Chapter 3
Generalized Sampling in $L^2(\mathbb{R}^d)$ Shift-Invariant Subspaces with Multiple Stable Generators

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Abstract In order to avoid most of the problems associated with classical Shannon's sampling theory, nowadays, signals are assumed to belong to some shift-invariant subspace. In this work we consider a general shift-invariant space V_Φ^2 of $L^2(\mathbb{R}^d)$ with a set Φ of r stable generators. Besides, in many common situations, the available data of a signal are samples of some filtered versions of the signal itself taken at a sub-lattice of $\mathbb{R}^d$. This leads to the problem of generalized sampling in shift-invariant spaces. Assuming that the ℓ^2-norm of the generalized samples of any $f \in V_\Phi^2$ is stable with respect to the $L^2(\mathbb{R}^d)$-norm of the signal f, we derive frame expansions in the shift-invariant subspace allowing the recovery of the signals in V_Φ^2 from the available data. The mathematical technique used here mimics the Fourier duality technique which works for classical Paley–Wiener spaces.

3.1 By Way of Introduction

The classical Whittaker–Shannon–Kotel'nikov sampling theorem (WSK sampling theorem) [23, 52] states that any function f band-limited to $[-1/2, 1/2]$, i.e., $f(t) = \int_{-1/2}^{1/2} \widehat{f}(w) \mathrm{e}^{2\pi i t w} \mathrm{d}w$ for each $t \in \mathbb{R}$, may be reconstructed from the sequence of samples $\{f(n)\}_{n\in\mathbb{Z}}$ as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}, \quad t \in \mathbb{R}.$$

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Thus, the Paley–Wiener space $PW_{1/2}$ of band-limited functions to $[-1/2,1/2]$ is generated by the integer shifts of the cardinal sine function, $\operatorname{sinc}(t) := \sin \pi t/\pi t$. A simple proof of this result is given by using the Fourier duality technique which uses that the Fourier transform

$$\mathscr{F} : PW_{1/2} \longrightarrow L^2[-1/2,1/2]$$
$$f \longmapsto \widehat{f}$$

is a unitary operator from the Paley–Wiener space $PW_{1/2}$ of band-limited functions to $[-1/2,1/2]$ onto $L^2[-1/2,1/2]$. Thus, applying the inverse Fourier transform $\mathscr{F}^{-1}$ to the Fourier series $\widehat{f} = \sum_{n=-\infty}^{\infty} f(n)\,\mathrm{e}^{-2\pi i n w}$ of $\widehat{f}$ in $L^2[-1/2,1/2]$ one gets

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} f(n)\mathscr{F}^{-1}\left[\mathrm{e}^{-2\pi i n w}\chi_{[-1/2,1/2]}(w)\right](t) \\ &= \sum_{n=-\infty}^{\infty} f(n)\,\frac{\sin \pi(t-n)}{\pi(t-n)} \ \text{ in } \ L^2(\mathbb{R})\,. \end{aligned}$$

The pointwise convergence comes from the fact that $PW_{1/2}$ is a reproducing kernel Hilbert space (written shortly as RKHS) where convergence in norm implies pointwise convergence (which is, in this case, uniform on $\mathbb{R}$); this comes out from the inequality $|f(t)| \le \|f\|$ for each $t \in \mathbb{R}$ and $f \in PW_{1/2}$ (for the RKHS's theory and applications, see, for instance, [37]).

The WSK theorem has its d-dimensional counterpart. Any function f band-limited to the d-dimensional cube $[-1/2,1/2]^d$, i.e., $f(t) = \int_{[-1/2,1/2]^d} \widehat{f}(x)\mathrm{e}^{2\pi i x^\top t}dx$ for each $t \in \mathbb{R}^d$ (here we are using the notation $x^\top t := x_1t_1 + \cdots + x_dt_d$ identifying elements in $\mathbb{R}^d$ with column vectors), may be reconstructed from the sequence of samples $\{f(\alpha)\}_{\alpha\in\mathbb{Z}^d}$ as

$$f(t) = \sum_{\alpha\in\mathbb{Z}^d} f(\alpha)\frac{\sin \pi(t_1-\alpha_1)}{\pi(t_1-\alpha_1)}\cdots\frac{\sin \pi(t_d-\alpha_d)}{\pi(t_d-\alpha_d)}\,, \quad t = (t_1,\ldots,t_d) \in \mathbb{R}^d\,,$$

where $\alpha = (\alpha_1,\ldots,\alpha_d)$. Although Shannon's sampling theory has had an enormous impact, it has a number of problems, as pointed out by Unser in [44,45]: It relies on the use of ideal filters; the band-limited hypothesis is in contradiction with the idea of a finite duration signal; the band-limiting operation generates Gibbs oscillations; and finally, the sinc function has a very slow decay at infinity which makes computation in the signal domain very inefficient. Besides, in several dimensions, it is also inefficient to assume that a multidimensional signal is band-limited to a d-dimensional interval. Moreover, many applied problems impose different a priori constraints on the type of signals. For this reason, sampling and reconstruction problems have been investigated in spline spaces, wavelet spaces, and general shift-invariant spaces; signals are assumed to belong to some shift-invariant space of the form $V_\varphi^2 := \overline{\text{span}}_{L^2}\{\varphi(t-\alpha) : \alpha \in \mathbb{Z}^d\}$ where the function φ in $L^2(\mathbb{R}^d)$ is called the generator of V_φ^2. See, for instance, [1,3,4,6,7,10,24,45,47,49–51,53] and references therein.

In this new context, the analogous of the WSK sampling theorem in a shift-invariant space V_φ^2 was first time proved by Walter in [47].

3.1.1 Walter's Sampling Theorem in Shift-Invariant Spaces

Let $\varphi \in L^2(\mathbb{R})$ be a stable generator for the shift-invariant space V_φ^2 which means that the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ^2. A Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Any Riesz basis $\{x_n\}_{n=1}^\infty$ has a unique biorthogonal (dual) Riesz basis $\{y_n\}_{n=1}^\infty$, i.e., $\langle x_n, y_m\rangle_{\mathscr{H}} = \delta_{n,m}$, such that the expansions

$$x = \sum_{n=1}^\infty \langle x, y_n\rangle_{\mathscr{H}}\, x_n = \sum_{n=1}^\infty \langle x, x_n\rangle_{\mathscr{H}}\, y_n$$

hold for every $x \in \mathscr{H}$ (see [11] for more details and proofs). Recall that the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence, i.e., a Riesz basis for V_φ^2 (see, for instance, [11, p 143]) if and only if there exist two positive constants $0 < A \le B$ such that

$$A \le \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(w+k)|^2 \le B, \quad \text{a.e. } w \in [0,1].$$

Thus, we have that $V_\varphi^2 = \{\sum_{n\in\mathbb{Z}} a_n\, \varphi(\cdot - n) : \{a_n\} \in \ell^2(\mathbb{Z})\} \subset L^2(\mathbb{R})$.

We assume that the functions in the shift-invariant space V_φ^2 are continuous on $\mathbb{R}$. This is equivalent to say that the generator φ is continuous on $\mathbb{R}$ and the function $\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ is uniformly bounded on $\mathbb{R}$ (see [42]). Thus, any $f \in V_\varphi^2$ is defined on $\mathbb{R}$ as the pointwise sum $f(t) = \sum_{n\in\mathbb{Z}} a_n \varphi(t-n)$ for each $t \in \mathbb{R}$.

On the other hand, the space V_φ^2 is the image of the Hilbert space $L^2[0,1]$ by means of the isomorphism

$$\mathscr{T}_\varphi : L^2[0,1] \longrightarrow V_\varphi^2$$
$$\{\mathrm{e}^{-2\pi i n x}\}_{n\in\mathbb{Z}} \longmapsto \{\varphi(t-n)\}_{n\in\mathbb{Z}},$$

which maps the orthonormal basis $\{\mathrm{e}^{-2\pi i n w}\}_{n\in\mathbb{Z}}$ for $L^2[0,1]$ onto the Riesz basis $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ for V_φ^2. For any $F \in L^2[0,1]$ we have

$$\begin{aligned}\mathscr{T}_\varphi F(t) &= \sum_{n\in\mathbb{Z}} \left\langle F, \mathrm{e}^{-2\pi i n x}\right\rangle \varphi(t-n) \\ &= \left\langle F, \sum_{n\in\mathbb{Z}} \overline{\varphi(t-n)}\mathrm{e}^{-2\pi i n x}\right\rangle = \langle F, K_t\rangle_{L^2[0,1]}, \quad t \in \mathbb{R},\end{aligned}$$

where, for each $t \in \mathbb{R}$, the function $K_t \in L^2[0,1]$ is given by

$$K_t(x) = \sum_{n\in\mathbb{Z}} \overline{\varphi(t-n)}\mathrm{e}^{-2\pi inx} = \overline{\sum_{n\in\mathbb{Z}} \varphi(t+n)\mathrm{e}^{-2\pi inx}} = \overline{Z\varphi(t,x)}\,.$$

Here, $Z\varphi(t,x) := \sum_{n\in\mathbb{Z}} \varphi(t+n)\mathrm{e}^{-2\pi inx}$ denotes the Zak transform of the function φ. See [11, 22] for properties and uses of the Zak transform.

As a consequence, the samples in $\{f(a+m)\}_{m\in\mathbb{Z}}$ of $f \in V_\varphi^2$, where $a \in [0,1)$ is fixed, can be expressed as

$$f(a+m) = \langle F, K_{a+m}\rangle = \langle F, \mathrm{e}^{-2\pi imx}K_a\rangle\,, \quad m \in \mathbb{Z} \text{ where } F = \mathscr{T}_\varphi^{-1} f\,.$$

Thus, the stable recovery of $f \in V_\varphi^2$ from the sequence of its samples $\{f(a+m)\}_{m\in\mathbb{Z}}$ reduces to the study of the sequence $\left\{\mathrm{e}^{-2\pi imx}K_a(x)\right\}_{m\in\mathbb{Z}}$ in $L^2[0,1]$. Recall that the operator $m_F : L^2[0,1] \to L^2[0,1]$ given as the product $m_F(f) = Ff$ is well defined if and only if $F \in L^\infty[0,1]$, and then, it is bounded with norm $\|m_F\| = \|F\|_\infty$. As a consequence, the following result comes out:

Theorem 1. *The sequence of functions* $\left\{\mathrm{e}^{-2\pi imx}K_a(x)\right\}_{m\in\mathbb{Z}}$ *is a Riesz basis for* $L^2[0,1]$ *if and only if the inequalities* $0 < \|K_a\|_0 \le \|K_a\|_\infty < \infty$ *hold, where* $\|K_a\|_0 := \operatorname{ess\,inf}_{x\in[0,1]} |K_a(x)|$ *and* $\|K_a\|_\infty := \operatorname{ess\,sup}_{x\in[0,1]} |K_a(x)|$. *Moreover, its biorthogonal Riesz basis is* $\left\{\mathrm{e}^{-2\pi imx}/\overline{K_a(x)}\right\}_{m\in\mathbb{Z}}$.

In particular, the sequence $\left\{\mathrm{e}^{-2\pi imx}K_a(x)\right\}_{m\in\mathbb{Z}}$ is an orthonormal basis in $L^2[0,1]$ if and only if $|K_a(x)| = 1$ a.e. in $[0,1]$.

Let a be a real number in $[0,1)$ such that $0 < \|K_a\|_0 \le \|K_a\|_\infty < \infty$; next, we prove Walter's sampling theorem for V_φ^2 in [47]. Given $f \in V_\varphi^2$, we expand the function $F = \mathscr{T}_\varphi^{-1} f \in L^2[0,1]$ with respect to the Riesz basis $\left\{\mathrm{e}^{-2\pi inx}/\overline{K_a(x)}\right\}_{n\in\mathbb{Z}}$. Thus, we get

$$F = \sum_{n\in\mathbb{Z}} \langle F, K_{a+n}\rangle \frac{\mathrm{e}^{-2\pi inx}}{\overline{K_a(x)}} = \sum_{n\in\mathbb{Z}} f(a+n)\frac{\mathrm{e}^{-2\pi inx}}{\overline{K_a(x)}} \quad \text{in } L^2[0,1]\,.$$

Applying the operator $\mathscr{T}_\varphi$ to the above expansion we obtain

$$f = \sum_{n\in\mathbb{Z}} f(a+n)\mathscr{T}_\varphi\left(\mathrm{e}^{-2\pi inx}/\overline{K_a(x)}\right) = \sum_{n\in\mathbb{Z}} f(a+n)S_a(\cdot - n) \quad \text{in } L^2(\mathbb{R})\,,$$

where we have used the shifting property $\mathscr{T}_\varphi(\mathrm{e}^{-2\pi inx}F)(t) = (\mathscr{T}_\varphi F)(t-n)$, $t \in \mathbb{R}$, and $n \in \mathbb{Z}$, satisfied by the isomorphism $\mathscr{T}_\varphi$ for the particular function $S_a := \mathscr{T}_\varphi(1/\overline{K_a}) \in V_\varphi^2$. As in the Paley–Wiener case, the shift-invariant space V_φ^2 is a RKHS. Indeed, for each $t \in \mathbb{R}$, the evaluation functional at t is bounded:

$$|f(t)| \le \|F\|\|K_t\| \le \|\mathscr{T}_\varphi^{-1}\|\|K_t\|\|f\| = \|\mathscr{T}_\varphi^{-1}\|\left(\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2\right)^{1/2}\|f\|, \quad f \in V_\varphi^2.$$

Therefore, the L^2-convergence implies pointwise convergence which here is uniform on $\mathbb{R}$. The convergence is also absolute due to the unconditional convergence of a Riesz expansion. Thus, for each $f \in V_\varphi^2$, we get the sampling formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(a+n) S_a(t-n), \quad t \in \mathbb{R}. \tag{3.1}$$

This mathematical technique, which mimics the Fourier duality technique for Paley–Wiener spaces [23], has been successfully used in deriving sampling formulas in other sampling settings [14, 16, 17, 19, 21, 25, 31, 32]. In this work, it will be used for obtaining generalized sampling formulas in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators.

