Chapter 1
Convergence of Classical Cardinal Series

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Abstract We consider symmetric partial sums of the classical cardinal series and record necessary and sufficient conditions for convergence. Included are growth conditions on the coefficients that imply analogous asymptotic behavior of the function represented by the series. Several relatively immediate corollaries are also recorded, including sampling-type theorems.

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1.1 Introduction

The classical cardinal series with coefficients $\{a(n): \; n = 0, \pm 1, \pm 2, \ldots\}$ is defined by

$$f(z) = \sum_{n=-\infty}^{\infty} a(n) \frac{\sin \pi(z-n)}{\pi(z-n)}, \tag{1.1}$$

where the variable z is often restricted to the real line but, in general, can take on complex values. The coefficients of course are in general complex.

Under suitable restrictions on the coefficients $\{a(n): \; n = 0, \pm 1, \pm 2, \ldots\}$ the series (1.1) provides a solution to the interpolation problem of finding an entire function $f(z)$ of exponential type no greater than π that satisfies

$$f(n) = a(n), \qquad n = 0, \pm 1, \pm 2, \ldots \,. \tag{1.2}$$

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The cardinal series (1.1) is a well-known and highly celebrated solution to the interpolation problem (1.2). Indeed the list [2–8, 10, 11, 15–17, 19, 20] is but a small sampling of the many articles and books that are devoted to or significantly treat the subject. We assume that the reader is familiar with what are now relatively widely well-known facts concerning the cardinal series (1.1) that are associated with the theory commonly referred to as the W-K-S sampling theorem and that can be found, for example, in [11, Lecture 20] or [16, Chap. 9].

In this note we are concerned with the convergence of the symmetric partial sums of (1.1), more specifically, with conditions on the coefficients $\{a(n)\}$ that insure the convergence of the sequence $\{f_N(z) :\ N = 1,2,\ldots\}$ of symmetric partial sums

$$f_N(z) = \sum_{n=-N}^{N} a(n)\frac{\sin\pi(z-n)}{\pi(z-n)}. \tag{1.3}$$

We use standard notation and only alert the reader to the fact that E_π denotes the class of entire functions of exponential type no greater than π that have no greater than polynomial growth along the real axis. In view of the distributional variant of the Paley–Wiener theorem, for example, see [9, Theorem 1.7.7], E_π consists of the Fourier transforms of distributions with support in the interval $[-\pi,\pi]$.

The main results, including some explanatory material, are given in Sect. 1.2. All the details, including necessary technical lemmas, are given in Sect. 1.3. Section 1.4 is devoted to certain miscellany that is a relatively immediate consequence of the development in Sects. 1.2 and 1.3; Corollary 6 here is an example of a sampling-type theorem mentioned in the introduction.

1.2 Results

We make use of the fact that the partial sums $f_N(z)$ defined by (1.3) can be re-expressed as

$$f_N(z) = \frac{\sin\pi z}{\pi}\sum_{n=-N}^{N}\frac{(-1)^n a(n)}{(z-n)}. \tag{1.4}$$

It follows from (1.4) that when the sequence of coefficients $\{a(n)\}$ is even, namely $a(-n) = a(n)$ for $n = 1,2,\ldots,$ we may write

$$f_N(z) = \frac{\sin\pi z}{\pi}\left\{\frac{a(0)}{z} + 2z\sum_{n=1}^{N}\frac{(-1)^n a(n)}{(z^2-n^2)}\right\}. \tag{1.5}$$

From (1.5) it is clear that when $\{a(n)\}$ is an even sequence then the convergence of $\sum(-1)^n a(n)/n^2$ is a sufficient condition for the convergence of the partial sums $\{f_N(z)\}$. This condition is also necessary. Furthermore, the limiting function $f(z)$

is a solution within a certain class of entire functions to the interpolation problem $f(n) = a(n)$, $n = 0, \pm 1, \pm 2, \ldots$. We formulate this more precisely as follows.

