_{j_0j}}{\tilde{q}^{i_0}_{j_0j_0}}\right).$$

Then the irreducibility of $\widetilde{Q}^{i_0}$ and Lemma A.39 imply that

$$v(x,s_{i_0j_0})=v(x,s_{i_0j})\text{ for all }j=1,\dots,m_{i_0}.\tag{8.13}$$

This completes the proof. □

Remark 8.7. If $v^{\varepsilon}(x,\alpha)$ is differentiable with respect to x, following the Lipschitz continuity of $v^{\varepsilon}(x,\alpha)$, the partial derivative $(\partial v^{\varepsilon}(x,\alpha)/\partial x)$ is uniformly bounded by $K(1+|x|^{\kappa})$. In this case, the proof of the lemma can be much simplified by using the method in the proof of Lemma 7.5. Actually, it can be shown that for $k=1,\dots,l$,

$$\widetilde{Q}^k\begin{pmatrix}v(x,s_{k1})\\ \vdots\\ v(x,s_{km_k})\end{pmatrix}\ge 0,$$

which implies (8.13).

Now define another control problem $\mathcal{P}^0$ with averaged dynamics weighted by the quasi-stationary distribution and cost function in which $\alpha^\varepsilon(t)$ is replaced by the "limit" of its aggregated process. To be more precise, let $\nu^k = (\nu_1^k, \ldots, \nu_{m_k}^k)$ denote the stationary distribution of $\widehat{Q}^k$. Recall that $\tilde{\mathbb{1}} = \text{diag}(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_l})$. Let

$$\overline{Q} = \text{diag}(\nu^1, \ldots, \nu^l)\widehat{Q}\tilde{\mathbb{1}}$$

and $\overline{\alpha}(t)$, $t \geq 0$, be a Markov chain generated by $\overline{Q}$. Define a control set for the limit problem as

$$\Gamma_0 = \Big\{(U^1, \ldots, U^l) : \ U^k = (u^{k1}, \ldots, u^{km_k}), \ u^{kj} \in \Gamma, \ \text{for } k = 1, \ldots, l\Big\}.$$

Consider a class of controls $\mathcal{A}^0$,

$$\mathcal{A}^0 := \ \Big\{U(t) = (U^1(t), \cdots, U^l(t)) \in \Gamma_0 : \\ U(t) \text{ is progressively measurable w.r.t. } \sigma\{\overline{\alpha}(s) : \ s \leq t\}\Big\}.$$

For any $U \in \Gamma_0$, define

$$\overline{f}(x, k, U) = \sum_{j=1}^{m_k} \nu_j^k f(x, s_{kj}, u^{kj}),$$
$$\overline{G}(x, k, U) = \sum_{j=1}^{m_k} \nu_j^k G(x, s_{kj}, u^{kj}).$$

The limit control problem is

$$\mathcal{P}^0 : \begin{cases} \text{minimize: } J^0(x, k, U(\cdot)) \\ \qquad = E\displaystyle\int_0^\infty e^{-\rho t}\overline{G}(\overline{x}(t), \overline{\alpha}(t), U(t))dt, \\ \text{subject to: } \dfrac{d\overline{x}(t)}{dt} = \overline{f}(\overline{x}(t), \overline{\alpha}(t), U(t)), \\ \qquad \overline{x}(0) = x, \ \overline{\alpha}(0) = k, \ U(\cdot) \in \mathcal{A}^0, \\ \text{value function: } v^0(x, k) = \displaystyle\inf_{U(\cdot) \in \mathcal{A}^0} J^0(x, k, U(\cdot)). \end{cases} \tag{8.14}$$

Note that the HJB equations associated with the original problem $\mathcal{P}^\varepsilon$ consist of m equations, whereas the HJB equations for the limit problem $\mathcal{P}^0$ contain only l equations. By using the aggregation/decomposition

approach, we are able to achieve substantial reduction of system dimensionality when m is large and l is small.

Now we are in a position to show that the sequence of value functions v^ε of $\mathcal{P}^\varepsilon$ converges to that of $\mathcal{P}^0$ as $\varepsilon \to 0$. Let

$$\Omega_{H^0} = \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^l \times \mathbb{R}^n.$$

For $\mathcal{P}^0$, define the Hamiltonian H^0 (a function on Ω_{H^0}) as

$$H^0(x,k,v(x,\cdot),p) = \min_{U\in\Gamma_0}\left\{\overline{f}(x,k,U)p + \overline{G}(x,k,U) + \overline{Q}v(x,\cdot)(k)\right\} - \rho v(x,k).$$

The HJB equation for $\mathcal{P}^0$ is

$$H^0\left(x,k,v(x,\cdot),\frac{\partial v(x,k)}{\partial x}\right) = 0. \tag{8.15}$$

Similar to Lemma 8.5, it can be shown that the value function $v^0(x,k)$ for $\mathcal{P}^0$ is locally Lipschitz in x, and is the unique viscosity solution to (8.15).

Next we show that the value function $v^\varepsilon(x,\alpha)$ converges to a limit $v(x,k)$ that satisfies the HJB equation (8.15). Then the uniqueness of the solution of (8.15) implies $v(x,k) = v^0(x,k)$. More precisely, the following theorem holds.

Theorem 8.8. *Assume* (A8.1)–(A8.3). *Then for all* $\alpha \in \mathcal{M}_k$ *and* $k = 1,\dots,l$,

$$\lim_{\varepsilon\to 0} v^\varepsilon(x,\alpha) = v^0(x,k).$$

Proof: By Lemma 8.5, for each sequence of $\{\varepsilon \to 0\}$, there exists a further subsequence (still indexed by ε) such that $v^\varepsilon(x,\alpha)$ converges. Denote the limit by $v(x,\alpha)$. Then by Lemma 8.6, $v(x,\alpha) = v(x,k)$; the exact value of α is unimportant and only k counts.

Fix $k = 1,\dots,l$. For any $\alpha = s_{kj} \in \mathcal{M}_k$, let $v(x,k)$ be a limit of $v^\varepsilon(x,s_{kj})$ for some subsequence of ε. In view of Lemma A.25, take a function $\phi(\cdot) \in C^1(\mathbb{R}^n)$ such that $v(x,k) - \phi(x)$ has a strictly local maximum at x_0 in a neighborhood $N(x_0)$. Choose $x_j^\varepsilon \in N(x_0)$ such that for each $\alpha = s_{kj} \in \mathcal{M}_k$,

$$v^\varepsilon(x_j^\varepsilon, s_{kj}) - \phi(x_j^\varepsilon) = \max_{x\in N(x_0)}\{v^\varepsilon(x,s_{kj}) - \phi(x)\}.$$

Then it follows that $x_j^\varepsilon \to x_0$ as $\varepsilon \to 0$. Moreover,

$$\sum_{j=1}^{m_k} \nu_j^k H\left(x_j^\varepsilon, s_{kj}, v^\varepsilon, \frac{\partial\phi}{\partial x}\right) \geq 0. \tag{8.16}$$

Observe that

$$\begin{aligned}
&\sum_{j=1}^{m_k} \nu_j^k \left(\widetilde{Q}^k v^\varepsilon(x_j^\varepsilon, \cdot)(\alpha)\right) \\
&\quad = \sum_{j=1}^{m_k} \sum_{i \neq j} \nu_j^k \tilde{q}_{ji}^k [v^\varepsilon(x_j^\varepsilon, s_{ki}) - v^\varepsilon(x_j^\varepsilon, s_{kj})] \\
&\quad \leq \sum_{j=1}^{m_k} \sum_{i \neq j} \nu_j^k \tilde{q}_{ji}^k [(v^\varepsilon(x_i^\varepsilon, s_{ki}) - \phi(x_i^\varepsilon)) - (v^\varepsilon(x_j^\varepsilon, s_{kj}) - \phi(x_j^\varepsilon))] \\
&\quad = \sum_{j=1}^{m_k} \sum_{i=1}^{m_k} \nu_j^k \tilde{q}_{ji}^k [v^\varepsilon(x_i^\varepsilon, s_{ki}) - \phi(x_i^\varepsilon)] \\
&\quad = \sum_{i=1}^{m_k} [v^\varepsilon(x_i^\varepsilon, s_{ki}) - \phi(x_i^\varepsilon)] \sum_{j=1}^{m_k} \nu_j^k \tilde{q}_{ji}^k = 0.
\end{aligned}$$

The above inequality follows from

$$v^\varepsilon(x_i^\varepsilon, s_{ki}) - \phi(x_i^\varepsilon) \geq v^\varepsilon(x_j^\varepsilon, s_{ki}) - \phi(x_j^\varepsilon).$$

Then (8.16) leads to the inequalities

$$\begin{aligned}
&\sum_{j=1}^{m_k} \nu_j^k H\left(x_0, s_{kj}, v, \frac{\partial \phi}{\partial x}\right) \\
&\qquad \geq \lim_{\varepsilon \to 0} \sum_{j=1}^{m_k} \nu_j^k H\left(x_j^\varepsilon, s_{kj}, v^\varepsilon, \frac{\partial \phi}{\partial x}\right) \geq 0.
\end{aligned}$$

Therefore, we have

$$H^0\left(x_0, k, v, \frac{\partial \phi}{\partial x}\right) = \sum_{j=1}^{m_k} \nu_j^k H\left(x_0, s_{kj}, v, \frac{\partial \phi}{\partial x}\right) \geq 0.$$

Thus $v(x, k)$ is a viscosity subsolution to (8.15).

Similarly, v is also a viscosity supersolution to (8.15). Moreover, the uniqueness of solution of (8.15) implies $v(x, k) = v^0(x, k)$, the value for $\mathcal{P}^0$. Thus, for any subsequence of ε (indexed also by ε), $v^\varepsilon(x, \alpha) \to v^0(x, k)$. The desired result thus follows. □

Remark 8.9. The linearity in u in (A8.1) is not essential for Theorem 8.8. In fact, the theorem holds for more general models. However, assuming $f(x, \alpha, u)$ to be linear in u enables us to derive the optimal feedback control law for the limit problem. One can then use such a control law as a guide to construct feedback controls for the original problem $\mathcal{P}^\varepsilon$. The Lipschitz property is needed to ensure the system has a unique solution in the viscosity sense.

8.4 Asymptotic Optimal Controls

Using the optimal control policy for the limit problem $\mathcal{P}^0$, we aim at deriving a feedback control policy for the original problem $\mathcal{P}^\varepsilon$. Moreover, we study the asymptotics of such a control policy as $\varepsilon \to 0$, and obtain the asymptotic optimality.

For $k = 1, \ldots, l$, write

$$U^*(x,k) = (u^{*1}(x,k), \ldots, u^{*m_k}(x,k)),$$

and denote the minimizer of the left-hand side of (8.15) by

$$U^*(x) = (U^*(x,1), \ldots, U^*(x,l)) \in \Gamma_0.$$

The condition below guarantees that $U^*(x)$ is locally Lipschitz.

(**A8.4**) The function $f(x,\alpha,u)$ depends on (α,u) only in that $f(x,\alpha,u) = f(\alpha,u)$ and the cost function $G(x,\alpha,u)$ is twice differentiable with respect to u such that

$$\frac{\partial^2 G(x,\alpha,u)}{\partial u^2} \geq c_0 I > 0,$$

for some constant c_0. There exists a constant K such that

$$\left| G(x+y,\alpha,u) - G(x,\alpha,u) - \left\langle \frac{\partial}{\partial x} G(x,\alpha,u), y \right\rangle \right| \leq K(1+|x|^\kappa)|y|^2.$$

Lemma 8.10. *Assume* (A8.1)–(A8.4). *Then*

(a) $v^0(x,k)$ *is convex and continuously differentiable.*

(b) $U^*(x)$ *is locally Lipschitz in that there exists a constant* K *such that*

$$|U^*(x) - U^*(y)| \leq K(1+|x|^\kappa + |y|^\kappa)|x-y|. \tag{8.17}$$

(c) $U^*(x)$ *is an optimal feedback control.*

Proof: The convexity of $v^0(x,k)$ is due to the convergence of $v^\varepsilon \to v^0$ and the convexity of $v^\varepsilon(x,\alpha)$ in Lemma 8.5.

The continuous differentiability of $v^0(x,k)$ can be obtained as in Sethi and Zhang [192, Lemma 5.6.1]. Moreover, the value function has Lipschitz partial derivatives in x. Then we apply Lemma A.32 to derive (b), which yields (c). □

For each $j = 1, \ldots, m_k$ and $k = 1, \ldots, l$, let $\tilde{f}_{kj}(x) = f(x, s_{kj}, u^{*j}(x, k))$. Then $\tilde{f}_{kj}(x)$ is locally Lipschitz. Therefore, under the optimal feedback control law $U^*(x)$ for $\mathcal{P}^0$, the following system

$$\frac{dx^*(t)}{dt} = \sum_{k=1}^{l} \sum_{j=1}^{m_k} \nu_j^k I_{\{\overline{\alpha}(t)=k\}} \tilde{f}_{kj}(x^*(t)), \ x^*(0) = x \tag{8.18}$$

has a unique solution.

Now construct a feedback control $u(x, \alpha)$ for $\mathcal{P}^\varepsilon$ by

$$u(x, \alpha) = \sum_{k=1}^{l} \sum_{j=1}^{m_k} I_{\{\alpha=s_{kj}\}} u^{*j}(x, k). \tag{8.19}$$

Let $u^\varepsilon(t) = u(x^\varepsilon(t), \alpha^\varepsilon(t))$, $t \geq 0$. It is easily seen that the system equation for $\mathcal{P}^\varepsilon$ given by

$$\frac{dx^\varepsilon(t)}{dt} = \sum_{k=1}^{l} \sum_{j=1}^{m_k} I_{\{\alpha^\varepsilon(t)=s_{kj}\}} \tilde{f}_{kj}(x^\varepsilon(t)), \ x^\varepsilon(0) = x$$

has a unique solution $x^\varepsilon(t)$. Next we show that $u^\varepsilon(\cdot)$ is asymptotically optimal in the sense that

$$|J^\varepsilon(x, \alpha, u^\varepsilon(\cdot)) - v^\varepsilon(x, \alpha)| \to 0.$$

This step is realized by introducing an auxiliary process. Let $\overline{x}^\varepsilon(t)$ denote an intermediate process defined by

$$\frac{d\overline{x}^\varepsilon(t)}{dt} = \sum_{k=1}^{l} \sum_{j=1}^{m_k} \nu_j^k I_{\{\overline{\alpha}^\varepsilon(t)=k\}} \tilde{f}_{kj}(\overline{x}^\varepsilon(t)), \ \overline{x}^\varepsilon(0) = x,$$

where $\overline{\alpha}^\varepsilon(\cdot)$ is the aggregated process of $\alpha^\varepsilon(\cdot)$ defined in Section 5.3.2. By comparing $x^\varepsilon(\cdot)$ with $\overline{x}^\varepsilon(\cdot)$, we establish an estimate of $|x^\varepsilon(t) - \overline{x}^\varepsilon(t)|$. Note that

$$\begin{aligned} x^\varepsilon(t) - \overline{x}^\varepsilon(t) = & \sum_{k=1}^{l} \sum_{j=1}^{m_k} \int_0^t [\tilde{f}_{kj}(x^\varepsilon(s)) - \tilde{f}_{kj}(\overline{x}^\varepsilon(s))] I_{\{\alpha^\varepsilon(s)=s_{kj}\}} ds \\ & + \sum_{k=1}^{l} \sum_{j=1}^{m_k} \int_0^t \left(I_{\{\alpha^\varepsilon(s)=s_{kj}\}} - \nu_j^k I_{\{\overline{\alpha}^\varepsilon(s)=k\}} \right) \tilde{f}_{kj}(\overline{x}^\varepsilon(s)) ds. \end{aligned}$$

In view of the Lipschitz property of $\tilde{f}_{kj}(x)$ and (8.10),

$$\begin{aligned} E|x^\varepsilon(t) - \overline{x}^\varepsilon(t)| \leq & \quad K \int_0^t (1 + t^\kappa + |x|^\kappa) E|x^\varepsilon(s) - \overline{x}^\varepsilon(s)| ds \\ & + \sum_{k=1}^{l} \sum_{j=1}^{m_k} E \left| \int_0^t \left(I_{\{\alpha^\varepsilon(s)=s_{kj}\}} - \nu_j^k I_{\{\overline{\alpha}^\varepsilon(s)=k\}} \right) \tilde{f}_{kj}(\overline{x}^\varepsilon(s)) ds \right|. \end{aligned}$$

Let $\tilde{f}(s) = \tilde{f}_{kj}(\overline{x}^\varepsilon(s))$. Then $\tilde{f}(s)$ is locally Lipschitz in s. As a result, $\tilde{f}(s)$ is differentiable almost everywhere with bounded derivative. Integration by parts yields

$$
\begin{aligned}
&E\left|\int_0^t \left(I_{\{\alpha^\varepsilon(s)=s_{kj}\}} - \nu_j^k I_{\{\overline{\alpha}^\varepsilon(s)=k\}}\right)\tilde{f}(s)ds\right| \\
&\quad \le E\left|\tilde{f}(t)\int_0^t \left(I_{\{\alpha^\varepsilon(s)=s_{kj}\}} - \nu_j^k I_{\{\overline{\alpha}^\varepsilon(s)=k\}}\right)ds\right| \\
&\qquad + E\left|\int_0^t \left(\int_0^s \left(I_{\{\alpha^\varepsilon(\tau)=s_{kj}\}} - \nu_j^k I_{\{\overline{\alpha}^\varepsilon(\tau)=k\}}\right)d\tau\right)\left(\frac{d\tilde{f}(s)}{ds}\right)ds\right|.
\end{aligned}
$$

As $\varepsilon \to 0$, by virtue of Theorem 5.25,

$$
E\left|\int_0^t \left(I_{\{\alpha^\varepsilon(s)=s_{kj}\}} - \nu_j^k I_{\{\overline{\alpha}^\varepsilon(s)=k\}}\right)\tilde{f}(s)ds\right| \to 0.
$$

An application of Gronwall's inequality yields

$$
E|x^\varepsilon(t) - \overline{x}^\varepsilon(t)| \to 0 \text{ as } \varepsilon \to 0. \tag{8.20}
$$

We next show $E|\overline{x}^\varepsilon(t) - x^*(t)| \to 0$, as $\varepsilon \to 0$. For each $T > 0$, let

$$
\overline{\alpha}_T^\varepsilon(\cdot) = \{\overline{\alpha}^\varepsilon(t) :\ t \le T\} \text{ and } \overline{\alpha}_T(\cdot) = \{\overline{\alpha}(t) :\ t \le T\}.
$$

Following from Theorem 5.27, one has

$$
\overline{\alpha}_T^\varepsilon(\cdot) \to \overline{\alpha}_T(\cdot) \text{ in distribution.}
$$

In view of the Skorohod representation (see Theorem A.11) without changing notation, we may assume $\overline{\alpha}_T^\varepsilon(\cdot) \to \overline{\alpha}_T(\cdot)$ w.p.1. Therefore, there exists a measurable set Ω_1 with $P(\Omega_1) = 1$ such that

$$
d(\overline{\alpha}_T^\varepsilon(\cdot)(\omega), \overline{\alpha}_T(\cdot)(\omega)) \to 0 \text{ for all } \omega \in \Omega_1, \text{ as } \varepsilon \to 0,
$$

where $d(\cdot,\cdot)$ is the distance under Skorohod topology on $D[0,T]$.

Let

$$
\Omega_2 = \Big\{\overline{\alpha}_T(\cdot) \text{ jumps at most countably many times in } [0,T]\Big\}.
$$

Then $P(\Omega_2) = 1$. Let $\Omega_0 = \Omega_1 \cap \Omega_2$. Then $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$,

$$
|\overline{\alpha}_T^\varepsilon(\cdot)(\omega) - \overline{\alpha}_T(\cdot)(\omega)| \to 0
$$

for all but at most countably many $t \in [0,T]$. Thus for all $\omega \in \Omega_0$,

$$
I_{\{\overline{\alpha}_T^\varepsilon(t)(\omega)=k\}} \to I_{\{\overline{\alpha}_T(t)(\omega)=k\}}
$$

for all but at most countably many $t \in [0,T]$. It follows that for $\varepsilon \to 0$,

$$\overline{x}^{\varepsilon}(t) \to x^*(t) \text{ w.p.1.} \tag{8.21}$$

Thus, for all $t \in [0,T]$, (8.21) and the dominated convergence theorem imply that $E|\overline{x}^{\varepsilon}(t) - x^*(t)| \to 0$ as $\varepsilon \to 0$. Finally, by virtue of (8.20) and the above estimates, as $\varepsilon \to 0$,

$$E|x^{\varepsilon}(t) - x^*(t)| \le E|x^{\varepsilon}(t) - \overline{x}^{\varepsilon}(t)| + E|\overline{x}^{\varepsilon}(t) - x^*(t)| \to 0. \tag{8.22}$$

Since T is arbitrary, (8.22) holds for all $t \ge 0$. □

Theorem 8.11. *Assume* (A8.1)–(A8.4). *Then*

$$u^{\varepsilon}(t) = u(x^{\varepsilon}(t), \alpha^{\varepsilon}(t))$$

constructed in (8.19) *is asymptotically optimal, i.e.,*

$$\lim_{\varepsilon \to 0} |J^{\varepsilon}(x, \alpha, u^{\varepsilon}(\cdot)) - v^{\varepsilon}(x, \alpha)| = 0.$$

Proof: For $\alpha \in \mathcal{M}_k$, $k = 1, \ldots, l$,

$$\begin{aligned} 0 \le\ & J^{\varepsilon}(x, \alpha, u^{\varepsilon}(\cdot)) - v^{\varepsilon}(x, \alpha) \\ =\ & (J^{\varepsilon}(x, \alpha, u^{\varepsilon}(\cdot)) - v^0(x, k)) + (v^0(x, k) - v^{\varepsilon}(x, \alpha)). \end{aligned}$$

In view of Theorem 8.8, we have

$$\left|v^{\varepsilon}(x, \alpha) - v^0(x, k)\right| \to 0.$$

Thus to establish the assertion, it suffices to show that

$$\left|J^{\varepsilon}(x, \alpha, u^{\varepsilon}(\cdot)) - v^0(x, k)\right| \to 0.$$

Let

$$\widetilde{G}_{kj}(x) = G(x, s_{kj}, u^{*j}(x, k)) \quad \text{and} \quad U^*(t) = U^*(x^*(t)).$$

Then

$$\begin{aligned} v^0(x, k) =\ & J^0(x, k, U^*(\cdot)) \\ =\ & E \int_0^{\infty} e^{-\rho t} \sum_{k=1}^{l} \sum_{j=1}^{m_k} \nu_j^k I_{\{\overline{\alpha}(t)=k\}} \widetilde{G}_{kj}(x^*(t)) dt, \end{aligned}$$

and

$$J^{\varepsilon}(x, \alpha, u^{\varepsilon}(\cdot)) = E \int_0^{\infty} e^{-\rho t} \sum_{k=1}^{l} \sum_{j=1}^{m_k} I_{\{\alpha^{\varepsilon}(t)=s_{kj}\}} \widetilde{G}_{kj}(x^{\varepsilon}(t)) dt.$$

It follows that

$$
\begin{aligned}
&J^\varepsilon(x,\alpha,u^\varepsilon(\cdot)) - v^0(x,k) \\
&\quad = E\int_0^\infty e^{-\rho t}\sum_{k=1}^{l}\sum_{j=1}^{m_k}\left(\widetilde{G}_{kj}(x^\varepsilon(t)) - \widetilde{G}_{kj}(\overline{x}^\varepsilon(t))\right) I_{\{\alpha^\varepsilon(t)=s_{kj}\}}dt \\
&\qquad + E\int_0^\infty e^{-\rho t}\sum_{k=1}^{l}\sum_{j=1}^{m_k}\left(\widetilde{G}_{kj}(\overline{x}^\varepsilon(t)) - \widetilde{G}_{kj}(x^*(t))\right) I_{\{\alpha^\varepsilon(t)=s_{kj}\}}dt \\
&\qquad + E\int_0^\infty e^{-\rho t}\sum_{k=1}^{l}\sum_{j=1}^{m_k}\widetilde{G}_{kj}(x^*(t))\nu_j^k\left(I_{\{\overline{\alpha}^\varepsilon(t)=k\}} - I_{\{\overline{\alpha}(t)=k\}}\right)dt \\
&\qquad + E\int_0^\infty e^{-\rho t}\sum_{k=1}^{l}\sum_{j=1}^{m_k}\widetilde{G}_{kj}(x^*(t))\left(I_{\{\alpha^\varepsilon(t)=s_{kj}\}} - \nu_j^k I_{\{\overline{\alpha}^\varepsilon(t)=k\}}\right)dt.
\end{aligned}
$$

The terms in the integrands of line 2 through line 4 above go to 0. In view of the Lebesgue dominated convergence theorem, the boundedness and the locally Lipschitz property of $\widetilde{G}_{kj}$, these terms go to zero. It can be shown by integration by parts as in the proof of (8.20) the last line also goes to zero. Therefore,

$$\left|J^\varepsilon(x,\alpha,u^\varepsilon(\cdot)) - v^0(x,k)\right| \to 0 \tag{8.23}$$

Hence,

$$|J^\varepsilon(x,\alpha,u^\varepsilon(\cdot)) - v^\varepsilon(x,\alpha)| \to 0$$

when $\varepsilon \to 0$ as desired. □

8.5 Convergence Rate

This section takes up the issue of rate of convergence of the sequence of value functions $v^\varepsilon(x,\alpha)$, which indicates how well $v^0(x,k)$ approximates $v^\varepsilon(x,\alpha)$. Owing to the complexity arising from the weak and strong interactions in the singularly perturbed Markov chain, the convergence rate of $v^\varepsilon(x,\alpha)$ is generally difficult to obtain. In this section, we confine ourselves to the case in which $\widetilde{Q}$ is irreducible, namely, $\widetilde{Q}$ consisting of a single irreducible block. Let $\mathcal{M} = \{1,\ldots,m\}$. Then $\Gamma_0 = \{U = (u^1,\ldots,u^m) :\ u^j \in \Gamma\}$. We impose additional assumptions below.

(**A8.5**) $\widetilde{Q}$ is irreducible.

(**A8.6**) There exist bounded functions $B_1(\alpha), B_2(\alpha)$ such that

$$f(x,\alpha,u) = B_1(\alpha)u + B_2(\alpha).$$

Moreover, there exist functions $G_1(x)$, $G_2(\alpha,u)$, that are convex in x and u, respectively, such that

$$G(x,\alpha,u) = G_1(x) + G_2(\alpha,u).$$

Under these additional assumptions, we will be able to estimate the convergence rate of the value function $v^\varepsilon(x,\alpha)$ to $v^0(x)$. Here $v^0(x)$ depends only on x because $\widetilde{Q}$ is irreducible.

Theorem 8.12. *Assume* (A8.1)–(A8.6). *Then there are positive constants* K *and* κ *such that*

$$|v^\varepsilon(x,\alpha) - v^0(x)| \le K(1+|x|^\kappa)\sqrt{\varepsilon}.$$

Proof: The essence is to compare the trajectories of $x^\varepsilon(\cdot)$ and that of the limit system. Let

$$U(t) = (u^1(t), \dots, u^m(t)) \in \mathcal{A}^0,$$

and define a control $u^\varepsilon(t)$ by

$$u^\varepsilon(t) = \sum_{j=1}^m I_{\{\alpha^\varepsilon(t)=j\}} u^j(t).$$

Clearly $u^\varepsilon(t) \in \mathcal{A}^\varepsilon$. Let $x^\varepsilon(t)$ and $\overline{x}(t)$ be the trajectories of systems $\mathcal{P}^\varepsilon$ and $\mathcal{P}^0$ under the controls $u^\varepsilon(t)$ and $u(t)$, respectively. Then

$$\begin{aligned}
\frac{dx^\varepsilon(t)}{dt} &= B_1(\alpha^\varepsilon(t))u^\varepsilon(t) + B_2(\alpha^\varepsilon(t)),\ x^\varepsilon(0) = x,\\
\frac{d\overline{x}(t)}{dt} &= \sum_{j=1}^m \nu_j(B_1(j)u^j(t) + B_2(j)),\ \overline{x}(0) = x.
\end{aligned}$$

Take the difference of the two equations above to obtain

$$\begin{aligned}
\frac{d}{dt}(x^\varepsilon(t) - \overline{x}(t)) = &\sum_{j=1}^m B_1(j)u^j(t)\left(I_{\{\alpha^\varepsilon(t)=j\}} - \nu_j\right)\\
&+ \sum_{j=1}^m B_2(j)\left(I_{\{\alpha^\varepsilon(t)=j\}} - \nu_j\right).
\end{aligned} \tag{8.24}$$

It follows from Corollary 5.21 that

$$E|x^\varepsilon(t) - \overline{x}(t)|^2 = O(\varepsilon t^2).$$

In addition, using the asymptotic expansion in Theorem 4.5, we have

$$\begin{aligned}
&EG_2(\alpha^\varepsilon(t), u^\varepsilon(t)) \\
&= E\sum_{j=1}^{m} G_2(j, u^j(t)) I_{\{\alpha^\varepsilon(t)=j\}} \\
&= \sum_{j=1}^{m} G_2(j, u^j(t)) P(\alpha^\varepsilon(t) = j) \\
&= \sum_{j=1}^{m} \nu_j G_2(j, u^j(t)) + \sum_{j=1}^{m} G_2(j, u^j(t))(P(\alpha^\varepsilon(t) = j) - \nu_j) \\
&= \sum_{j=1}^{m} \nu_j G_2(j, u^j(t)) + O\left(\varepsilon + \exp\left(-\frac{\kappa_0 t}{\varepsilon}\right)\right).
\end{aligned}$$

Then we have

$$\begin{aligned}
J^\varepsilon(x, \alpha, u^\varepsilon(\cdot)) = &\ E\int_0^\infty e^{-\rho t}(G_1(x^\varepsilon(t)) + G_2(\alpha^\varepsilon(t), u^\varepsilon(t)))dt \\
\le &\ E\int_0^\infty (G_1(\overline{x}(t)) + K(1 + t^\kappa + |x|^\kappa)|x^\varepsilon(t) - \overline{x}(t)|)dt \\
&+ E\int_0^\infty e^{-\rho t}\sum_{j=1}^{m} \nu_j G_2(j, u^j(t)) + K\sqrt{\varepsilon} \\
= &\ J^0(x, U(\cdot)) + K(1 + |x|^\kappa)\sqrt{\varepsilon}.
\end{aligned}$$

Recall that $U(\cdot) \in \mathcal{A}^0$ is arbitrary. It follows that

$$v^\varepsilon(x, \alpha) - v^0(x) \le K(1 + |x|^\kappa)\sqrt{\varepsilon}. \tag{8.25}$$

To obtain the convergence rate, it suffices to establish the reverse inequality. To this end, we show that for any control $u(t) \in \mathcal{A}^\varepsilon$, there exists a control $U(t) \in \mathcal{A}^0$ such that the corresponding system states under these controls are "close." In fact, for each $j \in \mathcal{M}$, let

$$u^j(t) = E[u^\varepsilon(t)|\alpha^\varepsilon(t) = j].$$

Clearly $U(t) = (u^1(t), \ldots, u^m(t)) \in \mathcal{A}^0$.

Let $x^\varepsilon(t)$ and $\overline{x}(t)$ denote the system states for $\mathcal{P}^\varepsilon$ and $\mathcal{P}^0$ under controls $u^\varepsilon(t) \in \mathcal{A}^\varepsilon$ and $U(t) \in \mathcal{A}^0$, respectively. Then

$$\begin{aligned} Ex^\varepsilon(t) - \overline{x}(t) &= \int_0^t \sum_{j=1}^m (B_1(j)u^j(s) + B_2(j))(P(\alpha^\varepsilon(s) = j) - \nu_j)ds \\ &= \int_0^t \sum_{j=1}^m (B_1(j)u^j(s) + B_2(j))O\left(\varepsilon + \exp\left(-\frac{\kappa_0 s}{\varepsilon}\right)\right)ds \\ &= O(\varepsilon). \end{aligned}$$

By the convexity, the local Lipschitz continuity and the *a priori* estimates of $\overline{x}(t)$ and $x^\varepsilon(t)$,

$$\begin{aligned} EG_1(x^\varepsilon(t)) &\geq G_1(Ex^\varepsilon(t)) \\ &\geq G_1(\overline{x}(t)) - K(1 + |\overline{x}(t)|^\kappa + |Ex^\varepsilon(t)|^\kappa)|Ex^\varepsilon(t) - \overline{x}(t)| \\ &= G_1(\overline{x}(t)) + (1 + |x|^\kappa)O(\varepsilon). \end{aligned}$$

Also, in view of the convexity of $G_2(\cdot)$, we have

$$\begin{aligned} EG_2(\alpha^\varepsilon(t), u^\varepsilon(t)) &= \sum_{j=1}^m E[G_2(j, u^\varepsilon(t))|\alpha^\varepsilon(t) = j]P(\alpha^\varepsilon(t) = j) \\ &\geq \sum_{j=1}^m G_2(E[u^\varepsilon(t)|\alpha^\varepsilon(t) = j], j)P(\alpha^\varepsilon(t) = j) \\ &= \sum_{j=1}^m G_2(j, u^j(t))\nu_j + O\left(\varepsilon + \exp\left(-\frac{\kappa_0 t}{\varepsilon}\right)\right). \end{aligned}$$

Then for any $u^\varepsilon(t) \in \mathcal{A}^\varepsilon$, it follows that

$$J^\varepsilon(x, \alpha, u^\varepsilon(\cdot)) \geq J^0(x, U(\cdot)) - K(1 + |x|^\kappa)\varepsilon.$$

Thus, we obtain

$$v^\varepsilon(x, \alpha) - v^0(x) \geq -K(1 + |x|^\kappa)\varepsilon.$$

The proof of Theorem 8.12 is completed. □

Remark 8.13. Theorem 8.12 indicates that the convergence rate of v^ε to v^0 is of the order $\sqrt{\varepsilon}$. In fact, this is the best rate possible. In Sethi and

Zhang [192], it was shown, by a one-dimensional example, that for any $\alpha \in \mathcal{M}$ and $x > 0$, there exists $c_1 > 0$ such that

$$\frac{|v^\varepsilon(x,\alpha) - v^0(x)|}{\sqrt{\varepsilon}} \geq c_1 > 0.$$

This means that the order $\sqrt{\varepsilon}$ cannot be improved.

We now establish the approximation error bound via the constructed controls. Let $U^*(t) = (u^{*1}(t), \ldots, u^{*m}(t)) \in \mathcal{A}^0$ denote an open-loop optimal control for $\mathcal{P}^0$. Using $U^*(t)$, construct a control

$$u^\varepsilon(t) = \sum_{j=1}^{m} I_{\{\alpha^\varepsilon(t)=j\}} u^{*j}(t)$$

for $\mathcal{P}^\varepsilon$. Then similar to the proof of Theorem 8.12,

$$|J^\varepsilon(x,\alpha,u^\varepsilon(\cdot)) - v^\varepsilon(x,\alpha)| \leq K(1+|x|^\kappa)\sqrt{\varepsilon}.$$

Such an error bound is good for open-loop controls. For a feedback control $U^*(x)$, the constructed control $u^\varepsilon(x,\alpha)$ provides an asymptotically optimal control for $\mathcal{P}^\varepsilon$, but so far the corresponding error bound has only been obtained under additional conditions such as ρ being large enough. We refer to Sethi and Zhang [192, Chapter 5] for further details.

Example 8.14 (Cont.). We continue our study of Example 8.3. Consider the case in which the demand fluctuates more rapidly than the capacity process. In this case, $z^\varepsilon(t)$ is the fast changing process, and $c^\varepsilon(t) = c(t)$ is the slowly varying capacity process being independent of ε. The idea is to derive a limit problem in which the fast fluctuating demand is replaced by its average. Thus one may ignore the detailed changes in the demand when making an average production planning decision.

Let

$$\mathcal{M} = \{s_{11}, s_{12}, s_{21}, s_{22}\} = \{(1,z_1),(1,z_2),(0,z_1),(0,z_2)\}.$$

Consider the generator Q^ε given by

$$Q^\varepsilon = \frac{1}{\varepsilon}\begin{pmatrix} -\lambda_z & \lambda_z & 0 & 0 \\ \mu_z & -\mu_z & 0 & 0 \\ 0 & 0 & -\lambda_z & \lambda_z \\ 0 & 0 & \mu_z & -\mu_z \end{pmatrix} + \begin{pmatrix} -\lambda_c & 0 & \lambda_c & 0 \\ 0 & -\lambda_c & 0 & \lambda_c \\ \mu_c & 0 & -\mu_c & 0 \\ 0 & \mu_c & 0 & -\mu_c \end{pmatrix},$$

where λ_z is the jump rate of the demand from z_1 to z_2 and μ_z is the rate from z_2 to z_1; λ_c and μ_c are the breakdown and repair rates, respectively.

In this example,

$$\overline{Q} = \begin{pmatrix} -\lambda_c & \lambda_c \\ \mu_c & -\mu_c \end{pmatrix}.$$

It follows that $\overline{\alpha}(t) = c(t)$. Moreover, the control set

$$\Gamma_0 = \{(u^{11}, u^{12}, 0, 0) : 0 \le u^{11}, u^{12} \le 1\},$$

since when $c(t) = 0$ the system is independent of the values of u^{21} and u^{22}. Furthermore, since $G(\cdot)$ is independent of u, we have $\overline{G}(\cdot) = G(\cdot)$. Therefore, the system equation in the limit problem $\mathcal{P}^0$ is given by

$$\frac{dx(t)}{dt} = c(t)u(t) - \overline{z},\ x(0) = x,$$

where $u = \nu_1^1 z_1 + \nu_2^1 z_2$,

$$\nu^1 = (\nu_1^1, \nu_2^1) = \left(\frac{\mu_z}{\lambda_z + \mu_z}, \frac{\lambda_z}{\lambda_z + \mu_z} \right),$$

and $\overline{z} = \nu_1^1 z_1 + \nu_2^1 z_2$. Recall that $0 < z_1 < z_2 < 1$. It follows that $\overline{z}$ is less than 1. Then Theorem 8.8 implies that $v^\varepsilon(x, \alpha) \to v^0(x, k)$, for $\alpha \in \mathcal{M}_k$.

Let

$$A_1 = \begin{pmatrix} -\dfrac{\rho + \mu_c}{\overline{z}} & \dfrac{\mu_c}{\overline{z}} \\ -\dfrac{\lambda_c}{1 - \overline{z}} & \dfrac{\rho + \lambda_c}{1 - \overline{z}} \end{pmatrix}.$$

It is easy to see that A_1 has two real eigenvalues, one greater than 0 and the other less than 0. Let $a_- < 0$ denote the negative eigenvalue of the matrix A_1 and define

$$x^* = \max\left(0, \frac{1}{a_-} \log\left[\frac{c^+}{c^+ + c^-}\left(1 + \frac{\rho\overline{z}}{\lambda_c\overline{z} - (\rho + \mu_c + \overline{z}a_-)(1 - \overline{z})}\right)\right]\right).$$

The optimal control for $\mathcal{P}^0$ is given by

$$\begin{aligned} &\text{If } c(t) = 0,\ u^*(x) = 0, \text{ and} \\ &\text{if } c(t) = 1,\ u^*(x) = \begin{cases} 0, & \text{if } x > x^*, \\ \overline{z}, & \text{if } x = x^*, \\ 1, & \text{if } x < x^*. \end{cases} \end{aligned}$$

Let

$$U^*(x) = (u^{*11}(x), u^{*12}(x), u^{*21}(x), u^{*22}(x))$$

denote the optimal control for $\mathcal{P}^0$. Note that $(u^{*11}(x), u^{*12}(x))$ corresponds to $c(t) = 1$ and $(u^{*21}(x), u^{*22}(x))$ corresponds to $c(t) = 0$. Naturally,

$(u^{*21}(x), u^{*22}(x)) = 0$, since, when $c(t) = 0$, there should be no production. When $c(t) = 1$, let $\nu_1^1 u^{*11}(x) + \nu_2^1 u^{*12}(x) = u^*(x)$. It should be pointed out that in this case the solution $(u^{*11}(x), u^{*12}(x))$ is not unique.