3.1.2 Statement of the General Problem

Assume that our functions (signals) belong to some shift-invariant space of the form

$$V_\Phi^2 := \overline{\operatorname{span}}_{L^2(\mathbb{R}^d)} \{\varphi_k(t-\alpha) : k = 1, 2, \dots, r \text{ and } \alpha \in \mathbb{Z}^d\},$$

where the functions in $\Phi := \{\varphi_1, \dots, \varphi_r\}$ in $L^2(\mathbb{R}^d)$ are called a set of generators for V_Φ^2. Assuming that the sequence $\{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d,\, k=1,2,\dots,r}$ is a Riesz basis for V_Φ^2, the shift-invariant space V_Φ^2 can be described as

$$V_\Phi^2 = \left\{ \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} d_k(\alpha)\ \varphi_k(t-\alpha) : d_k \in \ell^2(\mathbb{Z}^d), k = 1, 2, \dots, r \right\}. \tag{3.2}$$

See [8, 9, 36] for the general theory of shift-invariant spaces and their applications. These spaces and the scaling functions $\Phi = \{\varphi_1, \dots, \varphi_r\}$ appear in the multiwavelet setting. Multiwavelets lead to multiresolution analyses and fast algorithms just as scalar wavelets, but they have some advantages: they can have short support coupled with high smoothness and high approximation order, and they can be both symmetric and orthogonal (see, for instance, [29]). Classical sampling in multiwavelet subspaces has been studied in [38, 43].

On the other hand, in many common situations, the available data are samples of some filtered versions $f * \mathsf{h}_j$ of the signal f itself, where the average function h_j reflects the characteristics of the acquisition device. This leads to generalized sampling (also called average sampling) in V_Φ^2 (see, among others, [1, 5, 14, 16, 17, 30, 34, 35, 40, 41, 43]).

Suppose that s convolution systems (linear time-invariant systems or filters in engineering jargon) $\mathscr{L}_j$, $j = 1, 2, \dots, s$, are defined on the shift-invariant subspace V_Φ^2 of $L^2(\mathbb{R}^d)$. Assume also that the sequence of samples $\{(\mathscr{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ for f in V_Φ^2 is available, where the samples are taken at the sub-lattice $M\mathbb{Z}^d$ of $\mathbb{Z}^d$, where M denotes a matrix of integer entries with positive determinant. If we sample any function $f \in V_\Phi^2$ on $M\mathbb{Z}^d$, we are using the sampling rate $1/r(\det M)$ and, roughly speaking, we will need, for the recovery of $f \in V_\Phi^2$, the sequence of generalized samples $\{(\mathscr{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ coming from $s \ge r(\det M)$ convolution systems $\mathscr{L}_j$.

Assume that the sequences of generalized samples satisfy the following stability condition: There exist two positive constants $0 < A \le B$ such that

$$A\|f\|^2 \le \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} |\mathscr{L}_j f(M\alpha)|^2 \le B\|f\|^2 \quad \text{for all } f \in V_\Phi^2 .$$

In [5] the set of systems $\{\mathscr{L}_1, \mathscr{L}_2, \dots, \mathscr{L}_s\}$ is said to be an M-stable filtering sampler for V_Φ^2. The aim of this work is to obtain sampling formulas in V_Φ^2 having the form

$$f(t) = (\det M) \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathscr{L}_j f)(M\alpha) S_j(t - M\alpha), \quad t \in \mathbb{R}^d, \tag{3.3}$$

such that the sequence of reconstruction functions $\{S_j(\cdot - M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a frame for the shift-invariant space V_Φ^2. This will be done in the light of the frame theory for separable Hilbert spaces, by using a similar mathematical technique as in the above section.

Recall that a sequence $\{x_n\}$ is a frame for a separable Hilbert space $\mathscr{H}$ if there exist two constants $A, B > 0$ (frame bounds) such that

$$A\|x\|^2 \le \sum_n |\langle x, x_n\rangle|^2 \le B\|x\|^2 \quad \text{for all } x \in \mathscr{H} .$$

Given a frame $\{x_n\}$ for $\mathscr{H}$ the representation property of any vector $x \in \mathscr{H}$ as a series $x = \sum_n c_n x_n$ is retained, but, unlike the case of Riesz bases, the uniqueness of this representation (for overcomplete frames) is sacrificed. Suitable frame coefficients c_n, depending linearly and continuously on x, are obtained by using the dual frames $\{y_n\}$ of $\{x_n\}$, i.e., the sequence $\{y_n\}$ is another frame for $\mathscr{H}$ such that, for each $x \in \mathscr{H}$, the expansions $x = \sum_n \langle x, y_n\rangle x_n = \sum_n \langle x, x_n\rangle y_n$ hold. For more details on the frame theory see the superb monograph [11] and the references therein.

3.2 Preliminaries on $L^2(\mathbb{R}^d)$ Shift-Invariant Subspaces

Let $\Phi := \{\varphi_1, \varphi_2, \dots, \varphi_r\}$ be a set of functions, where $\varphi_k \in L^2(\mathbb{R}^d)$ $k = 1, 2, \dots, r$, such that the sequence $\{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d,\ k=1,2\dots,r}$ is a Riesz basis for the shift-invariant space

$$V_\Phi^2 := \left\{ \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} d_k(\alpha)\ \varphi_k(t-\alpha) \,:\, d_k \in \ell^2(\mathbb{Z}^d), k = 1, 2\dots, r \right\} \subset L^2(\mathbb{R}^d) .$$

There exists a necessary and sufficient condition involving the Gramian matrix function

$$G_{\boldsymbol{\Phi}}(w) := \sum_{\alpha\in\mathbb{Z}^d} \widehat{\boldsymbol{\Phi}}(w+\alpha)\overline{\widehat{\boldsymbol{\Phi}}(w+\alpha)}^{\top}, \text{ where } \widehat{\boldsymbol{\Phi}} := (\widehat{\varphi}_1, \widehat{\varphi}_2, \dots, \widehat{\varphi}_r)^{\top},$$

which assures that the sequence $\{\varphi_k(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d,\, k=1,2\dots,r}$ is a Riesz basis for $V^2_{\boldsymbol{\Phi}}$, namely (see, for instance, [5]): There exist two positive constants c and C such that

$$c\,\mathbb{I}_r \le G_{\boldsymbol{\Phi}}(w) \le C\,\mathbb{I}_r \quad \text{a.e. } w\in[0,1)^d. \tag{3.4}$$

We assume throughout this chapter that the functions in the shift-invariant space $V^2_{\boldsymbol{\Phi}}$ are continuous on $\mathbb{R}^d$. As in the case of one generator, this is equivalent to the generators $\boldsymbol{\Phi}$ being continuous on $\mathbb{R}^d$ with $\sum_{\alpha\in\mathbb{Z}^d} |\boldsymbol{\Phi}(t-\alpha)|^2$ uniformly bounded on $\mathbb{R}^d$. Thus, any $f\in V^2_{\boldsymbol{\Phi}}$ is defined on $\mathbb{R}^d$ as the pointwise sum

$$f(t) = \sum_{k=1}^{r} \sum_{\alpha\in\mathbb{Z}^d} d_k(\alpha)\; \varphi_k(t-\alpha), \quad t\in\mathbb{R}^d. \tag{3.5}$$

Besides, the space $V^2_{\boldsymbol{\Phi}}$ is an RKHS since the evaluation functionals, $E_t f := f(t)$, are bounded on $V^2_{\boldsymbol{\Phi}}$. Indeed, for each fixed $t\in\mathbb{R}^d$, we have

$$\begin{aligned}
|f(t)|^2 &= \left| \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} d_k(\alpha)\; \varphi_k(t-\alpha) \right|^2 \\
&\le \left(\sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} |d_k(\alpha)|^2 \right) \left(\sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} |\varphi_k(t-\alpha)|^2 \right) \\
&= \left(\sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} |d_k(\alpha)|^2 \right) \left(\sum_{\alpha\in\mathbb{Z}^d} |\boldsymbol{\Phi}(t-\alpha)|^2 \right) \\
&\le \frac{\|f\|^2}{c} \sum_{\alpha\in\mathbb{Z}^d} |\boldsymbol{\Phi}(t-\alpha)|^2, \quad f\in V^2_{\boldsymbol{\Phi}},
\end{aligned}$$

where we have used Cauchy–Schwarz's inequality in (3.5), and the inequality satisfied for any lower Riesz bound c of the Riesz basis $\{\varphi_k(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d,\, k=1,2\dots,r}$ for $V^2_{\boldsymbol{\Phi}}$, i.e., $c\sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r} |d_k(\alpha)|^2 \le \|f\|^2$.

Thus, the convergence in $V^2_{\boldsymbol{\Phi}}$ in the $L^2(\mathbb{R}^d)$ sense implies pointwise convergence which is uniform on $\mathbb{R}^d$.

The product space

$$L^2_r[0,1)^d := \left\{ \mathbf{F} = (F_1, F_2, \dots, F_r)^{\top} : F_k \in L^2[0,1)^d,\ k = 1,2,\dots,r \right\}$$

with its usual inner product

$$\langle \mathbf{F},\mathbf{H}\rangle_{L^2_r[0,1)^d} := \sum_{k=1}^{r} \langle F_k,H_k\rangle_{L^2[0,1)^d} = \int_{[0,1)^d} \mathbf{H}^*(w)\mathbf{F}(w)\mathrm{d}w$$

becomes a Hilbert space. Similarly, we introduce the product Banach space $L^\infty_r[0,1)^d$.

The system $\{\mathrm{e}^{-2\pi i\alpha^\top w}\mathbf{e}_k\}_{\alpha\in\mathbb{Z}^d,\, k=1,2,\dots,r}$, where $\mathbf{e}_k$ denotes the vector of $\mathbb{R}^r$ with all the components null except the kth component which is equal to one, is an orthonormal basis for $L^2_r[0,1)^d$.

The shift-invariant space V^2_Φ is the image of $L^2_r[0,1)^d$ by means of the isomorphism

$$\mathscr{T}_\Phi : L^2_r[0,1)^d \longrightarrow V^2_\Phi$$

$$\{\mathrm{e}^{-2\pi i\alpha^\top w}\mathbf{e}_k\}_{\alpha\in\mathbb{Z}^d,\, k=1,2,\dots,r} \longmapsto \{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d,\, k=1,2,\dots,r},$$

which maps the orthonormal basis $\{\mathrm{e}^{-2\pi i\alpha^\top w}\mathbf{e}_k\}_{\alpha\in\mathbb{Z}^d,\, k=1,2,\dots,r}$ for $L^2_r[0,1)^d$ onto the Riesz basis $\{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d,\, k=1,2,\dots,r}$ for V^2_Φ. For each $\mathbf{F}=(F_1,\dots,F_r)^\top \in L^2_r[0,1)^d$ we have

$$\mathscr{T}_\Phi\mathbf{F}(t) := \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r} \left\langle F_k,\mathrm{e}^{-2\pi i\alpha^\top \cdot}\right\rangle_{L^2[0,1)^d}\varphi_k(t-\alpha)\,, \quad t\in\mathbb{R}^d\,. \tag{3.6}$$

The isomorphism $\mathscr{T}_\Phi$ can also be expressed by

$$f(t) = \mathscr{T}_\Phi\mathbf{F}(t) = \langle \mathbf{F},\mathbf{K}_t\rangle_{L^2_r[0,1)^d}\,, \quad t\in\mathbb{R}^d\,,$$

where the kernel transform $\mathbb{R}^d \ni t \mapsto \mathbf{K}_t \in L^2_r[0,1)^d$ is defined as $\mathbf{K}_t(x) := \overline{\mathbf{Z}\Phi}(t,x)$, and $\mathbf{Z}\Phi$ denotes the Zak transform of Φ, i.e.,

$$(\mathbf{Z}\Phi)(t,w) := \sum_{\alpha\in\mathbb{Z}^d} \Phi(t+\alpha)\mathrm{e}^{-2\pi i\alpha^\top w}\,.$$

Note that $(\mathbf{Z}\Phi) = (Z\varphi_1,\dots,Z\varphi_r)^\top$ where Z denotes the usual Zak transform.

The following shifting property of $\mathscr{T}_\Phi$ will be used later: For $\mathbf{F}\in L^2_r[0,1)^d$ and $\alpha\in\mathbb{Z}^d$, we have

$$\mathscr{T}_\Phi\left[\mathbf{F}(\cdot)\mathrm{e}^{-2\pi i\alpha^\top\cdot}\right](t) = \mathscr{T}_\Phi\mathbf{F}(t-\alpha)\,, \quad t\in\mathbb{R}^d\,. \tag{3.7}$$

3.2.1 The Convolution Systems $\mathscr{L}_j$ on V^2_Φ

We consider s convolution systems $\mathscr{L}_j f = f * \mathsf{h}_j$, $j=1,2,\dots,s$, defined for $f\in V^2_\Phi$ where each impulse response h_j belongs to one of the following three types:

(a) The impulse response h_j is a linear combination of partial derivatives of shifted delta functionals, i.e.,

$$\big(\mathscr{L}_j f\big)(t) := \sum_{|\beta|\le N_j} c_{j,\beta} D^\beta f(t+d_{j,\beta}), \quad t\in\mathbb{R}^d .$$

If there is a system of this type, we also assume that $\sum_{\alpha\in\mathbb{Z}^d} |D^\beta \varphi(t-\alpha)|^2$ is uniformly bounded on $\mathbb{R}^d$ for $|\beta|\le N_j$.