Theorem 1. *Suppose the sequence of coefficients $\{a(n)\}$ is even, namely $a(-n) = a(n)$ for $n = 1, 2, \ldots$.*

1. *If*

$$\sum (-1)^n \frac{a(n)}{n^2} \quad \textit{converges,} \tag{1.6}$$

then the partial sums $f_N(z)$, $N = 1, 2, \ldots$, converge uniformly on compact subsets of the complex plane $\mathbb{C}$. The limiting function

$$f(z) = \frac{\sin \pi z}{\pi} \left\{ \frac{a(0)}{z} + 2z \sum_{n=1}^{\infty} \frac{(-1)^n a(n)}{(z^2 - n^2)} \right\} \tag{1.7}$$

is even, is in E_π, satisfies

$$|f(z)| \mathrm{e}^{-\pi |\mathrm{Im}\, z|} = O(|z|^2 \log |z|) \quad \textit{as} \quad |z| \to \infty, \tag{1.8}$$

and solves the interpolation problem (1.2).

2. *If (1.6) fails to hold then the sequence $f_N(z)$ fails to converge as $N \to \infty$ at every point z that is not an integer.*

The statement concerning convergence of the partial sums $f_N(z)$ can be reformulated as follows: There is an entire function $f(z)$ such that for every positive number R,

$$\lim_{N \to \infty} \sup_{|z| \le R} \left| f(z) - \sum_{n=-N}^{N} a(n) \frac{\sin \pi (z-n)}{\pi (z-n)} \right| .$$

In view of the function

$$z \sin \pi z,$$

any solution of the interpolation problem (1.2) that is even, is in E_π, and enjoys (1.8) cannot be unique. Additional restrictions on the coefficients $\{a(n)\}$ are required to ensure that the solution given by (1.7) is unique within an appropriate class of entire functions in analogy with the celebrated sampling theorem, for example, [11, Lecture 20, Theorem 1].

Theorem 2. *Suppose the sequence of coefficients $\{a(n)\}$ is even and satisfies property (1.6). If, in addition, for some p that satisfies $0 \le p \le 2$ we have*

$$a(n) = O(n^p) \quad \textit{as} \quad n \to \infty, \tag{1.9}$$

then the limiting function $f(z)$ defined by (1.7) satisfies

$$|f(z)| \mathrm{e}^{-\pi |\mathrm{Im}\, z|} = O(|z|^p \log |z|) \quad \textit{as} \quad |z| \to \infty. \tag{1.10}$$

If $0 \leq p < 1$ *then the limiting function* $f(z)$ *defined by* (1.7) *is the unique solution to the interpolation problem* (1.2) *that is in* E_π*, is even, and satisfies* (1.10).

Note that condition (1.9) on the growth of the coefficients $\{a(n)\}$ does not imply that the solution (1.7) of the interpolation problem (1.2) has the same order of growth. However, an additional restriction, on what amounts to the oscillatory nature of the coefficients, will ensure that the solution (1.7) has the same order of growth as the coefficients (1.9).

Theorem 3. *Suppose the sequence of coefficients* $\{a(n)\}$ *is even and satisfies property* (1.6)*. If, in addition, for some* p *that satisfies* $0 \leq p \leq 2$ *we have*

$$a(n+1) - a(n) = O(n^{p-1}) \quad \text{as} \quad n \to \infty \quad \text{when} \quad 0 < p \leq 2 \tag{1.11}$$

and

$$\sum_{n=1}^{\infty} |a(n+1) - a(n)| < \infty \quad \text{when} \quad p = 0, \tag{1.12}$$

then the limiting function $f(z)$ *defined by* (1.7) *satisfies*

$$|f(z)| e^{-\pi |\operatorname{Im} z|} = O(|z|^p) \quad \text{as} \quad |z| \to \infty. \tag{1.13}$$

When the sequence of coefficients $\{a(n)\}$ is odd, namely $a(-n) = -a(n)$ for $n = 1, 2, \ldots$, in view of (1.4) we have

$$f_N(z) = \frac{2 \sin \pi z}{\pi} \sum_{n=1}^{N} \frac{(-1)^n n a(n)}{(z^2 - n^2)}. \tag{1.14}$$

From (1.14) it should be clear that the conditions required of $\{a(n)\}$ in this case will be somewhat more restrictive than in the even case. Nevertheless, with relatively minor modifications, the analogues of Theorems 1–3 remain valid and can be formulated as follows.