Using $u^{*11}(x)$ and $u^{*12}(x)$, we construct a control for $\mathcal{P}^0$ as

$$\begin{aligned} u^\varepsilon(x,c,z) = {} & I_{\{c=1\}} \left(I_{\{z=z_1\}} u^{*11}(x) + I_{\{z=z_2\}} u^{*12}(x) \right) \\ & + I_{\{c=0\}} \left(I_{\{z=z_1\}} u^{*21}(x) + I_{\{z=z_2\}} u^{*22}(x) \right) \\ = {} & I_{\{c=1\}} \left(I_{\{z=z_1\}} u^{*11}(x) + I_{\{z=z_2\}} u^{*12}(x) \right). \end{aligned}$$

Note that in this example, the optimal control $U^*(x)$ is not Lipschitz. Therefore the conditions in Theorem 8.11 are not satisfied. However, we can still show, as in Sethi and Zhang [192, Chapter 5], that

$$u^\varepsilon(t) = u^\varepsilon(x(t), c(t), z^\varepsilon(t))$$

is asymptotically optimal.

One may also consider the case in which the capacity process changes rapidly, whereas the random demand is relatively slowly varying. Similar to the previous case, assume $c^\varepsilon(\cdot)$ is the capacity process and $z^\varepsilon(\cdot) = z(\cdot)$ is the demand. Using exactly the same approach, one may resolve this problem. The discussion is analogous to the previous case; the details are omitted.

8.6 Weak Convergence Approach

Treating asymptotic optimality of systems involving singularly perturbed Markov chains, the previous sections all focus on the dynamical programming methods. We switch gears and use a weak convergence approach to study nearly optimal control problems in this section. As in the previous sections, the main idea is that in lieu of dealing with the original complex system, consider its corresponding limit and apply the optimal or nearly optimal controls of the limit problem to that of the original problem. The goal is to show that such a procedure leads to near optimality of the original problems. The weak convergence approach in conjunction with the relaxed control representation allows us to use weaker conditions. Under the relaxed control formulation, the system is linear in the control variable, and the compactness is also easily obtainable. To maintain the continuity of presentation, definition of relaxed controls and the notation of weak convergence are relegated to Section A.3 in Appendix.

8.6.1 Problem Setup

Let $\alpha^\varepsilon(t)$ be a singularly perturbed Markov chain with a finite-state space $\mathcal{M}$ generated by $Q^\varepsilon(t)$, which is allowed to be a function of t. Consider a stochastic dynamical system with the state $x^\varepsilon(t) \in \mathbb{R}^n$, and control $u(t) \in \Gamma \subset \mathbb{R}^{n_1}$. Let

$$f(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathcal{M} \times \mathbb{R}^{n_1} \mapsto \mathbb{R}^n,$$

$$G(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathcal{M} \times \mathbb{R}^{n_1} \mapsto \mathbb{R}.$$

The problem of interest is

$$\begin{aligned} &\text{minimize: } J^\varepsilon(u(\cdot)) = E\int_0^T G(x^\varepsilon(t), \alpha^\varepsilon(t), u(t))dt, \\ &\text{subject to: } \frac{dx^\varepsilon(t)}{dt} = f(x^\varepsilon(t), \alpha^\varepsilon(t), u(t)),\ x^\varepsilon(0) = x. \end{aligned} \tag{8.26}$$

As in the previous section, $f(x,\alpha,u)$ represents the dynamics of the system and $G(x,\alpha,u)$ is the running cost. The expected cost is evaluated over a finite horizon. Such formulation allows one to treat nonstationary systems. Note that $Q^\varepsilon(t)$ is time dependent, and so is $\overline{Q}(t)$, the generator of the limit of $\overline{\alpha}^\varepsilon(t)$. Using the same approach as this section with modifications, infinite horizon problems may also be incorporated into our formulation. To proceed, let us set up the problem by using the relaxed control representation.

8.6.2 Relaxed Control Formulation

Let $\widetilde{m}(\cdot)$ be a relaxed control representation (see Section A.3) for the control $u(\cdot)$ in (8.26). Rewrite (8.26) as

$$\mathcal{P}^\varepsilon : \begin{cases} \text{minimize: } J^\varepsilon(\widetilde{m}^\varepsilon) = E\int_0^T\int G(x^\varepsilon(t), \alpha^\varepsilon(t), \varrho)\widetilde{m}_t^\varepsilon(d\varrho)dt, \\ \text{subject to: } \dfrac{dx^\varepsilon(t)}{dt} = \displaystyle\int f(x^\varepsilon(t), \alpha^\varepsilon(t), \varrho)\widetilde{m}_t^\varepsilon(d\varrho),\ x^\varepsilon(0) = x, \end{cases} \tag{8.27}$$

where $\widetilde{m}_t$ is the "derivative" of $\widetilde{m}$ with respect to t (see Section A.3 for definition and discussions). The cost function and system equation can be written as

$$\begin{aligned} J^\varepsilon(\widetilde{m}^\varepsilon) &= E\left(\sum_{i=1}^{l}\sum_{j=1}^{m_i}\int_0^T\int G(x^\varepsilon(t), s_{ij}, \varrho)\widetilde{m}_t^\varepsilon(\varrho)dt\right), \\ x^\varepsilon(t) &= x + \sum_{i=1}^{l}\sum_{j=1}^{m_i}\int_0^t\int f(x^\varepsilon(s), s_{ij}, \varrho)m_s^\varepsilon(d\varrho)I_{\{\alpha^\varepsilon(s)=s_{ij}\}}ds. \end{aligned}$$

The integral equation for the dynamical system is convenient in the analysis to follow.

8.6.3 Near Optimality

This subsection is devoted to the study of $\mathcal{P}^\varepsilon$ in (8.27). It begins with a set of assumptions, proceeds with two preparatory lemmas, and ends with the theorems on weak convergence and near optimality.

Denote the σ-algebra of Borel subsets of any set S by $\mathcal{B}(S)$. Let

$$\mathbb{M} = \Big\{\widetilde{m}(\cdot);\ \widetilde{m}(\cdot) \text{ is a measure on } \mathcal{B}(\Gamma \times [0,\infty)) \text{ satisfying } \widetilde{m}(\Gamma \times [0,t]) = t \text{ for all } t \geq 0\Big\}.$$

Given a filtration $\mathcal{F}_t$. A random $\mathbb{M}$-valued measure $\widetilde{m}(\cdot)$ is an admissible relaxed control if for each $B \in \mathcal{B}(\Gamma)$, the function defined by

$$\widetilde{m}(B,t) \equiv \widetilde{m}(B \times [0,t]) \text{ is } \mathcal{F}_t - \text{adapted}.$$

A relaxed control representation $\widetilde{m}^\varepsilon(\cdot)$ is admissible for $\mathcal{P}^\varepsilon$ if $\widetilde{m}^\varepsilon(\cdot) \in \mathbb{M}$ and is $\mathcal{F}_t^\varepsilon = \sigma\{x^\varepsilon(s), \alpha^\varepsilon(s) :\ s \leq t\}$ adapted.

Use $\mathcal{R}^\varepsilon$ to denote the set of all admissible controls, i.e.,

$$\mathcal{R}^\varepsilon = \{\widetilde{m}^\varepsilon(\cdot) \in \mathbb{M};\ \widetilde{m}^\varepsilon(\cdot) \text{ is } \mathcal{F}_t^\varepsilon \text{ adapted}\}.$$

To proceed, we need the following conditions.

(**A8.7**) $\alpha^\varepsilon(\cdot)$ is a Markov chain having state space

$$\mathcal{M} = \Big\{s_{11}, \ldots, s_{1m_1}, \ldots, s_{l1} \ldots, s_{lm_l}\Big\},$$

and generator

$$Q^\varepsilon(t) = \frac{1}{\varepsilon}\widetilde{Q}(t) + \widehat{Q}(t),$$

where $\widetilde{Q}(t) = \text{diag}(\widetilde{Q}^1(t), \ldots, \widetilde{Q}^l(t))$ such that, for $t \in [0,T]$, $\widehat{Q}(t)$ and $\widetilde{Q}^k(t)$ for $k = 1, \ldots, l$, are themselves generators of appropriate dimensions. For each $k = 1, \ldots, l$, $\widetilde{Q}^k(t)$ is weakly irreducible.

(**A8.8**) The control space Γ is a compact set. The function $f(\cdot)$ is continuous on $\mathbb{R}^n \times \Gamma \times \mathcal{M}$. For each (ϱ, α), $f(\cdot, \alpha, \varrho)$ satisfies a linear growth condition, and is Lipschitz continuous. In addition, for each x and α the set $f(x,\alpha,\Gamma) = \{f(x,\alpha,u);\ u \in \Gamma\}$ is convex (see Definition A.33).

(**A8.9**) $G(\cdot)$ is bounded and continuous.

Remark 8.15. A few words about the assumptions are in order. For the dynamic programming approach used in the previous sections, the generator is independent of time. In this section, we are able to deal with time-dependent $\widetilde{Q}(t)$ under weaker conditions. In view of the definition of a generator (Definition 2.2), $\widetilde{Q}(\cdot)$ and $\widehat{Q}(\cdot)$ are both bounded. We only require that each subgenerator $\widetilde{Q}^i(t)$ is weakly irreducible, and only need the weak convergence of $\overline{\alpha}^\varepsilon(\cdot)$ to $\overline{\alpha}(\cdot)$ and certain moment estimates, which are guaranteed by our results in Theorem 5.57. It should be pointed out that there is no smoothness assumption on $Q^\varepsilon(\cdot)$. Measurability is sufficient. Note that the function $f(\cdot)$ depends on the variable u nonlinearly. One of the advantages of the use of the relaxed control representation is that it allows us to "convert" the nonlinear dependence on u to a situation where the function depends on the control linearly. As for the differential equation, the Lipschitz condition, together with the convexity of $f(x, \alpha, \Gamma)$ implies the existence of the unique solution (see Roxin [185]). The boundedness of the running cost in (A8.9) is not essential; it allows simpler exposition, however.

Corresponding to (8.27), there is an associate limit problem $\mathcal{P}^0$:

$$\mathcal{P}^0 : \begin{cases} \text{minimize:} \\ \quad J(m) = E\sum_{i=1}^{l}\sum_{j=1}^{m_i}\int_0^T \int G(x(t), s_{ij}, \varrho) m_t(d\varrho)\nu_j^i(t) I_{\{\overline{\alpha}(t)=i\}} dt, \\ \text{subject to:} \\ \quad x(t) = x + \sum_{i=1}^{l}\sum_{j=1}^{m_i}\int_0^t \int f(x(s), s_{ij}, \varrho) m_s(d\varrho)\nu_j^i(s) I_{\{\overline{\alpha}(s)=i\}} ds. \end{cases}$$

Denote by $\mathcal{R}^0$ the set of admissible controls for the limit problem, that is,

$$\mathcal{R}^0 = \{m(\cdot) \in \mathbb{M};\ m(\cdot) \text{ is } \mathcal{F}_t \text{ adapted}\},$$

where $\mathcal{F}_t = \sigma\{x(s), \overline{\alpha}(s);\ s \le t\}$.

To prepare us for the subsequent studies, we state two lemmas. The first one reveals properties of the limit problem and gives *a priori* bounds for both $\mathcal{P}^\varepsilon$ and $\mathcal{P}^0$. The *a priori* bounds are easily obtained by examining the defining ordinary differential equations and utilizing the linear growth condition, the Lipschitz continuity together with Gronwall's inequality; the proof of the last assertion of the following lemma is similar to the corresponding results in Fleming [62], Kushner and Runggaldier [142], Kushner [140]. The second lemma is a version of the chattering lemma and can be proved similar to that of [140, 142].

Lemma 8.16. *The following assertions hold.*

(a) *The solution $x^\varepsilon(\cdot)$ of the system equation in $\mathcal{P}^\varepsilon$,*

$$\sup_{0\le t\le T}|x^\varepsilon(t)| \le K(1+|x|)\ w.p.1. \tag{8.28}$$

where $x = x^\varepsilon(0)$ is the initial condition.

(b) *Let $m(\cdot)$ be an admissible relaxed control for $\mathcal{P}^0$. Then there is an $\mathcal{F}_t = \sigma\{x(s), \overline{\alpha}(s);\ s \le t\}$ adapted solution $x(\cdot)$ of the system equation in $\mathcal{P}^0$ such that*

$$\sup_{0\le t\le T}|x(t)| \le K(1+|x|)\ w.p.1,$$

where $x = x(0)$ is the initial condition.

(c) *Let $m^\eta(\cdot) \Rightarrow m(\cdot)$, where $m^\eta(\cdot)$ are admissible and η belongs to an index set $\mathcal{I} \subset \mathbb{R}$ such that $\eta \to 0$. Suppose $x^\eta(\cdot)$ is the solution to the differential equation in $\mathcal{P}^0$ with $m(\cdot)$ replaced by $m^\eta(\cdot)$. Then $x^\eta(\cdot) \Rightarrow x(\cdot)$ such that $m(\cdot)$ is admissible.*

Lemma 8.17. *The following assertions hold.*

(a) *There is an optimal relaxed control in $\mathcal{R}^0$.*

(b) *For each $\delta > 0$, there is an admissible $\overline{u}^\delta(\cdot)$ for the limit problem which is δ-optimal for $\mathcal{P}^0$, i.e.,*

$$J(\overline{u}^\delta) \le \inf_{m\in\mathcal{R}^0} J(m) + \delta.$$

(c) *There exists a piecewise-constant (in t) and locally Lipschitz continuous in x (uniformly in t) control $u^\delta(\cdot)$ such that*

$$J(u^\delta) \le \inf_{m\in\mathcal{R}^0} J(m) + \delta.$$

Now we are in a position to obtain the weak convergence result. The following theorem indicates that the system of interest, namely $\mathcal{P}^\varepsilon$, is close to the limit problem $\mathcal{P}^0$ in an appropriate sense.

Theorem 8.18. *Assume* (A8.7)–(A8.9). *Let $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$ and let $\widetilde{m}^\varepsilon(\cdot)$ be a δ_ε-optimal admissible relaxed control for $\mathcal{P}^\varepsilon$. Then the following assertions hold.*

(a) *$\{x^\varepsilon(\widetilde{m}^\varepsilon, \cdot), \widetilde{m}^\varepsilon(\cdot)\}$ is tight in $D^n[0,T] \times \mathbb{M}$.*

(b) *If $\widetilde{m}^\varepsilon(\cdot) \Rightarrow \widetilde{m}(\cdot)$ as $\varepsilon \to 0$, then $\widetilde{m}(\cdot) \in \mathcal{R}^0$ and the limit of any weakly convergent subsequence of $\{x^\varepsilon(\widetilde{m}^\varepsilon, \cdot), \widetilde{m}^\varepsilon(\cdot)\}$ satisfies the system equation in $\mathcal{P}^0$ with m replaced by $\widetilde{m}$.*

(c) *For the relaxed controls* $\widetilde{m}^\varepsilon(\cdot)$ *and* $\widetilde{m}(\cdot) \in \mathbb{M}$ *given above,*

$$J^\varepsilon(\widetilde{m}^\varepsilon) \to J(\widetilde{m}) \text{ as } \varepsilon \to 0.$$

Proof: We divide the proof into several steps. Using the weak convergence method and averaging techniques developed by Kushner [139], the proof proceeds by a series of approximations, each one simplifying the process a little more to eventually obtain the desired result.

Step 1: This step focuses on the tightness of the underlying process. Owing to the compactness of Γ, $\Gamma \times [0,T]$ is compact. As a result, $\{\widetilde{m}^\varepsilon(\cdot)\}$ is tight in $\mathbb{M}$. By virtue of the *a priori* bounds in Lemma 8.16, $\{x^\varepsilon(\cdot)\}$ is tight and all limits have continuous paths w.p.1 by virtue of Kushner [139, Lemma 7, p. 51]. This yields the desired tightness of $\{x^\varepsilon(\cdot), \widetilde{m}^\varepsilon(\cdot)\}$.

Step 2: Since $\{x^\varepsilon(\cdot), \widetilde{m}^\varepsilon(\cdot)\}$ is tight, using Prohorov's theorem, we may extract a convergent subsequence. For ease of presentation, we still use ε as its index. Suppose the limit is $(x(\cdot), \widetilde{m}(\cdot))$. In view of the Skorohod representation, without changing notation, suppose $x^\varepsilon(\cdot)$ converges to $x(\cdot)$ w.p.1, and the convergence is uniform on any bounded time interval.

First, for each Borel set B, $\widetilde{m}\{B \times [0,t]\}$ depends on (ω, t) and is absolutely continuous uniformly in (ω, t). This implies that the "derivative"

$$\widetilde{m}_t(B) = \lim_{\Delta \to 0^+} \frac{1}{\Delta}\Big(\widetilde{m}\{B \times [0,t]\} - \widetilde{m}\{B \times [0,t-\Delta]\}\Big)$$

exists for almost all (ω, t) with $t > 0$. Moreover $\widetilde{m}_t(\cdot)$ is (ω,t)-measurable such that $\widetilde{m}_t(\Gamma) = 1$, and for each bounded and continuous function $\widetilde{\rho}(\cdot)$,

$$\int_0^t \int \widetilde{\rho}(s,\varrho)\widetilde{m}_t(d\varrho)ds = \int_0^t \int \widetilde{\rho}(s,\varrho)\widetilde{m}(d\varrho \times ds).$$

Thus $\widetilde{m}(\cdot)$ is admissible.

To proceed, write $x^\varepsilon(\cdot)$ as

$$\begin{aligned} x^\varepsilon(t) = x + \sum_{i=1}^{l}\sum_{j=1}^{m_i}\Bigg[&\int_0^t \int f(x(s), s_{ij}, \varrho)\widetilde{m}_s(d\varrho)\nu_j^i(s)I_{\{\overline{\alpha}^\varepsilon(s)=i\}}ds \\ &+ \int_0^t \int [f(x^\varepsilon(s), s_{ij}, \varrho) - f(x(s), s_{ij}, \varrho)]\widetilde{m}_s(d\varrho)\nu_j^i(s)I_{\{\overline{\alpha}^\varepsilon(s)=i\}}ds \\ &+ \int_0^t \int f(x^\varepsilon(s), s_{ij}, \varrho)[I_{\{\alpha^\varepsilon(s)=s_{ij}\}} - \nu_j^i(s)I_{\{\overline{\alpha}^\varepsilon(s)=i\}}]\widetilde{m}_s(d\varrho)ds \\ &+ \int_0^t \int f(x^\varepsilon(s), s_{ij}, \varrho)[\widetilde{m}_s^\varepsilon(d\varrho) - \widetilde{m}_s(d\varrho)]I_{\{\alpha^\varepsilon(s)=s_{ij}\}}ds\Bigg]. \end{aligned} \tag{8.29}$$

We show that the second, the third, and the fourth lines in (8.29) go to 0 as $\varepsilon \to 0$ in probability uniformly in t. First of all, in view of the continuity of

$f(\cdot)$, the weak convergence of $x^\varepsilon(\cdot)$ to $x(\cdot)$ and the Skorohod representation infer that by enlarging the probability space,

$$\lim_{\varepsilon\to 0}\sup_{0\le t\le T} E\left|\int_0^t\int [f(x^\varepsilon(s),s_{ij},\varrho)-f(x(s),s_{ij},\varrho)] \times\widetilde{m}_s(d\varrho)\nu_j^i(s)I_{\{\overline{\alpha}^\varepsilon(s)=i\}}ds\right| = 0.$$

The second line of (8.29) goes to zero in probability.

To estimate the terms in the third line, using the *a priori* bound of $x^\varepsilon(\cdot)$, the continuity of $f(\cdot)$, and Theorem 5.52, an integration by parts leads to

$$\begin{aligned}\lim_{\varepsilon\to 0}&\left(\sup_{0\le t\le T} E\left|\int_0^t\int f(x^\varepsilon(s),s_{ij},\varrho)\right.\right.\\ &\left.\left.\times[I_{\{\alpha^\varepsilon(s)=s_{ij}\}}-\nu_j^i(s)I_{\{\overline{\alpha}^\varepsilon(s)=i\}}]\widetilde{m}_s(d\varrho)ds\right|^2\right)\\ &\le \lim_{\varepsilon\to 0}K\left(\sup_{0\le t\le T}\left|\int_0^t[I_{\{\alpha^\varepsilon(s)=s_{ij}\}}-\nu_j^i(s)I_{\{\overline{\alpha}^\varepsilon(s)=i\}}]ds\right|^2\right)=0.\end{aligned}$$

The last line above follows from Theorem 5.52. Hence the third line of (8.29) also goes to 0 in probability.

In view of the convergence of $\widetilde{m}^\varepsilon(\cdot)$ to $\widetilde{m}(\cdot)$, and the continuity and the boundedness of $f(\cdot)$,

$$\lim_{\varepsilon\to 0}\left(\sup_{0\le t\le T} E\left|\int_0^t\int f(x^\varepsilon(s),s_{ij},\varrho) \times[\widetilde{m}_s^\varepsilon(d\varrho)-\widetilde{m}_s(d\varrho)]I_{\{\alpha^\varepsilon(s)=s_{ij}\}}ds\right|\right)=0.$$

This implies that the fourth line of (8.29) goes to 0 in probability.

Therefore,

$$x^\varepsilon(t)=x+\sum_{i=1}^{l}\sum_{j=1}^{m_i}\int_0^t\int f(x(s),s_{ij},\varrho)\widetilde{m}_s(d\varrho)\nu_j^i(s)I_{\{\overline{\alpha}^\varepsilon(s)=i\}}ds+o(1), \tag{8.30}$$

where $o(1)\to 0$ in probability uniformly in $t\in[0,T]$ as $\varepsilon\to 0$. Note that by virtue of Theorem 5.53, $\overline{\alpha}^\varepsilon(\cdot)$ converges weakly to $\overline{\alpha}(\cdot)$ and hence $I_{\{\overline{\alpha}^\varepsilon(s)=i\}}$ converges to $I_{\{\overline{\alpha}(s)=i\}}$ weakly. As a result, the limit $x(\cdot)$ satisfies

$$x(t)=x+\sum_{i=1}^{l}\sum_{j=1}^{m_i}\int_0^t\int f(x(s),s_{ij},\varrho)\widetilde{m}_s(d\varrho)\nu_j^i(s)I_{\{\overline{\alpha}(s)=i\}}ds.$$

Owing to the nature of the D space, there can be at most countable number of points t at which $P\{x(t) \neq x(t^-)\} > 0$ (see Kushner [139, p. 32]). Let T_p denote the complement of this set, and let $t_{\kappa_1} < t < t + s$ with t, t_{κ_1}, $t + s \in T_p$, i.e., they are in the set of continuity points of $x(\cdot)$. Let $h(\cdot)$ be any bounded and continuous function, and $F(\cdot)$ be any continuously differentiable function with compact support. Let $p_{\kappa_2}(\cdot)$ be an arbitrary bounded and continuous function. Note that

$$\begin{aligned}\langle p_{\kappa_2}, \widetilde{m}^\varepsilon\rangle_t &= \int_0^t \int p_{\kappa_2}(s,\varrho)\widetilde{m}^\varepsilon_s(d\varrho)ds \\ &\to \int_0^t \int p_{\kappa_2}(s,\varrho)\widetilde{m}_s(d\varrho)ds = \langle p_{\kappa_2}, \widetilde{m}\rangle_t,\end{aligned}$$

as $\varepsilon \to 0$. Let i_1 and j_1 be arbitrary positive integers. Then by virtue of the weak convergence and the Skorohod representation (without changing notation),

$$\begin{aligned}&E\big(h(x^\varepsilon(t_{\kappa_1}), \langle p_{\kappa_2}, \widetilde{m}^\varepsilon\rangle_{t_{\kappa_1}}, \kappa_1 \le i_1, \kappa_2 \le j_1) \\ &\qquad\qquad \times (F(x^\varepsilon(t+s)) - F(x^\varepsilon(t)))\big) \\ &\to E\big(h(x(t_{\kappa_1}), \langle p_{\kappa_2}, \widetilde{m}\rangle_{t_{\kappa_1}}, \kappa_1 \le i_1, \kappa_2 \le j_1) \\ &\qquad\qquad \times (F(x(t+s)) - F(x(t)))\big).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\lim_{\varepsilon\to 0} Eh(x^\varepsilon(t_{\kappa_1}), \langle p_{\kappa_2}, \widetilde{m}^\varepsilon\rangle_{t_{\kappa_1}}, \kappa_1 \le i_1, \kappa_2 \le j_1) \\ &\quad \times \left[F(x^\varepsilon(t+s)) - F(x^\varepsilon(t)) - \int_t^{t+s} A^\varepsilon F(x^\varepsilon(\tau))d\tau\right] = 0,\end{aligned}$$

where

$$A^\varepsilon F(x) = \left(\frac{\partial F(x)}{\partial x}\right)' \left(\sum_{i=1}^{l}\sum_{j=1}^{m_i} \int f(x, s_{ij}, \varrho)\widetilde{m}^\varepsilon_t(d\varrho) I_{\{\alpha^\varepsilon(t)=s_{ij}\}}\right).$$

Consequently, using (8.30),

$$\begin{aligned}&Eh(x(t_{\kappa_1}), \langle p_{\kappa_2}, \widetilde{m}\rangle_{t_{\kappa_1}}, \kappa_1 \le i_1, \kappa_2 \le j_1) \\ &\quad \times \left(F(x(t+s)) - F(x(t)) - \int_t^{t+s} AF(x(\tau))d\tau\right) = 0,\end{aligned} \tag{8.31}$$

where

$$AF(x) = \left(\frac{\partial F(x)}{\partial x}\right)' \left(\sum_{i=1}^{l}\sum_{j=1}^{m_i}\int f(x, s_{ij}, \varrho)\widetilde{m}_t(d\varrho)\nu_j^i(t)I_{\{\overline{\alpha}(t)=i\}}\right).$$

The arbitrariness of i_1, j_1, $F(\cdot)$, $h(\cdot)$, $p_{\kappa_2}(\cdot)$, t_{κ_1}, t, s, together with (8.31), implies that $x(\cdot)$ solves the martingale problem with operator A, that is,

$$F(x(t)) - F(x(0)) - \int_0^t AF(x(s))ds$$

is a martingale for each bounded real-valued function $F(\cdot)$ being continuously differentiable with compact support. Equivalently, $x(\cdot)$ satisfies the limit problem, and $x(\cdot)$ has continuous paths with probability one. Furthermore, $\widetilde{m}(\cdot)$ is an admissible relaxed control for the limit problem $\mathcal{P}^0$.
Step 3: The weak convergence of $(x^\varepsilon(\cdot), \widetilde{m}^\varepsilon(\cdot), \overline{\alpha}^\varepsilon(\cdot))$ to $(x(\cdot), \widetilde{m}(\cdot), \overline{\alpha}(\cdot))$, the continuity of $G(\cdot)$, and detailed estimates similar to those leading to (8.30) imply $J^\varepsilon(\widetilde{m}^\varepsilon) \to J(\widetilde{m})$ as $\varepsilon \to 0$. □

Remark 8.19 The proof above is in the spirit of the direct averaging method (see Kushner [139, Chapter 5]). That is, by viewing the chain $\alpha^\varepsilon(\cdot)$ as a "noise," one averages it out and derives the limit system via a direct approach. Note that in our system, the limit distributions, namely $\nu^i(\cdot)$, for $i = 1, \ldots, l$, are time dependent. In general, nonstationarity is very hard to deal with. The asymptotic properties of the aggregated chain $\overline{\alpha}^\varepsilon(\cdot)$, obtained in Chapter 5, especially the weak convergence of $\overline{\alpha}^\varepsilon(\cdot)$ and Theorem 5.52 enables us to overcome the difficulty and obtain the desired result.

To proceed, we aim at deriving a limit result for the approximation of $\mathcal{P}^\varepsilon$ via $\mathcal{P}^0$. Denote by v^ε and v^0 the value functions of $\mathcal{P}^\varepsilon$ and $\mathcal{P}^0$, respectively. The asymptotically near optimality is in the theorem below, which indicates a nearly optimal control for the original problem $\mathcal{P}^\varepsilon$ can be obtained via a δ-optimal control of $\mathcal{P}^0$.

Theorem 8.20. *Assume* (A8.7)–(A8.9). *Then*

$$\lim_{\varepsilon\to 0} v^\varepsilon = v^0. \tag{8.32}$$

Moreover, for each $\delta > 0$, there exists a Lipschitz continuous feedback control $\overline{u}^\delta(x,t)$, which is δ-optimal for $\mathcal{P}^0$ such that for the cost function $J^\varepsilon(\cdot)$ in $\mathcal{P}^\varepsilon$,

$$\limsup_{\varepsilon\to 0} \left|J^\varepsilon(\overline{u}^\delta) - v^\varepsilon\right| \le \delta.$$

Remark 8.21. This theorem indicates a nearly optimal control for the original problem $\mathcal{P}^\varepsilon$ can be obtained by solving the limit problem $\mathcal{P}^0$. Since $\delta > 0$ is arbitrary, $\overline{u}^\delta$ can be chosen to approximate the optimal solution with desired accuracy.

Proof of Theorem 8.20: By virtue of Lemma 8.17, for each $\delta > 0$, a δ-optimal control $\overline{u}^\delta(x,t)$ for Problem $\mathcal{P}^0$ exists. The weak convergence results of Theorem 8.18 then yield

$$x^\varepsilon(\overline{u}^\delta,\cdot) \Rightarrow x(\overline{u}^\delta,\cdot) \quad \text{and} \quad J^\varepsilon(\overline{u}^\delta) \to J(\overline{u}^\delta). \tag{8.33}$$

Since $\overline{u}^\delta$ is a δ-optimal control for $\mathcal{P}^0$,

$$J(\overline{u}^\delta) \le v^0 + \delta.$$

In view of (8.33), we have

$$J^\varepsilon(\overline{u}^\delta) = J(\overline{u}^\delta) + \Delta_1(\varepsilon) \le v^0 + \delta + \Delta_1(\varepsilon), \tag{8.34}$$

where $\Delta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Since v^ε is the value function for $\mathcal{P}^\varepsilon$, $v^\varepsilon \le J^\varepsilon(\overline{u}^\delta)$. By virtue of Theorem 8.18, choose $\widetilde{m}^\varepsilon \in \mathcal{R}^\varepsilon$ such that $v^\varepsilon \ge J^\varepsilon(\widetilde{m}^\varepsilon) - \varepsilon$. Since $\mathcal{R}^\varepsilon$ is relatively compact, there exists a subsequence $\{\widetilde{m}^\varepsilon(\cdot)\}$ such that $\widetilde{m}^\varepsilon(\cdot) \Rightarrow \widetilde{m}(\cdot)$. It follows from Theorem 8.18 again that

$$v^0 \le J(\widetilde{m}) = v^\varepsilon + \Delta_2(\varepsilon),$$

for some $\Delta_2(\varepsilon) \to 0$ as $\varepsilon \to 0$,

$$\begin{aligned} v^\varepsilon &\le J^\varepsilon(\overline{u}^\delta) \le v^0 + \delta + \Delta_1(\varepsilon) \\ &\le v^\varepsilon + \delta + \Delta_1(\varepsilon) + \Delta_2(\varepsilon). \end{aligned} \tag{8.35}$$

Sending $\varepsilon \to 0$ leads to

$$\limsup_{\varepsilon\to 0} |v^\varepsilon - v^0| \le \delta.$$

Since δ is arbitrary, $\lim_{\varepsilon\to 0} v^\varepsilon = v^0$.

Clearly $J^\varepsilon(\overline{u}^\delta) - v^\varepsilon \ge 0$. By virtue of (8.35),

$$0 \le J^\varepsilon(\overline{u}^\delta) - v^\varepsilon \le \delta + \Delta_1(\varepsilon) + \Delta_2(\varepsilon). \tag{8.36}$$

Taking lim sup in (8.36) yields

$$\limsup_{\varepsilon\to 0} \left| J^\varepsilon(\overline{u}^\delta) - v^\varepsilon \right| \le \delta.$$

The proof of the theorem is thus completed. □

8.7 Notes

This chapter is based on Zhang, Yin, and Boukas [256], and Yin and Zhang [233, 234]. As far as the dynamic programming approach (Sections 8.2–8.5) is concerned, Theorem 8.8 can be extended to models in a finite-time horizon with a cost function

$$E\int_s^T G(x(\tau),\alpha^\varepsilon(\tau),u(\tau))d\tau,$$

and an initial condition $x(s)=x$ of the dynamic system (see Zhang and Yin [251] for a related work.) However, in this case, the optimal control U^* would be a function of (s,x). Note that the proof of Theorem 8.11 requires that $(d\tilde{f}(s)/ds)$ be bounded, which demands $(\partial v^0(s,x)/\partial s)$ be Lipschitz in s to make $U^*(s,x)$ Lipschitz in s. The Lipschitz property of $(\partial v^0(s,x)/\partial s)$ is difficult to verify. Nevertheless, when $\widetilde{Q}$ is irreducible (with a single block), the Lipschitz condition on $(\partial v^0(s,x)/\partial s)$ is not needed; see Zhang [247] for details.

In this chapter we only considered the structure of the two-level hierarchy, i.e., the original problem $\mathcal{P}^\varepsilon$ vs. the limit problem $\mathcal{P}^0$. Similar analysis can be carried out for multi-level hierarchy in which the reduction of complexity can be achieved step-by-step. This procedure can also be viewed as a multi-resolution approach in the sense of "increment in information" as in the wavelets theory; see Daubechies [40] for related discussions. Suppose in a given system, the frequencies of different events can be characterized by positive numbers $1/\varepsilon_1$, $1/\varepsilon_2$, $1/\varepsilon_3$, …, $1/\varepsilon_{n_0}$ for some positive integer n_0, where $\varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \cdots \ll \varepsilon_{n_0}$. Name the given problem $\mathcal{P}^{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{n_0}}$. If this system is too large to deal with, to find numerical solutions, one needs to work with an approximate problem that can be handled numerically (termed workable in what follows) and that provides a good approximation to the given system. To begin, one examines the problem $\mathcal{P}^{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{n_0}}$. If it is workable, solve it, otherwise, average out the variables corresponding to ε_1 and obtain a problem $\mathcal{P}^{0,\varepsilon_2,\ldots,\varepsilon_{n_0}}$. Then check if the resulting problem is workable. If the answer is yes, solve it; if not, average out the variables corresponding to ε_2. Continue in this way, we eventually obtain a system which is workable. This procedure leads to the smallest k such that $\mathcal{P}^{0,\ldots,0,\varepsilon_k,\varepsilon_{k+1},\ldots,\varepsilon_{n_0}}$ is workable.

Naturally, it is interesting to ask the following questions. If one can go, for example, from

$$\mathcal{P}^{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{n_0}} \text{ to } \mathcal{P}^{0,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{n_0}}, \text{ then to } \mathcal{P}^{0,0,\varepsilon_3,\ldots,\varepsilon_{n_0}},$$

will $\mathcal{P}^{0,0,\varepsilon_3,\ldots,\varepsilon_{n_0}}$ provide a good approximation to $\mathcal{P}^{\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{n_0}}$ when $(\varepsilon_1,\varepsilon_2)$ as a pair is small? In fact, such an approach was demonstrated in Sethi and Zhang [193] in the context of marketing and production planning; see also Sethi and Zhang [192]. Thus, one need only consider two-level

hierarchy and move from one level approximation to the next level approximation step-by-step till reaches a workable system.

In the context of manufacturing systems, optimal production policies were considered by many people including E. K. Boukas, A. Haurie, W. Fleming, S. Gershwin, J. Jiang, S. P. Sethi, H. M. Soner, G. Yin, Q. Zhang, and X. Y. Zhou; see Boukas and Haurie [19], Fleming, Sethi, and Soner [65] and the books Gershwin [71], Sethi and Zhang [192], Yin and Zhang [235, 236] for a review of the literature. Other related issues in singular perturbations can be found in the survey by Saksena, O'Reilly, and Kokotovic [186] and references therein.

In conjunction to the weak convergence approach, in Section 8.6, one may wish to investigate the infinite horizon counterpart. In this case, the objective function can be either a discounted cost

$$J^\varepsilon(\widetilde{m}^\varepsilon) = E\int_0^\infty e^{-\rho t}\int G(x^\varepsilon(t),\alpha^\varepsilon(t),\varrho)\widetilde{m}_t^\varepsilon(d\varrho)dt$$

for some $\rho > 0$, or an average cost per unit time

$$J^\varepsilon(\widetilde{m}^\varepsilon) = \limsup_{T\to\infty}\frac{1}{T}E\int_0^T\int G(x^\varepsilon(t),\alpha^\varepsilon(t),\varrho)\widetilde{m}_t^\varepsilon(d\varrho)dt.$$

The origin of the weak convergence method to nearly optimal controls is in the work of Kushner and Runggaldier [142] for systems driven by wide bandwidth processes. Singularly perturbed controlled diffusion is treated in Bensoussan [8] using partial differential equation methods and the related singularly perturbed systems (both control and filtering problems) with wideband noise are studied extensively in Kushner [140]. Many people have contributed to the literature of singular perturbation in control theory for deterministic and stochastic systems. The article of Kokotovic [126] contains a detailed survey on the subject, and the reference of Kokotovic, Bensoussan, and Blankenship [127] collects a large number of references. Systems with fast-changing processes and unknown parameters are treated in the paper of Yin and Zhang [233], where the limit system is a controlled diffusion. This chapter complements the previous work by providing near optimality for systems with Markovian driving processes of nondiffusion type and/or controlled Markov chains.

Related literature in control theory can be found in the books of Fleming and Rishel [63], Fleming and Soner [64], and the reference therein. The book by Sethi and Thompson [191] is a good source for examples in applications of management science. Note that in Lemma 8.10, we obtained a verification theorem when the value function is differentiable. A version of such a verification theorem can be derived without the differentiability condition; related results can be found in Zhou [258].

Warga [214] (see also Berkovitz [9]) initiated the relaxed control formulation for deterministic systems under the framework of variational problems.

Then Fleming [62] extended the result to that of stochastic systems. Such a representation is quite useful for many problems arising in a wide variety of applications (see Kushner [140] and Kushner and Runggaldier [142]). An extensive survey on this and related topics can be found in Kushner [140] among others.

In [175] (see also the references therein), Phillips and Kokotovic take asymptotic expansions of the cost function to treat near optimality. It will be interesting to see if an asymptotic expansion of the cost function can be derived under the formulation of the current chapter.

Recently, Costa and Dufour [33] studied the problem under the framework of a piecewise deterministic Markov process. They established the convergence of the value functions to the associated limit value function. Such property is obtained by showing that the liminf and limsup of some value functions satisfy some inequalities as $\varepsilon \to 0$ to relax the Lipschitz continuity condition; see also Costa and Dufour [34] for results in connection with discrete-time Markov decision processes (see Yin and Zhang [238]) in a general state space.

Another potential application of the approach in this chapter is to analyze hybrid filtering problems; see the papers of Blom and Bar-Shalom [17], Li [146], Haussmann and Zhang [82, 83], Zhang [250], and the book by Elliott, Aggoun, and Moore [57] for related literature. In target tracking and nonlinear filtering, an effective algorithm for dealing with Markovian switching systems is that of the Interacting Multiple Model (IMM) (see Blom and Bar-Shalom [17]). However, up to now, there is no theoretical justification for the desired optimality or near optimality of the IMM algorithm; see a recent survey by Li [146] for details. It is interesting from both theoretical and practical points of view to study the optimality of the IMM algorithm. It is conceivable that the singular perturbation approach discussed in this chapter can be used to obtain near optimality of the IMM filtering under suitable formulations.

9

Numerical Methods for Control and Optimization

9.1 Introduction

One of the main techniques presented in Chapters 7 and 8 is to reduce the complexity of singularly perturbed systems by studying the corresponding limit systems that are easier to handle than the original problems. The optimal or nearly optimal controls of the limit problems can be used to construct nearly optimal controls of the original systems. Although the limit systems are substantially simpler than the original pre-limit ones, very often closed-form solutions are still difficult to obtain, except in special cases. For example, in the context of stochastic manufacturing systems, a closed-form solution for optimal production planning is obtained for a system with one-machine and one-part-type by Akella and Kumar [2] for a discounted cost problem, and Zhang and Yin [251] for a finite horizon counterpart. Such closed-form solutions do not seem possible for more general manufacturing systems such as flowshops and jobshops (see Sethi and Zhang [192]). For many applications, one has to resort to a viable alternative – numerical methods.