(b) The impulse response h_j of $\mathscr{L}_j$ belongs to $L^2(\mathbb{R}^d)$. Thus, for any $f\in V_\varphi^2$, we have

$$\big(\mathscr{L}_j f\big)(t) := [f*\mathsf{h}_j](t) = \int_{\mathbb{R}^d} f(x)\mathsf{h}_j(t-x)\mathrm{d}x, \quad t\in\mathbb{R}^d.$$

(c) The function $\widehat{\mathsf{h}}_j \in L^\infty(\mathbb{R}^d)$ whenever $H_{\varphi_k}(x) := \sum_{\alpha\in\mathbb{Z}^d} |\widehat{\varphi}_k(x+\alpha)| \in L^2[0,1)^d$ for all $k=1,2,\dots,r$.

Lemma 1. *Let $\mathscr{L}$ be a convolution system of the type (b) or (c). Then, for each fixed $t\in\mathbb{R}^d$ the sequence $\{\big(\mathscr{L}\varphi_k\big)(t+\alpha)\}_{\alpha\in\mathbb{Z}^d}$ belongs to $\ell^2(\mathbb{Z}^d)$ for each $k=1,\dots,r$.*

Proof. First assume that $\mathsf{h}\in L^2(\mathbb{R}^d)$; then, we have

$$\sum_{\alpha\in\mathbb{Z}^d} |\mathscr{L}\varphi_k(t+\alpha)|^2 = \Big\| \sum_{\alpha\in\mathbb{Z}^d} \mathscr{L}\varphi_k(t+\alpha)\mathrm{e}^{-2\pi i\alpha^\top x}\Big\|^2_{L^2[0,1)^d} = \|Z\mathscr{L}\varphi_k(t,x)\|^2_{L^2[0,1)^d}$$
$$= \Big\| \sum_{\alpha\in\mathbb{Z}^d} \big(\widehat{\mathscr{L}\varphi_k}\big)(x+\alpha)\mathrm{e}^{2\pi i(x+\alpha)^\top t}\Big\|^2_{L^2[0,1)^d},$$

where, in the last equality, we have used a version of the Poisson summation formula [20, Lemma 2.1]. Notice that $\widehat{\varphi}_k, \widehat{\mathsf{h}}\in L^2(\mathbb{R}^d)$ implies, by Cauchy–Schwarz's inequality, that $\widehat{\varphi}_k\widehat{\mathsf{h}} = \widehat{\mathscr{L}\varphi_k}\in L^1(\mathbb{R}^d)$. Now,

$$\Big\| \sum_{\alpha\in\mathbb{Z}^d} \Big(\widehat{\mathscr{L}\varphi_k}\Big)(x+\alpha)\mathrm{e}^{2\pi i(x+\alpha)^\top t}\Big\|^2_{L^2[0,1)^d}$$
$$= \Big\| \sum_{\alpha\in\mathbb{Z}^d} \widehat{\varphi}_k(x+\alpha)\widehat{\mathsf{h}}(x+\alpha)\mathrm{e}^{2\pi i(x+\alpha)^\top t}\Big\|^2_{L^2[0,1)^d}$$
$$\le \Big\| \Big(\sum_{\alpha\in\mathbb{Z}^d} |\widehat{\varphi}_k(x+\alpha)|^2\Big)^{1/2} \Big(\sum_{\alpha\in\mathbb{Z}^d} |\widehat{\mathsf{h}}(x+\alpha)|^2\Big)^{1/2}\Big\|^2_{L^2[0,1)^d} \le C^{1/2}\|\mathsf{h}\|^2_{L^2[0,1)^d},$$

where we have used (3.4) and the fact that $\|\mathsf{h}\|^2_{L^2(\mathbb{R}^d)} = \|\sum_{\alpha\in\mathbb{Z}^d} |\widehat{\mathsf{h}}(x+\alpha)|^2\|_{L^1[0,1)^d}$. Finally, assume that $H_{\varphi_k} \in L^2[0,1)^d$; since $\widehat{\varphi}_k \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, we obtain that $\widehat{\mathscr{L}\varphi_k} = \widehat{\varphi}_k\widehat{\mathsf{h}} \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. Since $\sum_{\alpha\in\mathbb{Z}^d} |\widehat{\mathscr{L}\varphi_k}(x+\alpha)| \le \|\widehat{\mathsf{h}}\|_{L^\infty(\mathbb{R}^d)} H_{\varphi_k}(x)$, using again [20, Lemma 2.1], we get

$$\sum_{\alpha\in\mathbb{Z}^d} |\mathscr{L}\varphi_k(t+\alpha)|^2 = \left\| \sum_{\alpha\in\mathbb{Z}^d} \left(\widehat{\mathscr{L}\varphi_k}\right)(x+\alpha)\mathrm{e}^{2\pi i(x+\alpha)^\top t} \right\|^2_{L^2[0,1)^d}$$
$$\le \left\| \sum_{\alpha\in\mathbb{Z}^d} |\widehat{\mathscr{L}\varphi_k}(x+\alpha)| \right\|^2_{L^2[0,1)^d} \le \left\|\widehat{\mathsf{h}}\right\|^2_{L^\infty(\mathbb{R}^d)} \|H_{\varphi_k}\|^2_{L^2[0,1)^d}.$$

□

Lemma 2. *Let $\mathscr{L}$ be a convolution system of the type (a), (b), or (c). Then, for each $f\in V^2_\Phi$, we have*

$$(\mathscr{L}f)(t) = \left\langle \mathbf{F}, \left(\overline{\mathbf{Z}\mathscr{L}\boldsymbol{\Phi}}\right)(t,\cdot)\right\rangle_{L^2_r[0,1)^d}, \quad \textit{where} \quad \mathbf{F} = \mathscr{T}_{\boldsymbol{\Phi}}^{-1} f.$$

Proof. Assume that $\mathscr{L}$ is a convolution system of type (a). Under our hypothesis on $\mathscr{L}$, for $m = 0,1,2\ldots,N$, we have that

$$f^{(m)}(t) = \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^r \left\langle F_k, \mathrm{e}^{-2\pi i\alpha^\top\cdot}\right\rangle \varphi_k^{(m)}(t-\alpha).$$

Having in mind we have assumed that $\sum_{\alpha\in\mathbb{Z}^d} |\Phi^{(m)}(t-\alpha)|^2$ is uniformly bounded on $\mathbb{R}^d$, we obtain that

$$(\mathscr{L}f)(t) = \sum_{m=0}^N c_m f^{(m)}(t+d_m) = \sum_{m=0}^N c_m \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^r \left\langle F_k, \mathrm{e}^{-2\pi i\alpha^\top\cdot}\right\rangle \varphi_k^{(m)}(t+d_m-\alpha)$$
$$= \sum_{k=1}^r \left\langle F_k, \sum_{m=0}^N \overline{c_m} \sum_{\alpha\in\mathbb{Z}^d} \overline{\varphi_k}^{(m)}(t+d_m-\alpha)\mathrm{e}^{-2\pi i\alpha^\top\cdot}\right\rangle_{L^2[0,1)^d}$$
$$= \sum_{k=1}^r \left\langle F_k, \sum_{\alpha\in\mathbb{Z}^d} \overline{\mathscr{L}\varphi_k}(t-\alpha)\mathrm{e}^{-2\pi i\alpha^\top\cdot}\right\rangle = \sum_{k=1}^r \left\langle F_k, (\overline{\mathrm{Z}\mathscr{L}\varphi_k})(t,\cdot)\right\rangle_{L^2[0,1)^d}.$$

Assume now that $\mathscr{L}$ is a convolution system of the type (b) or (c). For each $t\in\mathbb{R}^d$, considering the function $\psi(x) := \overline{\mathsf{h}(-x)}$, we have

$$(\mathscr{L}f)(t) = \langle f, \psi(\cdot - t)\rangle_{L^2(\mathbb{R}^d)} = \left\langle \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^r \left\langle F_k, \mathrm{e}^{-2\pi i\alpha^\top\cdot}\right\rangle \varphi_k(\cdot-\alpha), \psi(\cdot - t)\right\rangle_{L^2(\mathbb{R}^d)}$$

$$= \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}\left\langle F_k, \mathrm{e}^{-2\pi i\alpha^\top\cdot}\right\rangle_{L^2[0,1)^d}\langle\varphi_k,\psi(\cdot-t+\alpha)\rangle_{L^2(\mathbb{R}^d)}$$

$$= \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}\left\langle F_k, \mathrm{e}^{-2\pi i\alpha^\top\cdot}\right\rangle_{L^2[0,1)^d}\mathscr{L}\varphi_k(t-\alpha)\,.$$

Since the sequence $\{(\mathscr{L}\varphi_k)(t+\alpha)\}_{\alpha\in\mathbb{Z}^d}\in\ell^2(\mathbb{Z}^d)$, Parseval's equality gives

$$(\mathscr{L}f)(t) = \sum_{k=1}^{r}\left\langle F_k, \sum_{\alpha\in\mathbb{Z}^d}\overline{\mathscr{L}\varphi_k}(t-\alpha)\,\mathrm{e}^{-2\pi i\alpha^\top\cdot}\right\rangle_{L^2[0,1)^d}$$
$$= \left\langle \mathbf{F}, (\overline{\mathbf{Z}\mathscr{L}\boldsymbol{\Phi}})(t,\cdot)\right\rangle_{L^2_r(0,1)},$$

which ends the proof. □

3.2.2 Sampling at a Lattice of $\mathbb{Z}^d$: An Expression for the Samples

Given a nonsingular matrix M with integer entries, we consider the lattice in $\mathbb{Z}^d$ generated by M, i.e.,

$$\Lambda_M := \{M\alpha : \alpha\in\mathbb{Z}^d\}\subset\mathbb{Z}^d\,.$$

Without loss of generality, we can assume that $\det M>0$; otherwise, we can consider $M' = ME$ where E is some $d\times d$ integer matrix satisfying $\det E=-1$. Trivially, $\Lambda_M = \Lambda'_M$. We denote by $M^\top$ and $M^{-\top}$ the transpose matrices of M and M^{-1}, respectively. The following useful generalized orthogonal relationship holds (see [46]):

$$\sum_{p\in\mathscr{N}(M^\top)}\mathrm{e}^{-2\pi i\alpha^\top M^{-T}p} = \begin{cases}\det M, & \alpha\in\Lambda_M\\ 0 & \alpha\in\mathbb{Z}^d\setminus\Lambda_M\end{cases} \tag{3.8}$$

where

$$\mathscr{N}(M^\top) := \mathbb{Z}^d\cap\{M^\top x : x\in[0,1)^d\}. \tag{3.9}$$

The set $\mathscr{N}(M^\top)$ has $\det M$ elements (see [46] or [48]). One of these elements is zero, say $i_1=0$; we denote the rest of elements by $i_2,\ldots,i_{\det M}$ ordered in any form; from now on, $\mathscr{N}(M^\top)=\{i_1=0,i_2,\ldots,i_{\det M}\}\subset\mathbb{Z}^d$.

Note that the sets, defined as $Q_l := M^{-\top}i_l + M^{-\top}[0,1)^d$, $l=1,2,\ldots,\det M$, satisfy (see [48, p 110])

$$Q_l\cap Q_{l'}=\emptyset \text{ if } l\neq l' \quad\text{and}\quad \mathrm{Vol}\left(\bigcup_{l=1}^{\det M}Q_l\right)=1\,.$$

Thus, $\int_{[0,1)^d} F(x)\mathrm{d}x = \sum_{l=1}^{\det M} \int_{Q_l} F(x)\mathrm{d}x$, for any function F integrable in $[0,1)^d$ and $\mathbb{Z}^d$-periodic. See also [39] and references therein for an abstract version of sampling in lattice invariant subspaces.

Now, assume that we sample the filtered versions $\mathscr{L}_j f$ of $f \in V_{\Phi}^2$, $j = 1,2,\dots,s$, at a lattice Λ_M. Having in mind Lemma 2, for $j = 1,2,\dots,s$ and $\alpha \in \mathbb{Z}^d$, we obtain that

$$\big(\mathscr{L}_j f\big)(M\alpha) = \langle \mathbf{F}, \overline{\mathbf{Z}\mathscr{L}_j \boldsymbol{\Phi}(M\alpha,\cdot)}\rangle = \Big\langle \mathbf{F}, \overline{\mathbf{Z}\mathscr{L}_j \boldsymbol{\Phi}}(0,\cdot)\mathrm{e}^{-2\pi i \alpha^\top M^\top \cdot} \Big\rangle_{L_r^2[0,1)^d}, \quad (3.10)$$

where $\mathbf{F} = \mathscr{T}_{\Phi}^{-1} f \in L_r^2[0,1)^d$. Denote

$$\mathbf{g}_j(x) := \mathbf{Z}\mathscr{L}_j \boldsymbol{\Phi}(0,x), \quad j = 1,2,\dots,s; \quad (3.11)$$

in other words, $\mathbf{g}_j^\top(x) := \big(g_{j,1}(x), g_{j,2}(x), \dots, g_{j,r}(x)\big)$, where $g_{j,k}(x) = Z\mathscr{L}_j \varphi_k(0,x)$ for $1 \le j \le s$ and $1 \le k \le r$.