Theorem 4. *Suppose the sequence of coefficients* $\{a(n)\}$ *is odd, namely* $a(-n) = -a(n)$ *for* $n = 1, 2, \ldots$.

1. If

$$\sum (-1)^n \frac{a(n)}{n} \quad \text{converges}, \tag{1.15}$$

then the partial sums $f_N(z)$, $N = 1, 2, \ldots$, *converge uniformly on compact subsets of the complex plane* $\mathbb{C}$*. The limiting function*

$$f(z) = \frac{2 \sin \pi z}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n n a(n)}{(z^2 - n^2)} \tag{1.16}$$

is odd, is in E_π*, satisfies*

$$|f(z)|\mathrm{e}^{-\pi|\operatorname{Im} z|} = O(|z|\log|z|) \quad as \quad |z| \to \infty, \tag{1.17}$$

and solves the interpolation problem (1.2).

2. *If* (1.15) *fails to hold, then the sequence* $f_N(z)$ *fails to converge as* $N \to \infty$ *at every point* z *that is not an integer.*

In view of the function

$$\sin \pi z,$$

any solution of the interpolation problem (1.2) that is odd, is in E_π, and enjoys (1.17) cannot be unique. In this case I am unaware of any conditions on the coefficients other than the decay conditions implied by the sampling-type theorems, for example, [11, Lecture 20, Theorems 1 and 2] or Corollary 6 in Sect. 1.4, that will ensure that the solution (1.16) is unique within some appropriate class of entire functions. The statements in Theorem 4 concerning convergence are also implied by [18, Theorem 1].

Theorem 5. *Suppose the sequence of coefficients* $\{a(n)\}$ *is odd and satisfies property* (1.15). *If, in addition, for some* p *that satisfies* $0 \le p \le 1$ *we have*

$$a(n) = O(n^p) \quad as \quad n \to \infty, \tag{1.18}$$

then the limiting function $f(z)$ *defined by* (1.16) *satisfies*

$$|f(z)|\mathrm{e}^{-\pi|\operatorname{Im} z|} = O(|z|^p \log|z|) \quad as \quad |z| \to \infty. \tag{1.19}$$

Note that condition (1.18) on the growth of the coefficients $\{a(n)\}$ does not imply that the solution (1.16) of the interpolation problem (1.2) has the same order of growth. However, as in the earlier case, an additional restriction, on what amounts to the oscillatory nature of the coefficients, will ensure that the solution (1.16) has the same order of growth as the coefficients (1.18).

Theorem 6. *Suppose the sequence of coefficients* $\{a(n)\}$ *is odd and satisfies property* (1.15). *If, in addition, for some* p *that satisfies* $0 \le p \le 1$ *we have*

$$a(n+1) - a(n) = O(n^{p-1}) \quad as \quad n \to \infty \quad when \quad 0 < p \le 1 \tag{1.20}$$

and

$$\sum_{n=1}^{\infty} |a(n+1) - a(n)| < \infty \quad when \quad p = 0, \tag{1.21}$$

then the limiting function $f(z)$ *defined by* (1.16) *satisfies*

$$|f(z)|\mathrm{e}^{-\pi|\operatorname{Im} z|} = O(|z|^p) \quad as \quad |z| \to \infty. \tag{1.22}$$

1.3 Details

In what follows the symbol C, with or without a subscript, is used to denote certain generic constants whose specific value can vary from one occurrence to another.

1.3.1 Proof of Theorem 1

In view of (1.5) we may re-express $f_N(z)$ as

$$\begin{aligned} f_N(z) &= \frac{\sin \pi z}{\pi}\left\{\frac{a(0)}{z} + 2z\sum_{n=1}^{N}\left[\left(\frac{1}{(z^2-n^2)}+\frac{1}{n^2}\right)(-1)^n a(n) - (-1)^n\frac{a(n)}{n^2}\right]\right\} \\ &= \frac{\sin \pi z}{\pi}\left\{\frac{a(0)}{z} + 2z\sum_{n=1}^{N}\left(\frac{(-1)^n z^2}{n^2(z^2-n^2)}\right)a(n) - 2z\sum_{n=1}^{N}(-1)^n\frac{a(n)}{n^2}\right\} \\ &= \phi_N(z) + \psi_N(z), \end{aligned}$$