As a complement to our discussion of singularly perturbed control problems for Markov chains, this chapter focuses on numerical methods for solutions of the control problems. This is a necessary step for many control and optimization problems and alleviates considerably the difficulties encountered. In fact, such a step often plays a crucial role in applications. To take up this issue, we examine the underlying problems from two different

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angles, namely, numerical approximation of optimal control problems and stochastic optimization formulation for finding optimal controls under threshold policies.

Treating the optimal control problems, we use the finite difference approximation method developed by Kushner (see Kushner [138], Kushner and Dupuis [141], and the references therein), which has been proven to be very useful for various stochastic systems. Having in our mind a wide variety of applications, we formulate the problem as a nonlinear controlled Markov chain. Our setup is general enough to include, for example, many problems in manufacturing models as special cases. The results obtained are applicable to various dynamical systems and controlled piecewise-deterministic processes.

For various control and optimization problems with long-run average costs, one is often content with a nearly optimal or suboptimal solution. One of the most easily implementable and monitoring strategies in practice is the class of threshold control policies, which provides an enticing alternative. Kimemia and Gershwin brought in the idea of the use of hedging (threshold) policies. Further work along this line may be found in Caramanis and Liberopoulos [24] among others. Under the threshold policy, a control problem can conveniently be transferred to an optimization procedure. The idea is to develop a systematic procedure for finding the optimal threshold values. The essence is to utilize stochastic approximation/optimization methods to resolve the problem. By focusing our attention to the class of threshold controls and considering the expected cost as a function of the threshold levels, we generate a sequence of noisy gradient estimates and update the estimate of the optimal threshold values by use of stochastic recursive algorithms.

The rest of the chapter is arranged as follows. In Section 9.2, we develop a finite difference approximation procedure. Section 9.3 concentrates on the stochastic optimization methods for long-run average cost under threshold policies. Further discussions and citation of related references are in Section 9.4.

9.2 Numerical Methods for Optimal Control

Consider numerical solutions for solving the following control problem:

$$\mathcal{P}: \begin{cases} \text{minimize: } J(x,\alpha,u(\cdot)) = E\displaystyle\int_0^\infty e^{-\rho t} G(x(t),\alpha(t),u(t))dt, \\ \text{subject to: } \dfrac{dx(t)}{dt} = f(x(t),\alpha(t),u(t)), \\ \qquad x(0)=x,\ u(\cdot)\in\mathcal{A},\ \alpha(0)=\alpha, \\ \text{value function: } v(x,\alpha) = \displaystyle\inf_{u(\cdot)\in\mathcal{A}} J(x,\alpha,u(\cdot)), \end{cases}$$

where $\alpha(\cdot)$ is a finite-state Markov chain generated by Q, and $\mathcal{A}$ denotes the set of all admissible controls (i.e., controls that are progressively measurable with respect to $\mathcal{F}(t) = \sigma\{\alpha(s) : \ s \le t\}$ and $u(t) \in \Gamma$, a compact subset of $\mathbb{R}^{n_1}$).

The HJB equation of the control problem $\mathcal{P}$ is

$$\rho v(x,\alpha) = \min_{u\in\Gamma}\Bigg\{ f(x,\alpha,u)\frac{\partial v(x,\alpha)}{\partial x} + G(x,\alpha,u)\Bigg\} + Qv(x,\cdot)(\alpha), \tag{9.1}$$

where as noted in Remark 8.4, $f(x,\alpha,u)(\partial v/\partial x)$ means $\langle f, (\partial v/\partial x)\rangle$, the usual inner product of f and $(\partial v/\partial x)$.

In view of the verification theorem (Theorem A.31), to find an optimal control for the problem, the dynamic programming approach requires a solution to the associated HJB equation. However, more often than not, a closed-form solution of the corresponding HJB equation is not obtainable. Thus, it is necessary to develop numerical algorithms to resolve the problem. In this section, we adopt Kushner's numerical methods for stochastic controls. Our approach consists of using an approximation method for the partial derivatives of the value function $v(x,\alpha)$ within a finite grid of the state vector x and a finite grid for the control vector, which transforms the original optimization problem to an auxiliary discounted Markov decision process. This transformation allows us to apply the well-known techniques, such as a successive approximation or the policy improvement, to solve the HJB equations and then the underlying optimization problems.

Let $\Delta x_i > 0$ denote the length of the finite difference interval of the variables x_i for $i = 1,\ldots,n$. Using this finite difference interval, approximate the value function $v(x,\alpha)$ by a sequence of functions $v^{\Delta}(x,\alpha)$ and the partial derivatives $(\partial v(x,\alpha)/\partial x_i)$ by

$$\begin{cases} \dfrac{1}{\Delta x_i}(v^{\Delta}(x(\Delta x_i,+),\alpha) - v^{\Delta}(x,\alpha)), & \text{if } f_i(x,\alpha,u) \ge 0, \\ \dfrac{1}{\Delta x_i}(v^{\Delta}(x,\alpha) - v^{\Delta}(x(\Delta x_i,-),\alpha)), & \text{if } f_i(x,\alpha,u) < 0, \end{cases}$$

where $f(x,\alpha,u) = (f_1(x,\alpha,u),\ldots,f_n(x,\alpha,u))'$ and

$$x(\Delta x_i,+) = (x_1,\ldots,x_{j-1},x_i+\Delta x_i,x_{j+1},\ldots,x_n)',$$

$$x(\Delta x_i,-) = (x_1,\ldots,x_{j-1},x_i-\Delta x_i,x_{j+1},\ldots,x_n)'.$$

This leads to

$$\begin{aligned} f_i(x,\alpha,u)\frac{\partial}{\partial x_i}v(x,\alpha) \doteq\ & \frac{|f_i(x,\alpha,u)|}{\Delta x_i}v^\Delta(x(\Delta x_i,+),\alpha)I_{\{f_i(x,\alpha,u)\geq 0\}} \\ & +\frac{|f_i(x,\alpha,u)|}{\Delta x_i}v^\Delta(x(\Delta x_i,-),\alpha)I_{\{f_i(x,\alpha,u)<0\}} \\ & -\frac{|f_i(x,\alpha,u)|}{\Delta x_i}v^\Delta(x,\alpha). \end{aligned}$$

With these approximations, we can "rewrite" the HJB equation (9.1) in terms of $v^\Delta(x,\alpha)$ as

$$\begin{aligned} v^\Delta(x,\alpha) =& \min_{u\in\Gamma}\left(\rho+|q_{\alpha\alpha}|+\sum_{i=1}^n\frac{|f_i(x,\alpha,u)|}{\Delta x_i}\right)^{-1} \\ & \times\left\{\sum_{i=1}^n\frac{|f_i(x,\alpha,u)|}{\Delta x_i}\left(v^\Delta(x(\Delta x_i,+),\alpha)I_{\{f_i(x,\alpha,u)\geq 0\}}\right.\right. \\ & \left. +v^\Delta(x(\Delta x_i,-),\alpha)I_{\{f_i(x,\alpha,u)<0\}}\right) \\ & \left.+G(x,\alpha,u)+\sum_{\beta\neq\alpha}q_{\alpha\beta}v^\Delta(x,\beta)\right\}. \end{aligned} \tag{9.2}$$

The theorem below shows that $v^\Delta(x,\alpha)$ converges to $v(x,\alpha)$ as the step size Δx_i goes to zero. For simplicity, we only consider the case that

$$\Delta x_1=\Delta x_2=\cdots=\Delta x_n=\Delta>0.$$

Theorem 9.1. *Assume* (A9.1) *and* (A9.2). *Suppose that* $v^\Delta(x,\alpha)$ *is a solution to* (9.2) *and*

$$0\leq v^\Delta(x,\alpha)\leq K(1+|x|^\kappa),$$

for some constants $K>0$ *and* $\kappa>0$. *Then*

$$\lim_{\Delta\to 0}v^\Delta(x,\alpha)=v(x,\alpha). \tag{9.3}$$

Proof: We only give a brief sketch here; for a detailed account, see Kushner and Dupuis [141]. Note that (9.2) can be written as

$$v^\Delta(x,\alpha)=\mathcal{T}v^\Delta(x,\alpha), \tag{9.4}$$

for an operator $\mathcal{T}$. The problem becomes a fixed point iteration procedure. It is not difficult to check that for each $\Delta>0$, the operator $\mathcal{T}$ is a contraction mapping. The contraction mapping principle then implies that (9.2) has a unique solution $v^\Delta(x,\alpha)$.

To proceed, define a sequence $\{v_k^\Delta(x,\alpha)\}$ as

$$v_0^\Delta(x,\alpha) = 0, \text{ and } v_{k+1}^\Delta(x,\alpha) := \mathcal{T} v_k^\Delta(x,\alpha), \ k \geq 0.$$

Using this sequence, we can show that the solution to (9.4) is continuous.

For any fixed positive Δ ($= \Delta x_i$, for $i = 1, \ldots, n$) and $\alpha \in \mathcal{M}$,

$$0 < \rho \leq \rho + |q_{\alpha\alpha}| + \sum_{i=1}^{n} \frac{|f_i(x,\alpha,u)|}{\Delta x_i} \leq K_1(1 + \Delta^{-1}),$$

for some constant K_1. As a result, (9.2) is equivalent to

$$\begin{aligned} 0 \ = \min_{u\in\Gamma}\Bigg\{ & \sum_{i=1}^{n} \frac{|f_i(x,\alpha,u)|}{\Delta x_i} \Big([v^\Delta(x(\Delta x_i,+),\alpha) - v^\Delta(x,\alpha)] I_{\{f_i(x,\alpha,u)\geq 0\}} \\ & + [v^\Delta(x(\Delta x_i,-),\alpha) - v^\Delta(x,\alpha)] I_{\{f_i(x,\alpha,u)<0\}} \Big) \\ & + G(x,\alpha,u) + \sum_{\beta\neq\alpha} q_{\alpha\beta}[v^\Delta(x,\beta) - v^\Delta(x,\alpha)] - \rho v^\Delta(x,\alpha) \Bigg\}. \end{aligned}$$

For each $x \in \mathbb{R}^n$ and $\alpha \in \mathcal{M}$, let

$$v^*(x,\alpha) := \limsup_{\delta\to 0} \Big(\limsup_{\Delta\to 0} \big[\sup\{v^\Delta(\tilde{x},\alpha) : |x - \tilde{x}| \leq \delta\} \big] \Big)$$

and

$$v_*(x,\alpha) := \liminf_{\delta\to 0} \Big(\liminf_{\Delta\to 0} \big[\inf\{v^\Delta(\tilde{x},\alpha) : |x - \tilde{x}| \leq \delta\} \big] \Big).$$

It is clear that $v^*(x,\alpha) \geq v_*(x,\alpha)$. Moreover, it can be shown that $v^*(x,\alpha)$ is upper semicontinuous and $v_*(x,\alpha)$ is lower semicontinuous.

To obtain the convergence result, it remains to derive the reverse inequality, $v^*(x,\alpha) \leq v_*(x,\alpha)$. In fact, we need only show that $v^*(x,\alpha)$ and $v_*(x,\alpha)$ are viscosity subsolution and viscosity supersolution to (9.1), respectively. This can be done as in Kushner and Dupuis [141, Theorem 14.3.1]. Consequently, by virtue of the uniqueness of the viscosity solution to the HJB equation (see Theorem A.24), $v^*(x,\alpha) \leq v_*(x,\alpha)$. Hence,

$$v^*(x,\alpha) = v_*(x,\alpha) = v(x,\alpha)$$

as desired. □

Remark 9.2. To obtain an optimal control via the dynamic programming approach, one needs to use the corresponding value function as in the verification theorem (see Theorem A.31). Usually, the numerical scheme produces only an approximate value function, which can be regarded as a perturbation of the true value function. The rationale is that by using the

approximate value function in the verification theorem, one can construct a feedback control policy that is approximately optimal. In fact, under fairly mild conditions and using a viscosity solution approach, Yan and Zhang [221] have shown that the control policy obtained using an approximate value function is indeed nearly optimal as the perturbations go to 0.

Remark 9.3. In view of the discussion above, there is nothing so special about the problem $\mathcal{P}$. The same approach can equally be applied to the singularly perturbed problem $\mathcal{P}^\varepsilon$ defined in Chapter 8. However, following our previous consideration, for a large and complex system, one would be better off to obtain a "reduced-order" system (limit system) first and to apply the numerical method only to the limit problem. The proof of Theorem 9.1 uses viscosity solution techniques. An alternative approach is to apply the method of weak convergence via Markov chain approximation techniques as in the setup of Kushner [138] or Kushner and Dupuis [141].

9.3 Optimization under Threshold Policy

This section consists of several subsections. First an optimal control problem is reformulated as a stochastic optimization problem. The next subsection gives the convergence proof of the recursive algorithm followed by a couple of examples in production planning with unreliable machines. The last subsection derives the estimation error for the approximation.

9.3.1 Stochastic Optimization Formulation

As in the previous section, suppose that $\alpha(\cdot)$ is a finite-state Markov chain with stationary transition probability or, equivalently, the generator $Q(\cdot) = Q$, a constant matrix. Let $x(t) \in \mathbb{R}^n$, $u(t) \in \Gamma$, a compact subset of $\mathbb{R}^{n_1}$, $f(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \Gamma \times \mathcal{M} \mapsto \mathbb{R}^n$, and $G(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \Gamma \times \mathcal{M} \mapsto \mathbb{R}$. Consider the following controlled dynamic system

$$\frac{dx(t)}{dt} = f(x(t), \alpha(t), u(t)), \;\; x(0) = x^0, \tag{9.5}$$

with a long-run average cost function

$$J(u) = \lim_{T_1 \to \infty} \frac{1}{T_1} E \int_0^{T_1} G(x(t), \alpha(t), u(t)) dt. \tag{9.6}$$

Instead of seeking optimal controls of the system given above, we reformulate it as a stochastic optimization problem. The main idea lies in concentrating on a class of controls of the threshold type. Under such a setting, our effort is to develop an easily implementable algorithm to approximate the optimal threshold levels. Here and hereafter, the terms threshold values and threshold levels will be used interchangeably.

Definition 9.4. A control policy $u(t) \in \Gamma$ is of threshold type with constant threshold levels if there are sets $A_i \subset \mathbb{R}^n$ and constants $c_i \in \Gamma \subset \mathbb{R}^{n_1}$ for $i = 1, \ldots, n_0$ such that for some integer n_0,

$$u(t) = \sum_{i=1}^{n_0} c_i I_{\{x(t) \in A_i\}}.$$

Typically, the sets A_i depend on some parameter $\theta \in \mathbb{R}^n$. To illustrate, consider the following example. This is an analytically solvable case, and describes the salient features of the threshold type of control policies.

Example 9.5. Consider a failure-prone manufacturing system with production capacity $\alpha(\cdot)$, that is a Markov chain with finite-state space $\mathcal{M}$. For simplicity, assume $\mathcal{M} = \{\alpha_1, \alpha_2\}$, where α_1 means the machine is up and α_2 means that the machine is down. Suppose that the breakdown and repair times are independent and exponentially distributed with parameters λ and μ, respectively. Denote the inventory level and the production rate of the system by $x(t)$, $u(t) \in \mathbb{R}$, respectively. For convenience, let $\alpha_1 = 1$ and $\alpha_2 = 0$. Then, the production constraints are given as

$$0 \le u(t) \le u_{\max}\alpha(t), \ t \ge 0,$$

where $u_{\max}$ is the maximum production rate (since $\alpha(t) = 0$ or 1, $u_{\max}$ is also the maximum capacity) of the machine. Our objective is to find the optimal control $u(\cdot)$ to

$$\begin{cases} \text{minimize: } J(u) = \lim\limits_{T_1 \to \infty} \dfrac{1}{T_1} E \displaystyle\int_0^{T_1} [c^+ x^+(t) + c^- x^-(t)]dt, \\ \text{subject to: } \dfrac{dx(t)}{dt} = u(t) - z, \ \ x(0) = x_0, \end{cases}$$

where z is a constant demand rate, $x^+ = \max\{0, x\}$ and $x^- = \max\{0, -x\}$, and c^+ and c^- are nonnegative constants. By means of dynamical programming equation approach, Bielecki and Kumar [12] derived the optimal control explicitly, and showed that the optimal control is of threshold type given by

$$u^*(t) = \begin{cases} u_{\max} I_{\{\alpha(t)=1\}}, & \text{if } x(t) < \theta^*, \\ z I_{\{\alpha(t)=1\}}, & \text{if } x(t) = \theta^*, \\ 0, & \text{if } x(t) > \theta^*, \end{cases}$$

where θ^* is the optimal threshold value given by

$$\theta^* = \begin{cases} 0, \quad \text{if} \quad \dfrac{u_{\max}\lambda(c^+ + c^-)}{c^+(u_{\max} - z)(\lambda + \mu)} \le 1 \ \text{ and } \ \dfrac{u_{\max} - z}{\lambda} > \dfrac{z}{\mu}, \\ \infty, \quad \text{if} \quad \dfrac{u_{\max} - z}{\lambda} \le \dfrac{z}{\mu}, \\ \dfrac{z(u_{\max} - z)}{\mu(u_{\max} - z) - \lambda z} \log\left(\dfrac{u_{\max}\lambda(c^+ + c^-)}{c^+(u_{\max} - z)(\lambda + \mu)}\right), \quad \text{otherwise.} \end{cases}$$

Moreover, the optimal cost is

$$J(u^*) = \begin{cases} \dfrac{c^- \lambda u_{\max}}{(\lambda+\mu)(\mu u_{\max} - \lambda z - \mu z)}, & \text{if } \theta^* = 0, \\ \dfrac{c^+ z}{\lambda+\mu} + \dfrac{c^+ z(u_{\max} - z)}{\mu(u_{\max} - z) - \lambda z} \log\left(\dfrac{u_{\max}\lambda(c^+ + c^-)}{c^+(u_{\max} - z)(\lambda+\mu)}\right), & \\ & \text{if } \theta^* > 0. \end{cases}$$

Since the optimal control depends on the threshold parameter θ, the expected cost can also be viewed as a function of the threshold. Intuitively, the optimal policy can be described as follows. If the inventory level is below the optimal threshold, one should produce at a full speed; if the inventory level is above the threshold, one should produce nothing; if the inventory level is at the threshold, one should produce exactly the same as the demand.

In view of (9.6), focusing our attention to the class of controls of threshold type, the cost $J(\cdot)$ becomes a function of the threshold levels (i.e., $J = J(\theta)$). Threshold types of control policies have drawn renewed attention lately, since the idea is appealing and the principle is easy to apply. First such policies are fairly simple in form and easily implementable so that they are particularly attractive in applications. Once a threshold value is determined, a controller or an operator can ignore detailed variations and concentrate only on adjusting controls according to the threshold levels. The corresponding control procedure is simpler as compared with the optimal control policies, since only a monitoring device/procedure is needed to keep track of the performance of the underlying system. In lieu of solving the HJB equations, only a few parameters need to be tuned. Moreover, in various situations, one is often content with suboptimality owing to the cost consideration and other limitations. Frequently, a suboptimal control is nearly or virtually as valuable as an optimal control. Furthermore, in many cases, threshold control policies are indeed optimal as in Example 9.5.

Upon transferring the problem to an optimization task, the foremost important task is to locate the optimal threshold values. This dictates the development of stochastic recursive algorithms. Our aim is to develop a systematic approach to approximate the threshold values.

Throughout the rest of the chapter, $\theta \in \mathbb{R}^n$ denotes a column vector. Denote $\xi(t,\theta) = (x(t), \alpha(t))$, and use $\eta > 0$, a small parameter, to represent the step size. The stochastic optimization algorithm takes the form

$$\begin{aligned} \theta_{k+1} &= \theta_k - \eta\,(\text{gradient estimate of } J(\cdot) \text{ at } \theta_k) \\ &= \theta_k - \frac{\eta}{T}\int_{kT}^{(k+1)T} g(\theta_k, \xi(t,\theta_k))dt, \end{aligned} \tag{9.7}$$

for each integer $k \geq 0$, where $g(\cdot)$ is an appropriate function. Example 9.11 and Example 9.12 in what follows, present two examples of such gradient estimates.

For fixed $T > 0$, the gradient estimate of $J(\theta)$ at the kth iterate is of the form

$$\frac{1}{T}\int_{kT}^{(k+1)T} g(\theta_k, \xi(t, \theta_k))dt.$$

In what follows, for notational convenience, we often suppress the θ dependence and write $\xi(t, \theta)$ as $\xi(t)$. Although the gradient estimate of $J(\cdot)$ in (9.7) can be obtained via finite difference approximation in a straightforward way, various alternatives exist. The infinitesimal perturbation analysis (IPA) approach (see Ho and Cao [87] and Glasserman [74]) provides a better alternative, however. While it is more efficient, this approach is application dependent. That is, one needs to figure out the gradient estimate for each application; there are no general forms of the gradient estimates available. We use a constant step size since an iterative algorithm with constant step size has the ability to track slight variation of the parameter and is more robust with respect to the random errors.

Using the IPA approach, for Example 9.5, the gradient estimate takes the form

$$\frac{1}{T}\int_0^T g(\theta_k, \xi(t, \theta_k))dt = \frac{1}{T}\int_0^T \left(c^+ I_{\{x(t)>0\}} - c^- I_{\{x(t)<0\}}\right)dt.$$

Example 9.12 gives an illustration for a two-machine system.

Remark 9.6. Equation (9.7) is not a standard stochastic approximation algorithm since averaging is used in the scheme together with continuous time random processes. In Yin, Yan, and Lou [228], with the goal of obtaining an asymptotically unbiased gradient estimator, T is chosen so that $T = T_\eta \to \infty$ as $\eta \to 0$. However, as noted in Kushner and Vázquez-Abad [143], and Kushner and Yin [145], one need not choose T so large. To guarantee the convergence of the algorithm, it is not necessary to use unbiased (or asymptotically unbiased) estimators of the gradient. In fact, large T may result in inefficient performance of the algorithms. A little bias would not and should not concern us.

9.3.2 Convergence

This subsection is devoted to investigating the convergence of the proposed algorithms. To proceed, the following assumptions are needed. For simplicity, assume the initial approximation θ_0 to be nonrandom.

(**A9.1**) For each θ and each k_1,

$$\frac{1}{kT}\sum_{j=k_1}^{k+k_1-1} E^{\mathcal{F}_{k_1}} \int_{jT}^{(j+1)T} g(\theta, \xi(t))dt \to \nabla J(\theta)$$

in probability, as $k \to \infty$, where $E^{\mathcal{F}_{k_1}}$ denotes the conditional expectation on $\mathcal{F}_{k_1 T} = \{\xi(s),\ s \le k_1 T\}$, for an integer $k_1 \ge 0$.

(**A9.2**) For each $T_1 < \infty$, $t \in [0, T_1]$,

$$\lim_{\delta \to 0} E\left(\sup_{|\theta - \tilde{\theta}| < \delta} |g(\theta, \xi(t)) - g(\tilde{\theta}, \xi(t))| \right) = 0.$$

(**A9.3**) For each $N < \infty$, the set of functions

$$\left\{ \sup_{|\theta| \le N} |g(\theta, \xi(t))| \right\} \text{ is uniformly integrable.}$$

Remark 9.7. These assumptions originate from particular applications of manufacturing models. Dealing with specific applications, these conditions can often be verified (see Yan, Yin, and Lou [220]). Condition (A9.1) is an ergodicity condition in the sense of convergence in probability, and is a basic averaging condition. If $\xi(\cdot)$ is a ϕ-mixing process with $E|\xi(t)| < \infty$, then it is a strongly ergodic process and hence (A9.1) holds. In fact, in this case, the convergence is in the sense of with probability one.

Condition (A9.2) indicates that the function $g(\cdot, \xi)$ may not be continuous, but its expectation is continuous such as for the case that $g(\cdot, \xi)$ is an indicator function or a combination of indicator functions.

In various applications, the function $g(\theta, \xi)$ is often bounded. In such a case, (A9.3) is verified. Condition (A9.3) allows us to deal with more complex situations. For example, if

$$|g(\theta, \xi)| \le h_0(\theta)\widetilde{g}_1(\xi) + \widetilde{g}_2(\xi),$$

where $h_0(\theta)$ is a continuous function, and $E|\tilde{g}_i(\xi)|^{1+\gamma} < \infty$, $i = 1, 2$, for some $\gamma > 0$, then condition (A9.3) is also satisfied.

To proceed, we work with continuous time interpolated processes. Let $\theta^\eta(\cdot)$ be defined by $\theta^\eta(0) = \theta_0$ and $\theta^\eta(t) = \theta_k$ for $t \in [k\eta, (k+1)\eta)$. Under the framework of weak convergence (see Kushner [139], and Kushner and Yin [145]), it will be shown that the following limit theorem holds.

Theorem 9.8. *Suppose that* (A9.1)–(A9.3) *are satisfied and the differential equation*

$$\frac{d\theta(t)}{dt} = -\nabla J(\theta) \tag{9.8}$$

has a unique solution for each initial condition θ_0. Assume, for simplicity, that $\theta^\eta(0) = \theta_0$ is independent of η. Then $\{\theta^\eta(t)\}$ is tight in $D([0, \infty); \mathbb{R}^n)$. Every weakly convergent subsequence has the same limit $\theta(\cdot)$ that satisfies the differential equation (9.8).

Remark 9.9. Recall that $D([0,\infty);\mathbb{R}^n)$ denotes the space of $\mathbb{R}^n$-valued functions that are right continuous and have left-hand limits, endowed with the Skorohod topology; see Section A.2 in Appendix A. In lieu of choosing $\theta^\eta(0) = \theta_0$, independent of η, one may use $\theta_0 = \theta_0^\eta$ and hence $\theta^\eta(0) = \theta_0^\eta$ (depending on η). Under an additional condition $\theta_0^\eta \Rightarrow \theta_0$, the result still holds.

Proof of Theorem 9.8: To avoid possible unboundedness, a truncation device will be used (see (A.8) in Appendix for a definition). For each $N < \infty$, let $\theta^{\eta,N}(\cdot)$ be the N-truncation of $\theta^\eta(\cdot)$ such that $\theta^{\eta,N}(t) = \theta^\eta(t)$ up until the first exit from the N-sphere $S_N = \{\theta; |\theta| \le N\}$. A pertinent use of the truncation device requires the use of a truncation function $q_N(\cdot)$, which is a smooth function defined as

$$q_N(\theta) = \begin{cases} 1 & \text{for } |\theta| \le N, \\ 0 & \text{for } |\theta| \ge N+1. \end{cases}$$

One then replace $g(\theta,\xi)$ below by $g_N(\theta,\xi) = g(\theta,\xi)q_N(\theta)$. For notational simplicity, we shall omit the truncation function in what follows, however.

In view of the definition of the interpolation (without loss of generality, assume that t/η and $(t+s)/\eta$ are integers) and choosing a sequence of integers $\{k_\eta\}$ such that $k_\eta \to \infty$ as $\eta \to 0$ and $\eta k_\eta = \delta_\eta \to 0$, we have

$$\begin{aligned} \theta^{\eta,N}(t) &= \theta^{\eta,N}(0) - \frac{\eta}{T}\sum_{j=0}^{t/\eta-1}\int_{jT}^{(j+1)T} g(\theta_j^{\eta,N},\xi(\upsilon))d\upsilon \\ &= \theta^{\eta,N}(0) - \sum_{0\le i\delta_\eta\le t}\frac{\delta_\eta}{k_\eta T} \\ &\qquad \times \sum_{ik_\eta\le j\le (i+1)k_\eta-1}\int_{jT}^{(j+1)T} g(\theta_j^{\eta,N},\xi(\upsilon))d\upsilon \\ &= \theta^{\eta,N}(0) - \int_0^t B^\eta(\tau)d\tau, \end{aligned} \tag{9.9}$$

where $B^\eta(\cdot)$ is a piecewise-constant function on $[i\delta_\eta,(i+1)\delta_\eta)$, that is,

$$B^\eta(t) = \frac{1}{k_\eta T}\sum_{ik_\eta\le j\le (i+1)k_\eta-1}\int_{jT}^{(j+1)T} g(\theta_j^{\eta,N},\xi(\upsilon))d\upsilon \tag{9.10}$$

for $t \in [i\delta_\eta,(i+1)\delta_\eta)$. It follows from (9.9) that

$$\frac{d\theta^{\eta,N}(t)}{dt} = -B^\eta(t).$$

Condition (A9.3) implies that

$$\left\{\frac{1}{T}\int_{jT}^{(j+1)T} g(\theta_j^N,\xi(\upsilon))d\upsilon : \ j = 1,2,\ldots\right\} \text{ is uniformly integrable.}$$

Then by virtue of Theorem A.15, $\{\theta^{\eta,N}(\cdot), B^{\eta}(\cdot)\}$ is tight and the limit of any weakly convergent subsequence has continuous paths with probability one.

Pick out an arbitrary convergent subsequence and denote the limit by $(\theta^N(\cdot), \overline{B}(\cdot))$. By the Skorohod representation (without changing notation), we may assume that

$$(\theta^{\eta,N}(\cdot), B^{\eta}(\cdot)) \to (\theta^N(\cdot), \overline{B}(\cdot)) \text{ w.p.1}$$

and the convergence is uniform on any finite time interval.

Define

$$M^N(t) = \theta^N(t) - \theta^N(0) + \int_0^t \overline{B}(\theta^N(\upsilon))d\upsilon. \tag{9.11}$$

It will be seen in what follows that $\overline{B}(\cdot)$ is equal to $\nabla J(\cdot)$. If we can show that $M^N(t)$ is a continuous martingale, the limit theorem will hold for the truncated process. Note that $M^N(0) = 0$ and $M^N(t)$ is Lipschitz continuous. If it is a martingale, it must satisfy $M^N(t) \equiv 0$ (see Theorem A.21). Therefore, we need only verify the martingale property.

To verify the martingale property, let $h(\cdot)$ be any bounded and continuous function, κ be any positive integer, and t_{i_1} be such that $t_{i_1} < t < t + s$ for $i_1 \le \kappa$. In view of the weak convergence and the Skorohod representation, we have

$$\begin{aligned} &\lim_{\eta\to 0} Eh(\theta^{\eta,N}(t_{i_1}), i_1 \le \kappa)\left(\theta^{\eta,N}(t+s) - \theta^{\eta,N}(t)\right) \\ &\quad = Eh(\theta^N(t_{i_1}), i_1 \le \kappa)\left(\theta^N(t+s) - \theta^N(t)\right). \end{aligned} \tag{9.12}$$

Recall that $E^{\mathcal{F}_j}$ denotes the conditional expectation with respect to the σ-algebra $\mathcal{F}_{jT} = \sigma\{\xi(t), t \le jT\}$. Using the recursion (9.9), we have

$$\begin{aligned} &\lim_{\eta\to 0} Eh(\theta^{\eta,N}(t_{i_1}), i_1 \le \kappa)\left(\theta^{\eta,N}(t+s) - \theta^{\eta,N}(t)\right) \\ &\quad = \lim_{\eta\to 0} Eh(\theta^{\eta,N}(t_{i_1}), i_1 \le \kappa) \\ &\qquad \times\left(-\sum_{i\delta_\eta=t}^{t+s} \frac{\delta_\eta}{k_\eta T} \sum_{ik_\eta \le j \le (i+1)k_\eta - 1} E^{\mathcal{F}_j} \int_{jT}^{(j+1)T} g(\theta_j^{\eta,N}, \xi(\upsilon))d\upsilon\right) \\ &\quad = \lim_{\eta\to 0} Eh(\theta^{\eta,N}(t_{i_1}), i_1 \le \kappa) \\ &\qquad \times\left(-\sum_{i\delta_\eta=t}^{t+s} \frac{\delta_\eta}{k_\eta T} \sum_{ik_\eta \le j \le (i+1)k_\eta - 1} E^{\mathcal{F}_j} \int_{jT}^{(j+1)T} g(\theta^{\eta,N}(\tau), \xi(\upsilon))d\upsilon\right). \end{aligned} \tag{9.13}$$

The last equality above follows from the weak convergence, the Skorohod representation, (A9.2), and $\eta j \to \tau$ as $\eta \to 0$ for j satisfying $ik_\eta \le j \le (i+1)k_\eta$.

Now for any $\Delta > 0$, there exists a function $\theta^{N,\Delta}(\cdot)$ that takes only finitely many values (say $\overline{\theta}_1, \ldots, \overline{\theta}_{n_0}$) such that

$$|\theta^N(\tau) - \theta^{N,\Delta}(\tau)| < \Delta.$$

Consequently, by applying (A9.2), the limit in (9.13) is the same as that of

$$\lim_{\eta\to 0} Eh(\theta^{\eta,N}(t_{i_1}), i_1 \le \kappa) \times \left(-\sum_{i\delta_\eta = t}^{t+s} \frac{\delta_\eta}{k_\eta T} \sum_{ik_\eta \le j \le (i+1)k_\eta - 1} E^{\mathcal{F}_j} \int_{jT}^{(j+1)T} g(\theta^{N,\Delta}(\tau), \xi(\upsilon)) d\upsilon \right).$$

By virtue of (9.10), the limit of $B^\eta(\tau)$ is the same as that of

$$\begin{aligned} &\frac{1}{k_\eta T} \sum_{ik_\eta \le j \le (i+1)k_\eta - 1} E^{\mathcal{F}_j} \int_{jT}^{(j+1)T} g(\theta^{N,\Delta}(\tau), \xi(\upsilon)) d\upsilon \\ &= \sum_{i_2=1}^{n_0} \frac{1}{k_\eta T} \sum_{ik_\eta \le j \le (i+1)k_\eta - 1} E^{\mathcal{F}_j} \int_{jT}^{(j+1)T} g(\overline{\theta}_{i_2}, \xi(\upsilon)) d\upsilon \, I_{\{\theta^{N,\Delta}(\tau) = \overline{\theta}_{i_2}\}} \\ &\to \sum_{i_2=1}^{n_0} \nabla J(\overline{\theta}_{i_2}) I_{\{\theta^{N,\Delta}(\tau) = \overline{\theta}_{i_2}\}} \quad \text{in probability} \\ &= \nabla J(\theta^{N,\Delta}(\tau)). \end{aligned}$$

Since $\Delta > 0$ is arbitrary,

$$\frac{1}{k_\eta T} \sum_{ik_\eta \le j \le (i+1)k_\eta - 1} E^{\mathcal{F}_j} \int_{jT}^{(j+1)T} g(\theta^{N,\Delta}(\tau), \xi(\upsilon)) d\upsilon \to \nabla J(\theta^N(\tau))$$

in probability as $\eta \to 0$. Incorporating this with (9.12) and (9.13) yields

$$\begin{aligned} &\lim_{\eta\to 0} Eh(\theta^{\eta,N}(t_{i_1}), i_1 \le \kappa) \left(\theta^{\eta,N}(t+s) - \theta^{\eta,N}(t)\right) \\ &\quad = Eh(\theta^N(t_{i_1}), i_1 \le \kappa) \left(\theta^N(t+s) - \theta^N(t) + \int_t^{t+s} \nabla J(\theta^N(\tau)) d\tau \right). \end{aligned} \tag{9.14}$$

Combining (9.12) to (9.14), we arrive at

$$Eh(\theta^N(t_i), i \le \kappa) \left(\theta^N(t+s) - \theta^N(t) + \int_t^{t+s} \nabla J(\theta^N(\tau)) d\tau \right) = 0.$$

Hence $M^N(t)$ is a martingale.

Finally, use the idea of Kushner [139, Theorem 2.2 and the Corollary], to finish the proof. The main idea is outlined below. Let $P_{\theta(0)}(\cdot)$ (the subscript $\theta(0)$ signifies the dependence on the initial data) and $P^N(\cdot)$ be the measures induced by $\theta(\cdot)$ and $\theta^N(\cdot)$, respectively, on $\mathcal{B}$, the σ-algebra of Borel subsets of $D([0,\infty);\mathbb{R}^n)$. $P_{\theta(0)}(\cdot)$ is unique since there is a unique solution to the ordinary differential equation for the initial value $\theta(0)$. Thus, for each $T_1<\infty$,

$$P_{\theta(0)}(\theta(\cdot)\in A) = P^N(\theta^N(\cdot)\in A)$$

for each $A\in\mathcal{B}$ such that $\theta(t)$ takes values in S_N (the N-sphere). As a result,

$$\lim_{N\to\infty} P_{\theta(0)}\left(\sup_{t\le T_1}|\theta(t)|\le N\right)=1.$$

This, together with the weak convergence of $\theta^{\eta,N}(\cdot)$, implies that $\theta^\eta(\cdot)\Rightarrow\theta(\cdot)$. Since the limit is unique, it does not depend on the chosen subsequence. The proof of the theorem is completed. □

Theorem 9.8 is similar to the law of large numbers. It gives information on the location and/or distribution of $\theta^\eta(\cdot)$ for small η and for large but bounded t. There is a natural connection between the recursive procedure and the corresponding ordinary differential equation. The optimal threshold values sought are stable points of the differential equation (9.8).

Theorem 9.10. *Assume that the conditions of Theorem 9.8 hold. Suppose the differential equation in (9.8) has a unique asymptotically stable point θ^* (in the sense of Liapunov stability) and the set*

$$\{\theta_k;\ k<\infty,\eta>0\} \tag{9.15}$$

is bounded in probability in that for each $\Delta>0$, there is a $\kappa_\Delta>0$ such that for all $\eta>0$, and all k,

$$P(|\theta_k|\ge\kappa_\Delta)\le\Delta.$$

Let $t_\eta\to\infty$ as $\eta\to0$. Then $\theta^\eta(t_\eta+\cdot)$ is tight in $D([0,\infty;\mathbb{R}^n)$ and any weak limit is equal to θ^*.

Equation (9.15) can be established by using a perturbed Liapunov function method (see Kushner [139], and Kushner and Yin [145]). Theorem 9.10 can be deduced analogously as in the aforementioned reference (see also Kushner and Yin [144, Theorem 5.1]). We give the main idea below. Let $T_1>0$, and consider the pair $\{\theta^\eta(t_\eta+\cdot),\theta^\eta(t_\eta-T_1+\cdot)\}$, which is tight. Choose a weakly convergent subsequence (still indexed by η) with limit denoted by $(\theta(\cdot),\theta_{T_1}(\cdot))$. Then $\theta(0)=\theta_{T_1}(T_1)$. The "initial value" $\theta_{T_1}(0)$ may not be known, but all possible values of $\theta_{T_1}(0)$ belong to a set that is bounded in probability for all T_1 and all convergent subsequences.

The asymptotic stability then implies that for each $\Delta > 0$, there is a $T_\Delta > 0$ such that for all $T_1 > T_\Delta$,

$$P(|\theta_{T_1}(T_1) - \theta^*| > \Delta) < \Delta.$$

Hence the theorem follows.

9.3.3 Examples

To illustrate the idea of approximation of threshold control policies, we consider two manufacturing models in this subsection. The reasons include: (a) Demonstrate that for the example treated below, threshold control policies are indeed optimal. (b) Illustrate the use of stochastic optimization procedure for the long-run average cost criteria.

In the first example, our approximation results compare well with those of Bielecki and Kumar [12]. The second example deals with a two-machine system, in which no closed-form solution (analytic solution or explicit formula) has been found up to date. To reformulate the problem using optimization formulation, we develop stochastic algorithms to estimate the optimal threshold values.

Example 9.11. Return to Example 9.5. Choose $\lambda = 0.1$, $\mu = 0.125$, $z = 1.0$, $c^+ = 2.0$, $c^- = 9.0$, and $u_{\max} = 2.0$. Applying the result of [12], the optimal threshold level and the optimal cost are $\theta^* = 66.96$ and $J(\theta^*) = 142.89$, respectively. Using our algorithm with step size $\eta = 0.5$ and initial value $\theta_0 = 20$, and taking averages of 100 replications, the approximation method gives $\tilde{\theta}^* = 67.23$ (with a 95% confidence interval $[66.64, 67.80]$), and $J(\tilde{\theta}^*) = 139.43$. Different initial conditions yield equally good approximation results. The already existing analytical result allows us to compare the performance of the approximation algorithm with the closed-form solution. We would like to note that even if the explicit solution is available, various parameters (λ, μ, etc.) may not be known; these parameters are not required in our approach. Thus the stochastic optimization approach provides a viable alternative and effective procedure.