As a consequence of expression (3.10) for generalized samples, a challenging problem is to study the completeness, Bessel, frame, or Riesz basis properties of any sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i \alpha^\top M^\top x}\big\}_{\alpha \in \mathbb{Z}^d,\ j=1,2,\dots,s}$ in $L_r^2[0,1)^d$. To this end we introduce the $s \times r(\det M)$ matrix of functions

$$\mathbb{G}(x) := \begin{bmatrix} \mathbf{g}_1^\top(x) & \mathbf{g}_1^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_1^\top(x + M^{-\top} i_{\det M}) \\ \mathbf{g}_2^\top(x) & \mathbf{g}_2^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_2^\top(x + M^{-\top} i_{\det M}) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{g}_s^\top(x) & \mathbf{g}_s^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_s^\top(x + M^{-\top} i_{\det M}) \end{bmatrix} \quad (3.12)$$

and its related constants

$$A_{\mathbb{G}} := \operatorname*{ess\,inf}_{x \in [0,1)^d} \lambda_{\min}[\mathbb{G}^*(x)\mathbb{G}(x)], \quad B_{\mathbb{G}} := \operatorname*{ess\,sup}_{x \in [0,1)^d} \lambda_{\max}[\mathbb{G}^*(x)\mathbb{G}(x)],$$

where $\mathbb{G}^*(x)$ denotes the transpose conjugate of the matrix $\mathbb{G}(x)$ and $\lambda_{\min}$ (respectively $\lambda_{\max}$), the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix $\mathbb{G}^*(x)\mathbb{G}(x)$. Observe that $0 \le A_{\mathbb{G}} \le B_{\mathbb{G}} \le \infty$. Note that in the definition of the matrix $\mathbb{G}(x)$ we are considering the $\mathbb{Z}^d$-periodic extension of the involved functions $\mathbf{g}_j$, $j = 1,2,\dots,s$.

We now present a general result valid for functions $\mathbf{g}_j$ in $L_r^2[0,1)^d$, $j = 1,2,\dots,s$, even if they are not given by (3.11).

Lemma 3. *Let $\mathbf{g}_j$ be in $L_r^2[0,1)^d$ for $j = 1,2,\dots,s$ and let $\mathbb{G}(x)$ be its associated matrix as in (3.12). Then:*

(a) The sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i \alpha^\top M^\top x}\big\}_{\alpha \in \mathbb{Z}^d,\ j=1,2,\dots,s}$ is a complete system for $L_r^2[0,1)^d$ if and only if the rank of the matrix $\mathbb{G}(x)$ is $r(\det M)$ a.e. in $[0,1)^d$.

(b) The sequence $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ *is a Bessel sequence for* $L^2_r[0,1)^d$ *if and only if* $\mathbf{g}_j\in L^\infty_r[0,1)^d$ *(or equivalently* $B_{\mathbb{G}}<\infty$*). In this case, the optimal Bessel bound is* $B_{\mathbb{G}}/(\det M)$.

(c) The sequence $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ *is a frame for* $L^2_r[0,1)^d$ *if and only if* $0<A_{\mathbb{G}}\le B_{\mathbb{G}}<\infty$. *In this case, the optimal frame bounds are* $A_{\mathbb{G}}/(\det M)$ *and* $B_{\mathbb{G}}/(\det M)$.

(d) The sequence $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ *is a Riesz basis for* $L^2_r[0,1)^d$ *if and only if it is a frame and* $s=r(\det M)$.

Proof. For any $\mathbf{F}\in L^2_r[0,1)^d$, we have

$$
\begin{aligned}
&\left\langle \mathbf{F}(x),\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\right\rangle_{L^2_r[0,1)^d}\\
&=\int_{[0,1)^d}\sum_{k=1}^{r}F_k(x)g_{j,k}(x)\mathrm{e}^{2\pi i\alpha^\top M^\top x}\,\mathrm{d}x\\
&=\sum_{k=1}^{r}\sum_{l=1}^{\det M}\int_{Q_l}F_k(x)g_{j,k}(x)\mathrm{e}^{2\pi i\alpha^\top M^\top x}\,\mathrm{d}x\\
&=\sum_{k=1}^{r}\int_{M^{-\top}[0,1)^d}\sum_{l=1}^{\det M}F_k\left(x+M^{-\top}i_l\right)g_{j,k}\left(x+M^{-\top}i_l\right)\,\mathrm{e}^{2\pi i\alpha^\top M^\top x}\,\mathrm{d}x\\
&=\int_{M^{-\top}[0,1)^d}\sum_{k=1}^{r}\sum_{l=1}^{\det M}F_k\left(x+M^{-\top}i_l\right)g_{j,k}\left(x+M^{-\top}i_l\right)\,\mathrm{e}^{2\pi i\alpha^\top M^\top x}\,\mathrm{d}x\\
&=\int_{M^{-\top}[0,1)^d}\sum_{l=1}^{\det M}\mathbf{g}_j^\top\left(x+M^{-\top}i_l\right)\mathbf{F}\left(x+M^{-\top}i_l\right)\,\mathrm{e}^{2\pi i\alpha^\top M^\top x}\,\mathrm{d}x, \qquad (3.13)
\end{aligned}
$$

where we have considered the $\mathbb{Z}^d$-periodic extension of $\mathbf{F}$. By using that the sequence $\{\mathrm{e}^{2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d}$ is an orthogonal basis for $L^2(M^{-\top}[0,1)^d)$ we obtain

$$
\begin{aligned}
&\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\left|\left\langle \mathbf{F}(x),\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\right\rangle_{L^2_r[0,1)^d}\right|^2\\
&\quad=\frac{1}{\det M}\sum_{j=1}^{s}\left\|\sum_{l=1}^{\det M}\mathbf{g}_j^\top(x+M^{-\top}i_l)\mathbf{F}(x+M^{-\top}i_l)\right\|^2_{L^2_r(M^{-\top}[0,1)^d)}.
\end{aligned}
$$

Denoting $\mathbb{F}(x):=\left[\mathbf{F}^\top(x),\mathbf{F}^\top(x+M^{-\top}i_2),\ldots,\mathbf{F}^\top(x+M^{-\top}i_{\det M})\right]^\top$, the equality above reads

$$\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\left|\left\langle \mathbf{F}(x),\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\right\rangle_{L^2_r[0,1)^d}\right|^2=\frac{1}{\det M}\left\|\mathbb{G}(x)\mathbb{F}(x)\right\|^2_{L^2_s(M^{-\top}[0,1)^d)}. \tag{3.14}$$

On the other hand, using that the function $\mathbf{g}_j$ is $\mathbb{Z}^d$-periodic, we obtain that the set $\{\mathbf{g}_j(x+M^{-\top}i_l+M^{-\top}i_1),\mathbf{g}_j(x+M^{-\top}i_l+M^{-\top}i_2),\dots,\mathbf{g}_j(x+M^{-\top}i_l+M^{-\top}i_{\det M})\}$ has the same elements as $\{\mathbf{g}_j(x+M^{-\top}i_1),\mathbf{g}_j(x+M^{-\top}i_2),\dots,\mathbf{g}_j(x+M^{-\top}i_{\det M})\}$. Thus, the matrix $\mathbb{G}(x+M^{-\top}i_l)$ has the same columns of $\mathbb{G}(x)$, possibly in a different order. Hence, $\operatorname{rank}\mathbb{G}(x)=r(\det M)$ a.e. in $[0,1)^d$ if and only if $\operatorname{rank}\mathbb{G}(x)=r(\det M)$ a.e. in $M^{-\top}[0,1)^d$. Moreover,

$$A_{\mathbb{G}}=\operatorname*{ess\,inf}_{x\in M^{-\top}[0,1)^d}\lambda_{\min}[\mathbb{G}^*(x)\mathbb{G}(x)],\quad B_{\mathbb{G}}=\operatorname*{ess\,sup}_{x\in M^{-\top}[0,1)^d}\lambda_{\max}[\mathbb{G}^*(x)\mathbb{G}(x)]. \tag{3.15}$$

To prove (a), assume that there exists a set $\Omega\subseteq M^{-\top}[0,1)^d$ with positive measure such that $\operatorname{rank}\mathbb{G}(x)<r(\det M)$ for each $x\in\Omega$. Then, there exists a measurable function $v(x)$, $x\in\Omega$, such that $\mathbb{G}(x)v(x)=0$ and $\|v(x)\|_{L^2_{r(\det M)}(M^{-\top}[0,1)^d)}=1$ in Ω. This function can be constructed as in [28, Lemma 2.4]. Define $\mathbf{F}\in L^2_r[0,1)^d$ such that $\mathbb{F}(x)=v(x)$ if $x\in\Omega$ and $\mathbb{F}(x)=0$ if $x\in M^{-\top}[0,1)^d\setminus\Omega$. Hence, from (3.14), we obtain that the system is not complete. Conversely, if the system is not complete, by using (3.14), we obtain an $\mathbb{F}(x)$ different from 0 in a set with positive measure such that $\mathbb{G}(x)\mathbb{F}(x)=0$. Thus, $\operatorname{rank}\mathbb{G}(x)<r(\det M)$ on a set with positive measure. To prove (b) notice that

$$\begin{aligned}\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\left|\left\langle \mathbf{F}(x),\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\right\rangle_{L^2_r[0,1)^d}\right|^2&=\frac{1}{\det M}\left\|\mathbb{G}(x)\mathbb{F}(x)\right\|^2_{L^2_s(M^{-\top}[0,1)^d)}\\&=\frac{1}{\det M}\int_{M^{-\top}[0,1)^d}\mathbb{F}^*(x)\mathbb{G}^*(x)\mathbb{G}(x)\mathbb{F}(x)\mathrm{d}x.\end{aligned} \tag{3.16}$$

If $B_{\mathbb{G}}<\infty$, then, for each $\mathbb{F}$, we have

$$\begin{aligned}\frac{1}{\det M}\int_{M^{-\top}[0,1)^d}\mathbb{F}^*(x)\mathbb{G}^*(x)\mathbb{G}(x)\mathbb{F}(x)\mathrm{d}x&\le\frac{B_{\mathbb{G}}}{\det M}\|\mathbb{F}\|^2_{L^2_{r(\det M)}(M^{-\top}[0,1)^d)}\\&=\frac{B_{\mathbb{G}}}{\det M}\|\mathbf{F}\|^2_{L^2_r[0,1)^d},\end{aligned} \tag{3.17}$$

from which the sequence $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a Bessel sequence and its optimal Bessel bound is less than or equal to $B_{\mathbb{G}}/(\det M)$.
Let $K<B_{\mathbf{G}}$; there exists a set $\Omega_K\subset M^{-\top}[0,1)^d$ with positive measure such that $\lambda_{\max_{x\in\Omega_K}}[\mathbb{G}^*(x)\mathbb{G}(x)]\ge K$. Let $\mathbf{F}\in L^2_r[0,1)^d$ such that its associated vector function

$\mathbb{F}$ is 0 if $x \in M^{-\top}[0,1)^d \setminus \Omega_K$ and $\mathbb{F}$ is an eigenvector of norm 1 associated with the largest eigenvalue of $\mathbb{G}^*(x)\mathbb{G}(x)$ if $x \in \Omega_K$. Using (3.16), we obtain

$$\sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} \left| \left\langle \mathbf{F}(x), \overline{\mathbf{g}_j(x)} \mathrm{e}^{-2\pi i \alpha^\top M^\top x} \right\rangle_{L^2_r[0,1)^d} \right|^2 \geq \frac{K}{\det M} \|\mathbf{F}\|^2_{L^2_r[0,1)^d}.$$

Therefore, if $B_{\mathbf{G}} = \infty$, the sequence $\left\{\overline{\mathbf{g}_j(x)} \mathrm{e}^{-2\pi i \alpha^\top M^\top x}\right\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ is not a Bessel sequence, and the optimal Bessel bound is $B_{\mathbb{G}}/(\det M)$.

To prove (c) assume first that $0 < A_{\mathbb{G}} \leq B_{\mathbb{G}} < \infty$. By using part (b), the sequence $\left\{\overline{\mathbf{g}_j(x)} \mathrm{e}^{-2\pi i \alpha^\top M^\top x}\right\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ is a Bessel sequence in $L^2_r[0,1)^d$. Moreover, using (3.16) and the Rayleigh–Ritz theorem (see [26, p 176]), for each $\mathbf{F} \in L^2_r[0,1)^d$, we obtain

$$\begin{aligned}\sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} \left| \left\langle \mathbf{F}(x), \overline{\mathbf{g}_j(x)} \mathrm{e}^{-2\pi i \alpha^\top M^\top x} \right\rangle_{L^2_r[0,1)^d} \right|^2 &\geq \frac{A_{\mathbb{G}}}{\det M} \|\mathbb{F}\|^2_{L^2_{r(\det M)}(M^{-\top}[0,1)^d)} \\ &= \frac{A_{\mathbb{G}}}{\det M} \|\mathbf{F}\|^2_{L^2_r[0,1)^d}. \qquad (3.18)\end{aligned}$$

Hence, the sequence $\left\{\overline{\mathbf{g}_j(x)} \mathrm{e}^{-2\pi i \alpha^\top M^\top x}\right\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ is a frame with optimal lower bound larger than or equal to $A_{\mathbb{G}}/(\det M)$.