where

$$\phi_N(z) = \frac{\sin \pi z}{\pi}\left\{\frac{a(0)}{z} + 2z\sum_{n=1}^{N}\left(\frac{(-1)^n z^2}{n^2(z^2-n^2)}\right)a(n)\right\}$$

and

$$\psi_N(z) = \frac{-2z\sin \pi z}{\pi}\sum_{n=1}^{N}(-1)^n\frac{a(n)}{n^2}.$$

In view of (1.6) the sequence $\psi_N(z)$ converges uniformly on compacta as $N \to \infty$. Condition (1.6) also implies that $\lim_{n\to\infty} a(n)/n^2 = 0$ and hence $\sum_{n=1}^{\infty} a(n)/n^4$ converges absolutely. It follows that the series

$$\sum_{n=1}^{\infty}\left(\frac{2z^3\sin \pi z}{\pi(z^2-n^2)}\right)(-1)^n\frac{a(n)}{n^2}$$

converges absolutely and uniformly on compacta. This means, of course, that $\phi_N(z)$ converges uniformly on compacta as $N \to \infty$.

From the last expression for $f_N(z)$ it follows that $f_N(z)$ converges uniformly on compacta as $N \to \infty$ since both $\phi_N(z)$ and $\psi_N(z)$ do so.

We may express the limiting function $f(z)$ as

$$f(z) = \frac{\sin \pi z}{\pi}\left\{\frac{a(0)}{z} + 2z\sum_{n=1}^{\infty}\frac{(-1)^n a(n)}{(z^2-n^2)}\right\} \tag{1.7}$$

since the series in fact converges to the entire function $f(z)$. In view of the above development we may also express $f(z)$ as

$$f(z) = \phi(z) + cz\sin\pi z, \tag{1.23}$$

where

$$\phi(z) = \frac{\sin\pi z}{\pi}\left\{\frac{a(0)}{z} + 2z\sum_{n=1}^{\infty}\left(\frac{(-1)^n z^2}{n^2(z^2-n^2)}\right)a(n)\right\}$$

and

$$c = \frac{-2}{\pi}\sum_{n=1}^{N}(-1)^n\frac{a(n)}{n^2}.$$

An efficient way of arguing that $f(z)$ is in E_π is to observe that this is an immediate consequence of (1.8).

To see that $f(z)$ satisfies (1.8) use representation (1.7) of $f(z)$, assume $|z| \geq 100$, and break up the series into a sum over $n \geq 2|z|$ and another over $n < 2|z|$. Thus

$$f(z) = A_N(z) + B_N(z),$$

where N is the greatest integer $\leq 2|z|$,

$$A_N(z) = \frac{\sin\pi z}{\pi}\left\{\frac{a(0)}{z} + 2z\sum_{n=1}^{N}\frac{(-1)^n a(n)}{z^2-n^2}\right\},$$

and

$$B_N(z) = \frac{\sin\pi z}{\pi}\left\{2z\sum_{n=N+1}^{\infty}\frac{(-1)^n a(n)}{z^2-n^2}\right\}.$$

To estimate $A_N(z)$ assume that Im z is positive so that $|\mathrm{e}^{i2\pi z}-1| \leq 2$ and note that

$$\left|\frac{\sin\pi z}{z\pm n}\right| = |\mathrm{e}^{-i\pi z}|\left|\frac{\mathrm{e}^{2\pi iz}-1}{z\pm n}\right| = \mathrm{e}^{\pi|\operatorname{Im} z|}\left|\frac{\mathrm{e}^{2\pi i(z\pm n)}-1}{z\pm n}\right|$$

and

$$\left|\frac{\mathrm{e}^{2\pi i(z\pm n)}-1}{z\pm n}\right| \leq \frac{C}{1+\big||n|-|z|\big|}.$$

Hence,

$$\frac{2z\sin\pi z}{z^2-n^2} = \left\{\frac{1}{z-n}+\frac{1}{z+n}\right\}\sin\pi z \leq \frac{C\mathrm{e}^{\pi|\operatorname{Im} z|}}{1+\big||n|-|z|\big|},$$

so that

$$|A_N(z)| \le C e^{\pi|\operatorname{Im} z|} \sum_{n=0}^{N} \frac{|a(n)|}{1+\big|n-|z|\big|}.$$