Example 9.12. The example to be presented was considered by Yan, Yin, and Lou [220], in which a combination of infinitesimal perturbation analysis initiated by Ho (see Ho and Cao [87]) and stochastic approximation was suggested. Kushner and Vázquez-Abad [143] further examined this model and relaxed the conditions for convergence. For $i = 1, 2$, use $x_i(t)$ to denote the inventory levels of machine i, and $u_i(t)$ the production rate of machine i. Since we are not solving the dynamic programming equations, the demand processes can be quite general. They do not have to be constants although a constant demand rate is used here for simplicity. In what follows, we formulate the surplus control model, and construct the approximation procedure.

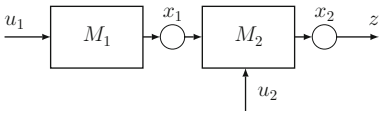

FIGURE 9.1. A Two-Machine System

The two machines are in a cascade form and the inventory levels are given by

$$\begin{aligned} \frac{dx_1(t)}{dt} &= u_1(t) - u_2(t), \\ \frac{dx_2(t)}{dt} &= u_2(t) - z, \\ x_1(t) &\geq 0,\ t \geq 0. \end{aligned} \tag{9.16}$$

For each $i = 1, 2$, let the machine capacity be $\alpha_i(t)$ with

$$\alpha_i(t) = \begin{cases} 1, & \text{machine } i \text{ is working;} \\ 0, & \text{otherwise.} \end{cases}$$

We then have

$$0 \leq u_i(t) \leq u_{i\max}\alpha_i(t),\ i = 1, 2,$$

where $u_{i\max}$ is the maximum capacity of machine i for $i = 1, 2$. Assume that

$$u_{1\max} > u_{2\max} > z.$$

This scenario is depicted in Figure 9.1.

Surplus at machine i is defined as the difference between accumulative production and accumulated demand, i.e., it is the inventory level (or work in progress) at machine i plus the inventory level of all downstream machines. Let $s_i(t)$ be the surplus for machine i, for $i = 1, 2$:

$$s_1(t) = x_1(t) + x_2(t) \text{ and } s_2(t) = x_2(t).$$

Note that the surplus can be positive or negative. A negative surplus means that there is a backlog. With these definitions, the system dynamics can also be written as

$$\begin{aligned} \frac{ds_1(t)}{dt} &= u_1(t) - z, \\ \frac{ds_2(t)}{dt} &= u_2(t) - z, \\ s_1(t) &\geq s_2(t),\ t \geq 0. \end{aligned} \tag{9.17}$$

Comparing (9.17) with the one-machine model in the work of Akella and Kumar [2], and Bielecki and Kumar [12], the surplus control policy is more or less like having two machines operating independently.

Let θ_i denote the surplus threshold levels of machine i. The control policy is given by

$$u_1(t) = \begin{cases} u_{1\max} I_{\{\alpha_1(t)=1\}}, & \text{if } s_1(t) < \theta_1, \\ z I_{\{\alpha_1(t)=1\}}, & \text{if } s_1(t) = \theta_1, \\ 0, & \text{if } s_1(t) > \theta_1; \end{cases}$$

$$u_2(t) = \begin{cases} u_{2\max} I_{\{\alpha_2(t)=1\}}, & \text{if } s_2(t) < \theta_2,\ s_1(t) - s_2(t) > 0, \\ z I_{\{\alpha_2(t)=1\}}, & \text{if } s_2(t) = \theta_2,\ s_1(t) - s_2(t) > 0, \\ 0, & \text{if } s_2(t) > \theta_2,\ s_1(t) - s_2(t) > 0. \end{cases}$$

The interpretation of the control policies is similar to that of the one machine case. The problem to be investigated is to find the optimal threshold value $\theta^* = (\theta_1^*, \theta_2^*)$ such that the cost functional

$$J(\theta) = \lim_{T\to\infty} \frac{1}{T} E \int_0^T (c_1 x_1(t) + c_2^+ x_2^+(t) + c_2^- x_2^-(t))dt \tag{9.18}$$

is minimized.

Define

$$\tau_0^1(\theta) = 0,\ \tau_0^2(\theta) = \inf\{t > 0;\ s_1(t,\theta) = \theta_1\},$$

and $\tau_k^i(\theta)$ for $i = 1, 2$ and $k > 0$, recursively by

$$\tau_k^1(\theta) = \min\{t \geq \tau_{k-1}^2(\theta); s_1(t,\theta) = s_2(t,\theta)\} \quad \text{and}$$

$$\tau_k^2(\theta) = \min\{t \geq \tau_k^1(\theta); s_2(t,\theta) = \theta_2\}.$$

Moreover, define $\gamma_0^2(\theta) = 0$ and

$$\gamma_k^1(\theta) = \min\{t \geq \gamma_{k-1}^2(\theta);\ s_2(t,\theta) = \theta_2\},$$

$$\gamma_k^2(\theta) = \min\{t \geq \gamma_k^1(\theta);\ s_1(t,\theta) = s_2(t,\theta)\}.$$

Furthermore, let

$$w_1(t,\theta) = \sum_{k=1}^{\infty} I_{\{\tau_k^1(\theta) \leq t \leq \tau_k^2(\theta)\}} \quad \text{and}$$

$$w_2(t,\theta) = \sum_{k=1}^{\infty} I_{\{\gamma_k^1(\theta) \leq t \leq \gamma_k^2(\theta)\}}.$$

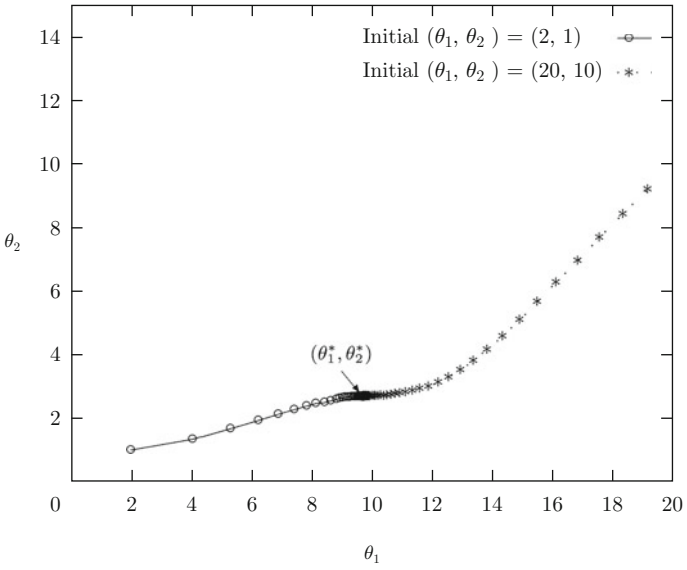

FIGURE 9.2. Convergence of the Iterates

Note that at most one of the indicator functions in the sums above can be positive at a given time t. The summation is thus well defined. Then the integrand of the gradient estimates can be written as

$$g_1(\theta, \xi(t)) = c_1 I_{\{t \geq \tau_0^2(\theta)\}} + c_2^+ w_1(t, \theta) I_{\{s_2(t,\theta) \geq 0\}} - c_2^- w_1(t, \theta) I_{\{s_2(t,\theta) < 0\}},$$

$$g_2(\theta, \xi(t)) = c_2^+ w_2(t, \theta) I_{\{s_2(t,\theta) \geq 0\}} - c_2^- w_2(t, \theta) I_{\{s_2(t,\theta) < 0\}}$$

via perturbation analysis (see Ho and Cao [87] and Glasserman [74]). The notation of the stopping times, suggested in Kushner and Vázquez-Abad [143], allows us to write the gradient estimates in a compact form. In the original paper of Yan, Yin, and Lou [220], some auxiliary processes were used in lieu of the stopping times. The ideas are the same, however.

Figure 9.2 demonstrates the performance of the algorithm for two machine case. One may generate contour curves via simulation for each set of threshold values, the approximation obtained in our algorithm can be seen to belong to the region of optimality. Our numerical results demonstrate that the initial conditions do not affect the algorithm much and the algorithm is robust with respect to the initial data.

9.3.4 Error Bounds

This subsection continues our investigation of Algorithm (9.7). We derive an error bound on the approximation sequence. The consideration of this subsection falls into the category of rates of convergence.

Theorem 9.13. *Assume that the conditions of Theorem 9.8 are satisfied and there is a twice continuously differentiable Liapunov function $V(\cdot)$ such that*

$$V(\theta) \geq 0,\ V(\theta) \to \infty \text{ as } |\theta| \to \infty,$$

$$(V_\theta(\theta))' \nabla J(\theta) \geq \kappa_0 V(\theta),$$

for some $\kappa_0 > 0$, and $V_{\theta\theta}(\cdot)$ is bounded, where $V_\theta(\cdot)$ and $V_{\theta\theta}(\cdot)$ denote the first and the second derivatives of $V(\cdot)$, respectively, and ζ' denotes the transpose of a vector $\zeta \in \mathbb{R}^{n\times 1}$. Suppose that for each θ,

$$\begin{aligned} &\left| \sum_{j=k}^{\infty} E^{\mathcal{F}_k} \frac{1}{T} \int_{jT}^{(j+1)T} (g(\theta, \xi(t)) - \nabla J(\theta))\, dt \right| \leq K \text{ and} \\ &\left| \sum_{j=k}^{\infty} E^{\mathcal{F}_k} \frac{1}{T} \int_{jT}^{(j+1)T} (g(\theta, \xi(t)) - \nabla J(\theta))_\theta\, dt \right| \leq K \end{aligned} \tag{9.19}$$

for some $K > 0$, where $E^{\mathcal{F}_k}$ denotes the conditional expectation with respect to $\mathcal{F}_{kT} = \sigma\{\xi(s), s \leq kT\}$. Assume that

$$|g(\theta, \xi)|^2 + |\nabla J(\theta)|^2 \leq K(1 + V(\theta)).$$

Then

$$\limsup_{k\to\infty} V(\theta_k) = O(\eta). \tag{9.20}$$

Remark 9.14. An alternative form of the first inequality in (9.19) is

$$\left| \int_{kT}^{\infty} E^{\mathcal{F}_k} (g(\theta, \xi(t)) - \nabla J(\theta))\, dt \right| \leq K, \tag{9.21}$$

and similar analogue holds for the second inequality in (9.19). It is readily seen that if $\xi(\cdot)$ is a ϕ-mixing process with mixing rate $\rho(\cdot)$ such that $\int_0^\infty \rho(t) < \infty$, the mixing inequality (see Kushner [139, p. 82]) implies that

$$\left| \int_{kT}^{\infty} E^{\mathcal{F}_k} (g(\theta, \xi(t)) - \nabla J(\theta))\, dt \right| \leq 2 \int_{kT}^{\infty} \rho(t - kT) dt \leq K,$$

with similar estimates regarding the second inequality in (9.19).

Outline of Proof of Theorem 9.13: We use a technique known as perturbed Liapunov function method (see Kushner and Yin [145] and the references therein). Since the proof is similar to that of [145, Chapter 10], only an outline of the idea is given.

By virtue of a Taylor expansion, direct calculation leads to

$$\begin{aligned} &E^{\mathcal{F}_k} V(\theta_{k+1}) - V(\theta_k) \\ &\quad = -\eta (V_\theta(\theta_k))' \nabla J(\theta_k) + O(\eta^2)(1 + V(\theta_k)) \\ &\qquad - \frac{\eta (V_\theta(\theta_k))'}{T} E^{\mathcal{F}_k} \int_{kT}^{(k+1)T} (g(\theta_k, \xi(t)) - \nabla J(\theta_k))\, dt. \end{aligned}$$

Define

$$V_1^\eta(k) = -\frac{\eta E^{\mathcal{F}_k} (V_\theta(\theta_k))'}{T} \sum_{j=k}^{\infty} \int_{jT}^{(j+1)T} (g(\theta_k, \xi(t)) - \nabla J(\theta_k))\, dt,$$

$$V^\eta(k) = V(\theta_k) + V_1^\eta(k).$$

It is easily seen that

$$|V_1^\eta(k)| \le \eta K (1 + V(\theta_k)). \tag{9.22}$$

Detailed calculation leads to

$$E^{\mathcal{F}_k} V^\eta(k+1) - V^\eta(k) \le -\eta \kappa_0 V(\theta_k) + O(\eta^2)(1 + V(\theta_k)).$$

Equation (9.22) then yields that

$$E^{\mathcal{F}_k} V^\eta(k+1) \le V^\eta(k) - \eta \kappa_0 V^\eta(k) + O(\eta^2)(1 + V^\eta(k)).$$

By choosing η small enough, we obtain

$$E^{\mathcal{F}_k} V^\eta(k+1) \le \left(1 - \frac{\eta \kappa_0}{2}\right) V^\eta(k) + O(\eta^2).$$

Iterating on the above inequality, taking expectation and lim sup as $k \to \infty$, and using (9.22), the desired result follows. □

Remark 9.15. If the Liapunov function is locally (near θ^*) quadratic, it can be shown that there is an N_η such that

$$\left\{ U_k = \frac{\theta_k - \theta^*}{\sqrt{\eta}} : \ k \ge N_\eta \right\}$$

is tight. Define

$$U^\eta(t) = U_k \text{ for } t \in [(k - N_\eta)\eta, (k - N_\eta + 1)\eta).$$

Under further conditions, one can obtain a local results in connection with a stochastic differential equation.

9.4 Notes

This chapter has been devoted to the numerical solutions of the control problems. It consists of numerical methods for solving the HJB equations and approximation for threshold control policies. The main techniques used are the finite difference approximation methods (see Kushner [138], Kushner and Dupuis [141]) and the stochastic optimization methods (see Kushner and Yin [145]).

The computational methods for the optimal control problem presented here are equivalent to methods of computing the optimal controls for discrete Markov chain models. For a general background and discussion on the method and many references for controlled diffusion and jump diffusion processes, we refer to Kushner [138] and Kushner and Dupuis [141]. For applications of such methods in manufacturing models, we refer the reader to Yan and Zhang [221] among others. In the implementation of the numerical method, one may use either "value iteration" that is essentially a fixed point iteration, or "policy iteration," and the variation and/or modification of the aforementioned procedures, such as Jacobi iteration, Gauss-Seidel method, and accelerated Jacobi and Gauss-Seidel methods (see [141] for a detailed discussion on this and related matters). In practice, one often wishes to use an accelerated procedure to speed up the computation. The recent advances on multigrid and domain decomposition methods give new hope for solving large-dimensional systems. As its deterministic counterpart, by and large, this is still a current research topic for stochastic systems.

Converting optimal control problems into optimization problems under threshold type of control policies is in Yin, Yan, and Lou [228], and Yan, Yin, and Lou [220]. Early work on developing hedging policies is in Kimemia and Gershwin [121]; related work along this direction is in Caramanis and Liberopoulos [24]. For systems arising in production planning see [220] where a combined approach of the infinitesimal perturbation analysis with stochastic optimization methods is utilized. The computation presented here was done by Houmin Yan. We are very grateful for his help. For further approaches on using stochastic approximation based algorithms for optimization with long-run average costs, see Kushner and Yin [145, Chapter 9] and the references therein. In applications, one may use a projection or truncation algorithm. Treatments of such algorithms and a comprehensive study on stochastic approximation algorithms can be found in [145].

10
Hybrid LQG Problems

10.1 Introduction

This chapter develops asymptotic optimal controls of a class of hybrid linear quadratic Gaussian (LQG) systems that consist of a collection of diffusions coupled by a finite-state Markov chain. It is well known that LQG systems are most popular in the control systems community, especially owing to their simple structure. In addition, many nonlinear systems can be linearized locally to simplify the analysis. Many LQG systems stem from various applications in speech recognition, pattern recognition, signal processing, telecommunications, and manufacturing. Owing to their importance, there has been a growing interest in studying the control and optimization of such systems.

In the traditional setting, feedback control design of linear systems is based on a plant with fixed parameters. This, however, prevents one from treating situations in which the actual systems differ from the assumed nominal model. Therefore, efforts have been made to design more "robust" controls such that certain requirements are met simultaneously for a set of plants. Owing to the needs in various applications, one is particularly interested in developing controls of hybrid systems.

To a large extent, a hybrid system shows both "continuous" and "discrete" characteristics. These system features can be seen from the following two aspects. In the formulation, apart from the usual Brownian noise, a random environment affects the dynamics through the system coefficient matrices. In contrast to the usual continuous dynamic systems,

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Stochastic Modelling 37, DOI 10.1007/978-1-4614-4346-9_10,

the random environment evolves as discrete events. It is thus natural to introduce a Markov chain model to represent the random environment. Enlarging the "state" by considering both the continuous state variables of the LQG problem and the Markov chain yields both continuous and discrete characteristics.

Unlike the traditional LQG problem in which the system coefficient matrices are fixed, we allow these matrices to depend on a Markov jump process with finite-state space. Thus the system displays different configuration corresponding to different states of the Markov chain. The situation is more involved as compared to the traditional setting. The value functions satisfy a set of Riccati equations involving the generator of the underlying Markov chain. In many applications, the state space of the Markov chain is often very large. It is thus difficult to obtain solutions to these Riccati equations. To overcome the difficulty, we use singular perturbation techniques in the modeling, control design, and optimization. The resulting systems naturally display certain two-time-scale behavior, a fast time scale and a slowly varying one. To put this in a manageable framework, we introduce a small parameter $\varepsilon > 0$, and model the underlying system as one involving two-time-scale Markov chains.

We use an averaging approach to analyze the system in which the underlying Markov chain involves weak and strong interactions. The idea is to aggregate the states according to their jump rates and replace the actual system with its average. Using the optimal control of the limit or averaged system as a guide, we then construct controls for the actual systems leading to feasible approximation schemes. Our investigation encompasses three cases, namely, recurrent Markov chains, inclusion of transient states, and inclusion of absorbing states. Although they are related, each of them has its distinct structure. We show that these approximation schemes give us nearly optimal controls. By focusing on approximate optimality, we are able to reduce the complexity of the underlying systems drastically. The reduction of dimensionality is the main advantage of the averaging approach. To demonstrate how the average schemes work, we provide a numerical example of a one-dimensional system.

The remainder of the chapter is organized as follows. In the next section, we present the basic formulation and the motivation for the two-time-scale problems. Section 10.3 proceeds with the study of the optimal LQG problem. Using a dynamic programming approach, we give the Hamilton-Jacobi-Bellman (HJB) equations satisfied by the value functions, and then derive the corresponding Riccati equations. Section 10.4 treats models involving recurrent Markov chains with fast and slow motions, and we obtain approximation schemes. Sections 10.5 and 10.6 continue our investigation by taking up the issues of inclusion of transient and absorbing states, respectively. Using probabilistic arguments and analytic techniques, the approximation schemes are shown to be nearly optimal. To further illustrate, Section 10.7 gives an example to demonstrate the

asymptotic properties and error bounds. The numerical results indicate that the approximation scheme performs quite well. Finally, we conclude the chapter with further thoughts and additional remarks in Section 10.9.

10.2 Problem Formulation

We consider a stationary finite-state Markov chain $\alpha(t) \in \mathcal{M} = \{1, \ldots, m\}$. Working with a finite horizon for some finite $T > 0$, consider the linear system

$$\begin{aligned} dx(t) &= [A(\alpha(t))x(t) + B(\alpha(t))u(t)]dt + \sigma dw(t), \\ x(s) &= x, \text{ for } s \le t \le T, \end{aligned} \tag{10.1}$$

where $x(t) \in \mathbb{R}^{n_1}$ is the state, $u(t) \in \mathbb{R}^{n_2}$ is the control, $A(i) \in \mathbb{R}^{n_1 \times n_1}$ and $B(i) \in \mathbb{R}^{n_1 \times n_2}$ are well defined and have finite values for $i \in \mathcal{M}$, and $w(\cdot)$ is a standard Brownian motion. Our objective is to find the optimal control $u(\cdot)$ such that the expected quadratic cost function

$$\begin{aligned} J(s, x, \alpha, u(\cdot)) = E\Big\{ \int_s^T & [x'(t)M(\alpha(t))x(t) \\ & + u'(t)N(\alpha(t))u(t)]dt + x'(T)Dx(T) \Big\} \end{aligned} \tag{10.2}$$

is minimized, where E is the expectation given $\alpha(s) = \alpha$ and $x(s) = x$, $M(i)$, $i = 1, \ldots, m$, are symmetric nonnegative definite matrices, and $N(i)$, $i = 1, \ldots, m$, and D are symmetric positive definite matrices.

In many applications, due to various sources of uncertainty, the Markov chain involved is often inevitably large. This brings about much of the difficulty. Moreover, these systems may be quite sensitive to small perturbations of the parameter values. Thus effort has been devoted to resolving the problem and to rendering a reasonable solution.

The idea is to take advantage of the intrinsic structural properties and to decompose the large-dimensional system into a number of subsystems each of which has a simpler structure. The underlying Markov chain has two time scales, i.e., $\alpha(t) = \alpha^\varepsilon(t)$, where the generator of $\alpha^\varepsilon(\cdot)$ Q^ε consists of two parts, a rapidly changing part and a slowly varying one, i.e.,

$$Q^\varepsilon = \frac{1}{\varepsilon}\widetilde{Q} + \widehat{Q}. \tag{10.3}$$

In the rest of this chapter, we consider the model (10.1) and (10.2) in which the process $\alpha(\cdot)$ is replaced by $\alpha^\varepsilon(\cdot)$. The resulting cost function is denoted by $J^\varepsilon(s, x, \alpha, u(\cdot))$. In addition, we assume the processes $\alpha^\varepsilon(\cdot)$ and $w(\cdot)$ to be independent.

10.3 Optimal Controls

This section studies the optimal LQG control problem using the dynamic programming approach, and derives the associated HJB and Riccati equations. Let

$$v^\varepsilon(s,x,i) = \inf_{u(\cdot)} J^\varepsilon(s,x,i,u(\cdot))$$

be the value function. Then v^ε satisfies the following system of HJB equations: for $0 \le s \le T$ and $i \in \mathcal{M}$,

$$\begin{aligned} 0 = \frac{\partial v^\varepsilon(s,x,i)}{\partial s} + \min_u \Big\{ & (A(i)x + B(i)u)' \frac{\partial v^\varepsilon(s,x,i)}{\partial x} \\ & + x'M(i)x + u'N(i)u + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma' \frac{\partial^2 v^\varepsilon(s,x,i)}{\partial x^2}\right) \\ & + Q^\varepsilon v^\varepsilon(s,x,\cdot)(i) \Big\}, \end{aligned} \tag{10.4}$$

with the boundary condition $v^\varepsilon(T,x,i) = x'Dx$, where

$$Q^\varepsilon f(s,x,\cdot)(i) = \sum_{j \ne i} q_{ij}^\varepsilon (f(s,x,j) - f(s,x,i)), \tag{10.5}$$

for a suitable $f(\cdot,\cdot,\cdot)$.

Intuitively, the solution to the system of HJB equations must be of quadratic form. We thus propose such a form and proceed with finding the related functions, which is to some extent like the method of undetermined coefficients. To proceed, let

$$v^\varepsilon(s,x,i) = x'K^\varepsilon(s,i)x + q^\varepsilon(s,i), \tag{10.6}$$

for some $m \times m$ matrix K^ε and a scalar function q^ε. Without loss of generality, we may assume K^ε to be symmetric. It follows that $(\partial/\partial x)v^\varepsilon = 2K^\varepsilon x$. Substituting (10.6) into (10.4) and comparing the coefficients of x leads to the following Riccati equations for $K^\varepsilon(s,i)$;

$$\begin{aligned} \dot{K}^\varepsilon(s,i) = & -K^\varepsilon(s,i)A(i) - A'(i)K^\varepsilon(s,i) - M(i) \\ & + K^\varepsilon(s,i)B(i)N^{-1}(i)B'(i)K^\varepsilon(s,i) - Q^\varepsilon K^\varepsilon(s,\cdot)(i), \end{aligned} \tag{10.7}$$

with $K^\varepsilon(T,i) = D$, where $Q^\varepsilon K^\varepsilon(s,\cdot)(i)$ is as in (10.5), and the equations for q^ε are

$$\dot{q}^\varepsilon(s,i) = -\mathrm{tr}(\sigma\sigma' K^\varepsilon(s,i)) - Q^\varepsilon q^\varepsilon(s,\cdot)(i), \tag{10.8}$$

with $q^\varepsilon(T,i) = 0$. Moreover, it can be shown (see, for example, Fleming and Rishel [63]) that the system of equations has a unique solution. In view of

the positive definite property of K^ε (see Lemma 10.2), the optimal control $u^{\varepsilon,*}$ has the form

$$u^{\varepsilon,*}(s,x,i) = -N^{-1}(i)B'(i)K^\varepsilon(s,i)x. \tag{10.9}$$

To find the optimal control, one has to solve the Riccati equations. However, in many problems in telecommunications and manufacturing, such solutions are difficult to obtain due to the large dimensionality. In this case, one has to resort to approximation schemes. We present an averaging approach in the next three sections.

10.4 Two-Time-Scale Approximation: Recurrent States

Here, we consider the cases that the states of the Markov chain are divisible into a number of groups such that it fluctuates very rapidly among different states within a group consisting of recurrent states, but jumps less frequently from one group to another. This is conveniently modeled in terms of the two-time-scale Markov chains as follows. Consider the generator of the Markov chain given by (10.3). Assume $\widetilde{Q}$ has a block-diagonal form

$$\widetilde{Q} = \text{diag}(\widetilde{Q}^1, \ldots, \widetilde{Q}^l), \tag{10.10}$$

where $\widetilde{Q}^k \in \mathbb{R}^{m_k \times m_k}$ are weakly irreducible, for $k = 1, \ldots, l$, and $\sum_{k=1}^{l} m_k = m$. Let

$$\mathcal{M}_k = \{s_{k1}, \ldots, s_{km_k}\} \quad \text{for} \quad k = 1, \ldots, l$$

denote the states corresponding to $\widetilde{Q}^k$ and let $\mathcal{M}$ denote the state space of the underlying chains. Then

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l \\ &= \{s_{11}, \ldots, s_{1m_1}, \ldots, s_{l1}, \ldots, s_{lm_l}\}. \end{aligned}$$

Since $\widetilde{Q}^k = (\tilde{q}^k_{ij})_{m_k \times m_k}$ and $\widehat{Q} = (\hat{q}_{ij})_{m \times m}$ are generators, for $k = 1, \ldots, l$, $\sum_{j=1}^{m_k} \tilde{q}^k_{ij} = 0$, for $i = 1, \ldots, m_k$, and $\sum_{j=1}^{m} \hat{q}_{ij} = 0$, for $i = 1, \ldots, m$. Note that $\widetilde{Q}$ governs the rapidly changing part and $\widehat{Q}$ describes the slowly varying components. The slow and fast components are intertwined through weak and strong interactions in the sense that the underlying Markov chain fluctuates rapidly within a single group $\mathcal{M}_k$ and jumps less frequently between any two groups $\mathcal{M}_k$ and $\mathcal{M}_j$ for $k \neq j$. More precisely, if we consider the states in $\mathcal{M}_k$ as a single "state," then all such "states" are

coupled through the matrix $\widehat{Q}$, and transitions from $\mathcal{M}_k$ to $\mathcal{M}_j$, $k \neq j$, are possible.

By aggregating all the states in $\mathcal{M}_k$ into a "super" state k, we obtain an aggregated process $\{\overline{\alpha}^\varepsilon(\cdot)\}$ defined by $\overline{\alpha}^\varepsilon(t) = k$ when $\alpha^\varepsilon(t) \in \mathcal{M}_k$. The process $\overline{\alpha}^\varepsilon(\cdot)$ is not necessarily Markovian. However, using a certain probabilistic argument, we have shown in Chapter 5 that $\overline{\alpha}^\varepsilon(\cdot)$ converges weakly to $\overline{\alpha}(\cdot)$ generated by

$$\overline{Q} = \text{diag}(\nu^1, \ldots, \nu^l)\widehat{Q}\text{diag}(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_l}),$$

where ν^k is the stationary distribution of $\widetilde{Q}^k$, $k = 1, \ldots, l$, and $\mathbb{1}_n = (1, \ldots, 1)' \in \mathbb{R}^n$. Moreover, for any bounded deterministic $\beta(\cdot)$,

$$E\left(\int_s^T [I_{\{\alpha^\varepsilon(t)=s_{kj}\}} - \nu_j^k I_{\{\overline{\alpha}^\varepsilon(t)=k\}}]\beta(t)dt\right)^2 = O(\varepsilon). \tag{10.11}$$

10.4.1 Limit Riccati Equations

The following theorem is concerned with the convergence of K^ε and q^ε. For any function F on $\mathcal{M}$, we define

$$\overline{F} = \sum_{j=1}^{m_k} \nu_j^k F(s_{kj}) \quad \text{for} \quad k = 1, \ldots, l.$$

Similarly, for F_1 and F_2, we define

$$\overline{F_1 F_2} = \sum_{j=1}^{m_k} \nu_j^k F_1(s_{kj}) F_2(s_{kj}).$$

Theorem 10.1. *For $k = 1, \ldots, l$ and $j = 1, \ldots, m_k$, $K^\varepsilon(s, s_{kj}) \to \overline{K}(s, k)$ and $q^\varepsilon(s, s_{kj}) \to \overline{q}(s, k)$, uniformly on $[0, T]$ as $\varepsilon \to 0$, where $\overline{K}(s, k)$ and $\overline{q}(s, k)$ are the unique solutions to the following differential equations: For $k = 1, \ldots, l$,*

$$\begin{aligned} \dot{\overline{K}}(s,k) &= -\overline{K}(s,k)\overline{A}(k) - \overline{A}'(k)\overline{K}(s,k) - \overline{M}(k) \\ &\quad + \overline{K}(s,k)\overline{BN^{-1}B'}(k)\overline{K}(s,k) - \overline{Q}\ \overline{K}(s,\cdot)(k), \end{aligned} \tag{10.12}$$

with $\overline{K}(T, k) = D$ and

$$\dot{\overline{q}}(s,k) = -\text{tr}\Big(\sigma\sigma'\overline{K}(s,k)\Big) - \overline{Q}\overline{q}(s,\cdot)(k), \tag{10.13}$$

with $\overline{q}(T, k) = 0$, respectively.

To proceed, we first establish three lemmas concerning the positive definiteness of K^ε, the a priori estimates of K^ε and q^ε, and their Lipschitz continuity.

Lemma 10.2. *The solution $K^\varepsilon(s,i)$ to (10.7) is positive definite for $i \in \mathcal{M}$ and $0 \le s \le T$.*

Proof: Let

$$S^\varepsilon(s,i) = N^{-1}(s,i)B'(s,i)K^\varepsilon(s,i) \quad \text{and}$$

$$F^\varepsilon(s,i) = A(s,i) - B(s,i)S^\varepsilon(s,i).$$

Then the optimal control $u^*(s,x,i)$ is equal to $-S^\varepsilon(s,i)x$ and the optimal trajectory $x^*(t)$ satisfies the equation

$$dx^*(t) = F^\varepsilon(t,\alpha^\varepsilon(t))x^*(t)dt + \sigma dw(t).$$

Let

$$\Sigma(t) = S^{\varepsilon,\prime}(t,\alpha^\varepsilon(t))N(\alpha^\varepsilon(t))S^\varepsilon(t,\alpha^\varepsilon(t)).$$

Then, we have

$$\begin{aligned} v^\varepsilon(s,x,i) &= J^\varepsilon(s,x,i,u^*(\cdot)) \\ &= E\Bigg\{\int_s^T (x^*(t))'[M(\alpha^\varepsilon(t)) + \Sigma(t)]x^{*,\prime}(t)dt \\ &\qquad + x^{*,\prime}(T)Dx^*(T)\Bigg\}, \end{aligned} \tag{10.14}$$

where $u^*(t) = u^*(t,\alpha^\varepsilon(t),x^*(t))$.

Let $\Psi(t,s)$ be the principal matrix solution to the ordinary differential equation

$$\dot{z}(t) = F^\varepsilon(t,\alpha^\varepsilon(t))z(t),$$

i.e., it satisfies the above differential equation with $z(t)$ replaced by $\Psi(t,s)$ and initial data $\Psi(s,s) = I$, for each sample path $\alpha^\varepsilon(\cdot)$. Then we can show that

$$\frac{\partial}{\partial x}x^*(t) = \Psi(t,s) \quad \text{and} \quad \frac{\partial^2}{\partial x^2}x^*(t) = 0.$$

Recall that $M(i)$, $N(i)$ and D are symmetric nonnegative definite. Differentiating both sides of equation (10.14) twice with respect to x and using the quadratic form of v leads to

$$\begin{aligned} K^\varepsilon(s,i) &= E\Big\{\int_s^T \Psi'(t,s)[M(\alpha^\varepsilon(t)) + \Sigma(t)]\Psi(t,s)dt \\ &\qquad + \Psi'(T,s)D\Psi(T,s)\Big\} \\ &\geq E\Big\{\Psi'(T,s)D\Psi(T,s)\Big\} > 0. \quad \square \end{aligned}$$

Lemma 10.3. *There exist constants C_0 and $k_0 > 0$ such that for all $s \in [0,T]$ and $i \in \mathcal{M}$,*

$$|K^\varepsilon(s,i)| \leq C_0 e^{k_0 T}(T+1),$$

$$|q^\varepsilon(s,i)| \leq C_0 e^{k_0 T}(T+1),$$

where $K^\varepsilon(s,i)$ and $q^\varepsilon(s,i)$ are solutions to equations (10.7) and (10.8), respectively.

Proof: We first show that

$$0 \leq v^\varepsilon(s,x,i) \leq C_0 e^{k_0 T}(|x|^2+1)(T+1). \tag{10.15}$$

It is clear that $v^\varepsilon(s,x,i) \geq 0$ because $J^\varepsilon(s,x,i,u(\cdot)) \geq 0$ for all admissible $u(\cdot)$. To derive the upper bound, let $u_0(t) = 0$. Then under such a control, we can show using Itô's formula that

$$E|x(t)|^2 \leq |x|^2 + k_0 \int_s^t (E|x(r)|^2 + 1)dr.$$

Let $\phi(t) = E|x(t)|^2 + 1$. Then we have

$$\phi(t) \leq (|x|^2+1) + k_0 \int_s^t \phi(r)dr.$$

In view of Gronwall's inequality, we have

$$E|x(t)|^2 \leq \phi(t) \leq e^{k_0 T}(|x|^2+1). \tag{10.16}$$

The above inequality holds for all $t \in [0,T]$. Now, for $0 \le s \le T$,

$$\begin{aligned} v^\varepsilon(s,x,i) &\le J^\varepsilon(s,x,i,u_0(\cdot)) \\ &\le E\left\{\int_s^T x'(t)M(\alpha^\varepsilon(t))x(t)dt + x'(T)Dx(T)\right\} \\ &\le C_0\left\{\int_s^T E|x(t)|^2dt + E|x(T)|^2\right\} \\ &\le C_0e^{k_0T}(|x|^2+1)(T-s+1) \\ &\le C_0e^{k_0T}(|x|^2+1)(T+1). \end{aligned}$$

Setting $x=0$ in (10.6) yields

$$v^\varepsilon(s,0,i) = q^\varepsilon(s,i) \quad \text{and}$$

$$0 \le q^\varepsilon(s,i) \le C_0e^{k_0T}(T+1).$$

We now show that $|K^\varepsilon(s,i)| \le C_0e^{k_0T}(T+1)$. In view of Lemma 10.2, the matrix $K^\varepsilon(s,i)$ is symmetric and positive definite. It follows from the definition of the matrix norm that

$$|K^\varepsilon(s,i)| = \max\{\text{eigenvalues of } K^\varepsilon(s,i)\}.$$

It suffices to show that for every unit vector ξ,

$$\xi'K^\varepsilon(s,i)\xi \le C_0e^{k_0T}(T+1). \tag{10.17}$$

In fact, in view of (10.15), we have by taking $x=a\xi$ with a a scalar,

$$\begin{aligned} a^2\xi'K^\varepsilon(s,i)\xi + q^\varepsilon(s,i) &= v^\varepsilon(s,a\xi,i) \\ &\le C_0e^{k_0T}(a^2+1)(T+1). \end{aligned}$$

Dividing both the left- and right-hand sides of this inequality by a^2 and sending $a \to \infty$, we obtain (10.17). □

Lemma 10.4. *For $i \in \mathcal{M}$, the solutions to (10.7) and (10.8), namely $K^\varepsilon(\cdot,i)$ and $q^\varepsilon(\cdot,i)$, are uniformly Lipschitz continuous on $[0,T]$.*

Proof: Let us divide the proof into two steps. In the first step, we show that the value function $v^\varepsilon(s,x,i)$ is uniformly Lipschitz. Then in the second step we prove the Lipschitz property of $K^\varepsilon(\cdot)$ and $q^\varepsilon(\cdot)$.

Step 1. We show that there exists a constant C (which may depend on T) such that for any $\delta > 0$ and $(s, x, i) \in [0, T] \times \mathbb{R}^{n_1} \times \mathcal{M}$,

$$|v^\varepsilon(s+\delta, x, i) - v^\varepsilon(s, x, i)| \le C(|x|^2 + 1)\delta. \tag{10.18}$$

In fact, for a given triple (s, x, i), we write the value function $v^\varepsilon(s, x, i)$ as follows:

$$\begin{aligned} v^\varepsilon(s, x, i) \ = E\Bigg\{ \int_s^T & [(x^*(t))' M(\alpha^\varepsilon(t)) x^*(t) \\ & + (u^*(t))' N(\alpha^\varepsilon(t)) u^*(t)] dt + (x^*(T))' D x^*(T) \Bigg\}, \end{aligned} \tag{10.19}$$

where $u^*(t) = u^*(t, \alpha^\varepsilon(t), x^*(t))$ is the optimal control defined in (10.9), $x^*(t)$ is the corresponding trajectory, and E is the conditional expectation given $(x^*(s), \alpha^\varepsilon(s)) = (x, i)$.