Conversely, if $\left\{\overline{\mathbf{g}_j(x)} \mathrm{e}^{-2\pi i \alpha^\top M^\top x}\right\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ is a frame for $L^2_r[0,1)^d$, we know by part (b) that $B_{\mathbb{G}} < \infty$. In order to prove that $A_{\mathbb{G}} > 0$, consider any constant $K > A_{\mathbb{G}}$. Then, there exists a set $\Omega_K \subset M^{-\top}[0,1)^d$ with positive measure such that $\lambda_{\min_{x\in\Omega_K}}[\mathbb{G}^*(x)\mathbb{G}(x)] \leq K$. Let $\mathbf{F} \in L^2_r[0,1)^d$ such that its associated $\mathbb{F}(x)$ is 0 if $x \in M^{-\top}[0,1)^d \setminus \Omega_K$ and $\mathbb{F}(x)$ is an eigenvector of norm 1 associated with the smallest eigenvalue of $\mathbb{G}^*(x)\mathbb{G}(x)$ if $x \in \Omega_K$. Since $\mathbb{F}$ is bounded, we have that $\mathbb{G}(x)\mathbb{F}(x) \in L^2_s(M^{-\top}[0,1)^d)$. From (3.16) we get

$$\begin{aligned}\sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} \left| \left\langle \mathbf{F}(x), \overline{\mathbf{g}_j(x)} \mathrm{e}^{-2\pi i \alpha^\top M^\top x} \right\rangle_{L^2_r[0,1)^d} \right|^2 &\leq \frac{K}{\det M} \|\mathbb{F}\|^2_{L^2_{r(\det M)}(M^{-\top}[0,1)^d)} \\ &= \frac{K}{\det M} \|\mathbf{F}\|^2_{L^2_r[0,1)^d}. \qquad (3.19)\end{aligned}$$

Denoting by A the optimal lower frame bound of $\left\{\overline{\mathbf{g}_j(x)} \mathrm{e}^{-2\pi i \alpha^\top M^\top x}\right\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$, we have obtained that $K/(\det M) \geq A$ for each $K > A_{\mathbb{G}}$; thus, $A_{\mathbb{G}}/(\det M) \geq A$ and, consequently, $A_{\mathbb{G}} > 0$. Moreover, under the hypotheses of part (c), we deduce that $A_{\mathbb{G}}/(\det M)$ and $B_{\mathbb{G}}/(\det M)$ are the optimal frame bounds.

The proof of (d) is based on the following result ([11, Theorem 6.1.1]): A frame is a Riesz basis if and only if it has a biorthogonal sequence. Assume that the sequence $\left\{\overline{\mathbf{g}_j(x)} \mathrm{e}^{-2\pi i \alpha^\top M^\top x}\right\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ is a Riesz basis for $L^2_r[0,1)^d$ being the sequence $\{\mathbf{h}_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ its biorthogonal sequence. Using (3.13) we get

$$\int_{M^{-\top}[0,1)^d} \sum_{l=1}^{\det M} \mathbf{g}_j^\top \left(x+M^{-\top}i_l\right) \mathbf{h}_{j',0}\left(x+M^{-\top}i_l\right) \,\mathrm{e}^{2\pi i\alpha^\top M^\top x}\mathrm{d}x$$
$$= \left\langle \mathbf{h}_{j',0}(\cdot), \overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top \cdot}\right\rangle = \delta_{j,j'}\delta_{\alpha,0}\,.$$

Therefore,

$$\sum_{l=1}^{\det M} \mathbf{g}_j^\top \left(x+M^{-\top}i_l\right) \mathbf{h}_{j',0}\left(x+M^{-\top}i_l\right) \,\mathrm{e}^{2\pi i\alpha^\top M^\top x}$$
$$= (\det M)\delta_{j,j'} \quad \text{a.e. in } M^{-\top}[0,1)^d\,.$$

Thus, the matrix $\mathbb{G}(x)$ has a right inverse a.e. in $M^{-\top}[0,1)^d$ and, in particular, $s \le r(\det M)$. On the other hand, $A_{\mathbb{G}} > 0$ implies that $\det[\mathbb{G}^*(x)\mathbb{G}(x)] > 0$, a.e. in $M^{-\top}[0,1)^d$, and there exists the matrix $[\mathbb{G}^*(x)\mathbb{G}(x)]^{-1}\mathbb{G}^*(x)$ a.e. in $M^{-\top}[0,1)^d$. This matrix is a left inverse of the matrix $\mathbb{G}(x)$ which implies $s \ge r(\det M)$. Thus, we obtain that $r(\det M) = s$.

Conversely, assume that $\left\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\right\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a frame for $L^2_r[0,1)^d$ and $r(\det M) = s$. In this case, $\mathbb{G}(x)$ is a square matrix, and $\det[\mathbb{G}(x)^*(x)\mathbf{G}(x)(x)] > 0$ a.e. in $M^{-\top}[0,1)^d$ implies that $\det\mathbb{G}(x) \ne 0$ a.e. in $M^{-\top}[0,1)^d$. Having in mind the structure of $\mathbb{G}(x)$ its inverse must be the $r(\det M)\times s$ matrix

$$\mathbb{G}^{-1}(x) = \begin{bmatrix} \mathbf{c}_1(x) & \dots & \mathbf{c}_s(x) \\ \mathbf{c}_1(x+M^{-\top}i_2) & \dots & \mathbf{c}_s(x+M^{-\top}i_2) \\ \vdots & & \vdots \\ \mathbf{c}_1(x+M^{-\top}i_{\det M}) & \dots & \mathbf{c}_s(x+M^{-\top}i_{\det M}) \end{bmatrix},$$

where, for each $j = 1,2,\dots,s$, the function $\mathbf{c}_j \in L^2_r[0,1)^d$.

It is easy to verify that the sequence $\left\{(\det M)\mathbf{c}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\right\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a biorthogonal sequence of $\left\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\right\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$, and therefore it is a Riesz basis for $L^2_r[0,1)^d$. □

3.3 Generalized Regular Sampling in V^2_Φ

In this section we prove that expression (3.10) allows us to obtain $\mathbf{F} = \mathscr{T}_\Phi^{-1} f$ from the generalized samples $\{\mathscr{L}_j f(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$; as a consequence, applying the isomorphism $\mathscr{T}_\Phi$, we recover the function f in V^2_Φ.

Assume that the functions $\mathbf{g}_j$ given in (3.11) belong to $\in L^\infty_r[0,1)^d$ for $j = 1,2,\ldots,s$; thus, $\mathbf{g}_j^\top(x)\mathbf{F}(x) \in L^2[0,1)^d$. Having in mind (3.8) and the expression (3.10) for the generalized samples, we have that

$$
\begin{aligned}
&(\det M)\sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_j f\big)(M\alpha)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\\
&= \sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_j f\big)(\alpha)\mathrm{e}^{-2\pi i\alpha^\top x}\sum_{p\in\mathscr{N}(M^\top)}\mathrm{e}^{-2\pi i\alpha^\top M^{-\top}p}\\
&= \sum_{p\in\mathscr{N}(M^\top)}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_j f\big)(\alpha)\mathrm{e}^{-2\pi i\alpha^\top(x+M^{-\top}p)}\\
&= \sum_{p\in\mathscr{N}(M^\top)}\sum_{\alpha\in\mathbb{Z}^d}\Big\langle \mathbf{F},\overline{\mathbf{g}_j(\cdot)}\mathrm{e}^{-2\pi i\alpha^\top M^\top\cdot}\Big\rangle_{L^2_r[0,1)^d}\mathrm{e}^{-2\pi i\alpha^\top(x+M^{-\top}p)}\\
&= \sum_{p\in\mathscr{N}(M^\top)}\sum_{\alpha\in\mathbb{Z}^d}\left(\int_{[0,1)^d}\sum_{k=1}^{r}F_k(y)g_{j,k}(y)\mathrm{e}^{-2\pi i\alpha^\top M^\top y}\mathrm{d}y\right)\mathrm{e}^{-2\pi i\alpha^\top(x+M^{-\top}p)}\\
&= \sum_{p\in\mathscr{N}(M^\top)}\sum_{k=1}^{r}F_k\Big(x+M^{-\top}p\Big)\,g_{j,k}(x+M^{-\top}p)\\
&= \sum_{p\in\mathscr{N}(M^\top)}\mathbf{g}_j^\top\Big(x+M^{-\top}p\Big)\;\mathbf{F}(x+M^{-\top}p)\,.
\end{aligned}
$$

Defining $\mathbb{F}(x) := \big[\mathbf{F}^\top(x),\mathbf{F}^\top(x+M^{-\top}i_2),\ldots,\mathbf{F}^\top(x+M^{-\top}i_{\det M})\big]^\top$, the above equality allows us to write, in matrix form, that $\mathbb{G}(x)\,\mathbb{F}(x)$ equals to

$$
(\det M)\left[\sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_1 f\big)(M\alpha)\mathrm{e}^{-2\pi i\alpha^\top M^\top x},\ldots,\sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_s f\big)(M\alpha)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\right]^\top.
$$

In order to recover the function $\mathbf{F} = \mathscr{T}_{\Phi}^{-1}f$, assume the existence of an $r\times s$ matrix $\mathbf{a}(x) := [\mathbf{a}_1(x),\ldots,\mathbf{a}_s(x)]$, with entries in $L^\infty[0,1)^d$, such that

$$
[\mathbf{a}_1(x),\ldots,\mathbf{a}_s(x)]\;\mathbb{G}(x) = [\mathbb{I}_r,\mathbb{O}_{(\det M-1)r\times r}]\quad \text{a.e. in } [0,1)^d\,.
$$

If we left multiply $\mathbb{G}(x)\mathbb{F}(x)$ by $\mathbf{a}(x)$, we get

$$
\mathbf{F}(x) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathscr{L}_j f)\,(M\alpha)\,\mathbf{a}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\quad \text{in } L^2_r[0,1)^d\,. \tag{3.20}
$$

Finally, the isomorphism $\mathscr{T}_\Phi$ gives

$$
f(t) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_j f\big)(M\alpha)(\mathscr{T}_\Phi\mathbf{a}_j)(t-M\alpha)\,,\quad t\in\mathbb{R}^d\,,
$$

where we have used the shifting property (3.7) and that the space V_Φ^2 is an RKHS. Much more can be said about the above sampling result. In fact, the following theorem holds:

Theorem 2. *Assume that the functions $\mathbf{g}_j$ given in (3.11) belong to $L_r^\infty[0,1)^d$ for each $j=1,2,\ldots,s$. Let $\mathbb{G}(x)$ be the associated matrix defined in $[0,1)^d$ as in (3.12). The following statements are equivalents:*

(a) $A_\mathbb{G} > 0$.

(b) *There exists an $r\times s$ matrix $\mathbf{a}(x) := \big[\mathbf{a}_1(x),\ldots,\mathbf{a}_s(x)\big]$ with columns $\mathbf{a}_j \in L_r^\infty[0,1)^d$ and satisfying*

$$\big[\mathbf{a}_1(x),\ldots,\mathbf{a}_s(x)\big]\mathbb{G}(x) = [\mathbb{I}_r, \mathbb{O}_{(\det M-1)r\times r}] \quad a.e.\ in\ [0,1)^d. \tag{3.21}$$

(c) *There exists a frame for V_Φ^2 having the form $\{S_{j,\mathbf{a}}(\cdot - M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ such that for any $f\in V_\Phi^2$*

$$f = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathscr{L}_j f)(M\alpha)\,S_{j,\mathbf{a}}(\cdot - M\alpha) \quad in\ L^2(\mathbb{R}^d). \tag{3.22}$$

(d) *There exists a frame $\{S_{j,\alpha}(\cdot)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ for V_Φ^2 such that for any $f\in V_\Phi^2$*

$$f = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathscr{L}_j f)(M\alpha)\,S_{j,\alpha} \quad in\ L^2(\mathbb{R}^d). \tag{3.23}$$

Proof. First we prove that (a) implies (b). As the determinant of the positive semidefinite matrix $\mathbb{G}^*(x)\mathbb{G}(x)$ is equal to the product of its eigenvalues, condition (a) implies that $\operatorname{ess\,inf}_{x\in\mathbb{R}^d}\det[\mathbb{G}^*(x)\mathbb{G}(x)] > 0$. Hence, there exists the left pseudo-inverse matrix $\mathbb{G}^\dagger(x) := [\mathbb{G}^*(x)\mathbb{G}(x)]^{-1}\mathbb{G}^*(x)$, a.e. in $[0,1)^d$, and it satisfies $\mathbb{G}^\dagger(x)\mathbb{G}(x) = \mathbb{I}_{r(\det M)}$. The first r rows of $\mathbb{G}^\dagger(x)$ form an $r\times s$ matrix $[\mathbf{a}_1(x),\ldots,\mathbf{a}_s(x)]$ which satisfies (3.21). Moreover, the functions $\mathbf{a}_j(x)$, $j=1,2,\ldots,s$, are essentially bounded since the condition $\operatorname{ess\,inf}_{x\in[0,1)^d}\det[\mathbb{G}^*(x)\mathbb{G}(x)] > 0$ holds.