Now,

$$|a(n)| \le C|z|^2$$

in the above sum, since for $n \ge 1$ (1.6) implies that $|a(n)|/n^2$ is bounded which in turn implies $|a(n)| \le Cn^2 \le CN^2$. The last two displayed inequalities imply that

$$|A_N(z)| \le C|z|^2 e^{\pi|\operatorname{Im} z|} \sum_{n=0}^{N} \frac{1}{1+\big|n-|z|\big|} \le C|z|^2 e^{\pi|\operatorname{Im} z|} \log|z|. \tag{1.24}$$

An analogous argument *mutatis mutandis* shows that (1.24) is still valid when $\operatorname{Im} z \le 0$ so that (1.24) holds whenever $|z|$ is sufficiently large.

To estimate $B_N(z)$ break it up into two terms analogous to $\phi_N(z)$ and $\psi_N(z)$ above. Namely, write

$$B_N(z) = \frac{\sin \pi z}{\pi} \left\{ 2 \sum_{n=N+1}^{\infty} \left(\frac{(-1)^n z^3}{n^2(z^2-n^2)} \right) a(n) - 2z \sum_{n=N+1}^{\infty} (-1)^n \frac{a(n)}{n^2} \right\}$$

and note that

$$\begin{aligned} \left| \sum_{n=N+1}^{\infty} \left(\frac{(-1)^n z^3}{n^2(z^2-n^2)} \right) a(n) \right| &\le |z|^3 \sum_{n=N+1}^{\infty} \frac{C_1|a(n)|}{n^4} \\ &\le |z|^3 \sum_{n=N+1}^{\infty} \frac{C_2}{n^2} \le C_3|z|^3 N^{-1} \le C|z|^2 \end{aligned}$$

and

$$\left| z \sum_{n=N+1}^{\infty} (-1)^n \frac{a(n)}{n^2} \right| \le C|z|.$$

The last expression for $B_N(z)$ together with the last two inequalities implies that

$$|B_N(z)| \le C|z|^2 e^{\pi|\operatorname{Im} z|}. \tag{1.25}$$

The desired result (1.8) follows from (1.24) and (1.25).

Now, suppose that the series in (1.6) diverges. The proof of item 2 can be reduced to two simple cases. (a) If the terms of the series in (1.6) are unbounded, then so are the terms of the series (1.7), and desired result follows. (b) If the terms of the series in (1.6) are bounded, then using representation (1.23) for $f(z)$, note that the series representing $\phi(z)$ converges while the series representing the constant c diverges, and the desired result follows.

1.3.2 *Proof of Theorem 2*

The proof of (1.10) is essentially analogous to the proof of (1.8) while making use of the additional restrictions on the coefficients $\{a(n)\}$. The only significant modification involves the estimation of $B_N(z)$ which requires the consideration of two cases depending on whether p is less than or ≥ 1.

Thus, use representation (1.7) of $f(z)$, assume $|z| \geq 100$, break up the series into a sum over $n \geq 2|z|$ and another over $n < 2|z|$, and write

$$f(z) = A_N(z) + B_N(z),$$

where N is the greatest integer $\leq 2|z|$ and both $A_N(z)$ and $B_N(z)$ are defined exactly the same as in the proof of (1.8). Then estimating $A_N(z)$ as before but using the fact that in this case $|a(n)| \leq C|z|^p$ results in

$$|A_N(z)| \leq C|z|^p e^{\pi|\operatorname{Im} z|} \log|z|. \tag{1.26}$$

As mentioned earlier the estimation of $B_N(z)$ depends on whether p is less than or ≥ 1.