By a change of variable $t \mapsto t + \delta$ in (10.19), we have

$$\begin{aligned} v^\varepsilon(s, x, i) = \ E\Bigg\{ \int_{s+\delta}^{T+\delta} & [(x^*(t-\delta))' M(\alpha^\varepsilon(t-\delta)) x^*(t-\delta) \\ & + (u^*(t-\delta))' N(\alpha^\varepsilon(t-\delta)) u^*(t-\delta)] dt \\ & + (x^*(T))' D x^*(T) \Bigg\}. \end{aligned}$$

For $t \in [s+\delta, T+\delta]$, let

$$\widetilde{x}(t) = x^*(t-\delta), \ \widetilde{\alpha}(t) = \alpha^\varepsilon(t-\delta), \quad \text{and} \quad \widetilde{u}(t) = u^*(t-\delta),$$

Clearly,

$$\widetilde{x}(s+\delta) = x^*(s) = x, \ \widetilde{\alpha}(s+\delta) = \alpha^\varepsilon(s) = i$$

and $\widetilde{\alpha}(\cdot)$ is also a Markov chain generated by Q^ε. Moreover,

$$\begin{aligned} v^\varepsilon(s, x, i) \ &= E\Bigg\{ \int_{s+\delta}^{T+\delta} [\widetilde{x}'(t) M(\widetilde{\alpha}(t)) \widetilde{x}(t) \\ &\qquad + \widetilde{u}'(t) N(\widetilde{\alpha}(t)) \widetilde{u}(t)] dt + \widetilde{x}'(T+\delta) D \widetilde{x}(T+\delta) \Bigg\} \\ &\ge E\Bigg\{ \int_{s+\delta}^{T} [\widetilde{x}'(t) M(\widetilde{\alpha}(t)) \widetilde{x}(t) + \widetilde{u}'(t) N(\widetilde{\alpha}(t)) \widetilde{u}(t)] dt \\ &\qquad + \widetilde{x}'(T+\delta) D \widetilde{x}(T+\delta) \Bigg\}. \end{aligned}$$

Differentiating $v^\varepsilon(t,\widetilde{x}(t),\widetilde{\alpha}(t))$, Dynkin's formula leads to

$$\begin{aligned} v^\varepsilon(s+\delta,x,i) \le E\Big\{\int_{s+\delta}^{T}[\widetilde{x}'(t)M(\widetilde{\alpha}(t))\widetilde{x}(t) \\ +\widetilde{u}'(t)N(\widetilde{\alpha}(t))\widetilde{u}(t)]dt+\widetilde{x}'(T)D\widetilde{x}(T)\Big\}. \end{aligned} \tag{10.20}$$

Applying Itô's formula to $\widetilde{x}'(t)D\widetilde{x}(t)$ and using (10.16) yields

$$E\Big(\widetilde{x}'(T)D\widetilde{x}(T)-\widetilde{x}'(T+\delta)D\widetilde{x}(T+\delta)\Big)\le C(|x|^2+1)\delta.$$

It follows that

$$v^\varepsilon(s+\delta,x,i)-v^\varepsilon(s,x,i)\le C(|x|^2+1)\delta. \tag{10.21}$$

We now derive the reverse inequality. Let $E=E_{x,i}$ denote the conditional expectation given $(x(s+\delta),\alpha^\varepsilon(s+\delta))=(x,i)$. That is, we suppress the x and i dependence in what follows for notational simplicity. Then we have, under the optimal control u^* and the corresponding trajectory x^*,

$$\begin{aligned} v^\varepsilon(s+\delta,x,i) &= E\Big\{\int_{s+\delta}^{T}[x^*(t))'M(\alpha^\varepsilon(t))x^*(t) \\ &\quad +(u^*(t))'N(\alpha^\varepsilon(t))u^*(t)]dt+(x^*(T))'Dx^*(T)\Big\} \\ &= E\Big\{\int_{s}^{T-\delta}[x^*(t+\delta))'M(\alpha^\varepsilon(t+\delta))x^*(t+\delta) \\ &\quad +(u^*(t+\delta))'N(\alpha^\varepsilon(t+\delta))u^*(t+\delta)]dt \\ &\quad +(x^*(T))'Dx^*(T)\Big\}. \end{aligned} \tag{10.22}$$

Let

$$\check{\alpha}(t)=\alpha^\varepsilon(t+\delta),\ \check{u}(t)=u^*(t+\delta),\ \text{ and }\ \check{x}(t)=x^*(t+\delta),\ \text{ for }\ t\in[s,T-\delta].$$

Note that $\check{\alpha}(\cdot)$ is a Markov chain generated by Q^ε with $\check{\alpha}(s)=\alpha^\varepsilon(s+\delta)=i$. We may extend the definition of $\check{\alpha}(\cdot)$ for $t\in(T-\delta,T]$. If we define $\check{u}(t)=0$ for $t\in(T-\delta,T]$, then $\check{x}(t)$ can be defined on $(T-\delta,T]$ as well by solving the corresponding system equation (10.1). Then

$$\begin{aligned} v^\varepsilon(s+\delta,x,i) = E\Big\{\int_{s}^{T-\delta}[\check{x}'(t)M(\check{\alpha}(t))\check{x}(t) \\ +\check{u}'(t)N(\check{\alpha}(t))\check{u}(t)]dt+\check{x}'(T-\delta)D\check{x}(T-\delta)\Big\}. \end{aligned}$$

Moreover, differentiating $v^\varepsilon(t,\check{x}(t),\check{\alpha}(t))$ and using Dynkin's formula results in

$$v^\varepsilon(s,x,i) \le E\Big\{\int_s^T [\check{x}'(t)M(\check{\alpha}(t))\check{x}(t) + \check{u}'(t)N(\check{\alpha}(t))\check{u}(t)]dt + \check{x}'(T)D\check{x}(T)\Big\}.$$

Thus, we have

$$\begin{aligned} & v^\varepsilon(s+\delta,x,i) - v^\varepsilon(s,x,i) \\ & \quad \ge -E\int_{T-\delta}^T [\check{x}'(t)M(\check{\alpha}(t)\check{x}(t) + \check{u}'(t)N(\check{\alpha}(t)\check{u}(t)]dt \\ & \qquad + E\Big\{\check{x}'(T-\delta)D\check{x}(T-\delta) - \check{x}'(T)D\check{x}(T)\Big\} \\ & \quad \ge -C(|x|^2+1)\delta. \end{aligned}$$

Step 2. We claim that the functions $K^\varepsilon(\cdot,i)$ and $q^\varepsilon(\cdot,i)$ are Lipschitz. Similarly to the proof of Lemma 10.3, taking $x=0$ in (10.18) yields

$$|q^\varepsilon(s+\delta,i) - q^\varepsilon(s,i)| \le C\delta.$$

Taking $x = a\xi$ with $|\xi|=1$ and sending $a\to\infty$, we obtain

$$|K^\varepsilon(s+\delta,i) - K^\varepsilon(s,i)| \le C\delta.$$

This completes the proof. □

Proof of Theorem 10.1: We prove only the convergence of K^ε because the proof for that of q^ε is similar.

Let $i = s_{kj} \in \mathcal{M}_k$. Then in view of Lemma 10.3 and Lemma 10.4, $\{K^\varepsilon(s,s_{kj})\}$ is equicontinuous and uniformly bounded. It follows from the Arzelà-Ascoli theorem that, for each sequence of $\{\varepsilon\to 0\}$, there exists a further subsequence (still indexed by ε) such that $K^\varepsilon(s,s_{kj})$ converges uniformly on $[0,T]$ to a continuous function, say $K^0(s,s_{kj})$.

First, we show that $K^0(s,s_{kj})$ is independent of j. In fact, writing the corresponding Riccati equation in its integral form (with time running backward and the terminal condition $K^\varepsilon(T,s_{kj}) = D$),

$$\begin{aligned} K^\varepsilon(s,s_{kj}) = D & + \int_s^T [K^\varepsilon(r,s_{kj})A(s_{kj}) + A'(s_{kj})K^\varepsilon(r,s_{kj}) \\ & + M(s_{kj}) - K^\varepsilon(r,s_{kj})B(s_{kj})N^{-1}(s_{kj})B'(s_{kj})K^\varepsilon(r,s_{kj}) \\ & + Q^\varepsilon K^\varepsilon(r,\cdot)(s_{kj})]dr, \end{aligned} \tag{10.23}$$

and noting that Q^ε is given by (10.3) and the uniform boundedness of K^ε, multiplying both sides of (10.23) by ε yields

$$\int_s^T \widetilde{Q}^k K^0(r,\cdot)(s_{kj})dr = \lim_{\varepsilon\to 0}\int_s^T \widetilde{Q}^k K^\varepsilon(r,\cdot)(s_{kj})dr = 0,$$

for $s \in [0,T]$.

Thus, in view of the continuity of $K^0(s,s_{kj})$, we obtain

$$\widetilde{Q}^k K^0(s,\cdot)(s_{kj}) = 0, \text{ for } s \in [0,T]. \tag{10.24}$$

Following the irreducibility of $\widetilde{Q}^k$, we have $K^0(s,s_{kj}) = K^0(s,k)$ which is independent of j.

We next show that

$$K^\varepsilon(s,s_{kj}) \to K^0(s,k) = \overline{K}(s,k).$$

For each $k = 1,\ldots,l$, we multiply $K^\varepsilon(s,s_{kj})$ by ν_j^k and then sum over the index j. Let

$$\overline{F}(r,k) = \sum_{j=1}^{m_k} \nu_j^k F(r,s_{kj}).$$

The corresponding Riccati equation has the form

$$\begin{aligned}\sum_{j=1}^{m_k}\nu_j^k K^\varepsilon(s,s_{kj}) = D &+ \int_s^T [\overline{K^\varepsilon A}(r,k) + \overline{A'K^\varepsilon}(r,k) + \overline{M}(k) \\ &-\overline{K^\varepsilon B N^{-1} B' K^\varepsilon}(r,k) + \sum_{j=1}^{m_k}\nu_j^k Q^\varepsilon K^\varepsilon(r,\cdot)(s_{kj})]dr.\end{aligned}$$

Sending $\varepsilon \to 0$ and noting the uniform convergence of $K^\varepsilon(s,s_{kj}) \to K^0(s,k)$, we have

$$\left(\sum_{j=1}^{m_k}\nu_j^k \widehat{Q}\mathbb{1}_{m_k}\right) K^0(s,\cdot)(k) = \overline{Q}K^0(s,\cdot)(k).$$

Since $\sum_{j=1}^{m_k}\nu_j^k = 1$, we obtain

$$\begin{aligned}K^0(s,k) = D &+ \int_s^T [K^0(r,k)\overline{A}(k) + \overline{A}'(k)K^0(r,k) + \overline{M}(k) \\ &-K^0(r,k)\overline{BN^{-1}B'}(k)K^0(r,k) + \overline{Q}K^0(r,\cdot)(k)]dr.\end{aligned}$$

Thus the uniqueness of the Riccati equation implies that

$$K^0(s,k) = \overline{K}(s,k).$$

As a result, it follows that

$$K^\varepsilon(s,s_{kj}) \to \overline{K}(s,k).$$

The proof is concluded. □

10.4.2 Nearly Optimal Controls

The convergence of $K^\varepsilon(s,i)$ and $q^\varepsilon(s,i)$ leads to that of

$$v^\varepsilon(s,x,i) = x'K^\varepsilon(s,i)x + q^\varepsilon(s,i),$$

where $K^\varepsilon(s,i)$ and $q^\varepsilon(s,i)$ denote the solutions to the differential equations (10.7) and (10.8), respectively. It follows that $v^\varepsilon(s,x,s_{kj}) \to v(s,x,k)$, for $j = 1,\dots,m_k,$ as $\varepsilon \to 0$, where

$$v(s,x,k) = x'\overline{K}(s,k)x + \overline{q}(s,k)$$

corresponds to the value function of a limit problem. Let $\mathcal{U}$ denote the control set for the limit problem

$$\mathcal{U} = \Big\{U = (U^1,\dots,U^l) :\ U^k = (u^{k1},\dots,u^{km_k}), u^{kj} \in \mathbb{R}^{n_2}\Big\}.$$

Define

$$f(s,x,k,U) = \overline{A}(k)x + \sum_{j=1}^{m_k} \nu_j^k B(s_{kj})u^{kj},$$
$$\widetilde{N}(k,U) = \sum_{j=1}^{m_k} \nu_j^k \Big(u^{kj,\prime} N(s_{kj})u^{kj}\Big).$$

Then it can be shown that $v(s,x,k)$ satisfies the following HJB equations:

$$\begin{aligned} 0 \ = \ & \frac{\partial v(s,x,k)}{\partial s} + \min_{U\in\mathcal{U}}\Big\{ f(s,x,k,U)\frac{\partial v(s,x,k)}{\partial x} \\ & + x'\overline{M}(k)x + \widetilde{N}(k,U) + \frac{1}{2}\mathrm{tr}\Big(\sigma\sigma'\frac{\partial^2 v(s,x,k)}{\partial x^2}\Big) \\ & + \overline{Q}v(s,\cdot,x)(k)\Big\}, \end{aligned} \tag{10.25}$$

with $v(T,x,k) = x'Dx$. The corresponding control problem is

$$\begin{cases} \text{Min } J(s,x,k,U(\cdot)) = E\Big\{\displaystyle\int_s^T [x'(t)\overline{M}(\overline{\alpha}(t))x(t) \\ \qquad\qquad + \widetilde{N}(\overline{\alpha}(t),U(t))]dt + x'(T)Dx(T)\Big\} \\ \text{s.t. } dx(t) = f(t,x(t),\overline{\alpha}(t),U(t))dt + \sigma dw(t), x(s) = x, \end{cases}$$

where $\overline{\alpha}(\cdot) \in \{1,\dots,l\}$ is a Markov chain generated by $\overline{Q}$ with $\overline{\alpha}(s) = k$.

The optimal control for this limit problem is

$$U^*(s,x,k) = (U^{1*}(s,x),\dots,U^{l*}(s,x))$$

with

$$U^{k*}(s,x) = (u^{k1*}(s,x), \ldots, u^{km_k*}(s,x)),$$

and

$$u^{kj*}(s,x) = -N^{-1}(s_{kj})B'(s_{kj})\overline{K}(s,k)x.$$

Using such controls (as in Sethi and Zhang [192] for manufacturing systems) we construct

$$u^{\varepsilon}(s,x,\alpha) = \sum_{k=1}^{l}\sum_{j=1}^{m_k} I_{\{\alpha=s_{kj}\}}u^{kj*}(s,x) \tag{10.26}$$

for the original problem. Note that this control can also be written as if $\alpha \in \mathcal{M}_k$, $u^{\varepsilon}(s,x,\alpha) = -N^{-1}(\alpha)B'(\alpha)\overline{K}(s,k)x$. Apparently, this control is identical to the optimal control in (10.9) except that K^{ε} is replaced by $\overline{K}$. We use $u^{\varepsilon}(t) = u^{\varepsilon}(t,\alpha^{\varepsilon}(t),x(t))$ for the original problem, which will be shown to be nearly optimal.

If $B(s_{kj}) = B(k)$ and $N(s_{kj}) = N(k)$ are independent of j, then, in view of (10.11), we may replace $I_{\{\alpha^{\varepsilon}(t)=s_{kj}\}}$ by $I_{\{\overline{\alpha}^{\varepsilon}(t)=k\}}\nu_j^k$ and consider

$$\begin{aligned}\overline{u}^{\varepsilon}(s,x,\alpha) &= \sum_{k=1}^{l}\sum_{j=1}^{m_k} I_{\{\alpha\in\mathcal{M}_k\}}\nu_j^k u^{kj*}(s,x)\\ &= -N^{-1}(k)B'(k)\overline{K}(s,k)x, \text{ if } \alpha\in\mathcal{M}_k.\end{aligned} \tag{10.27}$$

Thus, we can write $\overline{u}^{\varepsilon}(s,x,\alpha) = \overline{u}^{\varepsilon}(s,x,k)$. Note that the control $\overline{u}^{\varepsilon}$ needs only the information $\alpha^{\varepsilon}(t) \in \mathcal{M}_k$. Thus, we can use the control

$$\overline{u}^{\varepsilon}(t) = \overline{u}^{\varepsilon}(t,x(t),\overline{\alpha}^{\varepsilon}(t)) \tag{10.28}$$

instead.

Theorem 10.5. *The following assertions hold:*

(1) *The control $u^{\varepsilon}(t)$ defined in (10.26) is nearly optimal, i.e.,*

$$\lim_{\varepsilon\to 0}|J^{\varepsilon}(s,x,\alpha,u^{\varepsilon}(\cdot)) - v^{\varepsilon}(s,x,\alpha)| = 0.$$

(2) *Assume $B(s_{kj}) = B(k)$ and $N(s_{kj}) = N(k)$ independent of j. Then $\overline{u}^{\varepsilon}(t)$ defined in (10.28) is nearly optimal, i.e.,*

$$\lim_{\varepsilon\to 0}|J^{\varepsilon}(s,x,\alpha,\overline{u}^{\varepsilon}(\cdot)) - v^{\varepsilon}(s,x,\alpha)| = 0.$$

Proof: Recall that $\overline{\alpha}^{\varepsilon}(t) = k$ if $\alpha^{\varepsilon}(t) \in \mathcal{M}_k$. Then the constructed control u^{ε} can be written as follows:

$$u^{\varepsilon}(t) = -N^{-1}(\alpha^{\varepsilon}(t))B'(\alpha^{\varepsilon}(t))\overline{K}(t,\overline{\alpha}^{\varepsilon}(t))x^{\varepsilon}(t),$$

where $x^\varepsilon(t)$ is the corresponding trajectory governed by the differential equation:

$$dx^\varepsilon(t) = \Big(A(\alpha^\varepsilon(t)) - B(\alpha^\varepsilon(t))N^{-1}(\alpha^\varepsilon(t))B'(\alpha^\varepsilon(t)) \times \overline{K}(t, \overline{\alpha}^\varepsilon(t))\Big)x^\varepsilon(t)dt + \sigma dw(t),$$

with $x^\varepsilon(s) = x$. The cost function is given by

$$J^\varepsilon(s, x, \alpha, u^\varepsilon(\cdot)) = E\Big\{\int_s^T x^{\varepsilon,\prime}(t)\Big(M(\alpha^\varepsilon(t)) + \widehat{N}(t, \alpha^\varepsilon(t))\Big)x^\varepsilon(t)dt + x^{\varepsilon,\prime}(T)Dx^\varepsilon(T)\Big\},$$

where $\widehat{N}(t, s_{kj}) = \overline{K}(t,k)B(s_{kj})N^{-1}(s_{kj})B'(s_{kj})\overline{K}(t,k)$. Let $x(t)$ denote the optimal trajectory of the limit problem. Then

$$dx(t) = f(t, x(t), \overline{\alpha}(t), U^*(t))dt + \sigma dw(t),$$

with $x(s) = x$. Then using the weak convergence of $\overline{\alpha}^\varepsilon(\cdot)$ to $\overline{\alpha}(\cdot)$, the Skorohod representation, and (10.11), we can show that

$$E|x^\varepsilon(t) - x(t)| \to 0, \tag{10.29}$$

which leads to

$$|J^\varepsilon(s, x, \alpha, u^\varepsilon(\cdot)) - v(s, x, k)| \to 0. \tag{10.30}$$

This together with $v^\varepsilon \to v$ leads to

$$\lim_{\varepsilon \to 0} |J^\varepsilon(s, x, \alpha, u^\varepsilon(\cdot)) - v^\varepsilon(s, x, \alpha)| = 0$$

as desired.

To obtain the second part of the theorem, note that under the condition $B(s_{kj}) = B(k)$ and $N(s_{kj}) = N(k)$, we have

$$\overline{u}^\varepsilon(t) = -N^{-1}(\overline{\alpha}^\varepsilon(t))B'(\overline{\alpha}^\varepsilon(t))\overline{K}(t, \overline{\alpha}^\varepsilon(t))x^\varepsilon(t).$$

The corresponding trajectory is given by

$$dx^\varepsilon(t) = \Big(A(\alpha^\varepsilon(t)) - B(\overline{\alpha}^\varepsilon(t))N^{-1}(\overline{\alpha}^\varepsilon(t))B'(\overline{\alpha}^\varepsilon(t)) \times \overline{K}(t, \overline{\alpha}^\varepsilon(t))\Big)x^\varepsilon(t)dt + \sigma dw(t),$$

with $x^\varepsilon(s) = x$. The optimal trajectory $x(t)$ for the limit problem is

$$dx(t) = f(t, x(t), \overline{\alpha}(t), U^*(t))dt + \sigma dw(t),$$

where

$$f(t,x,k,U^*) = \overline{A}(k)x(t) - B(k)N^{-1}(k)B'(k)\overline{K}(t,k)x(t),$$

with $x(s) = x$. In this case, we can verify (10.29) and (10.30) in a similar way to complete the proof. □

Remark 10.6. This theorem indicates that the constructed control is almost as good as the optimal one if $\alpha^\varepsilon(\cdot)$ jumps rapidly in each of its recurrent groups. The most attractive feature of such an approximation scheme is that it requires much less computation effort. For instance, if the system dimension $n_1 = 3$ and the Markov chain has $m = 40$ states divided into 5 groups with each group consisting 8 states, then the optimal scheme requires compute the Riccati equations of dimension $(n_1(n_1+1)/2)\times m = 6\times 40 = 240$. The dimension of the limit Riccati equation is only $(n_1(n_1+1)/2)\times 5 = 30$. Thus the computational complexity can be substantially reduced.

Example 10.7. Consider a special case of the Markov chain $\alpha^\varepsilon(t) \in \mathcal{M} = \{1,\dots,m\}$ such that $\widetilde{Q}$ in (10.3) is weakly irreducible. Denote the stationary distribution of $\alpha^\varepsilon(\cdot)$ by $\nu = (\nu_1,\dots,\nu_m)$. We have that $K^\varepsilon(s,i) \to \overline{K}(s)$ and $q^\varepsilon(s,i) \to \overline{q}(s)$, as $\varepsilon \to 0$, where the limit functions $\overline{K}(\cdot)$ and $\overline{q}(\cdot)$ satisfy the following differential equations:

$$\dot{\overline{K}}(s) = -\overline{K}(s)\overline{A} - \overline{A}'\overline{K}(s) - \overline{M} + \overline{K}(s)\overline{BN^{-1}B'}\,\overline{K}(s) \tag{10.31}$$

with $\overline{K}(T) = D$ and

$$\dot{\overline{q}}(s) = -\mathrm{tr}\Big(\sigma\sigma'\overline{K}(s)\Big), \quad \text{with } \overline{q}(T) = 0. \tag{10.32}$$

The optimal control for this limit problem is

$$\begin{aligned} &U^*(s,x) = (u^{1*}(s,x),\dots,u^{m*}(s,x)) \text{ with} \\ &u^{i*}(s,x) = -N^{-1}(i)B'(i)\overline{K}(s)x,\ i = 1,\dots,m. \end{aligned} \tag{10.33}$$

Using U^*, we construct u^ε for the original problem:

$$u^\varepsilon(s,x,i) = -N^{-1}(i)B'(i)\overline{K}(s)x. \tag{10.34}$$

Then our result demonstrates that the control so constructed is nearly optimal.

Remark 10.8. In this chapter, we concentrate on the case that σ in (10.1) is a constant matrix for simplicity. Our approach can be extended to treat $\sigma = \sigma(\alpha^\varepsilon(t))$. In this case, $\mathrm{tr}(\sigma\sigma'(\partial^2/\partial x^2)v^\varepsilon(s,x,i))$ in (10.4) will be

replaced by $\text{tr}(\sigma(i)\sigma'(i)(\partial^2/\partial x^2)v^\varepsilon(s,x,i))$. Then $\text{tr}(\sigma\sigma'(\partial^2/\partial x^2)\, v(s,x,k))$ in (10.25) becomes

$$\text{tr}(\overline{\sigma\sigma'}(k)(\partial^2/\partial x^2)v(s,x,k))$$

and the term $\text{tr}(\sigma\sigma'\overline{K}(s))$ in (10.32) becomes $\text{tr}(\overline{\sigma\sigma'}(k)\overline{K}(s))$. Since the Riccati equations (10.12) and (10.31) do not involve σ, they remain to be unchanged, so are the optimal control $u^{\varepsilon,*}(s,x,i)$ and nearly optimal control $u^\varepsilon(s,x,\alpha)$.

In this case, the proof for the convergence of K^ε and q^ε is almost identical. The proof for the near optimality of u^ε needs to be modified by using

$$\left(\overline{\alpha}^\varepsilon(t), \int_0^t \sigma(\alpha^\varepsilon(s))dw(s)\right) \to \left(\overline{\alpha}(t), \int_0^t \overline{\sigma}(\overline{\alpha}(s))d\overline{w}(s)\right),$$

in distribution, where $\overline{\sigma}(k)$ is a matrix such that $\overline{\sigma}(k)\overline{\sigma}'(k) = \overline{\sigma\sigma'}(k)$ and $\overline{w}(\cdot)$ is a standard Brownian motion. The fuller account will require much more complex notation. It appears to be more instructive to present the main line of work without going through complex notation so we choose the current setting. Similar extensions hold for models with inclusions of transient and absorbing states to be discussed in the subsequent sections.

10.5 Two-Time-Scale Approximation: Inclusion of Transient States

In this section, we consider the case in which the Markov chain has transient states. To incorporate the transient states, we assume as in Chapter 4 that

$$\widetilde{Q} = \begin{pmatrix} \widetilde{Q}_r & 0 \\ \widetilde{Q}_0 & \widetilde{Q}_* \end{pmatrix}, \tag{10.35}$$

where $\widetilde{Q}_r = \text{diag}(\widetilde{Q}^1, \ldots, \widetilde{Q}^l)$, $\widetilde{Q}_0 = (\widetilde{Q}^1_*, \ldots, \widetilde{Q}^l_*)$ such that for each $k = 1, \ldots, l$, $\widetilde{Q}^k$ is a generator with dimension $m_k \times m_k$, $\widetilde{Q}_* \in \mathbb{R}^{m_* \times m_*}$ matrix, $\widetilde{Q}^k_* \in \mathbb{R}^{m_* \times m_k}$, and $m_1 + \cdots + m_l + m_* = m$. The state space of the underlying Markov chain is given by

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l \cup \mathcal{M}_* \\ &= \{s_{11}, \ldots, s_{1m_1}, \ldots, s_{l1}, \ldots, s_{lm_l}, s_{*1}, \ldots, s_{*m_*}\}, \end{aligned}$$

where $\mathcal{M}_* = \{s_{*1}, \ldots, s_{*m_*}\}$ consists of the transient states. Suppose that for $k = 1, \ldots, l$, $\widetilde{Q}^k$ are weakly irreducible, and $\widetilde{Q}_*$ has eigenvalues with negative real parts.

Write

$$\widehat{Q} = \begin{pmatrix} \widehat{Q}^{11} & \widehat{Q}^{12} \\ \widehat{Q}^{21} & \widehat{Q}^{22} \end{pmatrix}$$

where $\widehat{Q}^{11} \in \mathbb{R}^{(m-m_*)\times(m-m_*)}$, $\widehat{Q}^{12} \in \mathbb{R}^{(m-m_*)\times m_*}$, $\widehat{Q}^{21} \in \mathbb{R}^{m_*\times(m-m_*)}$, and $\widehat{Q}^{22} \in \mathbb{R}^{m_*\times m_*}$. We define

$$\overline{Q}_* = \text{diag}(\nu^1, \ldots, \nu^l)(\widehat{Q}^{11}\widetilde{\mathbb{1}} + \widehat{Q}^{12}(a_{m_1}, \ldots, a_{m_l})), \tag{10.36}$$

with $\tilde{\mathbb{1}} = \text{diag}(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_l})$, $\mathbb{1}_{m_j} = (1, \ldots, 1)' \in \mathbb{R}^{m_j\times 1}$, and for $j = 1, \ldots, l$,

$$a_{m_j} = (a_{m_j}(1), \ldots, a_{m_j}(l)) = -\widetilde{Q}_*^{-1}\widetilde{Q}_*^j \mathbb{1}_{m_j}. \tag{10.37}$$

It can be shown as in Chapter 5 that $a_{m_j} \geq 0$ and $\sum_{j=1}^l a_{m_j} = \mathbb{1}_{m_*}$.

Let ξ_j denote a random variable such that

$$P(\xi_j = i | \alpha^\varepsilon(t) = s_{*j}) = a_{m_j}(i).$$

Define

$$\overline{\alpha}^\varepsilon(t) = \begin{cases} k, & \text{if } \alpha^\varepsilon(t) \in \mathcal{M}_k \\ \xi_j, & \text{if } \alpha^\varepsilon(t) = s_{*j}. \end{cases}$$

Let $\overline{\alpha}(\cdot) \in \{1, \ldots, l\}$ be a Markov chain generated by $\overline{Q}_*$. Then it can be shown as in Chapter 5 that $\overline{\alpha}^\varepsilon(\cdot) \to \overline{\alpha}(\cdot)$ in distribution. Moreover, for $k = 1, \ldots, l$,

$$\begin{aligned} &E\left(\int_0^T [I_{\{\alpha^\varepsilon(t)=s_{kj}\}} - \nu_j^k I_{\{\overline{\alpha}^\varepsilon(t)=k\}}]\beta(t)dt\right)^2 = O(\varepsilon), \\ &E\left(\int_0^T I_{\{\alpha^\varepsilon(t)=s_{*j}\}} dt\right)^2 = O(\varepsilon^2). \end{aligned} \tag{10.38}$$

Theorem 10.9. *As $\varepsilon \to 0$, $K^\varepsilon(s, s_{kj}) \to \overline{K}(s,k)$ and $q^\varepsilon(s, s_{kj}) \to \overline{q}(s,k)$, for $k = 1, \ldots, l, j = 1, \ldots, m_k$, $K^\varepsilon(s, s_{*j}) \to \overline{K}_*(s,j)$ and $q^\varepsilon(s, s_{*j}) \to \overline{q}_*(s,j)$, for $j = 1, \ldots, m_*$ uniformly on $[0,T]$, where*

$$\overline{K}_*(s,j) = a_{m_1}(j)\overline{K}(s,1) + \cdots + a_{m_l}(j)\overline{K}(s,l),$$

$$\overline{q}_*(s,j) = a_{m_1}(j)\overline{q}(s,1) + \cdots + a_{m_l}(j)\overline{q}(s,l),$$

and $\overline{K}(s,k)$ and $\overline{q}(s,k)$ are the unique solutions to the following equations: for $k = 1, \ldots, l$,

$$\begin{aligned} \dot{\overline{K}}(s,k) \ &= -\overline{K}(s,k)\overline{A}(k) - \overline{A}'(k)\overline{K}(s,k) - \overline{M}(k) \\ &\quad + \overline{K}(s,k)\overline{BN^{-1}B'}(k)\overline{K}(s,k) - \overline{Q}_*\,\overline{K}(s,\cdot)(k), \end{aligned} \tag{10.39}$$

with $\overline{K}(T,k) = D$ *and*

$$\dot{\overline{q}}(s,k) = -\mathrm{tr}\Big(\sigma\sigma'\overline{K}(s,k)\Big) - \overline{Q}_*\overline{q}(s,\cdot)(k), \tag{10.40}$$

with $\overline{q}(T,k) = 0$.

Proof: For notational simplicity, we consider only the 1-dimensional case, i.e., $n_1 = 1$. In this case, K^ε is a scalar function. Following the proof of Theorem 10.1 up to equation (10.24), we have, for $s \in [0,T]$,

$$\widetilde{Q}^k K^0(s,\cdot)(s_{kj}) = 0, \text{ for } k = 1,\dots,l, j = 1,\dots,m_k,$$

$$(\widetilde{Q}^1_*,\dots,\widetilde{Q}^l_*,\widetilde{Q}_*)(K^0(s,s_{11}),\dots,K^0(s,s_{1m_1}),\dots,K^0(s,s_{l1}),$$

$$\dots,K^0(s,s_{lm_l}),K^0(s,s_{*1}),\dots,K^0(s,s_{*m_*}))' = 0.$$

Again the irreducibility of $\widetilde{Q}^k$ implies

$$(K^0(s,s_{k1}),\dots,K^0(s,s_{km_k}))' = K^0(s,k)\mathbb{1}_{m_k}.$$

Let

$$K_*(s) = (K^0(s,s_{*1}),\dots,K^0(s,s_{*m_*}))'.$$

Then we have

$$\widetilde{Q}^1_*\mathbb{1}_{m_1}K^0(s,1) + \cdots + \widetilde{Q}^l_*\mathbb{1}_{m_l}K^0(s,l) + \widetilde{Q}_*K_*(s) = 0.$$

Hence,

$$\begin{aligned} K_*(s) &= -\widetilde{Q}_*\left(\widetilde{Q}^1_*\mathbb{1}_{m_1}K^0(s,1) + \cdots + \widetilde{Q}^l_*\mathbb{1}_{m_l}K^0(s,l)\right) \\ &= a_{m_1}K^0(s,1) + \cdots + a_{m_l}K^0(s,l). \end{aligned}$$

The rest of the proof follows like that of Theorem 10.1 except that $\overline{Q}$ is replaced by $\overline{Q}_*$. □

The convergence of K^ε and q^ε leads to $v^\varepsilon(s,x,s_{kj}) \to v(s,x,k)$, for $k = 1,\dots,l$, $j = 1,\dots,m_k$, $v^\varepsilon(s,x,s_{*j}) \to v_*(s,x,j)$, for $j = 1,\dots,m_*$, where

$$v_*(s,x,j) = a_{m_1}(j)v(s,x,1) + \cdots + a_{m_l}(j)v(s,x,l).$$

and $v(s,x,k) = x'\overline{K}(s,k)x + \overline{q}(s,k)$. The control set for the limit problem is the same as that for the recurrent case and is given by

$$\mathcal{U} = \Big\{U = (U^1,\dots,U^l) : U^k = (u^{k1},\dots,u^{km_k}), u^{kj} \in \mathbb{R}^{n_2}\Big\}.$$

Then the corresponding control problem is

$$\begin{cases} \text{Min } J(s,x,k,U(\cdot)) = E\Big\{\displaystyle\int_s^T [x'(t)\overline{M}(\overline{\alpha}(t))x(t) \\ \qquad\qquad +\widetilde{N}(\overline{\alpha}(t),U(t))]dt + x'(T)Dx(T)\Big\} \\ \text{s.t. } dx(t) = f(t,x(t),\overline{\alpha}(t),U(t))dt + \sigma dw(t), x(s) = x, \end{cases}$$

where $\overline{\alpha}(\cdot) \in \{1,\dots,l\}$ is a Markov chain generated by $\overline{Q}_*$ with $\overline{\alpha}(s) = k$.

The optimal control for this limit problem is

$$U^*(s,x,k) = (U^{1*}(s,x),\dots,U^{l*}(s,x))$$

with

$$U^{k*}(s,x) = (u^{k1*}(s,x),\dots,u^{km_k*}(s,x))$$

and

$$u^{kj*}(s,x) = -N^{-1}(s_{kj})B'(s_{kj})\overline{K}(s,k)x.$$

Similar to the recurrent case, we construct

$$\begin{aligned} u^\varepsilon(s,x,\alpha) = &\sum_{k=1}^{l}\sum_{j=1}^{m_k} I_{\{\alpha=s_{kj}\}}u^{kj*}(s,x) \\ &+\sum_{j=1}^{m_*} I_{\{\alpha=s_{*j}\}}u^{*j*}(s,x) \end{aligned} \tag{10.41}$$

for the original problem, where

$$u^{*j*}(s,x) = -N^{-1}(s_{*j})B'(s_{*j})\overline{K}_*(s,j)x.$$

Assume, for $k = 1,\dots,l$, that $B(s_{kj}) = B(k)$ and $N(s_{kj}) = N(k)$ independent of j. We may also consider

$$\begin{aligned} \overline{u}^\varepsilon(s,x,\alpha) = &\sum_{k=1}^{l}\sum_{j=1}^{m_k} I_{\{\alpha\in\mathcal{M}_k\}}\nu_j^k u^{kj*}(s,x) \\ &+\sum_{j=1}^{m_*} I_{\{\alpha=s_{*j}\}}u^{*j*}(s,x). \end{aligned} \tag{10.42}$$

Note that the control $\overline{u}^\varepsilon$ needs only the information as to whether $\alpha^\varepsilon(t) \in \mathcal{M}_k$ for $k = 1,\dots,l$ and $\alpha^\varepsilon(t) = s_{*j}$.

Theorem 10.10. *The following assertions hold:*

(1) *The control $u^\varepsilon(t)$ defined in (10.41) is nearly optimal, i.e.,*

$$\lim_{\varepsilon\to 0}|J^\varepsilon(s,x,\alpha,u^\varepsilon(\cdot)) - v^\varepsilon(s,x,\alpha)| = 0.$$

(2) *If* $B(s_{kj}) = B(k)$ *and* $N(s_{kj}) = N(k)$ *are independent of* j, *then* $\overline{u}^\varepsilon(t)$ *defined in* (10.42) *is nearly optimal, i.e.,*

$$\lim_{\varepsilon\to 0} |J^\varepsilon(s,x,\alpha,\overline{u}^\varepsilon(\cdot)) - v^\varepsilon(s,x,\alpha)| = 0.$$

Proof: The proof is similar to that of Theorem 10.5 except for the use of (10.38) in lieu of (10.11) in verifying the convergence of the trajectories. □

10.6 Two-Time-Scale Approximation: Inclusion of Absorbing States

In this section, we consider the case in which the Markov chain has absorbing states. Let $\alpha^\varepsilon(t)$ be a Markov chain generated by Q^ε defined in (10.3) with

$$\widetilde{Q} = \text{diag}(\widetilde{Q}^1, \dots, \widetilde{Q}^l, 0_{m_a\times m_a}), \tag{10.43}$$

where $\widetilde{Q}^k \in \mathbb{R}^{m_k\times m_k}$, for $k = 1,\dots,l$, $0_{m_a\times m_a}$ is the $m_a \times m_a$ zero matrix and $m_1 + \cdots + m_l + m_a = m$. Let $\mathcal{M}_k = \{s_{k1}, \dots, s_{km_k}\}$ for $k = 1,\dots,l$ denote the states corresponding to recurrent states, and $\mathcal{M}_a = \{s_{a1}, \dots, s_{am_a}\}$ denote the set of absorbing states. Then the state space can be decomposed as

$$\begin{aligned}\mathcal{M} &= \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l \cup \mathcal{M}_a \\ &= \{s_{11}, \dots, s_{1m_1}, \dots, s_{l1}, \dots, s_{lm_l}, s_{a1}, \dots, s_{am_a}\}.\end{aligned}$$

Define $\tilde{\mathbb{1}}_a = \text{diag}(\mathbb{1}_{m_1}, \dots, \mathbb{1}_{m_l}, I_{m_a})$, where I_{m_a} is the $m_a \times m_a$ identity matrix. Define also

$$\overline{Q}_a = \text{diag}(\nu^1, \dots, \nu^l, I_{m_a})\widehat{Q}\tilde{\mathbb{1}}_a, \tag{10.44}$$

where ν^k is the stationary distribution of $\widetilde{Q}^k$, $k = 1,\dots,l$.

As in the previous sections, we assume $\widetilde{Q}^k$, for $k = 1,\dots,l$, to be weakly irreducible. Then, the aggregation leads to the definition of the following process:

$$\overline{\alpha}^\varepsilon(t) = \begin{cases} k, & \text{if } \alpha^\varepsilon(t) \in \mathcal{M}_k \\ j+l, & \text{if } \alpha^\varepsilon(t) = s_{aj} \in \mathcal{M}_a. \end{cases} \tag{10.45}$$

For all $k = 1,\dots,l$ and $j = 1,\dots,m_i$ corresponding to the recurrent states, it can be shown as in Chapter 2 that

$$E\left(\int_0^t \left(I_{\{\alpha^\varepsilon(s)=s_{kj}\}} - \nu_j^k I_{\{\overline{\alpha}^\varepsilon(s)=k\}}\right) ds\right)^2 = O(\varepsilon), \tag{10.46}$$

uniformly in $t \in [0,T]$. Moreover, $\overline{\alpha}^\varepsilon(\cdot)$ converges weakly to $\overline{\alpha}(\cdot)$ that is generated by $\overline{Q}_a$.

Theorem 10.11. *As $\varepsilon \to 0$, $K^\varepsilon(s, s_{kj}) \to \overline{K}(s,k)$ and $q^\varepsilon(s, s_{kj}) \to \overline{q}(s,k)$ for $k = 1, \ldots, l, j = 1, \ldots, m_k$; $K^\varepsilon(s, s_{aj}) \to \overline{K}(s, l+j)$ and $q^\varepsilon(s, s_{aj}) \to \overline{q}(s, l+j)$, for $j = 1, \ldots, m_a$ uniformly on $[0,T]$, where $\overline{K}(s,k)$ and $\overline{q}(s,k)$ are the unique solutions to the following equations:*

$$\begin{aligned}
\dot{\overline{K}}(s,k) &= -\overline{K}(s,k)\overline{A}(k) - \overline{A}'(k)\overline{K}(s,k) - \overline{M}(k) \\
&\quad + \overline{K}(s,k)\overline{BN^{-1}B'}(k)\overline{K}(s,k) - \overline{Q}_a\, \overline{K}(s,\cdot)(k), \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } k = 1, \ldots, l, \\
\dot{\overline{K}}(s,k) &= -\overline{K}(s,k)A(s_{a(k-l)}) - A'(s_{a(k-l)})\overline{K}(s,k) \\
&\quad + \overline{K}(s,k)B(s_{a(k-l)})N^{-1}(s_{a(k-l)})B'(s_{a(k-l)})\overline{K}(s,k) \\
&\quad - M(s_{a(k-l)}) - \overline{Q}_a\, \overline{K}(s,\cdot)(k), \ \text{for } k = l+1, \ldots, l+m_a,
\end{aligned} \tag{10.47}$$

with $\overline{K}(T,k) = D$, and for $k = 1, \ldots, l, l+1, \ldots, l+m_a$,

$$\dot{\overline{q}}(s,k) = -\mathrm{tr}\Big(\sigma\sigma'\overline{K}(s,k)\Big) - \overline{Q}_a\overline{q}(s,\cdot)(k), \tag{10.48}$$

with $\overline{q}(T,k) = 0$.