Next, we prove that (b) implies (c). For $j=1,2,\ldots,s$, let $\mathbf{a}_j(x)$ be a function in $L_r^\infty[0,1)^d$ and satisfying $[\mathbf{a}_1(x),\ldots,\mathbf{a}_s(x)]\mathbb{G}(x) = [\mathbb{I}_r, \mathbb{O}_{(\det M-1)r\times r}]$. In (3.20) we have proved that, for each $\mathbf{F} = \mathscr{T}_\Phi^{-1}(f)\in L_r^2[0,1)^d$, we have the expansion

$$\mathbf{F}(x) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_j f\big)(M\alpha)\,\mathbf{a}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x} \quad \text{in } L_r^2[0,1)^d,$$

from which

$$f = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathscr{L}_j f\big)(M\alpha)\,S_{j,\mathbf{a}}(\cdot - M\alpha) \quad \text{in } L^2(\mathbb{R}^d),$$

where $S_{j,\mathbf{a}} := \mathscr{T}_{\Phi}\mathbf{a}_j$ for $j = 1,2,\ldots,s$. Since we have assumed that $\mathbf{g}_j \in L_r^\infty[0,1)^d$ for each $j = 1,2,\ldots,s$, the sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ is a Bessel sequence in $L_r^2[0,1)^d$ by using part (b) in Lemma 3. The same argument proves that the sequence $\big\{(\det M)\mathbf{a}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ is also a Bessel sequence in $L_r^2[0,1)^d$. These two Bessel sequences satisfy for each $\mathbf{F} \in L_r^2[0,1)^d$

$$\mathbf{F}(x) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\Big\langle \mathbf{F}, \overline{\mathbf{g}_j}\mathrm{e}^{-2\pi i\alpha^\top M^\top \cdot}\Big\rangle \mathbf{a}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x} \quad \text{in } L_r^2[0,1)^d .$$

Hence, they are a pair of dual frames for $L_r^2[0,1)^d$ (see [11, Lemma 5.6.2]). Since $\mathscr{T}_{\Phi}$ is an isomorphism, the sequence $\big\{S_{j,\mathbf{a}}(t - M\alpha)\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ is a frame for V_{Φ}^2; hence, (b) implies (c). Statement (c) implies (d) trivially.

Assume condition (d), applying the isomorphism $\mathscr{T}_{\Phi}^{-1}$ to the expansion (3.23) we get

$$\mathbf{F}(x) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\Big\langle \mathbf{F}, \overline{\mathbf{g}_j}\mathrm{e}^{-2\pi i\alpha^\top M^\top \cdot}\Big\rangle \mathscr{T}_{\Phi}^{-1}(S_{j,\alpha})(x) \quad \text{in } L_r^2[0,1)^d , \quad (3.24)$$

where $\big\{\mathscr{T}_{\Phi}^{-1}S_{j,\alpha}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ is a frame for $L_r^2[0,1)^d$. By using Lemma 3, the sequence $\Big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\Big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ is a Bessel sequence; expansion (3.24) implies that is also a frame (see [11, Lemma 5.6.2]). Hence, by using again Lemma 3, condition (a) holds. □

In the case that the functions $\mathbf{g}_j$, $j = 1,2,\ldots,s$, are continuous on $\mathbb{R}^d$ (for instance, if the sequences of generalized samples $\big\{\mathscr{L}_j\varphi_k(\alpha)\big\}_{\alpha\in\mathbb{Z}^d}$ belongs to $\ell^1(\mathbb{Z}^d)$ for $1 \le j \le s$ and $1 \le k \le r$), the following corollary holds:

Corollary 1. *Assume that the functions $\mathbf{g}_j$, $j = 1,2,\ldots,s$, in (3.11) are continuous on $\mathbb{R}^d$. Then, the following assertions are equivalents:*

(a) *rank $\mathbb{G}(x) = r(\det M)$ for all $x \in \mathbb{R}^d$.*
(b) *There exists a frame $\{S_{j,\mathbf{a}}(\cdot - rn)\}_{n\in\mathbb{Z},\ j=1,2,\ldots,s}$ for V_{Φ}^2 satisfying the sampling formula (3.22).*

Proof. Whenever the functions $\mathbf{g}_j$, $j = 1,2,\ldots,s$, are continuous on $\mathbb{R}^d$, condition $A_{\mathbb{G}} > 0$ is equivalent to that $\det\big[\mathbb{G}^*(x)\mathbb{G}(x)\big] \neq 0$ for all $x \in \mathbb{R}^d$. Indeed, if $\det\mathbb{G}^*(x)\mathbb{G}(x) > 0$, then the r first rows of the matrix $\mathbb{G}^\dagger(x) := [\mathbb{G}^*(x)\mathbb{G}(x)]^{-1}\mathbb{G}^*(x)$ give an $r\times s$ matrix $\mathbf{a}(x) = [\mathbf{a}_1(x), \mathbf{a}_2(x), \ldots, \mathbf{a}_s(x)]$ satisfying statement (b) in Theorem 2, and therefore $A_{\mathbb{G}} > 0$.

The reciprocal follows from the fact that $\det\big[\mathbb{G}^*(x)\mathbb{G}(x)\big] \ge A_{\mathbb{G}}^{r(\det M)}$ for all $x \in \mathbb{R}^d$. Since $\det\big[\mathbb{G}^*(x)\mathbb{G}(x)\big] \neq 0$ is equivalent to rank $\mathbb{G}(x) = r(\det M)$ for all $x \in \mathbb{R}^d$, the result is a consequence of Theorem 2. □

The reconstruction functions $S_{j,\mathbf{a}}$, $j=1,2,\dots,s$, are determined from the Fourier coefficients of the components of $\mathbf{a}_j(x) := [a_{1,j}(x), a_{2,j}(x), \dots, a_{r,j}]^\top$, $j=1,2,\dots,s$. More specifically, if $\widehat{a}_{k,j}(\alpha) := \int_{[0,1)^d} a_{k,j}(x)\mathrm{e}^{2\pi i\alpha^\top x}\mathrm{d}x$, we get (see (3.6))

$$S_{j,\mathbf{a}}(t) = \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}\widehat{a}_{k,j}(\alpha)\varphi_k(t-\alpha), \quad t\in\mathbb{R}^d. \tag{3.25}$$

The Fourier transform in (3.25) gives $\widehat{S}_{j,\mathbf{a}}(x) = \sum_{k=1}^{r} a_{k,j}(x)\widehat{\varphi}_k(x)$.

Assume that the $r\times s$ matrix $\mathbf{a}(x) = \big[\mathbf{a}_1(x), \mathbf{a}_2(x), \dots, \mathbf{a}_s(x)\big]$ satisfies (3.21). We consider the periodic extension of $a_{k,j}$, i.e., $a_{k,j}(x+\alpha) = a_{k,j}(x)$, $\alpha\in\mathbb{Z}^d$. For all $x\in[0,1)^d$, the $r(\det M)\times s$ matrix

$$\mathbb{A}^\top(x) := \begin{bmatrix} \mathbf{a}_1(x) & \mathbf{a}_2(x) & \cdots & \mathbf{a}_s(x) \\ \mathbf{a}_1(x+M^{-\top}i_2) & \mathbf{a}_2(x+M^{-\top}i_2) & \cdots & \mathbf{a}_s(x+M^{-\top}i_2) \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_1(x+M^{-\top}i_{\det M}) & \mathbf{a}_2(x+M^{-\top}i_{\det M}) & \cdots & \mathbf{a}_s(x+M^{-\top}i_{\det M}) \end{bmatrix} \tag{3.26}$$

is a left inverse matrix of $\mathbb{G}(x)$, i.e., $\mathbb{A}^\top(x)\mathbb{G}(x) = \mathbb{I}_{r(\det M)}$.

Provided that condition (3.21) is satisfied, it can be easily checked that all matrices $\mathbf{a}(x)$ with entries in $L^\infty[0,1)^d$ and satisfying (3.21) correspond to the first r rows of the matrices of the form

$$\mathbb{A}^\top(x) = \mathbb{G}^\dagger(x) + \mathbb{U}(x)\left[\mathbb{I}_s - \mathbb{G}(x)\mathbb{G}^\dagger(x)\right], \tag{3.27}$$

where $\mathbb{U}(x)$ is any $r(\det M)\times s$ matrix with entries in $L^\infty[0,1)^d$, and $\mathbb{G}^\dagger$ denotes the left pseudo-inverse $\mathbb{G}^\dagger(x) := [\mathbb{G}^*(x)\mathbb{G}(x)]^{-1}\mathbb{G}^*(x)$.

Notice that if $s = r(\det M)$, there exists a unique matrix $\mathbf{a}(x)$, given by the first r rows of $\mathbb{G}^{-1}(x)$; if $s > r(\det M)$, there are infinitely many solutions according to (3.27).

Moreover, the sequence $\big\{(\det M)\mathbf{a}_j^\dagger(\cdot)\mathrm{e}^{-2\pi i\alpha^\top M^\top\cdot}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$, associated with the $r\times s$ matrix $[\mathbf{a}_1^\dagger(x), \mathbf{a}_2^\dagger(x), \dots, \mathbf{a}_s^\dagger(x)]$ obtained from the r first rows of $\mathbb{G}^\dagger(x)$, gives precisely the canonical dual frame of the frame $\big\{\overline{\mathbf{g}_j(\cdot)}\mathrm{e}^{-2\pi i\alpha^\top M^\top\cdot}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$. Indeed, the frame operator $\mathscr{S}$ associated to $\big\{\overline{\mathbf{g}_j(\cdot)}\mathrm{e}^{-2\pi i\alpha^\top M^\top\cdot}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is given by

$$\mathscr{S}\mathbf{F}(x) = \frac{1}{\det M}\left[\overline{\mathbf{g}_1(x)}, \overline{\mathbf{g}_2(x)}, \dots, \overline{\mathbf{g}_s(x)}\right]\mathbb{G}(x)\mathbb{F}(x), \quad \mathbf{F}\in L_r^2[0,1)^d,$$

from which one gets

$$\mathscr{S}\left[(\det M)\mathbf{a}_j^\dagger(\cdot)\mathrm{e}^{-2\pi i\alpha^\top M^\top\cdot}\right](x) = \overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}, \quad j=1,2,\dots,s \text{ and } \alpha\in\mathbb{Z}^d.$$

Something more can be said in the case where $s = r(\det M)$:

Theorem 3. *Assume that the functions* $\mathbf{g}_j$, $j=1,2,\ldots,s$, *given in* (3.11) *belong to* $L_r^\infty[0,1)^d$ *and* $s=r(\det M)$. *The following statements are equivalent:*

(a) $A_{\mathbb{G}}>0$.
(b) There exists a Riesz basis $\{S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ *for* V_Φ^2 *such that for any* $f\in V_\Phi^2$, *the expansion*

$$f=(\det M)\sum_{\alpha\in\mathbb{Z}^d}\sum_{j=1}^{s}\big(\mathscr{L}_j f\big)(M\alpha)\,S_{j,\alpha} \tag{3.28}$$

holds in $L^2(\mathbb{R}^d)$.

In case the equivalent conditions are satisfied, necessarily $S_{j,\alpha}(t)=S_{j,\mathbf{a}}(t-M\alpha)$, $t\in\mathbb{R}^d$, *where* $S_{j,\mathbf{a}}=\mathscr{T}_\Phi(\mathbf{a}_j)$, $j=1,2,\ldots,s$, *and the* $r\times s$ *matrix* $\mathbf{a}:=[\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_s]$ *is formed with the r first rows of the inverse matrix* $\mathbb{G}^{-1}$. *The sampling functions* $S_{j,\mathbf{a}}$, $j=1,2,\ldots,s$, *satisfy the interpolation property* $(\mathscr{L}_{j'}S_{j,\mathbf{a}})(M\alpha)=\delta_{j,j'}\delta_{\alpha,0}$, *where* $j,j'=1,2,\ldots,s$ *and* $\alpha\in\mathbb{Z}^d$.

Proof. Assume that $A_{\mathbb{G}}>0$; since $\mathbb{G}(x)$ is a square matrix, this implies that $\operatorname{ess\,inf}_{x\in\mathbb{R}^d}|\det\mathbb{G}(x)|>0$. Therefore, the r first rows of $\mathbb{G}^{-1}(x)$ gives a solution of the equation $[\mathbf{a}_1(x),\ldots,\mathbf{a}_s(x)]\mathbb{G}(x)=[\mathbb{I}_r,\mathbb{O}_{(\det M-1)r\times r}]$ with $\mathbf{a}_j\in L_r^\infty[0,1)^d$ for $j=1,2,\ldots,s$. According to Theorem 2, the sequence

$$\{S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}:=\{S_{j,\mathbf{a}}(t-M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}\,,$$

where $S_{j,\mathbf{a}}=\mathscr{T}_\Phi(\mathbf{a}_j)$, satisfies the sampling formula (3.28). Moreover, the sequence

$$\big\{(\det M)\mathbf{a}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}=\big\{\mathscr{T}_\Phi^{-1}S_{j,\mathbf{a}}(\cdot-M\alpha)\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$$

is a frame for $L_r^2[0,1)^d$. Since $r(\det M)=s$, according to Lemma 3, it is a Riesz basis for $L_r^2[0,1)^d$. Hence, the sequence $\{S_{j,\mathbf{a}}(t-M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ is a Riesz basis for V_Φ^2, and condition (b) is proved.