If $0 \leq p < 1$, simply recall that

$$B_N(z) = \frac{\sin \pi z}{\pi} \left\{ 2z \sum_{n=N+1}^{\infty} \frac{(-1)^n a(n)}{z^2 - n^2} \right\}$$

and observe that

$$\left| z \sum_{n \geq 2|z|} \frac{(-1)^n a(n)}{(z^2 - n^2)} \right| \leq |z| \sum_{n \geq 2|z|} \frac{4|a(n)|}{3n^2} \leq C_1 |z| \sum_{n \geq 2|z|} n^{p-2} \leq C_2 |z| \, |z|^{p-1}.$$

to conclude that

$$|B_N(z)| \leq C|z|^p e^{\pi|\operatorname{Im} z|}. \tag{1.27}$$

If $1 \leq p \leq 2$ estimate exactly as in the derivation of (1.8) but use the bound $|a(n)| \leq Cn^p$. This leads to (1.27) for this case.

Bounds (1.26) and (1.27) together imply the desired result (1.10).

To see the uniqueness statement we argue as follows: If $g(z)$ is another solution of the interpolation problem (1.2), is in E_π, is even, and satisfies (1.10) for some $p < 1$, then $h(z) = \big(f(z) - g(z)\big) / \sin \pi z$ is an entire function that is $o(|z|)$ as $|z| \to \infty$. Hence Cauchy's estimate, [1, p 122, identity (25) with $n = 1$] implies that $h(z)$ is a constant. In view of the fact that $h(z)$ is odd this constant must be zero. Thus $g(z) = f(z)$, which implies the desired result.

1.3.3 A Technical Lemma

Let

$$S_n(z) = \frac{\sin \pi z}{\pi} \sum_{k=-n}^{n} \frac{(-1)^k}{(z-k)}. \tag{1.28}$$

Then in view of the uniqueness statement in Theorem 2, it follows that

$$\lim_{n \to \infty} S_n(z) = 1$$

uniformly on compacta. Our proof of Theorem 3 uses the fact that $S_n(z)$ is uniformly bounded in n on strips parallel to the real axis, $\{z : |\operatorname{Im} z| < R < \infty\}$, which however does not follow from Theorem 2 and requires an additional tweak.

Lemma. *There is a positive constant C, independent of z and n, such that*

$$|S_n(z)| \le C e^{\pi |\operatorname{Im} z|}. \tag{1.29}$$

To see the lemma note that for positive k

$$\frac{(-1)^{2k-1}}{z-(2k-1)} + \frac{(-1)^{2k}}{z-2k} = \frac{1}{(z-2k+1)(z-2k)}$$

and that

$$\left| \frac{\sin \pi z}{(z-2k+1)(z-2k)} \right| \le \frac{C e^{\pi |\operatorname{Im} z|}}{1+|z-2k|^2}$$

with a similar estimate valid for negative k. Hence

$$|S_{2k}(z) - S_{2(k-1)}(z)| \le C e^{\pi |\operatorname{Im} z|} \left\{ \frac{1}{1+|z-2k|^2} + \frac{1}{1+|z+2k|^2} \right\}.$$

If n is even, $n = 2m$, then

$$S_{2m}(z) = S_0(z) + \sum_{k=1}^{m} \{ S_{2k}(z) - S_{2(k-1)}(z) \},$$

and if n is odd, $n = 2m+1$, then

$$S_{2m+1}(z) = S_{2m}(z) - \frac{2z \sin \pi z}{z^2 - (2m+1)^2}.$$

Hence

$$|S_{2m}(z)| = |S_0(z)| + \sum_{k=1}^{m} |S_{2k}(z) - S_{2(k-1)}(z)|$$
$$\leq C\mathrm{e}^{\pi|\operatorname{Im} z|} \left\{ \frac{1}{1+|z|} + \sum_{k=1}^{m} \left\{ \frac{1}{1+|z-2k|^2} + \frac{1}{1+|z+2k|^2} \right\} \right\}$$

which implies that (1.29) is valid when $n = 2m$ and since

$$\left| \frac{2z\sin \pi z}{z^2 - (2m+1)^2} \right| \leq C\mathrm{e}^{\pi|\operatorname{Im} z|}$$

inequality (1.29) follows for all n.

1.3.4 Proof of Theorem 3

As in the proof of Theorem 2, use representation of $f(z)$, assume $|z| \geq 100$, and break up the series into a sum over $n \geq 2|z|$ and another over $n < 2|z|$, and write

$$f(z) = A_N(z) + B_N(z),$$

where N is the greatest integer $\leq 2|z|$ and both $A_N(z)$ and $B_N(z)$ are defined exactly the same as before. Also note that the hypothesis on the coefficients $\{a(n)\}$ implies that $a(n) = O(n^p)$ as $n \to \infty$.