Proof: The proof is similar to that of Theorem 10.1. We only note the following differences.

(i) The irreducibility of $\widetilde{Q}^k$ implies $K^\varepsilon(s, s_{kj}) \to K^0(s,k)$ for $k = 1, \ldots, l$, and $K^\varepsilon(s, s_{aj}) \to K^0(s, j+l)$ for $j = 1, \ldots, m_a$.

(ii) Let $K_a^0(s) = (K^0(s, l+1), \ldots, K^0(s, l+m_a))$. Then we have

$$\begin{aligned}
&\mathrm{diag}\,(\nu^1, \ldots, \nu^l, I_{m_a}) \left(\frac{1}{\varepsilon}\widetilde{Q} + \widehat{Q}\right) \\
&\qquad \times (K^0(s,1)\mathbb{1}'_{m_1}, \ldots, K^0(s,l)\mathbb{1}'_{m_l}, K_a^0(s))' \\
&= \overline{Q}_a(K^0(s,1), \ldots, K^0(s,l), K_a^0(s))'. \quad \square
\end{aligned}$$

The convergence of $K^\varepsilon(s,i)$ and $q^\varepsilon(s,i)$ leads to

$$v^\varepsilon(s,x,s_{kj}) \to v(s,x,k), \ \text{for } k = 1, \ldots, l, j = 1, \ldots, m_k,$$

$$v^\varepsilon(s,x,s_{aj}) \to v(s,x,l+j), \ \text{for } j = 1, \ldots, m_a,$$

where, for $k = 1, \ldots, l, l+1, \ldots, l+m_a$,

$$v(s,x,k) = x'\overline{K}(s,k)x + \overline{q}(s,k).$$

The control set for the limit problem is given by

$$\mathcal{U} = \Big\{U = (U^1, \ldots, U^l, U^{l+1}, \ldots, U^{l+m_a})\Big\},$$

where

$$U^k = (u^{k1}, \ldots, u^{km_k}), u^{kj} \in \mathbb{R}^{n_2}, \quad k = 1, \ldots, l, j = 1, \ldots, m_k,$$

$$U^k = u^{a(k-l)} \in \mathbb{R}^{n_2},$$

$$k = l+1, \ldots, l+m_a.$$

Define

$$f_a(s,x,k,U) = \begin{cases} \overline{A}(k)x + \sum_{j=1}^{m_k} \nu_j^k B(s_{kj})u^{kj}, & \text{if } k = 1, \ldots, l, \\ A(s_{a(k-l)})x + B(s_{a(k-l)})u^{a(k-l)}, & \\ \qquad\qquad \text{if } k = l+1, \ldots, l+m_a, \end{cases}$$

$$\overline{M}_a(k) = \begin{cases} \sum_{j=1}^{m_k} \nu_j^k M(s_{kj}), & \text{if } k = 1, \ldots, l, \\ M(s_{a(k-l)}), & \text{if } k = l+1, \ldots, l+m_a, \end{cases}$$

and

$$\widetilde{N}_a(k,U) = \begin{cases} \sum_{j=1}^{m_k} \nu_j^k \Big(u^{kj,\prime} N(s_{kj}) u^{kj}\Big), & \text{if } k = 1, \ldots, l, \\ u^{a(k-l),\prime} N(s_{a(k-l)}) u^{a(k-l)}, & \\ \qquad\qquad \text{if } k = l+1, \ldots, l+m_a. \end{cases}$$

The corresponding control problem is

$$\begin{cases} \text{Min } J(s,x,k,U(\cdot)) = E\Big\{\displaystyle\int_s^T [x'(t)\overline{M}_a(\overline{\alpha}(t))x(t) \\ \qquad +\widetilde{N}_a(\overline{\alpha}(t),U(t))]dt + x'(T)Dx(T)\Big\} \\ \text{s.t. } dx(t) = f_a(t,\overline{\alpha}(t),x(t),U(t))dt + \sigma dw(t), \quad x(s) = x, \end{cases}$$

where $\overline{\alpha}(\cdot) \in \{1,\ldots,l,l+1,\ldots,l+m_a\}$ is a Markov chain generated by $\overline{Q}_a$ with $\overline{\alpha}(s) = k$.

The optimal control for this limit problem is

$$U^*(s,x,k) = (U^{1*}(s,x),\ldots,U^{l*}(s,x), \\ U^{(l+1)*}(s,x),\ldots,U^{(l+m_a)*}(s,x))$$

where

$$U^{k*}(s,x) = (u^{k1*}(s,x),\ldots,u^{km_k*}(s,x)) \text{ for } k = 1,\ldots,l,$$

$$U^{k*}(s,x) = u^{a(k-l)*}(s,x) \text{ for } k = l+1,\ldots,l+m_a,$$

with

$$u^{kj*}(s,x) = -N^{-1}(s_{kj})B'(s_{kj})\overline{K}(s,k)x, \text{ for } k = 1,\ldots,l,$$

$$u^{a(k-l)*}(s,x) = -N^{-1}(s_{a(k-l)})B'(s_{a(k-l)})\overline{K}(s,k)x, \\ \text{for } k = l+1,\ldots,l+m_a.$$

Construct

$$\begin{aligned} u^\varepsilon(s,x,\alpha) &= \sum_{k=1}^{l}\sum_{j=1}^{m_k} I_{\{\alpha=s_{kj}\}}u^{kj*}(s,x) + \sum_{j=1}^{m_a} I_{\{\alpha=s_{aj}\}}u^{aj*}(s,x) \\ &= \begin{cases} -N^{-1}(\alpha)B'(\alpha)\overline{K}(s,k)x & \text{if } \alpha \in \mathcal{M}_k \\ -N^{-1}(\alpha)B'(\alpha)\overline{K}(s,l+j)x & \text{if } \alpha \in \mathcal{M}_a \end{cases} \end{aligned} \tag{10.49}$$

for the original problem.

Moreover, if we assume $B(s_{kj}) = B(k)$ and $N(s_{kj}) = N(k)$ to be independent of j, for $k = 1, \ldots, l$, we may also consider

$$\overline{u}^{\varepsilon}(s,x,\alpha) = \begin{cases} -N^{-1}(k)B'(k)\overline{K}(s,k)x \text{ if } \alpha \in \mathcal{M}_k \\ -N^{-1}(\alpha)B'(\alpha)\overline{K}(s,l+j)x \\ \qquad\qquad \text{if } \alpha = s_{aj} \in \mathcal{M}_a \end{cases} \tag{10.50}$$

Such a control $\overline{u}^{\varepsilon}$ depends only on the information regarding whether $\alpha^{\varepsilon}(t) \in \mathcal{M}_k$ or $\alpha^{\varepsilon}(t) = s_{aj} \in \mathcal{M}_a$.

Theorem 10.12. *The following assertions hold:*

(1) *The control $u^{\varepsilon}(t)$ defined in (10.49) is nearly optimal, i.e.,*

$$\lim_{\varepsilon \to 0} |J^{\varepsilon}(s,x,\alpha,u^{\varepsilon}(\cdot)) - v^{\varepsilon}(s,x,\alpha)| = 0.$$

(2) *If $B(s_{kj}) = B(k)$ and $N(s_{kj}) = N(k)$ are independent of j, then $\overline{u}^{\varepsilon}(t)$ defined in (10.50) is nearly optimal, i.e.,*

$$\lim_{\varepsilon \to 0} |J^{\varepsilon}(s,x,\alpha,\overline{u}^{\varepsilon}(\cdot)) - v^{\varepsilon}(s,x,\alpha)| = 0.$$

Proof: Using (10.46) and the convergence of $\alpha^{\varepsilon}(\cdot) \to \overline{\alpha}(\cdot)$, the proof of these results is similar to that of Theorem 10.5. □

10.7 A Numerical Example

For the purpose of demonstration, this section deals with a numerical example with $\alpha^{\varepsilon}(t) \in \mathcal{M} = \{1, 2\}$, $t \geq 0$, a Markov chain generated by

$$Q^{\varepsilon} = \frac{1}{\varepsilon} \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}.$$

Consider the one-dimensional dynamic system model

$$dx^{\varepsilon}(t) = \Big(A(\alpha^{\varepsilon}(t))x^{\varepsilon}(t) + B(\alpha^{\varepsilon}(t))u(t)\Big)dt + \sigma dw(t), \tag{10.51}$$

with the following specifications: $x^{\varepsilon}(0) = 0$, $A(1) = 0.5$, $A(2) = -0.1$, $B(1) = 1$, $B(2) = 2$, $\sigma = 1$, $M(1) = M(2) = N(1) = N(2) = D = 1$.

We discretize the equations with step size h. The time horizon in the continuous-time model is $T = 5$, and that in the corresponding discrete-time setting is $T_h = 5/h$ with $h = 0.01$. The results below are based on

computations using 100 sample paths. Take $s = 0$, $\alpha(0) = 1$, and $x(0) = x = 0$. Let $v^\varepsilon = v^\varepsilon(0,0,1)$, $J^\varepsilon = J^\varepsilon(0,0,1,u^\varepsilon(\cdot))$, and $v = v(0,0)$. Define

$$|K^\varepsilon - \overline{K}| = \frac{1}{T_h}\sum_{j=1}^{T_h}\Big(|K^\varepsilon(jh,1) - \overline{K}(jh)| + |K^\varepsilon(jh,2) - \overline{K}(jh)|\Big),$$

$$|x^\varepsilon - \overline{x}^\varepsilon| = \frac{1}{T_h}\sum_{j=1}^{T_h}|x^\varepsilon(jh) - \overline{x}^\varepsilon(jh)|,$$

where $x^\varepsilon(\cdot)$ is the optimal trajectory and $\overline{x}^\varepsilon(\cdot)$ is the near-optimal trajectory under u^ε. Then for various ε we have the error bounds given in Table 10.1. Taking $\varepsilon = 0.1$, the sample paths of $K^\varepsilon(\cdot,1)$, $K^\varepsilon(\cdot,2)$, $\overline{K}(\cdot)$, and the

ε	$\lvert K^\varepsilon - \overline{K}\rvert$	$\lvert x^\varepsilon - \overline{x}^\varepsilon\rvert$	$\lvert v^\varepsilon - v\rvert$	$\lvert J^\varepsilon - v^\varepsilon\rvert$
0.1	2.17ε	0.10ε	4.23ε	$2.21\sqrt{\varepsilon}$
0.01	2.47ε	0.19ε	5.01ε	$0.36\sqrt{\varepsilon}$
0.001	2.50ε	0.13ε	5.09ε	$0.46\sqrt{\varepsilon}$
0.0001	2.50ε	0.12ε	5.10ε	$4.17\sqrt{\varepsilon}$

TABLE 10.1. Error bounds

difference $|K^\varepsilon - \overline{K}|$ are given in Figure 10.1. The sample paths of the trajectories of $\alpha^\varepsilon(\cdot)$, $x^\varepsilon(\cdot)$, $\overline{x}^\varepsilon(\cdot)$ and the difference $|x^\varepsilon(t) - \overline{x}(t)|$ are depicted in Figure 10.2.

Next, we decrease the value of ε by letting $\varepsilon = 0.01$. Then the fast variation of the Markov chain is much more pronounced. The corresponding sample paths of $K^\varepsilon(t,i)$ and $\overline{K}(t,i)$, as well as the errors are plotted in Figure 10.3.

The sample paths of $\alpha^\varepsilon(\cdot)$, $x^\varepsilon(\cdot)$, $\overline{x}(\cdot)$, and the difference $|x^\varepsilon(t) - \overline{x}(t)|$ are plotted in Figure 10.4.

It can be seen from these graphs that the smaller the size of ε, the more frequently $\alpha^\varepsilon(\cdot)$ jumps, and the better the approximations are. To summarize, the numerical simulations indicate that our algorithm gives a very good approximation to exact optimal solutions with only half of the computational effort.

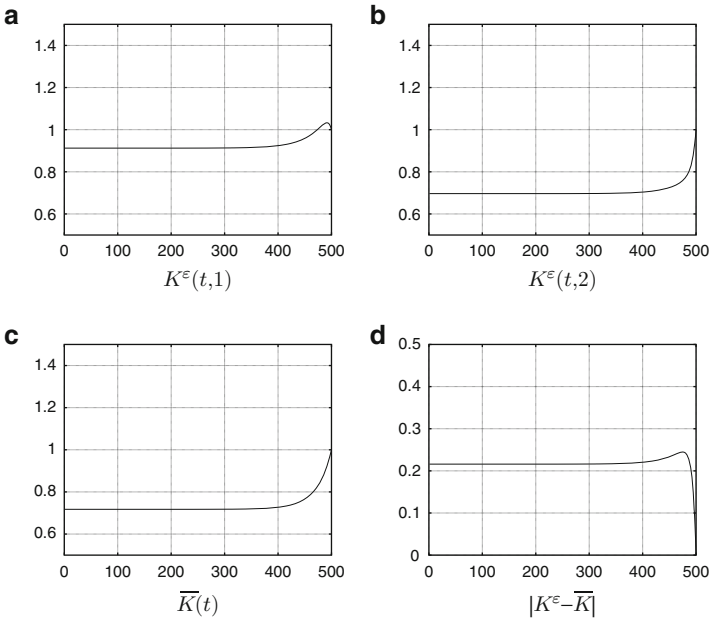

FIGURE 10.1. Riccati solutions with $\varepsilon = 0.1$

10.8 Remarks on Indefinite Control Weights

We have demonstrated how to construct nearly optimal controls for LQG problems. So far in our setup, the matrices associated with the control (often referred to as control weights), namely $N(\alpha)$ for $\alpha \in \mathcal{M}$ are assumed to be positive definite. For deterministic systems, if the control weights are not positive definite, the problem is not well posed. Nevertheless, for stochastic systems with the noise being Brownian motions, it has been shown recently that LQG with indefinite control weights could make sense if a certain balance were reached; see Chen, Li, and Zhou [26] and Yong and Zhou [246]. The control weights can be indefinite or even negative definite as long as they are not "too negative." The stochastic influence, to some extent, compensates the negative control weights to make the problem well posed. In this section, we briefly remark on treating near-optimal controls of switching diffusion systems with indefinite controls. For brevity, we will only formulate the problem and state the main results. For the interested reader, we refer to the paper by Liu, Yin, and Zhou [148] for the details.

Suppose that the switching process $\alpha^\varepsilon(\cdot)$ is as given in Section 10.4. That is, its generator is given by (10.3) with $\widetilde{Q}$ specified in (10.10). Consider a

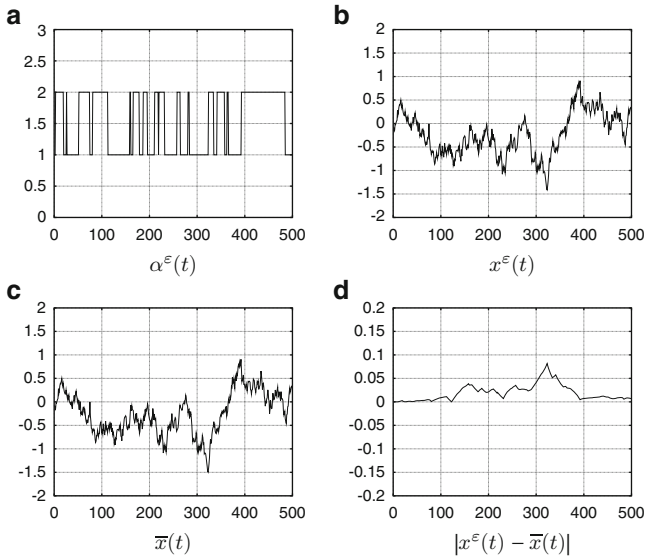

FIGURE 10.2. Various sample paths with $\varepsilon = 0.1$

switching diffusion system given by

$$\begin{aligned} dx(t) &= [A(\alpha^\varepsilon(t))x(t) + B(\alpha^\varepsilon(t))u(t)]dt \\ &\quad + C(\alpha^\varepsilon(t))u(t)dw(t), \quad \text{for} \quad s \le t \le T, \end{aligned} \tag{10.52}$$

with $x(s) = x$ and $\alpha^\varepsilon(s) = \alpha$. Our objective is to find the control $u(\cdot)$ to minimize

$$\begin{aligned} J^\varepsilon(s, x, \alpha, u(\cdot)) &= E_s\Big[\int_s^T [x'(t)M(\alpha^\varepsilon(t))x(t) \\ &\quad + u'(t)N(\alpha^\varepsilon(t))u(t)]dt + x'(T)D(\alpha^\varepsilon(T))x(T)\Big]. \end{aligned} \tag{10.53}$$

Note that in (10.52), the variance of the diffusion part is controlled. This in fact, is the main difference between the system given above and that considered in the previous sections. Consequently, one needs to use backward stochastic differential equation (SDE) techniques to treat the underlying system. For more details on backward SDEs, we refer the reader to Pardoux and Peng [171] and Yong and Zhou [246] for further reading.

With the help of the backward SDEs, we then proceed as in the previous sections. We can derive the limit system of Riccati equations and show that

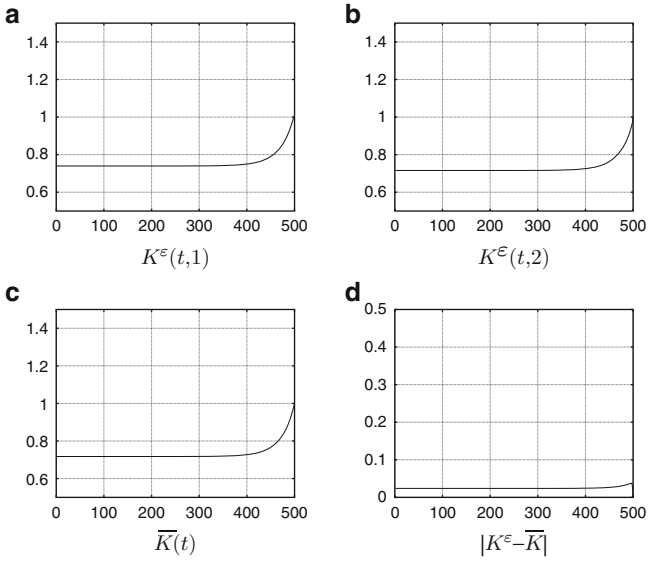

FIGURE 10.3. Riccati solutions with $\varepsilon = 0.01$

for $k = 1, \ldots, l$ and $j = 1, \ldots, m_k$, $K^\varepsilon(s, s_{kj}) \to \overline{K}(s,k)$ uniformly on $[0,T]$ as $\varepsilon \to 0$, where $\overline{K}(s,k)$ is the unique solution to

$$\begin{aligned}\dot{\overline{K}}(s,k) \; &= -\overline{K}(s,k)\overline{A}(k) - \overline{A}'(k)\overline{K}(s,k) - \overline{M}(k) \\ &\quad + \overline{K}(s,k)\overline{B(k)(N(k) + C'(k)K(s,k)C(k))^{-1}B'(k)} \\ &\quad \times \overline{K}(s,k) - \overline{Q}\,\overline{K}(s,\cdot)(k),\end{aligned} \tag{10.54}$$

with

$$\overline{K}(T,k) = \overline{D}(k) \stackrel{\text{def}}{=} \sum_{j=1}^{m_k} \nu_j^k D(s_{kj}).$$

We can then find the optimal control of the limit problem using backward SDE techniques. Denote such optimal controls by

$$U^*(s,x,k) = (U^{1*}(s,x), \ldots, U^{l*}(s,x)),$$

$$U^{k*}(s,x) = (u^{k1*}(s,x), \ldots, u^{km_k*}(s,x)),$$

$$u^{kj*}(s,x) = -\Phi^{-1}(s_{kj})B'(s_{kj})\overline{K}(s,k)x,$$

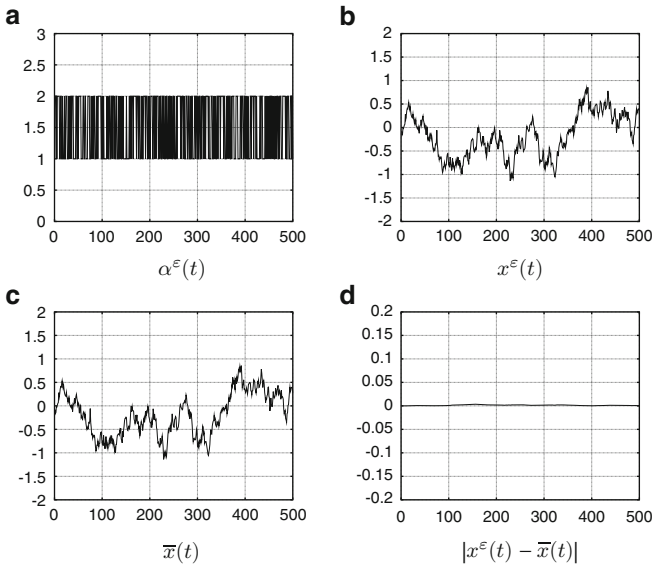

FIGURE 10.4. Various sample paths with $\varepsilon = 0.01$

where

$$\Phi^{-1}(s_{kj}) = \Phi^{-1}(s, s_{kj}) = (N(s_{kj}) + C'(s_{kj})K(s, s_{kj})C(s_{kj}))^{-1}.$$

We then construct controls

$$u^\varepsilon(s, x, \alpha) = \sum_{k=1}^{l} \sum_{j=1}^{m_k} I_{\{\alpha = s_{kj}\}} u^{*,kj}(s, x)$$

for the original problem. We can show that the $u^\varepsilon(t)$ given above is nearly optimal in that

$$|J^\varepsilon(s, x, \alpha, u^\varepsilon(\cdot)) - v^\varepsilon(s, x, \alpha)| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

10.9 Notes

LQ problems have a long and illustrious history. Nowadays, LQ control methodology is included in almost every standard textbook in control theory and is a "must" in undergraduate and graduate curricula. This chapter is based on Zhang and Yin [254]. A classical treatment of LQG control problems can be found in Fleming and Rishel [63], whereas much recent research effort has been devoted to the study of hybrid LQG systems for

over a decade; see, for example, Blair and Sworder [15], Caines and Chen [21], Mariton [154], Rishel [181], Ji and Chizeck [97, 98], Zhang [250], and the references therein. Related work on additive control of stochastic linear systems can also be found in Chow, Menaldi, and Robin [29].

The two-time-scale method and singular perturbation method for systems and controls can be traced back to the works of Delebecque [43], Delebecque and Quadrat [44], Phillips and Kokotovic [175], and Pan and Başar [164, 167], among others. Until very recently, the main focus for LQG problems has been on the case in which the control weight (the matrix associated with the control action) is positive definite. Recent advances in backward SDEs (see Pardoux and Peng [171] and also Yong and Zhou [246]) make it possible to treat problems with indefinite controls. There are ample applications of such control actions; see for example, Markowitz's mean-variance portfolio selection with regime switching considered in Zhou and Yin [259], Yin and Zhou [243], and many references therein.

Appendix A
Background Materials

This appendix collects a number of results used in the book. These results include generators of Markov chains, weak convergence methods, relaxed control representation, viscosity solutions of HJB equations, optimal controls, and a number of miscellaneous lemmas and theorems.

A.1 Properties of Generators

This section presents several properties of a generator Q (or $Q(t), t \geq 0$) of a continuous-time Markov chain. The first lemma can be found in Wilkinson [219].

Lemma A.1 (Gerschgorin's Theorem). *Let λ be an eigenvalue of an $m \times m$ matrix $Q = (q_{ij})$. Then there exists an index k such that*

$$|\lambda - q_{kk}| \leq \sum_{i \neq k} |q_{ki}|.$$

Proof: Use $\xi = (\xi_1, \ldots, \xi_m)' \in \mathbb{R}^{m \times 1}$ to denote a right-eigenvector of Q corresponding to the eigenvalue λ and use k to denote the index such that $|\xi_k| = \max_i |\xi_i|$. Then $Q\xi = \lambda\xi$ and, in particular,

$$\sum_{i=1}^{m} q_{ki}\xi_i = \lambda\xi_k.$$

G.G. Yin and Q. Zhang, *Continuous-Time Markov Chains and Applications*, 373
Stochastic Modelling 37, DOI 10.1007/978-1-4614-4346-9,

Note, by the choice of k, that $|\xi_i|/|\xi_k| \le 1$. Consequently,

$$|\lambda - q_{kk}| \le \sum_{i \ne k} |q_{ki}| \frac{|\xi_i|}{|\xi_k|} \le \sum_{i \ne k} |q_{ki}|. \quad \square$$

Lemma A.2. *Let Q be a weakly irreducible generator. Then*

(a) *zero is an eigenvalue with multiplicity one, and all other eigenvalues $\{\lambda_1, \ldots, \lambda_{m-1}\}$ have negative real parts;*

(b) *there exists a constant $K > 0$ such that*

$$\left| \exp(Qs) - \overline{P} \right| \le K \exp(-\widetilde{\kappa} s),$$

where

$$\widetilde{\kappa} = -\frac{1}{2} \left(\max_{1 \le i \le m-1} \mathrm{Re}(\lambda_i) \right) > 0,$$

$\overline{P} = \mathbb{1}(\nu_1, \cdots, \nu_m) \in \mathbb{R}^{m \times m}$, and $(\nu_1, \ldots, \nu_m)$ is the stationary distribution of the Markov process with generator Q.

Proof: Clearly, zero is an eigenvalue of Q with the corresponding eigenvector $\mathbb{1}$. Let $P(s) = \exp(Qs)$. Then $P(s)$ is the transition probability matrix of the Markov chain generated by Q satisfying the differential equation

$$\frac{dP(s)}{ds} = P(s)Q, \; P(0) = I.$$

By virtue of Theorem II.10.1 of Chung [31], $\lim_{s \to \infty} P(s)$ exists and is equal to a constant matrix $\overline{P}$. This in turn yields that

$$\lim_{s \to \infty} \exp(Qs) = \lim_{s \to \infty} P(s) = \overline{P} = (\overline{p}_{ij}). \tag{A.1}$$

Owing to the corollary of Theorem II.12.8 of [31],

$$\lim_{s \to \infty} \frac{d}{ds} (\exp(Qs)) = 0.$$

In view of the differential equation given above,

$$0 = \lim_{s \to \infty} \frac{dP(s)}{ds} = \lim_{s \to \infty} P(s)Q = \overline{P}Q. \tag{A.2}$$

For each $i = 1, \ldots, m$, denote the ith row of $\overline{P}$ by $\overline{P}_i$. The weak irreducibility of Q then implies that the system of equations

$$\overline{P}_i Q = 0, \quad \overline{P}_i \mathbb{1} = 1$$

has a unique solution. Since $\overline{P}$ is the limit of the transition matrix, $\overline{P}_i \ge 0$. As a result, $\overline{P}_i$ is the quasi-stationary distribution ν and $\overline{P}$ has identical rows with $\overline{P} = \mathbb{1}(\nu_1, \ldots, \nu_m)$.

Using the Jordan canonical form, there is a nonsingular matrix U such that

$$\exp(Qs) = U \text{ diag } \Big(\exp(J_0 s), \exp(J_1 s), \ldots, \exp(J_q s) \Big) U^{-1},$$

where J_0, J_1, ..., J_q are the Jordan blocks satisfying that J_0 is a diagonal matrix having appropriate dimension (if λ_i is a simple eigenvalue of Q, it appears in the block J_0), and that $J_k \in \mathbb{R}^{m_k \times m_k}$, $k = 1, \ldots, q$. Since $\lim_{s\to\infty} \exp(Qs)$ exists, all the nonzero eigenvalues λ_i, for $1 \le i \le m-1$, must have negative real parts. Moreover, in view of the weak irreducibility of Q, the eigenvalue zero is a simple eigenvalue (having multiplicity 1). Then it is easily seen that

$$\Big| \exp(Qs) - \overline{P} \Big| \le K \exp(-\widetilde{\kappa} s),$$

where $\widetilde{\kappa} = (-1/2) \max_{1\le i\le m-1} \text{Re}(\lambda_i)$. □

Remark A.3. The preceding lemma displays the spectrum property of a weakly irreducible generator. Such properties have been well studied in the literature; see Cox and Miller [36] and Iosifescu [95] (also the classical work of Doob [49]). The paper of Karlin and McGregor [104] gives criteria for classifying generalized birth and death processes. The treatment using Green's function approach is in Keilson [107]. Related issues on singularly perturbed autonomous systems can be found in Campbell [22], and Campbell and Rose [23]. The authors derived a necessary and sufficient condition for the convergence of $\exp(A + B/\varepsilon)t$ (for $t > 0$ as $\varepsilon \to 0$); they showed that the limit exists if and only if B is semistable.

Next we further our understanding of the properties of irreducible generators. The lemmas below exploit the nonnegativity and the connection of weak irreducibility and the rank of a generator, its eigenvalues, and quasi-stationary distributions.

Lemma A.4. *Given a generator Q, assume the system of equations*

$$\begin{cases} \nu Q = 0, \\ \nu \mathbb{1} = 1. \end{cases} \tag{A.3}$$

has a unique solution. Then the solution is nonnegative.

Proof: Let $P(t) := \exp(Qt)$. Then $P(t)$ is the transition matrix of a Markov chain generated by Q, since $P(t)$ is the unique solution to the forward equation (2.5). It follows that $P(t) \ge 0$. Consequently, as in the proof of Lemma A.2,

$$0 \le \lim_{t\to\infty} P(t) = \overline{P} = \mathbb{1}\nu.$$

Thus, $\overline{P}$ has identical rows ν and $\nu \geq 0$. Repeating the proof of Lemma A.2, similar to (A.2), $\overline{P}Q = 0$. As a result $\nu Q = 0$ and $\nu \mathbb{1} = 1$. Thus ν is a solution to (A.3). However, (A.3) has a unique solution. Therefore, ν is its only solution and it is nonnegative. □

Lemma A.5. *Let Q be an $m \times m$ generator. If the rank of Q is equal to $m-1$, then Q is weakly irreducible.*

Proof: In view of (A.1) and (A.2) in the proof of Lemma A.2, there exists a row vector $a = (a_1, \ldots, a_m)$ such that

$$aQ = 0 \text{ and } a\mathbb{1} = 1.$$

Note that the null space $N(Q') = \text{span}\{a\}$ because

$$\text{rank}(Q') = \text{rank}(Q) = m - 1.$$

Then $a = (\nu_1, \ldots, \nu_m)$ is the unique nonnegative solution to $aQ = 0$ and $a\mathbb{1} = 1$. Thus Q is weakly irreducible. □

Lemma A.6. *For each $0 \leq t \leq T$, let $Q(t)$ be a generator with $Q(t) \in \mathbb{R}^{m \times m}$. Assume $Q(t)$ is weakly irreducible and continuous on $[0,T]$. Then there exists a constant $\widehat{\kappa} > 0$ such that for any nonzero eigenvalue λ_t of $Q(t)$, $\text{Re}(\lambda_t) \leq -\widehat{\kappa}$ uniformly in $t \in [0,T]$.*

Proof: For each $t \in [0,T]$, let $h(\lambda, t) = \det(\lambda I - Q(t))$. Since $Q(t)$ has a zero eigenvalue with multiplicity 1, there exists a polynomial (in terms of the variable λ) $h_0(\lambda, t)$ such that its coefficients are continuous in $t \in [0,T]$, $h(\lambda, t) = \lambda h_0(\lambda, t)$, and $h_0(0,t) \neq 0$. If the lemma did not hold, then there would exist a sequence $\{t_n\}$ with $t_n \in [0,T]$ and nonzero eigenvalues λ_n such that $h_0(\lambda_n, t_n) = 0$ and $\text{Re}(\lambda_n) \to 0$.

In view of Gerschgorin's theorem (Lemma A.1) and the continuity of $Q(t)$ on the bounded interval $[0,T]$, the eigenvalues of $Q(t)$ lie in a compact set uniformly in $t \in [0,T]$. We may assume $t_n \to \overline{t}$ and $\lambda_n \to \overline{\lambda}$ as $n \to \infty$. It follows that $\text{Re}(\overline{\lambda}) = 0$ and $h_0(\overline{\lambda}, \overline{t}) = 0$. Thus $\overline{\lambda}$ is a nonzero eigenvalue of $Q(\overline{t})$ with zero real part, which contradicts (a) in Lemma A.2. □

A.2 Weak Convergence

The concept of weak convergence is a substantial generalization of convergence in distribution in elementary probability theory. This section gathers a number of definitions and results regarding weak convergence including tightness, martingale problem, Skorohod representation, Prohorov's theorem, and tightness criteria, etc.

Definition A.7 (Weak Convergence). Let P and P_n, $n = 1, 2, \ldots$, denote probability measures defined on a metric space $\mathbb{F}$. The sequence $\{P_n\}$ converges weakly to P if

$$\int f dP_n \to \int f dP$$

for every bounded and continuous function f on $\mathbb{F}$. Suppose that $\{X_n\}$ and X are random variables associated with P_n and P, respectively. The sequence X_n converges to X weakly if for any bounded and continuous function f on $\mathbb{F}$, $Ef(X_n) \to Ef(X)$ as $n \to \infty$.

Use $D([0,\infty);\mathbb{R}^r)$ to denote the space of $\mathbb{R}^r$-valued functions on $[0,\infty)$ that are right-continuous and that have left-hand limits. Let $\mathbb{L}$ denote a set of strictly increasing Lipschitz continuous functions $\phi : [0,\infty) \mapsto [0,\infty)$ such that the mapping is surjective with $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$, and

$$\gamma(\phi) := \sup_{0\le t<s} \left|\log\left(\frac{\phi(s)-\phi(t)}{s-t}\right)\right| < \infty.$$

Definition A.8 (Skorohod Topology). For $\xi, \eta \in D([0,\infty);\mathbb{R}^r)$, the *Skorohod topology* $d(\cdot,\cdot)$ on $D([0,\infty);\mathbb{R}^r)$ is defined as

$$d(\xi,\eta) = \inf_{\phi\in\mathbb{L}} \left\{\gamma(\phi) \vee \int_0^\infty e^{-\varrho} \sup_{t\ge 0}\Big(1 \wedge |\xi(t\wedge\varrho) - \eta(\phi(t)\wedge\varrho)|\Big) d\varrho\right\}.$$

We can define Skorohod topology analogously for either $D([0,T];\mathbb{R}^r)$ or $D([0,1];\mathbb{R}^r)$. For related references, see Ethier and Kurtz [59] and Billingsley [13]. In our study, we often work with $D([0,T];\mathbb{R}^r)$. The results to follow are often stated with respect to the space $D([0,\infty);\mathbb{R}^r)$, since this will allow us to apply them to the cases $t \in [0,T]$ for any $T > 0$.

Definition A.9 (Tightness). A family of probability measures $\mathcal{T}$ defined on a metric space $\mathbb{F}$ is *tight* if for each $\delta > 0$, there exists a compact set $K_\delta \subset \mathbb{F}$ such that

$$\inf_{P\in\mathcal{T}} P(K_\delta) \ge 1 - \delta.$$

The notion of tightness is closely related to compactness. The following theorem, known as Prohorov's theorem, accounts for such an implication. A complete proof can be found in Ethier and Kurtz [59].

Theorem A.10 (Prohorov's Theorem). *If $\mathcal{T}$ is tight, then $\mathcal{T}$ is relatively compact, i.e., every sequence of elements in $\mathcal{T}$ contains a weakly convergent subsequence. If the underlying metric space is complete and separable, the tightness is equivalent to relative compactness.*

Weak convergence techniques usually allow the use of much weaker conditions and results in more general setup. For purely analytic reasons, however, it is often more convenient to work with probability one convergence.

A device, known as Skorohod representation, provides us with such opportunities.

Theorem A.11 (The Skorohod Representation (Ethier and Kurtz [59])). *Let P_n and P denote probability measures on $D([0,\infty);\mathbb{R}^r)$ such that P_n converges weakly to P. Then there exists a probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{P})$ on which are defined $D([0,\infty);\mathbb{R}^r)$-valued random variables $\widetilde{X}_n$, $n=1,2,\ldots$, and $\widetilde{X}$ such that for any Borel set B and all $n<\infty$, $\widetilde{P}(\widetilde{X}_n\in B)=P_n(B)$, and $\widetilde{P}(\widetilde{X}\in B)=P(B)$ such that*

$$\lim_{n\to\infty}\widetilde{X}_n=\widetilde{X}\qquad w.p.1.$$

Let $C([0,\infty);\mathbb{R}^r)$ be the space of $\mathbb{R}^r$-valued continuous functions equipped with the sup-norm topology, C_0 be the set of real-valued continuous functions on $\mathbb{R}^r$ with compact support. Let C_0^k be the subset of C_0 functions that have continuous partial derivatives up to the order k.

Definition A.12. Let $\mathbb{F}$ denote a metric space and let A denote a linear operator on $B(\mathbb{F})$, the set of all Borel measurable functions defined on $\mathbb{F}$. Let $X(\cdot)=\{X(t):\ t\geq 0\}$ denote a right-continuous process with values in $\mathbb{F}$ such that

$$f(X(t))-\int_0^t Af(X(s))ds,$$

for each f in the domain of A, is a martingale with respect to the filtration $\sigma\{X(s):s\leq t\}$. Then $X(\cdot)$ is called a *solution of the martingale problem* for A.

Theorem A.13 (Ethier and Kurtz [59, p. 174]). *A right-continuous process $X(t)$, $t\geq 0$, is a solution of the martingale problem for the operator A if and only if*

$$E\left\{\left(f(X(t_{i+1}))-f(X(t_i))-\int_{t_i}^{t_{i+1}}Af(X(s))ds\right)\prod_{j=1}^{i}h_j(X(t_j))\right\}=0$$

whenever $0\leq t_1<t_2<\cdots<t_{i+1}$, $f\in\mathcal{D}(A)$, and $h_1,\ldots,h_i\in\mathcal{B}(\mathbb{F})$, where $\mathcal{D}(A)$ denotes the domain of A and $\mathcal{B}(\mathbb{F})$ denotes the Borel field of $\mathbb{F}$.

Theorem A.14 (Uniqueness of Martingale Problems). *Let $X(\cdot)$ and $Y(\cdot)$ denote two stochastic processes with paths in $D([0,T];\mathbb{R}^r)$. Let A denote an infinitesimal generator. If for any function f in the domain of A,*

$$f(X(t))-f(X(0))-\int_0^t Af(X(s))ds,\ t\geq 0,\quad \textit{and}$$
$$f(Y(t))-f(Y(0))-\int_0^t Af(Y(s))ds,\ t\geq 0,$$

are martingales and $X(t)$ and $Y(t)$ have the same distribution for each $t \geq 0$, $X(\cdot)$ and $Y(\cdot)$ have the same distribution on $D([0,\infty);\mathbb{R}^r)$.

Proof: The proof can be found in Ethier and Kurtz [59, p. 184], □

Theorem A.15. *Let $x^\varepsilon(\cdot)$ be a solution of the differential equation*

$$\frac{dx^\varepsilon(t)}{dt} = F^\varepsilon(t),$$

and for each $T < \infty$, $\{F^\varepsilon(t) : \ 0 \leq t \leq T\}$ be uniformly integrable. If the set of initial values $\{x^\varepsilon(0)\}$ is tight, then $\{x^\varepsilon(\cdot)\}$ is tight in $C([0,\infty);\mathbb{R}^r)$.