Conversely, assume now that $\{S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ is a Riesz basis for V_Φ^2 satisfying (3.28). From the uniqueness of the coefficients in a Riesz basis, we get that the interpolatory condition $(\mathscr{L}_{j'}S_{j,\alpha})(M\alpha')=\delta_{j,j'}\delta_{\alpha,\alpha'}$ holds for $j,j'=1,2,\ldots,s$ and $\alpha,\alpha'\in\mathbb{Z}^d$. Since $\mathscr{T}_\Phi^{-1}$ is an isomorphism, $\{\mathscr{T}_\Phi^{-1}S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ is a Riesz basis for $L_r^2[0,1)^d$. Expanding the function $\overline{\mathbf{g}_{j'}(x)}\mathrm{e}^{-2\pi i\alpha'^\top M^\top x}$ with respect to the dual basis of $\{\mathscr{T}_\Phi^{-1}S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$, denoted by $\{G_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$, we obtain

$$\begin{aligned}\overline{\mathbf{g}_{j'}(x)}\mathrm{e}^{-2\pi i\alpha'^\top M^\top x}&=\sum_{\alpha\in\mathbb{Z}^d}\sum_{j=1}^{s}\Big\langle\overline{\mathbf{g}_{j'}(\cdot)}\mathrm{e}^{-2\pi i\alpha'^\top M^\top\cdot},\mathscr{T}_\Phi^{-1}S_{j,\alpha}\Big\rangle_{L^2[0,1)^d}G_{j,\alpha}(x)\\&=\sum_{\alpha\in\mathbb{Z}^d}\overline{\mathscr{L}_{j'}S_{j,\alpha}}(M\alpha')G_{j,\alpha}(x)=G_{j',\alpha'}(x)\,.\end{aligned}$$

Therefore, the sequence $\{\overline{\mathbf{g}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ is the dual basis of the Riesz basis $\{\mathscr{T}_\Phi^{-1}S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$. In particular, it is a Riesz basis for $L_r^2[0,1)^d$, which implies, according to Lemma 3, that $A_\mathbf{G}>0$; this proves (a). Moreover, the sequence $\{\mathscr{T}_\Phi^{-1}S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ is necessarily the unique dual basis of the Riesz basis $\{\overline{\mathbf{g}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$. Therefore, this proves the uniqueness of the Riesz basis $\{S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ for V_Φ^2 satisfying (3.28). □

3.3.1 Reconstruction Functions with Prescribed Properties

A generalized sampling formula in the shift-invariant space V_Φ^2 as

$$f(t)=(\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathscr{L}_jf)(M\alpha)S_{j,\mathbf{a}}(t-M\alpha)\,,\quad t\in\mathbb{R}^d\,, \tag{3.29}$$

can be read as a filter bank. Indeed, introducing the expression for the sampling functions $S_{j,\mathbf{a}}(t)=\sum_{\beta\in\mathbb{Z}^d}\sum_{k=1}^{r}\widehat{a}_{k,j}(\beta)\varphi_k(t-\beta)$, $t\in\mathbb{R}^d$, the change $\gamma:=\beta+M\alpha$ in the summation's index gives

$$f(t)=(\det M)\sum_{k=1}^{r}\sum_{\gamma\in\mathbb{Z}^d}\left\{\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathscr{L}_jf)(M\alpha)\widehat{a}_{k,j}(\gamma-M\alpha)\right\}\varphi_k(t-\gamma)\,,\quad t\in\mathbb{R}^d\,.$$

Thus, the relevant data for the recovery of the signal $f\in V_\Phi^2$,

$$d_k(\gamma):=\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathscr{L}_jf)(M\alpha)\widehat{a}_{k,j}(\gamma-M\alpha)\,,\quad \gamma\in\mathbb{Z}^d\,,\quad 1\le k\le r\,,$$

is obtained by means of r filter banks whose impulse responses involve the Fourier coefficients of the entries of the $r\times s$ matrix $\mathbf{a}:=\left[\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_s\right]$ in (3.21), and the input is given by the sampling data.

Notice that reconstruction functions $S_{j,\mathbf{a}}$ with compact support in the above sampling formula implies low computational complexities and avoids truncation errors. This occurs whenever the generators φ_k have compact support and the sum in (3.25) is finite. These sums are finite if and only if the entries of the $r\times s$ matrix $\mathbf{a}$ are trigonometric polynomials. In this case, all the filter banks involved in the reconstruction process are finite impulse response (FIR) filters.

In order to give a necessary and sufficient condition assuring compactly supported reconstruction functions $S_{j,\mathbf{a}}$ in formula (3.29), we introduce first some complex notation, more convenient for this study. We denote $\mathbf{z}^\alpha:=z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_d^{\alpha_d}$ for $\mathbf{z}=(z_1,\ldots,z_d)\in\mathbb{C}^d$, $\alpha=(\alpha_1,\ldots,\alpha_d)\in\mathbb{Z}^d$, and the d-torus by $\mathbb{T}^d:=\{\mathbf{z}\in\mathbb{C}^d: |z_1|=|z_2|=\cdots=|z_d|=1\}$. For $1\le j\le s$ and $1\le k\le r$, we define

$$\mathsf{g}_{j,k}(\mathbf{z}) := \sum_{\mu\in\mathbb{Z}^d} \mathscr{L}_j\varphi_k(\mu)\mathbf{z}^{-\mu}, \quad \mathsf{g}_j^\top(\mathbf{z}) := \big(\mathsf{g}_{j,1}(\mathbf{z}), \mathsf{g}_{j,2}(\mathbf{z}), \ldots, \mathsf{g}_{j,r}(\mathbf{z})\big)$$

and the $s \times r(\det M)$ matrix

$$\mathsf{G}(\mathbf{z}) := \left[\mathsf{g}_j^\top\left(z_1 \mathrm{e}^{2\pi i m_1^\top i_l}, \ldots, z_d \mathrm{e}^{2\pi i m_d^\top i_l}\right)\right]_{\substack{j=1,2,\ldots,s \\ k=1,2,\ldots,r;\ l=1,2,\ldots,\det M}} \tag{3.30}$$

where $m_1, \ldots, m_d$ denote the columns of the matrix M^{-1}. Recall that $i_1, i_2, \ldots, i_{\det M}$ in $\mathbb{Z}^d$ are the elements of $\mathscr{N}(M^\top)$ defined in (3.9). Note also that for the values $x = (x_1, \ldots, x_d) \in [0,1)^d$ and $\mathbf{z} = (\mathrm{e}^{2\pi i x_1}, \ldots, \mathrm{e}^{2\pi i x_d}) \in \mathbb{T}^d$, we have $\mathbb{G}(x) = \mathsf{G}(\mathbf{z})$.

Provided that the functions $\mathbf{g}_j$ are continuous on $\mathbb{R}^d$, Corollary 1 can be reformulated as follows: There exists an $r \times s$ matrix $\mathsf{a}(\mathbf{z}) = \big[\mathsf{a}_1(\mathbf{z}), \ldots, \mathsf{a}_s(\mathbf{z})\big]$ with entries essentially bounded in the torus $\mathbb{T}^d$ and satisfying

$$\mathsf{a}(\mathbf{z})\mathsf{G}(\mathbf{z}) = [\mathbb{I}_r, \mathbb{O}_{(\det M-1)r\times r}] \quad \text{for all } \mathbf{z} \in \mathbb{T}^d \tag{3.31}$$

if and only if

$$\operatorname{rank} \mathsf{G}(\mathbf{z}) = r(\det M) \quad \text{for all } \mathbf{z} \in \mathbb{T}^d. \tag{3.32}$$

Denoting the columns of the matrix $\mathsf{a}(\mathbf{z})$ as $\mathsf{a}_j^\top(\mathbf{z}) = \big(\mathsf{a}_{1,j}(\mathbf{z}), \ldots, \mathsf{a}_{r,j}(\mathbf{z})\big)$, $j = 1, 2, \ldots, s$, the corresponding reconstruction functions $S_{j,\mathsf{a}}$ in sampling formula (3.29) are

$$S_{j,\mathsf{a}}(t) = \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} \widehat{\mathsf{a}}_{k,j}(\alpha)\varphi_k(t-\alpha), \quad t \in \mathbb{R}^d, \tag{3.33}$$

where $\widehat{\mathsf{a}}_{k,j}(\alpha)$, $\alpha \in \mathbb{Z}^d$, are the Laurent coefficients of the functions $\mathsf{a}_{k,j}(\mathbf{z})$, i.e.,

$$\mathsf{a}_{k,j}(\mathbf{z}) = \sum_{\alpha\in\mathbb{Z}^d} \widehat{\mathsf{a}}_{k,j}(\alpha)\mathbf{z}^{-\alpha}. \tag{3.34}$$

Note that, in order to obtain compactly supported reconstruction functions $S_{j,\mathsf{a}}$ in (3.29), we need an $r \times s$ matrix $\mathsf{a}(\mathbf{z})$ whose entries are Laurent polynomials, i.e., the sum in (3.34) is finite. The following result, which proof can be found in [16] under minor changes, holds:

Theorem 4. *Assume that the generators φ_k and the functions $\mathscr{L}_j\varphi_k$, $1 \le k \le r$ and $1 \le j \le s$, have compact support. Then, there exists an $r \times s$ matrix $\mathsf{a}(\mathbf{z})$ whose entries are Laurent polynomials and satisfying (3.31) if and only if*

$$\textit{rank}\ \mathsf{G}(\mathbf{z}) = r(\det M) \quad \textit{for all } \mathbf{z} \in (\mathbb{C}\setminus\{0\})^d.$$

The reconstruction functions $S_{j,\mathsf{a}}$, $j = 1, 2, \ldots, s$, obtained from such matrix $\mathsf{a}(\mathbf{z})$ through (3.33) have compact support.

From one of these $r \times s$ matrices, say $\widetilde{\mathsf{a}}(\mathbf{z}) = [\widetilde{\mathsf{a}}_1(\mathbf{z}), \dots, \widetilde{\mathsf{a}}_s(\mathbf{z})]$, we can get all of them. Indeed, it is easy to check that they are given by the r first rows of the $r(\det M) \times s$ matrices of the form

$$\mathsf{A}(\mathbf{z}) = \widetilde{\mathsf{A}}(\mathbf{z}) + \mathsf{U}(\mathbf{z})\left[\mathbb{I}_s - \mathsf{G}(\mathbf{z})\widetilde{\mathsf{A}}(\mathbf{z})\right], \tag{3.35}$$

where

$$\widetilde{\mathsf{A}}(\mathbf{z}) := \left[\widetilde{\mathsf{a}}_j(z_1 \mathrm{e}^{2\pi i m_1^\top i_l}, \dots, z_d \mathrm{e}^{2\pi i m_d^\top i_l})\right]_{\substack{k=1,2,\dots,r;\ l=1,2,\dots,\det M \\ j=1,2,\dots,s}},$$

and $\mathsf{U}(\mathbf{z})$ is any $r(\det M) \times s$ matrix with Laurent polynomial entries. Remember that $m_1, \dots, m_d$ denote the columns of the matrix M^{-1} and $i_1, \dots, i_{\det M}$ the elements of $\mathcal{N}(M^\top)$ defined in (3.9).

Next, we study the existence of reconstruction functions $S_{j,\mathsf{a}}$, $j = 1,2,\dots,s$, in (3.29) having exponential decay; it means that there exist constants $C > 0$ and $q \in (0,1)$ such that $|S_{j,\mathsf{a}}(t)| \le C q^{|t|}$ for each $t \in \mathbb{R}^d$. In so doing, we introduce the algebra $\mathscr{H}(\mathbb{T}^d)$ of all holomorphic functions in a neighborhood of the d-torus $\mathbb{T}^d$. Note that the elements in $\mathscr{H}(\mathbb{T}^d)$ are characterized as admitting a Laurent series where the sequence of coefficients decays exponentially fast [27].

The following theorem, which proof can be found in [16] under minor changes, holds:

Theorem 5. *Assume that the generators φ_k and the functions $\mathcal{L}_j\varphi_k$, $j = 1,2,\dots,s$ and $k = 1,2,\dots,r$, have exponential decay. Then, there exists an $r \times s$ matrix $\mathsf{a}(\mathbf{z}) = [\mathsf{a}_1(\mathbf{z}), \dots, \mathsf{a}_s(\mathbf{z})]$ with entries in $\mathscr{H}(\mathbb{T}^d)$ and satisfying (3.31) if and only if rank $\mathsf{G}(\mathbf{z}) = r(\det M)$ for all $\mathbf{z} \in \mathbb{T}^d$.*

In this case, all of such matrices $\mathsf{a}(\mathbf{z})$ are given as the first r rows of a $r(\det M) \times s$ matrix $\mathsf{A}(\mathbf{z})$ of the form

$$\mathsf{A}(\mathbf{z}) = \mathsf{G}^\dagger(\mathbf{z}) + \mathsf{U}(\mathbf{z})\left[\mathbb{I}_s - \mathsf{G}(\mathbf{z})\mathsf{G}^\dagger(\mathbf{z})\right], \tag{3.36}$$

where $\mathsf{U}(\mathbf{z})$ denotes any $r(\det M) \times s$ matrix with entries in the algebra $\mathscr{H}(\mathbb{T}^d)$ and $\mathsf{G}^\dagger(\mathbf{z}) := \left[\mathsf{G}^*(\mathbf{z})\mathsf{G}(\mathbf{z})\right]^{-1}\mathsf{G}^*(\mathbf{z})$. The corresponding reconstruction functions $S_{j,\mathsf{a}}$, $j = 1,2,\dots,s$, given by (3.33) have exponential decay.

3.3.2 Some Illustrative Examples

We include here some examples illustrating Theorem 4, a particular case of Theorem 2, by taking B-splines as generators; they certainly are important for practical purposes [44].

First notice that if the generator φ has compact support, the only situation when the reconstruction function S_a in formula (3.1) has compact support as well is the special case when φ is the linear B-spline $N_2(t) := \chi_{[0,1)} * \chi_{[0,1)}(t)$, where $\chi_{[0,1)}$ denotes the characteristic function of the interval $[0,1)$. For any $f \in V^2_{N_2}$, the following sampling formula holds:

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) N_2(t+1-n)\,, \quad t \in \mathbb{R}\,.$$

In this special case where $d = 1$ and $r = s = 1$, we have $\mathsf{G}(z) = z$, and consequently, $\mathsf{a}(z) = z^{-1}$ in Theorem 4.