Proof: The proof is essentially in Billingsley [13, Theorem 8.2] (see also Kushner [139, p. 51, Lemma 7]). □

Define the notion of "p-lim" and an operator A^ε as in Ethier and Kurtz [59]. Suppose that $z^\varepsilon(\cdot)$ are defined on the same probability space. Let $\mathcal{F}_t^\varepsilon$ be the minimal σ-algebra over which $\{z^\varepsilon(s), \xi^\varepsilon(s) : \ s \leq t\}$ is measurable and let E_t^ε denote the conditional expectation given $\mathcal{F}_t^\varepsilon$. Denote

$$\overline{M}^\varepsilon = \Big\{ f : \ f \text{ is real valued with bounded support and is progressively measurable w.r.t. } \{\mathcal{F}_t^\varepsilon\}, \ \sup_t E|f(t)| < \infty \Big\}.$$

Let $g(\cdot)$, $f(\cdot)$, $f^\Delta(\cdot) \in \overline{M}^\varepsilon$, For each $\Delta > 0$ and $t \leq T < \infty$, $f = p\text{-}\lim_\Delta f^\Delta$ if

$$\sup_{t,\Delta} E|f^\Delta(t)| < \infty,$$

then

$$\lim_{\Delta \to 0} E|f(t) - f^\Delta(t)| = 0 \text{ for each } t.$$

The function $f(\cdot)$ is said to be in the domain of A^ε, that is, $f(\cdot) \in \mathcal{D}(A^\varepsilon)$, and $A^\varepsilon f = g$, if

$$p - \lim_{\Delta \to 0} \left(\frac{E_t^\varepsilon f(t+\Delta) - f(t)}{\Delta} - g(t) \right) = 0.$$

If $f(\cdot) \in \mathcal{D}(A^\varepsilon)$, then Ethier and Kurtz [59] or Kushner [139, p. 39] implies that

$$f(t) - \int_0^t A^\varepsilon f(u)du \text{ is a martingale,}$$

and

$$E_t^\varepsilon f(t+s) - f(t) = \int_t^{t+s} E_t^\varepsilon A^\varepsilon f(u)du \quad \text{w.p.1.}$$

In applications, ϕ-mixing processes frequently arise, see [59] and [139]. The assertion below presents a couple of inequalities for uniform mixing processes. Further results on various mixing processes are in [59].

Lemma A.16 (Kushner [139, Lemma 4.4]). *Let $\xi(\cdot)$ be a ϕ-mixing process with mixing rate $\phi(\cdot)$ and let $h(\cdot)$ be a function of ξ that is bounded and measurable on $\mathcal{F}_t^\infty$. Then*

$$\left|E(h(t+s)|\mathcal{F}_0^t) - Eh(t+s)\right| \leq 2\phi(s).$$

If $t < u < v$, and $Eh(s) = 0$ for all s, then

$$\left|E(h(u)h(v)|\mathcal{F}_0^t) - Eh(u)h(v)\right| \leq 4\Big(\phi(v-u)\phi(u-t)\Big)^{\frac{1}{2}},$$

where $\mathcal{F}_\tau^t = \sigma\{\xi(s) : \tau \leq s \leq t\}$.

A crucial step in analyzing many limit problems is to obtain tightness of the sequences of interest. A sufficient condition known as the Kurtz' criterion appears to be rather handy to utilize.

Lemma A.17 (Kushner [139, Theorem 3, p. 47]). *Suppose that $\{Y^\varepsilon(\cdot)\}$ is a process with paths in $D([0,\infty);\mathbb{R}^r)$, and suppose that*

$$\lim_{K_1\to\infty}\left\{\limsup_{\varepsilon\to 0} P\left(\sup_{0\leq t\leq T}|Y^\varepsilon(t)| \geq K_1\right)\right\} = 0 \text{ for each } T < \infty, \tag{A.4}$$

and for all $0 \leq s \leq \Delta$, $t \leq T$,

$$\begin{aligned} &E_t^\varepsilon \min\left(1, |Y^\varepsilon(t+s) - Y^\varepsilon(t)|^2\right) \leq E_t^\varepsilon \gamma_\varepsilon(\Delta), \\ &\lim_{\Delta\to 0}\left\{\limsup_{\varepsilon\to 0} E\gamma_\varepsilon(\Delta)\right\} = 0. \end{aligned} \tag{A.5}$$

Then $\{Y^\varepsilon(\cdot)\}$ is tight in $D([0,\infty);\mathbb{R}^r)$.

Remark A.18. In lieu of (A.4), one may verify the following condition (see Kurtz [136, Theorem 2.7, p. 10]). Suppose that for each $\eta > 0$ and rational $t \geq 0$ there is a compact set $\Gamma_{t,\eta} \subset \mathbb{R}^r$ such that

$$\inf_\varepsilon P\left(Y^\varepsilon(t) \in \Gamma_{t,\eta}\right) > 1 - \eta. \tag{A.6}$$

The perturbed test function method is a useful technique in dealing with singularly perturbed stochastic systems. The next lemma, due to Kushner, gives a criterion for tightness of singularly perturbed systems via perturbed test function methods.

Lemma A.19 (Kushner [139, Theorem 3.4]). *Let $z^\varepsilon(\cdot) \in D([0,\infty);\mathbb{R}^r)$, $z^\varepsilon(0) = z_0$, and*

$$\lim_{\kappa_1\to\infty}\left\{\limsup_{\varepsilon\to 0} P\left(\sup_{t\leq T_1}|z^\varepsilon(t)| \geq \kappa_1\right)\right\} = 0 \tag{A.7}$$

for each $T_1 < \infty$. For each $f(\cdot) \in C_0^2$ and $T_1 < \infty$, let there be a sequence $\{f^\varepsilon(\cdot)\}$ such that $f^\varepsilon(\cdot) \in \mathcal{D}(A^\varepsilon)$ and that $\{A^\varepsilon f^\varepsilon(t) : \varepsilon > 0, t < T_1\}$ is uniformly integrable and

$$\lim_{\varepsilon \to 0} P\left(\sup_{t \le T_1} |f^\varepsilon(t) - f(z^\varepsilon(t))| \ge \kappa_2\right) = 0$$

for each $T_1 < \infty$ and each $\kappa_2 > 0$. Then $\{z^\varepsilon(\cdot)\}$ is tight in $D([0,\infty); \mathbb{R}^r)$.

The functions $f^\varepsilon(\cdot)$ above are the perturbed test functions. They are so constructed that they will be close to $f(z^\varepsilon(\cdot))$, and will result in desired cancelation in the averaging.

To apply Lemma A.19 for proving tightness, one needs to verify (A.7). Such verifications are usually nontrivial and involve complicated calculations. To overcome the difficulty, we utilize the device of N-truncation, defined as follows: For each $N > 0$, let $S_N = \{z : |z| \le N\}$ be the ball with radius N, let $z^{\varepsilon,N}(0) = z^\varepsilon(0)$, $z^{\varepsilon,N}(t) = z^\varepsilon(t)$ up until the first exit from S_N, and

$$\lim_{\kappa_1 \to \infty} \left\{ \limsup_{\varepsilon \to 0} P\left(\sup_{t \le T_1} |z^{\varepsilon,N}(t)| \ge \kappa_1\right)\right\} = 0 \tag{A.8}$$

for each $T_1 < \infty$. Then $z^{\varepsilon,N}(t)$ is said to be the N-truncation of $z^\varepsilon(\cdot)$.

Using the perturbed test function techniques, the lemma to follow provides sufficient conditions for weak convergence. Its proof is in Kushner [139].

Lemma A.20. *Suppose that $\{z^\varepsilon(\cdot)\}$ is defined on $[0,\infty)$. Let $\{z^\varepsilon(\cdot)\}$ be tight on $D([0,\infty); \mathbb{R}^r)$. Suppose that for each $f(\cdot) \in C_0^2$, and each $T_1 < \infty$, there exist $f^\varepsilon(\cdot) \in \mathcal{D}(A^\varepsilon)$ such that*

$$p - \lim_{\varepsilon \to 0} (f^\varepsilon(\cdot) - f(z^\varepsilon(\cdot))) = 0 \tag{A.9}$$

and

$$p - \lim_{\varepsilon \to 0} (A^\varepsilon f^\varepsilon(\cdot) - Af(z^\varepsilon(\cdot))) = 0. \tag{A.10}$$

Then $z^\varepsilon(\cdot) \Rightarrow z(\cdot)$.

The theorem below is useful in characterizing limit processes in weak convergence analysis. Its proof is in Kushner and Yin [145, Theorem 4.1.1].

Theorem A.21. *Let $M(t)$ be a continuous-time martingale whose paths are Lipschitz continuous with probability one on each bounded time interval. Then $M(t)$ is a constant with probability one.*

A.3 Relaxed Control

Let Γ be a compact set in some Euclidean space, which denotes the control space. Assume that $\mathcal{G}_t$ is any given filtration (for instance, $\mathcal{G}_t = \sigma\{\alpha(s) : 0 \le s \le t\}$, where $\alpha(\cdot)$ is a finite-state Markov chain). Denote the σ-algebra of Borel subsets of any set S by $\mathcal{B}(S)$. Let

$$\mathbb{M} = \Big\{\widetilde{m}(\cdot) : \widetilde{m}(\cdot) \text{ is a measure on } \mathcal{B}(\Gamma \times [0,\infty)) \text{ satisfying } \widetilde{m}(\Gamma \times [0,t]) = t \text{ for all } t \ge 0\Big\}.$$

A random $\mathbb{M}$-valued measure $\widetilde{m}(\cdot)$ is an admissible relaxed control if for each $B \in \mathcal{B}(\Gamma)$, the function defined by $\widetilde{m}(B,t) \equiv \widetilde{m}(B \times [0,t])$ is $\mathcal{G}_t$-progressively measurable. An equivalent formulation reads that $\widetilde{m}(\cdot)$ is a relaxed control if

$$\int_0^t h(s,\varrho)\widetilde{m}(ds \times d\varrho)$$

is progressively measurable with respect to $\{\mathcal{G}_t\}$ for each bounded and continuous function $h(\cdot)$.

If $\widetilde{m}(\cdot)$ is an admissible relaxed control, there is a measure-valued function $\widetilde{m}_t(\cdot)$ (the "derivative") such that $\widetilde{m}_t(d\varrho)dt = \widetilde{m}(dt \times d\varrho)$ and for smooth function $h(\cdot)$,

$$\int h(s,\varrho)\widetilde{m}(ds \times d\varrho) = \int ds \int h(s,\varrho)\widetilde{m}_s(d\varrho). \tag{A.11}$$

To proceed, topologize $\mathbb{M}$ as follows. Let $\{f_{n_i}(\cdot) : i < \infty\}$ be a countable dense (under the sup-norm topology) set of continuous functions on $\Gamma \times [0,n]$ for each n. Let

$$\langle \widetilde{m}, f \rangle = \int f(s,\varrho)\widetilde{m}(ds \times d\varrho), \tag{A.12}$$

and define

$$d(\widetilde{m}_1, \widetilde{m}_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(\widetilde{m}_1, \widetilde{m}_2),$$

where

$$d_n(\widetilde{m}_1, \widetilde{m}_2) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left(\frac{|(\widetilde{m}_1 - \widetilde{m}_2, f_{n_i})|}{1 + |(\widetilde{m}_1 - \widetilde{m}_2, f_{n_i})|} \right).$$

$\widetilde{m}_n(\cdot) \Rightarrow \widetilde{m}(\cdot)$ for a sequence of measures means the weak convergence in $\mathbb{M}$.

An ordinary admissible control $u(\cdot)$ is a feedback control for the system of interest if there is a Γ-valued Borel measurable function $u_0(\cdot)$ such that

$u(t) = u_0(x(t))$ for almost all ω, t. For each x, let $\widehat{m}(x, \cdot)$ be a probability measure on $(\Gamma, \mathcal{B}(\Gamma))$ and suppose that for each $B \in \mathcal{B}(\Gamma)$, $\widehat{m}(\cdot, B)$ is Borel measurable as a function of x. If for almost all ω and t, the derivative $\widetilde{m}_t(\cdot)$ of a relaxed control $\widetilde{m}(\cdot)$ can be written as $\widetilde{m}_t(\cdot) = \widehat{m}(x(t), \cdot)$, then $\widetilde{m}(\cdot)$ is said to be a relaxed feedback control.

Relaxed control formulation provide a convenient device with primary mathematical use; they can be approximated by ordinary controls through the following "chattering theorem." Its proof can be found in Kushner [140, Theorem 2.2, p. 50]. There is also a version of the chattering theorem for controlled diffusions. The interested reader is referred to [140, Chapter 3]; see also the related references cited there.

Theorem A.22. *Consider the differential equation*

$$\frac{dx(t)}{dt} = \int_\Gamma f(x(t), \varrho)\widetilde{m}_t(d\varrho), \quad x(0) = x,$$

and the cost functional

$$J(\widetilde{m}) = \int_0^T \int_\Gamma G(x(s), \varrho)\widetilde{m}_s(d\varrho)ds.$$

Suppose the differential equation has a unique solution for each initial condition $x(0) = x$ and each relaxed control $\widetilde{m}(\cdot)$. Then, for a given $\gamma > 0$, there is a finite set $\{\varrho_1^\gamma, \ldots, \varrho_{k_\gamma}^\gamma\} \equiv \Gamma^\gamma \subset \Gamma$ and a $\delta > 0$ such that for any admissible relaxed control $\widetilde{m}(\cdot)$, there is a Γ^γ-valued ordinary admissible control $u^\gamma(\cdot)$ (depending on $\widetilde{m}(\cdot)$) being constant on each interval $[i\delta, i\delta+\delta)$, for $i\delta + \delta \le T$ such that

$$\sup_{t \le T} \left| x(t, \widetilde{m}) - x(t, u^\gamma) \right| \le \gamma, \quad \left| J(x, \widetilde{m}) - J(x, u^\gamma) \right| \le \gamma,$$

where $x(t, \widetilde{m})$ and $x(t, u^\gamma)$ are the corresponding trajectories under $\widetilde{m}(\cdot)$ and u^γ, respectively.

A.4 Viscosity Solutions of HJB Equations

In control theory, typically the dynamic programming argument leads to an equation known as the Hamilton-Jacobi-Bellman (HJB) equation that the value function of the problem must satisfy. The equation formally involves partial derivatives of the value function, even though such a derivative may not exist at certain points. In these cases, a useful concept is the notion of viscosity solutions, which was introduced by Crandall and Lions [39]. In what follows, we briefly describe the ideas and some of the related results. For more information and discussion on viscosity solutions, the reader is

referred to the user's guide by Crandall, Ishii, and Lions [38] and the book of Fleming and Soner [64].

Let $\mathcal{M} = \{1, \ldots, m\}$ and $v : \mathbb{R}^n \times \mathcal{M} \to \mathbb{R}^1$ be a given function. Let $H(\cdot)$ denote a real-valued function on

$$\Omega_H := \mathbb{R}^n \times \mathcal{M} \times \mathbb{R}^m \times \mathbb{R}^n.$$

Consider the following equation

$$v(x,\alpha) - H\left(x, \alpha, v(x,\cdot), \frac{\partial v(x,\alpha)}{\partial x}\right) = 0. \tag{A.13}$$

Definition A.23 (Viscosity Solution). $v(x,\alpha)$ is a *viscosity solution* of Equation (A.13) if the following hold:

(a) $v(x,\alpha)$ is continuous in x and $|v(x,\alpha)| \le K(1+|x|^\kappa)$ for some $\kappa \ge 0$;

(b) for any $\alpha_0 \in \mathcal{M}$,

$$v(x_0,\alpha_0) - H\left(x_0, \alpha_0, v(x_0,\cdot), \frac{d\phi(x_0)}{dx}\right) \le 0,$$

whenever $\phi(x) \in C^1$ (i.e., continuously differentiable) and $v(x,\alpha_0) - \phi(x)$ has a local maximum at $x = x_0$; and

(c) for any $\alpha_0 \in \mathcal{M}$,

$$v(x_0,\alpha_0) - H\left(x_0, \alpha_0, v(x_0,\cdot), \frac{d\psi(x_0)}{dx}\right) \ge 0,$$

whenever $\psi(x) \in C^1$ and $v(x,\alpha_0) - \psi(x)$ has a local minimum at $x = x_0$.

If (a) and (b) (resp. (a) and (c)) hold, we say that v is a *viscosity subsolution* (resp. *viscosity supersolution*).

In this book, we mainly consider the HJB equation of the following form

$$\rho v(x,\alpha) = \min_{u\in\Gamma}\left\{b(x,\alpha,u)\frac{\partial v(x,\alpha)}{\partial x} + G(x,\alpha,u) + Qv(x,\cdot)(\alpha)\right\}, \tag{A.14}$$

where as indicated in Remark 8.4, $b(\partial/\partial x)v$ is understood to be the inner product of b and $(\partial/\partial x)v$, i.e., $\langle b, (\partial/\partial x)v\rangle$.

Theorem A.24 (Uniqueness Theorem). *Assume $b(x,\alpha,u)$ satisfies conditions in* (A8.1) *in Chapter 8 and for some positive constants K and κ,*

$$|G(x,\alpha,u)| \le K(1+|x|^\kappa) \text{ and}$$

$$|G(x_1,\alpha,u) - G(x_2,\alpha,u)| \le K(1+|x_1|^\kappa + |x_2|^\kappa)|x_1 - x_2|,$$

for all $\alpha \in \mathcal{M}$, x, x_1, and $x_2 \in \mathbb{R}^n$, and u in a compact and convex set Γ. Let Q be a generator to a Markov chain on $\mathcal{M}$. Then the HJB equation (A.14) has a unique viscosity solution.

Proof: The proof below is based on the argument of Ishii [96] and Soner [201]. Let $v_1(x,\alpha)$ and $v_2(x,\alpha)$ be two viscosity solutions to the HJB equation (A.14), and

$$\eta(x) = \exp(a(1+|x|^2)^{\frac{1}{2}}), \text{ where } a = \frac{\rho}{\sup_u |b(x,\alpha,u)|}.$$

For any $0 < \delta < 1$ and $0 < \gamma < 1$, consider a function defined on $\mathbb{R}^n \times \mathbb{R}^n \times \mathcal{M}$,

$$\Phi(x_1,x_2,\alpha) := v_1(x_1,\alpha) - v_2(x_2,\alpha) - \frac{1}{\delta}|x_1 - x_2|^2 - \gamma(\eta(x_1)+\eta(x_2)).$$

In view of the polynomial growth condition of v_1 and v_2 given in Definition A.23, it is easy to see that $\Phi(x_1,x_2,\alpha)$ has a global maximum at a point (x_1^0, x_2^0, α_0), since Φ is continuous and

$$\lim_{|x_1|+|x_2|\to\infty} \Phi(x_1,x_2,\alpha) = -\infty$$

for each $\alpha \in \mathcal{M}$. In particular,

$$\Phi(x_1^0,x_1^0,\alpha_0) + \Phi(x_2^0,x_2^0,\alpha_0) \le 2\Phi(x_1^0,x_2^0,\alpha_0).$$

Using the definition of Φ, we have

$$\frac{2}{\delta}|x_1^0 - x_2^0|^2 \le (v_1(x_1^0,\alpha_0) - v_1(x_2^0,\alpha_0)) + (v_2(x_1^0,\alpha_0) - v_2(x_2^0,\alpha_0)).$$

Again, the polynomial growth of v_1 and v_2 yields

$$|x_1^0 - x_2^0| \le \sqrt{\delta K(1+|x_1^0|^\kappa + |x_2^0|^\kappa)}. \tag{A.15}$$

Moreover, the choice of (x_1^0, x_2^0, α_0) implies $\Phi(0,0,\alpha_0) \le \Phi(x_1^0,x_2^0,\alpha_0)$. This yields

$$\begin{aligned}\gamma(\eta(x_1^0)+\eta(x_2^0)) \le\ & v_1(x_1^0,\alpha_0) - v_2(x_2^0,\alpha_0) - \frac{1}{\delta}|x_1^0 - x_2^0|^2 - \Phi(0,0,\alpha_0)\\ \le\ & K(1+|x_1^0|^\kappa + |x_2^0|^\kappa).\end{aligned}$$

Therefore, there exists a constant K_γ (independent of δ) such that

$$|x_1^0| + |x_2^0| \le K_\gamma. \tag{A.16}$$

Note that $x \mapsto \Phi(x, x_2^0, \alpha_0)$ takes its maximum at $x = x_1^0$. According to the definition of viscosity solutions (Definition A.23),

$$\begin{aligned}\rho v_1(x_1^0, \alpha_0) \leq \min_{u\in\Gamma}\Big\{ & b(x_1^0, \alpha_0, u)\left(\frac{2}{\delta}(x_1^0 - x_2^0) + \gamma\frac{d\eta(x_1^0)}{dx}\right) \\ & + G(x_1^0, \alpha_0, u) + Qv_1(x_1^0, \cdot)(\alpha_0)\Big\}.\end{aligned} \tag{A.17}$$

Similarly, note that $x \mapsto -\Phi(x_1^0, x, \alpha_0)$ takes its minimum at $x = x_2^0$. We can obtain the reverse inequality

$$\begin{aligned}\rho v_2(x_2^0, \alpha_0) \geq \min_{u\in\Gamma}\Big\{ & b(x_2^0, \alpha_0, u)\left(\frac{2}{\delta}(x_1^0 - x_2^0) - \gamma\frac{d\eta(x_2^0)}{dx}\right) \\ & + G(x_2^0, \alpha_0, u) + Qv_2(x_2^0, \cdot)(\alpha_0)\Big\}.\end{aligned} \tag{A.18}$$

Combining the two inequalities (A.17) and (A.18) yields

$$\begin{aligned}&\rho(v_1(x_1^0, \alpha_0) - v_2(x_2^0, \alpha_0)) \\ &\quad \leq \min_{u\in\Gamma}\Big\{ b(x_1^0, \alpha_0, u)\left(\frac{2}{\delta}(x_1^0 - x_2^0) + \gamma\frac{d\eta(x_1^0)}{dx}\right) \\ &\qquad + G(x_1^0, \alpha_0, u) + Qv_1(x_1^0, \cdot)(\alpha_0)\Big\} \\ &\qquad - \min_{u\in\Gamma}\Big\{ b(x_2^0, \alpha_0, u)\left(\frac{2}{\delta}(x_1^0 - x_2^0) - \gamma\frac{d\eta(x_2^0)}{dx}\right) \\ &\qquad + G(x_2^0, \alpha_0, u) + Qv_2(x_2^0, \cdot)(\alpha_0)\Big\} \\ &\quad \leq \sup_{u\in\Gamma}\Big\{ \big(b(x_1^0, \alpha_0, u) - b(x_2^0, \alpha_0, u)\big)\left(\frac{2}{\delta}(x_1^0 - x_2^0)\right) \\ &\qquad + \gamma\left(b(x_1^0, \alpha_0, u)\frac{\eta(x_1^0)}{dx} + b(x_2^0, \alpha_0, u)\frac{d\eta(x_2^0)}{dx}\right) \\ &\qquad + G(x_1^0, \alpha_0, u) - G(x_2^0, \alpha_0, u) \\ &\qquad + Qv_1(x_1^0, \cdot)(\alpha_0) - Qv_2(x_2^0, \cdot)(\alpha_0)\Big\}.\end{aligned} \tag{A.19}$$

In view of (A.15) and (A.16), there exists a subsequence of $\delta \to 0$ (still denoted by δ) and x_0 such that $x_1^0 \to x_0$ and $x_2^0 \to x_0$. Using this fact and

letting $\delta \to 0$ in (A.19), we have

$$\begin{aligned}
&\rho(v_1(x_0,\alpha_0) - v_2(x_0,\alpha_0)) \\
&\quad \le \sup_{u\in\Gamma}\Big\{ b(x_0,\alpha_0,u)\left(2\gamma\frac{d\eta(x_0)}{dx}\right) \\
&\qquad + Qv_1(x_0,\cdot)(\alpha_0) - Qv_2(x_0,\cdot)(\alpha_0)\Big\}.
\end{aligned} \tag{A.20}$$

On the other hand, $\Phi(x_1,x_2,\alpha_0)$ reaches a maximum at (x_1^0,x_2^0,α_0); it follows that for all x and all $\alpha\in\mathcal{M}$,

$$\begin{aligned}
&v_1(x,\alpha) - v_2(x,\alpha) - 2\gamma\eta(x) = \Phi(x,x,\alpha) \le \Phi(x_1^0,x_2^0,\alpha_0) \\
&\le v_1(x_1^0,\alpha_0) - v_2(x_2^0,\alpha_0) - \gamma(\eta(x_1^0)+\eta(x_2^0)).
\end{aligned}$$

Again, letting $\delta\to 0$ in the inequality above, and recalling that $x_1^0\to x_0$ and $x_2^0\to x_0$,

$$\begin{aligned}
&v_1(x,\alpha) - v_2(x,\alpha) - 2\gamma\eta(x) \\
&\qquad \le v_1(x_0,\alpha_0) - v_2(x_0,\alpha_0) - 2\gamma\eta(x_0).
\end{aligned} \tag{A.21}$$

In particular, taking $x = x_0$ in (A.21) leads to

$$v_1(x_0,\alpha) - v_2(x_0,\alpha) \le v_1(x_0,\alpha_0) - v_2(x_0,\alpha_0).$$

Thus,

$$\begin{aligned}
&Qv_1(x_0,\cdot)(\alpha_0) - Qv_2(x_0,\cdot)(\alpha_0) \\
&\quad = \sum_{\alpha\ne\alpha_0} q_{\alpha_0\alpha}(u)[v_1(x_0,\alpha) - v_1(x_0,\alpha_0) \\
&\qquad - v_2(x_0,\alpha) + v_2(x_0,\alpha_0)] \le 0.
\end{aligned} \tag{A.22}$$

Combining (A.20), (A.21), and (A.22) yields

$$\begin{aligned}
&v_1(x,\alpha) - v_2(x,\alpha) - 2\gamma\eta(x) \\
&\le \frac{1}{\rho}\sup_{u\in\Gamma}\left\{ b(x_0,\alpha,u)\left(2\gamma\frac{d\eta(x_0)}{dx}\right)\right\} - 2\gamma\eta(x_0) \\
&\le 2\gamma\eta(x_0) - 2\gamma\eta(x_0) = 0.
\end{aligned}$$

The second inequality above follows from the fact $(d/dx)\eta(x) \leq a\eta(x)$. Therefore,

$$v_1(x,\alpha) - v_2(x,\alpha) \leq 2\gamma\eta(x).$$

Letting $\gamma \to 0$, we arrive at

$$v_1(x,\alpha) - v_2(x,\alpha) \leq 0.$$

Using a similar argument, we can show

$$v_1(x,\alpha) - v_2(x,\alpha) \geq 0.$$

Thus $v_1(x,\alpha) = v_2(x,\alpha)$. This concludes the proof. □

The following lemma is used in the proof of Theorem 8.8. For a proof, see Crandall, Evans, and Lions [37, Lemma 1.1].

Lemma A.25. *Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set and $\eta \in C(\mathcal{O};\mathbb{R})$ be differentiable at $x_0 \in \mathcal{O}$. Then there exist $\phi_+ \in C^1(\mathcal{O};\mathbb{R})$ and $\phi_- \in C^1(\mathcal{O};\mathbb{R})$ such that*

$$\frac{\partial\phi_+(x_0)}{\partial x} = \frac{\partial\phi_-(x_0)}{\partial x} = \frac{\partial\eta(x_0)}{\partial x},$$

and $\eta - \phi_+$ (resp. $\eta - \phi_-$) has a strictly local maximum (resp. minimum) value of zero at x_0.

A.5 Value Functions and Optimal Controls

In this section we present a number of elementary properties of value functions and optimal feedback controls by considering a simple but representative model. Multidimensional analogues and models having more complex structures can be treated similarly.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\alpha(t) \in \mathcal{M} = \{1,\dots,m\}$, for $t \geq 0$, denote a Markov chain with generator Q. Consider the one-dimensional optimal control problem given below. Let $b(x,\alpha,u)$ be a function satisfying Condition (A8.1) in Chapter 8. The system equation and the control constraints are

$$\frac{dx(t)}{dt} = b(x(t),\alpha(t),u(t)),\ x(0) = x \text{ and } u(t) \in \Gamma,$$

where Γ is a convex and compact subset of $\mathbb{R}^{n_1}$.

Definition A.26. A control $u(\cdot) = \{u(t) \in \mathbb{R}^{n_1} : t \geq 0\}$ is called *admissible* with respect to the initial capacity α if

(a) $u(\cdot)$ is progressively measurable with respect to the filtration $\{\mathcal{F}_t\}$, where $\mathcal{F}_t = \sigma\{\alpha(s) : 0 \leq s \leq t\}$ and

(b) $u(t) \in \Gamma$ for all $t \geq 0$.

Let $\mathcal{A}$ denote the set of all admissible controls.

Definition A.27. A function $u(x, \alpha)$ is called an *admissible feedback control*, or simply *feedback control*, if

(a) for any given initial data x, the equation

$$\frac{dx(t)}{dt} = b(x(t), \alpha(t), u(x(t), \alpha(t))), \ x(0) = x$$

has a unique solution;

(b) $u(\cdot) = \{u(t) = u(x(t), \alpha(t)), \ t \geq 0\} \in \mathcal{A}$.

The problem of interest is to choose an admissible control $u(\cdot)$ so as to minimize the objective function

$$J(x, \alpha, u(\cdot)) = E \int_0^\infty e^{-\rho t} G(x(t), \alpha(t), u(t)) dt,$$

where x and α are the initial values of $x(t)$ and $\alpha(t)$, respectively, and $G(x, \alpha, u)$ is a function of x and u. Let $v(x, \alpha)$ denote the value function of the problem

$$v(x, \alpha) = \inf_{u(\cdot) \in \mathcal{A}} J(x, \alpha, u(\cdot)).$$

We would like to remind the reader that in what follows, K is a generic positive constant. In what follows, for simplicity, we work on $u(t)$ being real valued. The main techniques can be adopted to multidimensional cases with $u(t) \in \mathbb{R}^{n_1}$.

Lemma A.28. *The following assertions hold:*

(a) *If $G(x, \alpha, u)$ is locally Lipschitz in that*

$$|G(x_1, \alpha, u) - G(x_2, \alpha, u)| \leq K(1 + |x_1|^\kappa + |x_2|^\kappa)|x_1 - x_2|$$

for some constants K and κ, then $v(x, \alpha)$ is also locally Lipschitz in that

$$|v(x_1, \alpha) - v(x_2, \alpha)| \leq K(1 + |x_1|^\kappa + |x_2|^\kappa)|x_1 - x_2|.$$

(b) *If $G(x, \alpha, u)$ is jointly convex and $b(x, \alpha, u)$ is independent of x, then $v(x, \alpha)$ is convex in x for each $\alpha \in \mathcal{M}$.*

Proof: To verify (a), let $u(\cdot)$ denote an admissible control and let $x^1(\cdot)$ and $x^2(\cdot)$ denote the state trajectories under $u(\cdot)$ with initial values $x_1(0) = x_1$ and $x_2(0) = x_2$, respectively. Then there exists a constant K_1 such that

$$\left|x^1(t) - x^2(t)\right| \leq |x_1 - x_2|, \ \left|x^1(t)\right| \leq K_1(1 + |x_1|), \text{ and}$$

$$\left|x^2(t)\right| \leq K_1(1 + |x_2|).$$

In view of the local Lipschitz assumption on $G(\cdot)$, we can show that there exists a constant K_2 independent of $u(\cdot)$, x_1, and x_2 such that

$$|J(x_1, \alpha, u(\cdot)) - J(x_2, \alpha, u(\cdot))| \leq K_2(1 + |x_1|^\kappa + |x_2|^\kappa)|x_1 - x_2|.$$

It follows that

$$\begin{aligned} |v(x_1, \alpha) - v(x_2, \alpha)| &\leq \sup_{u(\cdot) \in \mathcal{A}} |J(x_1, \alpha, u(\cdot)) - J(x_2, \alpha, u(\cdot))| \\ &\leq K_2(1 + |x_1|^\kappa + |x_2|^\kappa)|x_1 - x_2|. \end{aligned}$$

We now prove (b). It suffices to show that $J(\cdot, \alpha, \cdot)$ is jointly convex. For any x_1 and x_2 and any admissible controls $u^1(\cdot)$ and $u^2(\cdot)$, let $x^1(t)$ and $x^2(t)$, $t \geq 0$, denote the trajectories corresponding to $(x_1, u^1(\cdot))$ and $(x_2, u^2(\cdot))$. Then for any $\gamma \in [0, 1]$,

$$\begin{aligned} &\gamma J(x_1, \alpha, u^1(\cdot)) + (1 - \gamma) J(x_2, \alpha, u^2(\cdot)) \\ &= E \int_0^\infty e^{-\rho t}[\gamma G(x^1(t), \alpha(t), u^1(t)) + (1 - \gamma) G(x^2(t), \alpha(t), u^2(t))]dt \\ &\geq E \int_0^\infty e^{-\rho t} G(x(t), \alpha(t), u(t))dt, \end{aligned}$$

where $u(t) := \gamma u^1(t) + (1 - \gamma)u^2(t)$ and $x(t)$, $t \geq 0$, denotes the trajectory with initial value $\gamma x_1 + (1 - \gamma)x_2$ and control $u(\cdot)$. Thus,

$$\begin{aligned} &\gamma J(x_1, \alpha, u^1(\cdot)) + (1 - \gamma) J(x_2, \alpha, u^2(\cdot)) \\ &\geq J(\gamma x_1 + (1 - \gamma)x_2, \alpha, \gamma u^1(\cdot) + (1 - \gamma)u^2(\cdot)). \end{aligned}$$

This means that $J(\cdot, \alpha, \cdot)$ is jointly convex. Therefore, $v(x, \alpha)$ is convex. □

The next lemma presents the dynamic programming principle (DPP). Such results are considered to be well known. However, it is not easy to find a convenient reference that is applicable to the class of stochastic control problems considered in this book. For completeness we provide a proof of the DPP.

Lemma A.29. *Let τ be an $\{\mathcal{F}_t\}$-stopping time. Then*

$$v(x,\alpha) = \inf_{u(\cdot)\in\mathcal{A}} E\left\{\int_0^\tau e^{-\rho t}G(x(t),\alpha(t),u(t))dt + e^{-\rho\tau}v(x(\tau),\alpha(\tau))\right\}.$$

Proof: We divide the proof into two steps.

Step 1. Consider the case when τ takes at most countably many values $\{t_1, t_2, \ldots\}$ with $t_{j+1} > t_j$ and $t_j \to \infty$ as $j \to \infty$. Let $u(\cdot)$ be an admissible control. On the set $\{\tau = t_j\}$, let $u_j(t) = u(t + t_j)$ for $t \geq 0$. Then for any fixed $\{\alpha(s) : \ s \leq t_j\}$, $u_j(\cdot)$ is admissible with respect to $\sigma\{\alpha(t) : \ t \geq t_j\}$.

Note that

$$\begin{aligned} &E\int_\tau^\infty e^{-\rho t}G(x(t),\alpha(t),u(t))dt \\ &\quad = \sum_{j=1}^\infty E\left(I_{\{\tau=t_j\}}\int_{t_j}^\infty e^{-\rho t}G(x(t),\alpha(t),u(t))dt\right). \end{aligned}$$

Changing of variable $t \longmapsto t + t_j$ leads to

$$\begin{aligned} &E\left(I_{\{\tau=t_j\}}\int_{t_j}^\infty e^{-\rho t}G(x(t),\alpha(t),u(t))dt\right) \\ &\quad = E\left(I_{\{\tau=t_j\}}e^{-\rho t_j}\int_0^\infty e^{-\rho t}G(x(t+t_j),\alpha(t+t_j),u(t+t_j))dt\right). \end{aligned} \tag{A.23}$$

Moreover, taking conditional expectation and using the Markov property yield

$$\begin{aligned} &E\left(\int_0^\infty e^{-\rho t}G(x(t+t_j),\alpha(t+t_j),u(t+t_j))dt\middle| x(s),\alpha(s) : s \leq t_j\right) \\ &\quad = E\left(\int_0^\infty e^{-\rho t}G(x(t+t_j),\alpha(t+t_j),u(t+t_j))dt\middle| x(t_j),\alpha(t_j)\right) \\ &\quad = J(x(t_j),\alpha(t_j),u_j(\cdot)). \end{aligned}$$

Combining this equality with (A.23) and noting that $I_{\{\tau=t_j\}}$ is $\mathcal{F}_{t_j}$ measurable, we obtain

$$\begin{aligned} &E\int_\tau^\infty e^{-\rho t}G(x(t),\alpha(t),u(t))dt \\ &\quad = \sum_{j=1}^\infty E\left(I_{\{\tau=t_j\}}e^{-\rho t_j}J(x(t_j),\alpha(t_j),u_j(t))\right) \\ &\quad \geq \sum_{j=1}^\infty E\left(I_{\{\tau=t_j\}}e^{-\rho t_j}v(x(t_j),\alpha(t_j))\right) \\ &\quad = E\left(e^{-\rho\tau}v(x(\tau),\alpha(\tau))\right). \end{aligned} \tag{A.24}$$

Since $v(x,\alpha) = \inf J(x,\alpha,u(\cdot))$ taken over all admissible $u(\cdot)$,

$$v(x,\alpha) \geq \inf_{u(\cdot)\in\mathcal{A}} E\left\{\int_0^\tau e^{-\rho t}G(x(t),\alpha(t),u(t))dt + e^{-\rho\tau}v(x(\tau),\alpha(\tau))\right\}.$$

We next derive the reverse inequality. Given an initial $x(0) = x$, the boundedness of $b(\cdot)$ implies that $|x(t)| \leq |x| + Kt$. Note that

$$|G(x(t),\alpha(t),u(t))| \leq K(1+t^\kappa)$$

and

$$\int_N^\infty e^{-\rho t}K(1+t^\kappa)dt \to 0$$

as $N \to \infty$. Moreover, on the set $\{\tau > N\}$,

$$e^{-\rho\tau}v(x(\tau),\alpha(\tau)) \leq e^{-\rho\tau}K(1+\tau^\kappa) \leq e^{-\rho N}K(1+N^\kappa) \to 0,$$

as $N \to \infty$. Without loss of generality, assume $\tau \leq N$ for N large enough. For $t_j \leq N$, $|x(t_j)| \leq |x| + KN \leq r_1$, for some r_1. Given $\delta > 0$, partition $\{|x| \leq r_1\}$ into intervals $I_1, I_2, \ldots, I_{l_0}$ of length $< \delta$ and choose $x_k \in I_k$ for $k = 1,2,\ldots,l_0$. For any $\eta > 0$ choose admissible $u_{ik}(\cdot)$ such that

$$J(x_k,i,u_{ik}(\cdot)) < v(x_k,i) + \eta. \tag{A.25}$$

Given an admissible $u(\cdot)$, define $\widetilde{u}(\cdot)$ by

$$\widetilde{u}(t) = \begin{cases} u(t), & \text{if } 0 \leq t \leq \tau, \\ \displaystyle\sum_{i,j,k} I_{\{\tau=t_j,\alpha(t_j)=i,x(t_j)\in I_k\}}u_{ik}(t-t_j), & \text{if } t > \tau. \end{cases} \tag{A.26}$$

Then $\widetilde{u}(\cdot)$ is admissible. It follows immediately that

$$E\int_0^\tau e^{-\rho t}G(x(t),\alpha(t),u(t))dt = E\int_0^\tau e^{-\rho t}G(x(t),\alpha(t),\widetilde{u}(t))dt.$$

Let $E_{ijk} = \{\tau = t_j, \alpha(t_j) = i, x(t_j) \in I_k\}$. Then as in the proof of (A.24),

$$\begin{aligned} &E\int_\tau^\infty e^{-\rho t}G(x(t),\alpha(t),\widetilde{u}(t))dt \\ &\quad = E\sum_{i,j,k} I_{E_{ijk}}\int_{t_j}^\infty e^{-\rho t}G(x(t),\alpha(t),\widetilde{u}(t))dt \\ &\quad = E\sum_{i,j,k} I_{E_{ijk}}e^{-\rho t_j}\int_0^\infty e^{-\rho t}G(x(t+t_j),\alpha(t+t_j),u_{ik}(t))dt \\ &\quad = E\sum_{i,j,k} I_{E_{ijk}}e^{-\rho t_j}J(x(t_j),i,u_{ik}(\cdot)). \end{aligned}$$

In view of the definition of $u_{ik}(\cdot)$, we have

$$\begin{aligned} & E\sum_{i,j,k} I_{E_{ijk}} e^{-\rho t_j} J(x(t_j), i, u_{ik}(\cdot)) \\ & \quad \le E\sum_{i,j,k} I_{E_{ijk}} e^{-\rho t_j} v(x(t_j), i) + F(\delta, \eta) \\ & \quad = Ee^{-\rho\tau} v(x(\tau), \alpha(\tau)) + F(\delta, \eta), \end{aligned}$$

where

$$\begin{aligned} F(\delta, \eta) = \eta & + E\sum_{i,j,k} I_{E_{ijk}} e^{-\rho t_j} (J(x(t_j), i, u_{ik}(\cdot)) - J(x_k, i, u_{ik}(\cdot)) \\ & - E\sum_{i,j,k} I_{E_{ijk}} e^{-\rho t_j} (v(x(t_j), i) - v(x_k, i)). \end{aligned}$$

Using the Lipschitz property of $G(\cdot)$ and $v(\cdot)$, it is easy to verify that $F(\delta, \eta) \to 0$ uniformly in $u(\cdot) \in \mathcal{A}$ as $\delta, \eta \to 0$. Therefore, for all $u(\cdot) \in \mathcal{A}$, we may define $\widetilde{u}(\cdot)$ as in (A.26) such that

$$\begin{aligned} & v(x, \alpha) \le J(x, \alpha, \widetilde{u}(\cdot)) \\ & \quad \le E\left\{\int_0^\tau e^{-\rho t} G(x(t), \alpha(t), u(t))dt + e^{-\rho\tau} v(x(\tau), \alpha(\tau))\right\} + F(\delta, \eta) \\ & \quad \to E\left\{\int_0^\tau e^{-\rho t} G(x(t), \alpha(t), u(t))dt + e^{-\rho\tau} v(x(\tau), \alpha(\tau))\right\}. \end{aligned}$$

Since $u(\cdot) \in \mathcal{A}$ is arbitrary, it follows that

$$v(x, \alpha) \le \inf_{u(\cdot)\in\mathcal{A}} E\left\{\int_0^\tau e^{-\rho t} G(x(t), \alpha(t), u(t))dt + e^{-\rho\tau} v(x(\tau), \alpha(\tau))\right\}.$$

Step 2. In this step, consider a general stopping time τ. For any $r = 1, 2, \ldots$, let

$$\tau_r = \sum_{j=1}^{\infty} \frac{j}{2^r} I_{\{(j-1)/2^r < \tau \le j/2^r\}}.$$

Then τ_r is also an $\{\mathcal{F}_t\}$-stopping time because

$$\{\tau_r \le t\} = \left\{\tau \le \frac{j_0}{2^r}\right\} \in \mathcal{F}_{j_0/2^r} \subset \mathcal{F}_t,$$

where $j_0 = \sup\{j : \ j/2^r \le t\}$. Moreover, for each r,

$$\tau \le \tau_r \le \tau + \frac{1}{2^r}.$$

For any $u(\cdot) \in \mathcal{A}$, let

$$\xi(\tau) = \int_0^\tau e^{-\rho t} G(x(t), \alpha(t), u(t))dt + e^{-\rho\tau} v(x(\tau), \alpha(\tau)).$$

Then it is easy to show that

$$\left| \inf_{u(\cdot)\in\mathcal{A}} E\xi(\tau) - \inf_{u(\cdot)\in\mathcal{A}} E\xi(\tau_r) \right| \leq \sup_{u(\cdot)\in\mathcal{A}} \left| E\xi(\tau) - E\xi(\tau_r) \right|. \tag{A.27}$$

It suffices to show that the right-hand side of (A.27) goes to 0 as $r \to \infty$. In view of $|x(t)| \leq |x| + Kt$ and the polynomial growth rate of G, there exists a constant K such that for all $u(\cdot) \in \mathcal{A}$,

$$E\left| \int_0^\tau e^{-\rho t} G(x(t), \alpha(t), u(t))dt - \int_0^{\tau_r} e^{-\rho t} G(x(t), \alpha(t), u(t))dt \right|$$
$$\leq \left| \int_\tau^{\tau_r} K e^{-\rho t}(1 + t^\kappa)dt \right| \to 0,$$

uniformly in $u(\cdot) \in \mathcal{A}$ as $r \to \infty$.

Next examine

$$\left| e^{-\rho\tau_r} v(x(\tau_r), \alpha(\tau_r)) - e^{-\rho\tau} v(x(\tau), \alpha(\tau)) \right|. \tag{A.28}$$

Note that as $N \to \infty$,

$$I_{\{\tau>N\}} \left\{ e^{-\rho\tau} v(x(\tau), \alpha(\tau)) \right\} \leq I_{\{\tau>N\}} e^{-\rho N} K(1 + N^\kappa)$$
$$\leq e^{-\rho N} K(1 + N^\kappa) \to 0.$$

Similarly, as $N \to \infty$,

$$I_{\{\tau_r>N\}} \left\{ e^{-\rho\tau_r} v(x(\tau_r), \alpha(\tau_r)) \right\} \leq e^{-\rho N} K(1 + N^\kappa) \to 0.$$

In view of the triangle inequality, given $\{\tau \leq N\}$ to estimate (A.28), it suffices to examine the following three terms

$$\left| v(x(\tau), \alpha(\tau)) - v(x(\tau), \alpha(\tau_r)) \right|,$$
$$\left| v(x(\tau), \alpha(\tau_r)) - v(x(\tau_r), \alpha(\tau_r)) \right|, \text{ and}$$
$$\left| (e^{-\rho\tau_r} - e^{-\rho\tau}) v(x(\tau_r), \alpha(\tau_r)) \right|.$$

Note that on the set $\{\tau \leq N\}$,

$$N - 1 \leq \tau_r \leq N.$$

In addition, $|x(\tau)| \le |x| + KN$ and $|x(\tau_r)| \le |x| + KN$. These imply $|v(x(\tau_r), \alpha(\tau_r))| \le K$ and

$$I_{\{\tau \le N\}} \left|(e^{-\rho\tau_r} - e^{-\rho\tau})v(x(\tau_r), \alpha(\tau_r))\right| \le K \left|e^{-\rho\tau_r} - e^{-\rho\tau}\right| \to 0,$$

as $r \to \infty$. Moreover,

$$\begin{aligned} &I_{\{\tau \le N\}} \Big|v(x(\tau), \alpha(\tau_r)) - v(x(\tau_r), \alpha(\tau_r))\Big| \\ &\qquad \le I_{\{\tau \le N\}} K \Big|x(\tau) - x(\tau_r)\Big| \le \frac{K}{2^r} \to 0. \end{aligned}$$

Finally, on the set $\{\tau \le N\}$, $|x(\tau)| \le r_2$ for some $r_2 > 0$. As in Step 1, we may partition the set $\{|x| \le r_2\}$ with intervals $\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_{l_1}$ of length $< \delta$. Then for $x_k \in \tilde{I}_k$, $k = 1, \ldots, l_1$,

$$\begin{aligned} &I_{\{\tau \le N\}} \Big|v(x(\tau), \alpha(\tau)) - v(x(\tau), \alpha(\tau_r))\Big| \\ &\quad = I_{\{\tau \le N\}} \sum_{k=1}^{l_1} I_{\{x(\tau) \in \tilde{I}_k\}} \Big|v(x(\tau), \alpha(\tau)) - v(x(\tau), \alpha(\tau_r))\Big| \\ &\quad \le I_{\{\tau \le N\}} \sum_{k=1}^{l_1} I_{\{x(\tau) \in \tilde{I}_k\}} \sup_{u(\cdot) \in \mathcal{A}} \Big|v(x_k, \alpha(\tau)) - v(x_k, \alpha(\tau)_r)\Big| + F_1(\delta), \end{aligned}$$

where

$$\begin{aligned} F_1(\delta) = I_{\{\tau \le N\}} \sum_{k=1}^{l_1} I_{\{x(\tau) \in \tilde{I}_k\}} \Bigg(&\Big|v(x(\tau), \alpha(\tau)) - v(x_k, \alpha(\tau))\Big| \\ &+ \Big|v(x(\tau), \alpha(\tau_r)) - v(x_k, \alpha(\tau_r))\Big| \Bigg). \end{aligned}$$

The Lipschitz property of v implies that $F_1(\delta) \to 0$ uniformly with respect to $u(\cdot) \in \mathcal{A}$ as $\delta \to 0$. Moreover, the right-continuity of $\alpha(\cdot)$, together with the Lebesgue dominated convergence theorem yields that, for $k = 1, \ldots, l_1$,

$$E\left\{ \sup_{u(\cdot) \in \mathcal{A}} \Big|v(x_k, \alpha(\tau)) - v(x_k, \alpha(\tau_r))\Big| \right\} \to 0.$$

Hence, (A.27) holds. This completes the proof. □

Theorem A.30. *The value function $v(x, \alpha)$ is the unique viscosity solution to the HJB equation*

$$\rho v(x, \alpha) = \min_{u \in \Gamma} \left\{ b(x, \alpha, u) \frac{\partial v(x, \alpha)}{\partial x} + G(x, \alpha, u) \right\} + Qv(x, \cdot)(\alpha). \quad \text{(A.29)}$$

Proof: We need only show that v is a viscosity solution to (A.29) because Theorem A.24 implies the uniqueness of $v(x,\alpha)$. In view of Definition A.23, it suffices to show that v is both a viscosity subsolution and a viscosity supersolution.

For any fixed $\alpha_0 \in \mathcal{M}$ and $x_0 \in \mathbb{R}^n$, let $\phi(\cdot) \in C^1(\mathbb{R}^n : \mathbb{R})$ be such that $v(x,\alpha_0) - \phi(x)$ attains its maximum at $x = x_0$ in a neighborhood $N(x_0)$. Let τ denote the first jump time of $\alpha(\cdot)$. Consider the control $u(t) = u$ for $0 \le t \le \tau$, where $u \in \Gamma$ being a constant. Moreover, let $\theta \in (0,\tau]$ be such that $x(t)$ starts at x_0 and stays in $N(x_0)$ for $0 \le t \le \theta$. Define

$$\psi(x,\alpha) = \begin{cases} \phi(x) + v(x_0,\alpha_0) - \phi(x_0), & \text{if } \alpha = \alpha_0, \\ v(x,\alpha), & \text{if } \alpha \neq \alpha_0. \end{cases}$$

Using Dynkin's formula and the fact that $\alpha(\theta) = \alpha_0$, for $0 \le t \le \theta$, we have

$$\begin{aligned} & Ee^{-\rho\theta}\psi(x(\theta),\alpha(\theta)) - v(x_0,\alpha_0) \\ &= E\int_0^\theta e^{-\rho t}\Big[-\rho\psi(x(t),\alpha_0) + b(x(t),\alpha_0,u(t))\frac{d\phi(x(t))}{dx} \\ &\qquad + Q\psi(x(t),\cdot)(\alpha_0)\Big]dt. \end{aligned} \tag{A.30}$$

Moreover, $x(t) \in N(x_0)$ for $0 \le t \le \theta$. Thus, by the definition of $\phi(\cdot)$,

$$\phi(x(t)) \ge v(x(t),\alpha_0) - (v(x_0,\alpha_0) - \phi(x_0)), \text{ for } 0 \le t \le \theta. \tag{A.31}$$

Replacing $\psi(x,\alpha_0)$ in (A.30) by the right-hand side of (A.31) and noting that $v(x_0,\alpha_0) - \phi(x_0)$ is a constant, we have

$$\begin{aligned} & Ee^{-\rho\theta}v(x(\theta),\alpha_0) - v(x_0,\alpha_0) \\ &\le E\int_0^\theta e^{-\rho t}\Big(-\rho v(x(t),\alpha_0) + b(x(t),\alpha_0,u(t))\frac{d\phi(x(t))}{dx} \\ &\qquad + Qv(x(t),\cdot)(\alpha_0)\Big)dt. \end{aligned} \tag{A.32}$$

Furthermore, by the dynamic programming principle in Lemma A.29,

$$\begin{aligned} v(x_0,\alpha_0) \le E\Big(&\int_0^\theta e^{-\rho t}G(x(t),\alpha(t),u(t))dt \\ &+ e^{-\rho\theta}v(x(\theta),\alpha(\theta))\Big). \end{aligned} \tag{A.33}$$

Combining (A.32) and (A.33),

$$0 \le E\int_0^\theta e^{-\rho t}\Big[G(x(t),\alpha(t),u(t)) - \rho v(x(t),\alpha_0) + \frac{d\phi(x(t))}{dx}b(x(t),\alpha_0,u(t)) + Qv(x(t),\cdot)(\alpha_0)\Big]dt.$$

Letting $\theta \to 0$ allows us to conclude

$$\min_{u\in\Gamma}\Big[b(x,\alpha_0,u)\frac{d\phi(x_0)}{dx} + G(x_0,\alpha_0,u)\Big] + Qv(x,\cdot)(\alpha_0) - \rho v(x_0,\alpha_0) \ge 0.$$

Thus, v is a viscosity subsolution.

Next we show that v is a viscosity supersolution. If it were not, then there would exist α_0, x_0, and $\delta_0 > 0$ such that for all $u \in \Gamma$,

$$b(x,\alpha_0,u)\frac{d\phi(x)}{dx} + G(x,\alpha_0,u) + Qv(x,\cdot)(\alpha_0) - \rho v(x,\alpha_0) \ge \delta_0 \qquad \text{(A.34)}$$

in a neighborhood $N(x_0)$, where $\phi(\cdot) \in C^1(\mathbb{R}^n;\mathbb{R})$ is such that $v(x,\alpha_0) - \phi(x)$ attains its minimum at x_0 in the neighborhood $N(x_0)$. Then for all $x \in N(x_0)$,

$$v(x,\alpha_0) \ge \phi(x) + (v(x_0,\alpha_0) - \phi(x_0)). \qquad \text{(A.35)}$$

Given $u \in \Gamma$, let θ_0 denote a number so small that starting at $x = x_0$, $x(t)$ stays in $N(x_0)$ for $0 \le t \le \theta_0$. Note that θ_0 depends on the control $u(\cdot)$. However, since $b(x(t),\alpha(t),u(t))$ is always bounded, there exists a constant $\theta_1 > 0$ such that $\theta_0 \ge \theta_1 > 0$. Let τ denote the first jump time of the process $\alpha(\cdot)$. Then for $0 \le \theta \le \min\{\theta_0,\tau\}$,

$$\begin{aligned} &J(x_0,\alpha_0,u(\cdot)) \\ &\ge E\Big\{\int_0^\theta e^{-\rho t}G(x(t),\alpha(t),u(t))dt + e^{-\rho\theta}v(x(\theta),\alpha(\theta))\Big\} \\ &\ge E\Big\{\int_0^\theta e^{-\rho t}[\delta_0 - b(x(t),\alpha(t),u(t))\frac{d\phi(x(t))}{dx} \\ &\qquad +\rho v(x(t),\alpha_0) - Qv(x(t),\cdot)(\alpha_0)]dt + e^{-\rho\theta}v(x(\theta),\alpha(\theta))\Big\}. \end{aligned}$$

The differentiability of ϕ together with (A.35) leads to

$$v(x_0,\alpha_0) \le E\Big\{\int_0^\theta e^{-\rho t}\Big[\rho v(x(t),\alpha(t)) - b(x(t),\alpha(t),u(t))\frac{d\phi(x(t))}{dx} - Qv(x(t),\cdot)(\alpha_0)\Big]dt + e^{-\rho t}v(x(\theta),\alpha(\theta)\Big\}.$$

It follows that

$$J(x_0, \alpha_0, u(\cdot)) \geq v(x_0, \alpha_0) + \delta_0 E \int_0^{\theta} e^{-\rho t} dt \geq v(x_0, \alpha_0) + \eta,$$

where $\eta = \delta_0 E \int_0^{\theta_1 \wedge \tau} e^{-\rho t} dt > 0$. This means that

$$v(x_0, \alpha_0) \geq v(x_0, \alpha_0) + \eta,$$

which is a contradiction. Therefore, $v(x, \alpha)$ is a viscosity supersolution. Consequently $v(x)$ is a viscosity solution of Equation (A.29). □

Theorem A.31 (Verification Theorem). *Let* $w(x, \alpha) \in C^1(\mathbb{R}^n; \mathbb{R})$ *such that* $|w(x, \alpha)| \leq K(1 + |x|^{\kappa})$ *and*

$$\rho w(x, \alpha) = \min_{u \in \Gamma} \left\{ b(x, \alpha, u) \frac{\partial w(x, \alpha)}{\partial x} + G(x, \alpha, u) + Qw(x, \cdot)(\alpha) \right\}.$$

Then the following assertions hold:

(a) $w(x, \alpha) \leq J(x, \alpha, u(\cdot))$ *for any* $u(t) \in \Gamma$.

(b) *Suppose that there are* $u^*(t)$ *and* $x^*(t)$ *satisfying*

$$\frac{dx^*(t)}{dt} = b(x^*(t), \alpha(t), u^*(t))$$

with $x^*(0) = x$, $r^*(t) = (\partial/\partial x) v(x^*(t), \alpha(t))$, *and*

$$\begin{aligned} &\min_{u \in \Gamma} \left\{ b(x, \alpha, u) r^*(t) + G(x^*(t), \alpha, u) + Qw(x^*(t), \cdot)(\alpha(t)) \right\} \\ &= b(x^*(t), \alpha(t), u^*(t)) r^*(t) + G(x^*(t), \alpha(t), u^*(t)) \\ &\qquad + Qw(x^*(t), \cdot)(\alpha(t)) \end{aligned}$$

almost everywhere in t *with probability one. Then*

$$w(x, \alpha) = v(x, \alpha) = J(x, \alpha, u^*(\cdot)).$$

Proof: Only a sketch of the proof is given. Further details can be found in Fleming and Rishel [63].

For $T < \infty$, the usual dynamic programming principle in Lemma A.29 yields that

$$\begin{aligned} v(x, \alpha) \leq E \Bigg\{ &\int_0^T e^{-\rho t} G(x(t), \alpha(t), u(t)) dt \\ &+ e^{-\rho T} v(x(T), \alpha(T)) \Bigg\}. \end{aligned} \tag{A.36}$$

Note that $|x(t)| = O(t)$. Taking limit as $T \to \infty$ yields (a). Using the polynomial growth condition, inequality (A.36) becomes an equality. □

Lemma A.32. *Let $c(u)$ be twice differentiable such that $(d^2/du^2)c(u) > 0$, and let $V(x)$ be locally Lipschitz, i.e.,*

$$|V(x_1) - V(x_2)| \le K(1 + |x_1|^\kappa + |x_2|^\kappa)|x_1 - x_2|.$$

Let $u^*(x)$ be the minimum of $F(x,u) := uV(x) + c(u)$. Then $u^*(x)$ is locally Lipschitz in that

$$|u^*(x_1) - u^*(x_2)| \le K(1 + |x_1|^\kappa + |x_2|^\kappa)|x_1 - x_2|.$$

Proof: It is easy to see that $F(x,u)$ satisfies the conditions of Lemma 6.3 in Fleming and Rishel [63], which implies that $u^*(x)$ is locally Lipschitz. □

A.6 Miscellany

This section consists of a number of miscellaneous results needed in this book. They include the notion of convex functions, Arzelà-Ascoli theorem, Fredholm alternative, and the proof of Theorem 4.41, among others.

Definition A.33 (Convex Sets and Convex Functions). A set $S \subset \mathbb{R}^r$ is convex if for any x and $y \in S$, $\gamma x + (1-\gamma)y \in S$ for any $0 \le \gamma \le 1$. A real-valued function f on S is *convex* if for any $x_1, x_2 \in S$ and $\gamma \in [0,1]$,

$$f(\gamma x_1 + (1-\gamma)x_2) \le \gamma f(x_1) + (1-\gamma)f(x_2).$$

If the inequality above is a strict inequality whenever $x_1 \ne x_2$ and $0 < \gamma < 1$, then f is *strictly convex.*

The definition above can be found in, for example, Fleming [61]. The lemma that follows establishes the connection of convex function with Lipschitz continuity and differentiability.

Lemma A.34 (Clarke [32, Theorem 2.5.1]). *Let f be a convex function on $\mathbb{R}^r$. Then*

(a) *f is locally Lipschitz and therefore continuous, and*

(b) *f is differentiable a.e.*

Theorem A.35 (Arzelà-Ascoli Theorem). *Let $S_N = \{x \in \mathbb{R}^r : |x| \le N\}$. Let $f_n(x)$ denote a sequence of continuous functions defined on S_N. If*

$$\sup_n \sup_{x \in S_N} |f_n(x)| < \infty$$

and

$$\sup_{n \ge 1, |x - x'| \le \delta} |f_n(x) - f_n(x')| \to 0 \text{ as } \delta \to 0,$$

then there exists a continuous function $f(x)$ defined on S_N and a subsequence $\{n_l\}$ such that

$$\sup_{x \in S_N} |f_{n_l}(x) - f(x)| \to 0 \text{ as } l \to \infty.$$

Proof: See Yosida [245] for a proof. □

Lemma A.36 (Yosida [245, p. 126]). *Let $\{u^k\}$ denote a sequence of elements in a Hilbert space $\mathcal{H}$. If u^k is bounded in norm, then there exists a subsequence of $\{u^k\}$ that converges to an element of $\mathcal{H}$ under the weak topology.*

The following Fredholm alternative, which provides a powerful method for establishing existence and uniqueness of solutions for various systems of equations, can be found in, for example, Hutson and Pym [90, p. 184].

Lemma A.37 (Fredholm Alternative). *Let $\mathbb{B}$ be a Banach space and $A : \mathbb{B} \to \mathbb{B}$ be a linear compact operator. Let $I : \mathbb{B} \to \mathbb{B}$ be the identity operator. Assume $\gamma \neq 0$. Then one of the two following alternatives holds.*

(a) *The homogeneous equation $(\gamma I - A)f = 0$ has only the zero solution, in which case $\gamma \in \rho(A)$-the resolvent set of A, $(\gamma I - A)^{-1}$ is bounded, and the inhomogeneous equation $(\gamma I - A)f = g$ has also one solution $f = (\gamma I - A)^{-1}g$, for each $g \in \mathbb{B}$.*

(b) *The homogeneous equation $(\gamma I - A)f = 0$ has a nonzero solution, in which case the inhomogeneous equation $(\gamma I - A)f = g$ has a solution iff $\langle g, f^* \rangle = 0$ for every solution f^* of the adjoint equation $\gamma f^* = A^* f^*$.*

This lemma is in a rather general form. When it is specialized to linear systems of algebraic equations, it can be written in the simple form below. Let B denote an $m \times m$ matrix. For any $\gamma \neq 0$, define an operator $A : \mathbb{R}^{1\times m} \to \mathbb{R}^{1\times m}$ as

$$Ay = y(\gamma I - B).$$

Note that in this case, I is just the $m \times m$ identity matrix I. Then the adjoint operator $A^* : \mathbb{R}^{m\times 1} \to \mathbb{R}^{m\times 1}$ is

$$A^* x = (\gamma I - B)x.$$

We have the following corollary.

Corollary A.38. *Suppose that b, $y \in \mathbb{R}^{1\times m}$ and $B \in \mathbb{R}^{m\times m}$. Consider the system $yB = b$. If the adjoint system $Bx = 0$ where $x \in \mathbb{R}^{m\times 1}$ has only the zero solution, then $yB = b$ has a unique solution given by $y = bB^{-1}$. If $Bx = 0$ has a nonzero solution x, then $yB = b$ has a solution iff $\langle b, x \rangle = 0$.*

Lemma A.39. *If a generator* $Q = (q_{ij})_{m\times m}$ *is irreducible and*

$$f(i) \le \sum_{j\neq i} \gamma_{ij} f(j), \tag{A.37}$$

where $\gamma_{ij} = q_{ij}/(-q_{ii})$, *then*

$$f(1) = f(2) = \cdots = f(m). \tag{A.38}$$

Proof: Let

$$Q_1 = \begin{pmatrix} -1 & \gamma_{12} & \cdots & \gamma_{1m} \\ \gamma_{21} & -1 & \cdots & \gamma_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{m1} & \gamma_{m2} & \cdots & -1 \end{pmatrix}.$$

Then (A.37) can be written in terms of Q_1 as

$$Q_1 f \ge 0, \text{ where } f = (f(1), \ldots, f(m))'. \tag{A.39}$$

Clearly, the irreducibility of Q implies that Q_1 is also irreducible. The irreducibility of Q_1 implies that the null space $N(Q_1)$ is a one-dimensional space spanned by $\mathbb{1} = (1, \ldots, 1)'$. Therefore, if (A.39) holds with an equality in place of the inequality, then (A.38) holds.

We show that the equality in (A.39) holds. Without loss of generality, assume

$$f(1) \ge f(2) \ge \cdots \ge f(m). \tag{A.40}$$

Let

$$\check{Q}_1 = \begin{pmatrix} -1 & \gamma_{12} \cdots \gamma_{1m} \\ 0_{m-1} & Q_2 \end{pmatrix},$$

where $0_{m-1} = (0, \ldots, 0)'$ is the $(m-1)$-dimensional zero vector and

$$Q_2 = \begin{pmatrix} -1 & \gamma_{23}^2 & \cdots & \gamma_{2m}^2 \\ \gamma_{32}^2 & -1 & \cdots & \gamma_{3m}^2 \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{m2}^2 & \gamma_{m3}^2 & \cdots & -1 \end{pmatrix}_{(m-1)\times(m-1)}$$

with $\gamma_{ij}^2 = (\gamma_{ij} + \gamma_{i1}\gamma_{1j})/(1 - \gamma_{i1}\gamma_{1i})$. It is easy to see that $\sum_{j\neq i} \gamma_{ij}^2 = 1$. Moreover, $\check{Q}_1$ can be obtained from Q_1 by the following procedure: (1) multiplying the first row of Q_1 by γ_{i1} and adding to the ith row; (2) normalizing the resulting matrix by dividing the ith row by $1 - \gamma_{i1}\gamma_{1i}$. Note

that the irreducibility of Q_1 implies that $1 - \gamma_{i1}\gamma_{1i} > 0$ and Q_2 is also irreducible.

Use the notation $Q_1 f \succ 0$ to represent $Q_1 f \geq 0$ and $Q_1 f \neq 0$. Let $f^2 = (f(2), \ldots, f(m))'$. Then by (A.40), $Q_1 f \succ 0$ implies $Q_2 f^2 \succ 0$. Take Q_2 as Q_1 and repeat the above procedure. After $(m-1)$ steps, we have $Q_m = \{0\}$ and $f^m = (f(m))$ such that $Q_m f^m \succ 0$, which is a contradiction. Thus the lemma follows. □

Lemma A.40. *Let $\beta(\cdot)$ be a Borel measurable function. For each $\eta(\cdot) \in D([0,T];\mathbb{R})$ define*

$$\phi(\eta)(t) = \int_0^t \beta(s)\eta(s)ds.$$

Then $\phi(\eta)$ is continuous on $D([0,T];\mathbb{R})$.

Proof: It suffices to show $\phi(\eta_n)(\cdot) \to 0$ as $\eta_n(\cdot) \to 0$ under the Skorohod topology. In fact, $\eta_n(\cdot) \to 0$ implies that $\eta_n(t) \to 0$ for all $t \in [0,T]$ (see Billingsley [13]). Hence, the Lebesgue dominated convergence theorem implies that $\phi(\eta_n)(\cdot) \to 0$. □

Lemma A.41. *Let $\eta_n(\cdot)$ and $\eta(\cdot)$ denote uniformly bounded functions in $D([0,T];\mathbb{R})$. Assume that $\eta(\cdot)$ has at most countably many of discontinuity points and $\eta_n(\cdot) \to \eta(\cdot)$ under the Skorohod topology. Then for all $t \in [0,T]$,*

$$\int_0^t \eta_n(s)ds \to \int_0^t \eta(s)ds.$$

Proof: In view of Billingsley [13, p. 112], $\eta_n(t) \to \eta(t)$ for all t at which $\eta(\cdot)$ is continuous. Since the Lebesgue measure of a countable set is 0, we have $\eta_n(\cdot) \to \eta(\cdot)$ a.e. The result follows from the Lebesgue dominated convergence theorem. □

Lemma A.42. *Let ξ and η denote two random variables on a probability space $(\Omega, \mathcal{F}, P)$. Assume η to be a discrete random variable taking values in $\{1, \ldots, m\}$. Then*

$$E[\xi|\eta] = \sum_{i=1}^{m} E[\xi|\eta = i] I_{\{\eta=i\}} \tag{A.41}$$

and for any event $A \in \mathcal{F}$,

$$P(A|\eta) = \sum_{i=1}^{m} P(A|\eta = i) I_{\{\eta=i\}}. \tag{A.42}$$

Proof: We only verify (A.41) because (A.42) can be obtained by taking $\xi = I_A$ in (A.41).

Let Ξ denote a class of $\mathcal{F}$ measurable random variables such that (A.41) holds, i.e.,

$$\Xi = \left\{ X : \ E[X|\eta] = \sum_{i=1}^{m} E[X|\eta = i] I_{\{\eta=i\}} \right\}.$$

Then it can be shown that Ξ is a monotone system and contains all indicator functions of a set in $\mathcal{F}$. Following from Chow and Teicher [30, Theorem 1.4.3], the set Ξ contains all $\mathcal{F}$ measurable random variables. □

Proof of Theorem 4.41

For the purpose of preparation, we establish a lemma first. It may be regarded as a comparison theorem, which compares two linear differential equations, one with constant coefficients (a constant matrix A) and the other being time-varying in nature. The rationale is that if the coefficient matrices are "close," then their solutions will be close as well.

Lemma A.43. *Let $A^\varepsilon(t) = (a_{ij}^\varepsilon(t))$ denote a matrix with*

$$a_{ij}^\varepsilon(t) \geq 0 \text{ for } i \neq j \text{ and } a_{ii}^\varepsilon(t) \leq -\sum_{j \neq i} a_{ij}^\varepsilon(t).$$

Assume

$$\left| A^\varepsilon(t) - A \right| \leq K\varepsilon(t+1), \tag{A.43}$$

for some stable matrix A with

$$\left| \exp(At) \right| \leq K \exp(-\kappa_0 t), \text{ for some } \kappa_0 > 0.$$

Let $Y(t,s)$ denote a fundamental matrix solution to

$$\frac{dY(t,s)}{dt} = Y(t,s)A^\varepsilon(t), \ Y(s,s) = I, \ 0 \leq s \leq t.$$

Then for $0 \leq s \leq t$,

$$\begin{aligned} \left| Y(t,s) - \exp(A(t-s)) \right| &\leq K\varepsilon \exp(-\kappa_0(t-s)) \\ &\times \left((t+1)^2 + \varepsilon(t+1)^4 \exp\left(K\varepsilon(t+1)^2\right) \right). \end{aligned} \tag{A.44}$$

Proof: As hinted in the remark preceding the lemma, consider an auxiliary problem

$$\frac{d\overline{Y}(t,s)}{dt} = \overline{Y}(t,s)A, \ Y(s,s) = I, \ 0 \leq s \leq t.$$

The solution is $\overline{Y}(t,s) = \exp(A(t-s))$. Denote $\widehat{Y}(t,s) = Y(t,s) - \overline{Y}(t,s)$. Then

$$\frac{d\widehat{Y}(t,s)}{dt} = \widehat{Y}(t,s)A + \widehat{Y}(t,s)(A^\varepsilon(t) - A) + \overline{Y}(t,s)(A^\varepsilon(t) - A),$$

with $\widehat{Y}(s,s) = 0$. It follows that

$$\begin{aligned}\widehat{Y}(t,s) = &\int_s^t \widehat{Y}(\tau,s)(A^\varepsilon(\tau) - A)\exp(A(t-\tau))d\tau \\ &+ \int_s^t \overline{Y}(\tau,s)(A^\varepsilon(\tau) - A)\exp(A(t-\tau))d\tau.\end{aligned}$$

Recall (A.43) and the fact that $|\overline{Y}(t,s)| \le K\exp(-\kappa_0(t-s))$. We obtain

$$\begin{aligned}\left|\widehat{Y}(t,s)\right| \le &\int_s^t K\varepsilon(\tau+1)\left|\widehat{Y}(\tau,s)\right|\exp(-\kappa_0(t-\tau))d\tau \\ &+ \int_s^t K\varepsilon(\tau+1)\exp(-\kappa_0(t-s))d\tau.\end{aligned}$$

Applying Gronwall's inequality to $\exp(\kappa_0(t-s))|\widehat{Y}(t,s)|$ yields (A.44). This concludes the proof. □

Proof of Theorem 4.41: First, write

$$\widehat{Q}(t) = \begin{pmatrix} \widehat{Q}_{11}(t) & \cdots & \widehat{Q}_{1l}(t) & \widehat{Q}_{1*}(t) \\ \vdots & \cdots & \vdots & \vdots \\ \widehat{Q}_{l1}(t) & \cdots & \widehat{Q}_{ll}(t) & \widehat{Q}_{l*}(t) \\ \widehat{Q}_{*1}(t) & \cdots & \widehat{Q}_{*l}(t) & \widehat{Q}_{*}(t) \end{pmatrix}, \tag{A.45}$$

where for all $i,j = 1,\ldots,l$, $\widehat{Q}_{ij}(t)$, $\widehat{Q}_{i*}(t)$, $\widehat{Q}_{*j}(t)$, and $\widehat{Q}_{*}(t)$, are $m_i \times m_j$, $m_i \times m_*$, $m_* \times m_j$, and $m_* \times m_*$ matrices, respectively.

In view of the differential equation governing $p^\varepsilon(\cdot)$, namely (4.40), and $\widetilde{Q}^k(t)\mathbb{1}_{m_k} = 0$, we have for $k = 1,\ldots,l$,

$$p^{\varepsilon,k}(\delta)\mathbb{1}_{m_k} = p^{0,k}\mathbb{1}_{m_k} + \frac{1}{\varepsilon}\int_0^\delta p^{\varepsilon,*}(s)\widetilde{Q}_*^k(s)ds\mathbb{1}_{m_k} + \int_0^\delta p^\varepsilon(s)\widehat{Q}^k(s)ds\mathbb{1}_{m_k},$$

where $\widehat{Q}^k(t)$ denotes the kth column of $\widehat{Q}(t)$ given in (A.45), that is,

$$\widehat{Q}^k(s) = \begin{pmatrix} \widehat{Q}_{1k}(t) \\ \vdots \\ \widehat{Q}_{lk}(t) \\ \widehat{Q}_{*k}(t) \end{pmatrix} \in \mathbb{R}^{m\times m_k}.$$

Since $p^\varepsilon(\cdot)$ and $\widehat{Q}^k(\cdot)$ are bounded,

$$\lim_{\delta\to 0}\left\{\limsup_{\varepsilon\to 0}\int_0^\delta p^\varepsilon(s)\left|\widehat{Q}^k(s)\right|ds\mathbb{1}_{m_k}\right\}=0,$$

and hence

$$\lim_{\delta\to 0}\left\{\limsup_{\varepsilon\to 0}\left|\int_0^\delta p^\varepsilon(s)\widehat{Q}^k(s)ds\mathbb{1}_{m_k}\right|\right\}=0.$$

It suffices to show

$$\frac{1}{\varepsilon}\int_0^\delta p^{\varepsilon,*}(s)\widetilde{Q}^k_*(s)ds\to p^{0,*}\int_0^\infty \exp(\widetilde{Q}_*(0)s)\widetilde{Q}^k_*(0)ds. \tag{A.46}$$

By changing of variable $s\to s/\varepsilon$, we have

$$\frac{1}{\varepsilon}\int_0^\delta p^{\varepsilon,*}(s)\widetilde{Q}^k_*(s)ds=\int_0^{\delta/\varepsilon} p^{\varepsilon,*}(\varepsilon s)\widetilde{Q}^k_*(\varepsilon s)ds.$$

Let $y(t)=p^{\varepsilon,*}(\varepsilon t)$ and $A^\varepsilon(t)=\widetilde{Q}_*(\varepsilon t)+\varepsilon\widehat{Q}_*(\varepsilon t)$. Then

$$\frac{dy(t)}{dt}=y(t)A^\varepsilon(t)+\varepsilon g^\varepsilon(t),\ y(0)=p^{0,*},$$

where $g^\varepsilon(t)=\sum_{k=1}^l p^{\varepsilon,k}(\varepsilon t)\widehat{Q}_{k*}(\varepsilon t)$. It is easy to see that $g^\varepsilon(t)$ is bounded and all of its components are nonnegative.

Let $Y(t,s)$ denote a fundamental matrix solution to

$$\frac{dY(t,s)}{dt}=Y(t,s)A^\varepsilon(t)\text{ with }Y(s,s)=I.$$

Then

$$p^{\varepsilon,*}(\varepsilon t)=y(t)=p^{0,*}Y(t,0)+\int_0^t\varepsilon g^\varepsilon(r)Y(t,r)dr.$$

Moreover, by Lemma A.43,

$$\begin{aligned}&p^{0,*}Y(t,0)=p^{0,*}\exp(\widetilde{Q}_*(0)t)+O(\Delta^\varepsilon(t,0)),\text{ and}\\&p^{\varepsilon,*}(\varepsilon t)=p^{0,*}\exp(\widetilde{Q}_*(0)t)+O(\Delta^\varepsilon(t,0))+\int_0^t\varepsilon g^\varepsilon(r)Y(t,r)dr,\end{aligned}$$

where

$$\Delta^\varepsilon(t,0)=\varepsilon\exp(-\kappa_0 t)((t+1)^2+\varepsilon(t+1)^4e^{K\varepsilon(t+1)^2}).$$

Note that

$$\begin{aligned}\int_0^{\delta/\varepsilon}p^{\varepsilon,*}(\varepsilon s)\widetilde{Q}^k_*(\varepsilon s)ds&=\int_0^{\delta/\varepsilon}p^{0,*}\exp(\widetilde{Q}_*(0)s)\widetilde{Q}^k_*(\varepsilon s)ds\\&+\int_0^{\delta/\varepsilon}O(\Delta^\varepsilon(s,0))\widetilde{Q}^k_*(\varepsilon s)ds+\int_0^{\delta/\varepsilon}g_1^\varepsilon(s)ds,\end{aligned} \tag{A.47}$$

where

$$g_1^\varepsilon(s) = \left(\int_0^s \varepsilon g^\varepsilon(r) Y(s,r) dr \right) \widetilde{Q}_*^k(\varepsilon s).$$

The stability of $\widetilde{Q}_*(0)$ and Lipschitz property of $\widetilde{Q}_*^k(t)$ imply that the first term on the right-hand side of (A.47) converges to

$$\int_0^\infty p^{0,*} \exp(\widetilde{Q}_*(0)s) \widetilde{Q}_*^k(0) ds.$$

To estimate the second term, note that for $t \le \delta/\varepsilon$,

$$\exp(-\kappa_0 t) \exp(K\varepsilon(t+1)^2) \le \exp(-(\kappa_0 - K(\delta+\varepsilon))t).$$

For ε and δ small enough so that $\kappa_0 - K(\varepsilon+\delta) > 0$, in view of the boundedness of $\widetilde{Q}_*^k(\cdot)$,

$$\int_0^{\delta/\varepsilon} O(\Delta^\varepsilon(s,0)) \widetilde{Q}_*^k(\varepsilon s) ds \le \varepsilon O(1) \to 0.$$

Finally, we show that

$$\lim_{\delta\to 0} \left\{ \limsup_{\varepsilon\to 0} \int_0^{\delta/\varepsilon} g_1^\varepsilon(s) ds \right\} = 0. \tag{A.48}$$

In view of the probabilistic interpretation and the structure of the matrix $\widetilde{Q}_*^k(\varepsilon t)$, all components of $g_1^\varepsilon(t)$ are nonnegative. It in turn implies that $\int_0^{\delta/\varepsilon} g_1^\varepsilon(s) ds \mathbb{1}_{m_*} \ge 0$. This together with

$$1 = \sum_{k=1}^l p^{\varepsilon,k}(\delta) \mathbb{1}_{m_k} + p^{\varepsilon,*}(\delta) \mathbb{1}_{m_*} = \sum_{k=1}^l \vartheta^{(0),k}(0)$$

then yields (A.48). The proof of the theorem is thus completed. □

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Index

G.G. Yin and Q. Zhang, *Continuous-Time Markov Chains and Applications*, Stochastic Modelling 37, DOI 10.1007/978-1-4614-4346-9,