3.3.2.1 The Case $d = 1, r = 1, M = 2$, and $s = 3$

Let $N_3(t) := \chi_{[0,1)} * \chi_{[0,1)} * \chi_{[0,1)}(t)$ be the quadratic B-spline, and let $\mathscr{L}_j$, $j = 1,2,3$, be the systems

$$\mathscr{L}_1 f(t) = f(t); \quad \mathscr{L}_2 f(t) = f\left(t + \frac{2}{3}\right) \quad \text{and} \quad \mathscr{L}_3 f(t) = f\left(t + \frac{4}{3}\right).$$

Since the functions $\mathscr{L}_j N_3$, $j = 1,2,3$, have compact support, then the entries of the 3×2 matrix $\mathsf{G}(z)$ in (3.30) are Laurent polynomials, and we can try to search a vector $\mathbf{a}(z) := [\mathbf{a}_1(z), \mathbf{a}_2(z), \mathbf{a}_3(z)]$ satisfying (3.31) with Laurent polynomials entries also. This implies reconstruction functions $S_{j,\mathsf{a}}$, $j = 1,2,3$, with compact support. Proceeding as in [14], we obtain that any function $f \in V^2_{N_3}$ can be recovered through the sampling formula

$$f(t) = \sum_{n \in \mathbb{Z}} \sum_{j=1}^{3} \mathscr{L}_j f(2n) \, S_{j,\mathbf{a}}(t-2n)\,, \quad t \in \mathbb{R}\,,$$

where the reconstruction functions, according to (3.33), are given by

$$\begin{aligned}
S_{1,\mathbf{a}}(t) &= \frac{1}{16}\left[N_3(t+3) - 3N_3(t+2) - 3N_3(t+1) + N_3(t)\right],\\
S_{2,\mathbf{a}}(t) &= \frac{1}{16}\left[27N_3(t+1) - 9N_3(t)\right],\\
S_{3,\mathbf{a}}(t) &= \frac{1}{16}\left[-9N_3(t+1) + 27N_3(t)\right], \quad t \in \mathbb{R}\,.
\end{aligned}$$

3.3.2.2 The Case $d = 1, r = 2, M = 1$, and $s = 3$

Consider the Hermite cubic splines defined as

$$\varphi_1(t) = \begin{cases} (t+1)^2(1-2t), & t \in [-1,0] \\ (1-t)^2(1+2t), & t \in [0,1] \\ 0, & |t| > 1 \end{cases} \quad \text{and} \quad \varphi_2(t) = \begin{cases} (t+1)^2 t, & t \in [-1,0] \\ (1-t)^2 t, & t \in [0,1] \\ 0, & |t| > 1 \end{cases}.$$

They are stable generators for the space $V^2_{\varphi_1,\varphi_2}$ (see [12]). Consider the sampling period $M = 1$ and the systems $\mathscr{L}_j$, $j = 1,2,3$, defined by

$$\mathscr{L}_1 f(t) := \int_t^{t+1/3} f(u)\mathrm{d}u, \quad \mathscr{L}_2 f(t) := \mathscr{L}_1 f\Big(t+\frac{1}{3}\Big), \quad \mathscr{L}_3 f(t) := \mathscr{L}_1 f\Big(t+\frac{2}{3}\Big).$$

Since the functions $\mathscr{L}_j\varphi_k$, $j = 1,2,3$ and $k = 1,2$, have compact support, then the entries of the 3×2 matrix $\mathsf{G}(z)$ in (3.30) are Laurent polynomials, and we can try to search an 2×3 matrix $\mathbf{a}(z) := [\mathbf{a}_1(z), \mathbf{a}_2(z), \mathbf{a}_3(z)]$ satisfying (3.31) with Laurent polynomials entries also. This leads to reconstruction functions $S_{j,\mathbf{a}}$, $j = 1,2,3$, with compact support. Proceeding as in [17], we obtain in $V^2_{\varphi_1,\varphi_2}$ the following sampling formula:

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{3} \mathscr{L}_j f(n)\, S_{j,\mathbf{a}}(t-n), \quad t \in \mathbb{R},$$

where the sampling functions, according to (3.33), are

$$\begin{aligned} S_{1,\mathbf{a}}(t) &:= \frac{85}{44}\varphi_1(t) + \frac{1}{11}\varphi_1(t-1) + \frac{85}{4}\varphi_2(t) - \varphi_2(t-1), \\ S_{2,\mathbf{a}}(t) &:= \frac{-23}{44}\varphi_1(t) - \frac{23}{44}\varphi_1(t-1) - \frac{23}{4}\varphi_2(t) + \frac{23}{4}\varphi_2(t-1), \\ S_{3,\mathbf{a}}(t) &:= \frac{1}{11}\varphi_1(t) + \frac{85}{44}\varphi_1(t-1) + \varphi_2(t) - \frac{85}{4}\varphi_2(t-1), \quad t \in \mathbb{R}. \end{aligned}$$

3.3.3 L^2-Approximation Properties

Consider an $r\times s$ matrix $\mathbf{a}(x) := \big[\mathbf{a}_1(x), \mathbf{a}_2(x), \ldots, \mathbf{a}_s(x)\big]$ with entries $a_{k,j} \in L^\infty[0,1)^d$, $1 \le k \le r$, $1 \le j \le s$, and satisfying (3.21). Let $S_{j,\mathbf{a}}$ be the associated reconstruction functions, $j = 1,2,\ldots,s$, given in Theorem 2. The aim of this section is to show that if the set of generators $\boldsymbol{\Phi}$ satisfies the Strang–Fix conditions of order ℓ, then the scaled version of the sampling operator

$$\Gamma_{\mathbf{a}} f(t) := \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathscr{L}_j f)(M\alpha) S_{j,\mathbf{a}}(t - M\alpha), \quad t \in \mathbb{R}^d,$$

gives L^2-approximation order ℓ for any smooth function f (in a Sobolev space). In so doing, we take advantage of the good approximation properties of the scaled space $\sigma_{1/h}V^2_{\Phi}$, where for $h > 0$, we are using the notation $\sigma_h f(t) := f(ht)$, $t \in \mathbb{R}^d$.

The set of generators $\Phi = \{\varphi_k\}_{k=1}^r$ is said to satisfy the Strang–Fix conditions of order ℓ if there exist r finitely supported sequences $b_k : \mathbb{Z}^d \to \mathbb{C}$ such that the function $\varphi(t) = \sum_{k=1}^r \sum_{\alpha\in\mathbb{Z}^d} b_k(\alpha)\varphi_k(t-\alpha)$ satisfies the Strang–Fix conditions of order ℓ, i.e.,

$$\widehat{\varphi}(0) \neq 0, \quad D^{\beta}\widehat{\varphi}(\alpha) = 0, \quad |\beta| < \ell, \quad \alpha \in \mathbb{Z}^d \setminus \{0\}. \tag{3.37}$$

We denote by $W_2^{\ell}(\mathbb{R}^d) := \{f : \|D^{\gamma} f\|_2 < \infty, |\gamma| \leq \ell\}$ the usual Sobolev space and by $|f|_{\ell,2} := \sum_{|\beta|=\ell} \|D^{\beta} f\|_2$ the corresponding seminorm of a function $f \in W_2^{\ell}(\mathbb{R}^d)$. When $2\ell > d$, we identify $f \in W_2^{\ell}(\mathbb{R}^d)$ with its continuous choice (see [2]).

It is well known that if Φ satisfies the Strang–Fix conditions of order ℓ and the generators φ_k satisfy a suitable decay condition, the space V^2_{Φ} provides L^2-approximation order ℓ for any function f regular enough. For instance, Lei et al. proved in [33, Theorem 5.2] the following result: If a set $\Phi = \{\varphi_k\}_{k=1}^r$ of stable generators satisfies the Strang–Fix conditions of order ℓ and the decay condition $\varphi_k(t) = O\big([1+|t|]^{-d-\ell-\varepsilon}\big)$ for each $k = 1, 2, \ldots, r$ and some $\varepsilon > 0$, then, for any $f \in W_2^{\ell}(\mathbb{R}^d)$, there exists a function $f_h \in \sigma_{1/h}V^2_{\Phi}$ such that

$$\|f - f_h\|_2 \leq C\,|f|_{\ell,2}\, h^{\ell}, \tag{3.38}$$

where the constant C does not depend on h and f.

In this section we assume that all the systems $\mathscr{L}_j$, $j = 1, 2, \ldots, s$, are of type (b), i.e., $\mathscr{L}_j f = f * \mathsf{h}_j$, belonging the impulse response h_j to the Hilbert space $\mathscr{L}^2(\mathbb{R}^d)$. Recall that a Lebesgue measurable function $\mathsf{h} : \mathbb{R}^d \longrightarrow \mathbb{C}$ belongs to the Hilbert space $\mathscr{L}^2(\mathbb{R}^d)$ if

$$|\mathsf{h}|_2 := \left(\int_{[0,1)^d} \Big(\sum_{\alpha\in\mathbb{Z}^d} |\mathsf{h}(t-\alpha)| \Big)^2 \mathrm{d}t \right)^{1/2} < \infty.$$

Notice that the space $\mathscr{L}^2(\mathbb{R}^d)$ coincides with the amalgam space $W(\ell^1, L^2)$ and that $\mathscr{L}^2(\mathbb{R}^d) \subset L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. For $f \in L^2(\mathbb{R}^d)$ and $\mathsf{h} \in \mathscr{L}^2(\mathbb{R}^d)$, the following inequality holds: $\big\|\{\mathsf{h} * f(\alpha)\}_{\alpha\in\mathbb{Z}^d}\big\|_2 \leq |\mathsf{h}|_2\, \|f\|_2$ (see [27, Theorem 3.1]); thus, the sequence of generalized samples $\{(\mathscr{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\ldots,s}$ belongs to $\ell^2(\mathbb{Z}^d)$ for any $f \in L^2(\mathbb{R}^d)$.

First we note that the operator $\Gamma_{\mathbf{a}} : \big(L^2(\mathbb{R}^d), \|\cdot\|_2\big) \longrightarrow \big(V_\Phi^2, \|\cdot\|_2\big)$ given by

$$(\Gamma_{\mathbf{a}} f)(t) := (\det M) \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathscr{L}_j f)(M\alpha) S_{j,\mathbf{a}}(t - M\alpha)\,, \quad t \in \mathbb{R}^d\,,$$

is a well-defined bounded operator onto V_Φ^2. Besides, $\Gamma_{\mathbf{d}} f = f$ for all $f \in V_\Phi^2$.

Under appropriate hypotheses we prove that the scaled operator $\Gamma_{\mathbf{a}}^h := \sigma_{1/h}\Gamma_{\mathbf{a}}\sigma_h$ approximates, in the L^2-norm sense, any function f in the Sobolev space $W_2^\ell(\mathbb{R}^d)$ as $h \to 0^+$. Specifically we have the following:

Theorem 6. *Assume $2\ell > d$ and that all the systems $\mathscr{L}_j$ satisfy $\mathscr{L}_j f = f * \mathsf{h}_j$ with $\mathsf{h}_j \in \mathscr{L}^2(\mathbb{R}^d)$, $j = 1,\dots,s$. Then,*

$$\|f - \Gamma_{\mathbf{a}}^h f\|_2 \le (1 + \|\Gamma_{\mathbf{a}}\|) \inf_{g\in\sigma_{1/h}V_\Phi^2} \|f - g\|_2, \quad f \in W_2^\ell(\mathbb{R}^d),$$

where $\|\Gamma_{\mathbf{a}}\|$ denotes the norm of the sampling operator $\Gamma_{\mathbf{a}}$. If the set of generators $\Phi = \{\varphi_k\}_{k=1}^r$ satisfies the Strang–Fix conditions of order ℓ and, for each $k = 1,2,\dots,r$, the decay condition $\varphi_k(t) = O\big([1+|t|]^{-d-\ell-\varepsilon}\big)$ for some $\varepsilon > 0$, then

$$\|f - \Gamma_{\mathbf{a}}^h f\|_2 \le C\,|f|_{\ell,2}\, h^\ell \quad \textit{for all } f \in W_2^\ell(\mathbb{R}^d),$$

where the constant C does not depend on h and f.

Proof. Using that $\Gamma_{\mathbf{a}}^h g = g$ for each $g \in \sigma_{1/h}V_\Phi^2$, then, for each $f \in L^2(\mathbb{R}^d)$ and $g \in \sigma_{1/h}V_\Phi^2$, Lebesgue's Lemma [13, p 30] gives

$$\|f - \Gamma_{\mathbf{a}}^h f\|_2 \le \|f - g\|_2 + \|\Gamma_{\mathbf{a}}^h g - \Gamma_{\mathbf{a}}^h f\|_2 \le (1 + \|\Gamma_{\mathbf{a}}\|) \inf_{g\in\sigma_{1/h}V_\Phi^2} \|f - g\|_2\,,$$

where we have used that $\|\Gamma_{\mathbf{a}}^h\| = \|\Gamma_{\mathbf{a}}\|$ for $h > 0$. Now, for each $f \in W_2^\ell(\mathbb{R}^d)$ and $h > 0$, there exists a function $f_h \in \sigma_{1/h}V_\Phi^2$ such that (3.38) holds, from which we obtain the desired result. □

More results on approximation by means of generalized sampling formulas can be found in [15, 18].

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