$B_N(z)$ can be estimated in exactly the same way as in the proof of Theorem 2 to get

$$|B_N(z)| \leq C|z|^p \mathrm{e}^{\pi|\operatorname{Im} z|}.$$

To estimate $A_N(z)$ use summation by parts to write

$$A_N(z) = \sum_{n=0}^{N-1} S_n(z) \big(a(n) - a(n+1)\big) + S_N(z)\, a(N),$$

where

$$S_0(z) = \frac{\sin \pi z}{\pi z}$$

and

$$S_n(z) = S_0(z) + \frac{\sin \pi z}{\pi} \left\{ 2z \sum_{k=1}^{n} \frac{(-1)^k}{z^2 - k^2} \right\}, \quad n = 1, 2, \ldots .$$

In view of inequality (1.29) the last expression for $A_N(z)$ allows us to write

$$|A_N(z)| \le C e^{\pi|\operatorname{Im} z|}\left\{\sum_{n=0}^{N-1}\big|a(n)-a(n+1)\big| + |a(N)|\right\}$$

which together with the hypothesis on the coefficients $\{a(n)\}$ implies that

$$|A_N(z)| \le C|z|^p e^{\pi|\operatorname{Im} z|}.$$

The bounds on $A_N(z)$ and $B_N(z)$ imply the desired result (1.22).

1.3.5 Proof of Theorems 4 and 5

The proofs of Theorems 4 and 5 are essentially the same as those of Theorems 1 and 2, *mutatis mutandis*.

The necessary modifications are evident by reexpressing (1.16) as

$$f(z) = \frac{2\sin\pi z}{\pi}\left\{\sum_{n=1}^{\infty}\left[\left(\frac{n}{(z^2-n^2)}+\frac{1}{n}\right)(-1)^n a(n) - (-1)^n\frac{a(n)}{n}\right]\right\}$$

which, in analogy with (1.23), can be written as

$$f(z) = \phi(z) + c\sin\pi z, \tag{1.30}$$

where

$$\phi(z) = \frac{2\sin\pi z}{\pi}\left\{\sum_{n=1}^{\infty}\left(\frac{(-1)^n z^2}{n(z^2-n^2)}\right)a(n)\right\}$$

and

$$c = \frac{-2}{\pi}\sum_{n=1}^{N}(-1)^n\frac{a(n)}{n}.$$

Also recall that

$$\left|\frac{2n\sin\pi z}{z^2-n^2}\right| = \left|\left\{\frac{1}{z-n}-\frac{1}{z+n}\right\}\sin\pi z\right| \le \frac{Ce^{\pi|\operatorname{Im} z|}}{1+\big||n|-|z|\big|}.$$

1.3.6 Another Technical Lemma

In analogy with (1.28) let

$$\operatorname{Sgn}_n(z) = \frac{\sin\pi z}{\pi}\sum_{k=-n}^{n}\frac{(-1)^k\operatorname{sgn}(k)}{(z-k)}, \tag{1.31}$$

where

$$\operatorname{sgn}(z) = \begin{cases} z/|z| & \text{if } z \neq 0, \\ 0 & \text{when } z = 0. \end{cases}$$

Then in view of Theorem 5 it follows that there is an entire function $\operatorname{Sgn}(z)$ in E_π such that

$$\lim_{n\to\infty} \operatorname{Sgn}_n(z) = \operatorname{Sgn}(z)$$

uniformly on compacta. Our proof of Theorem 6 is analogous to that of Theorem 3 and uses the fact that $\operatorname{Sgn}_n(z)$ is uniformly bounded in n on strips parallel to the real axis, $\{z : |\operatorname{Im} z| < R < \infty\}$, which does not follow from Theorem 5. The proof of the following lemma is completely analogous to the proof of (1.29).

Lemma. *There is a positive constant C, independent of z and n, such that*

$$|\operatorname{Sgn}_n(z)| \leq C e^{\pi|\operatorname{Im} z|}. \tag{1.32}$$

1.3.7 Proof of Theorem 6

Our proof of Theorem 6 is completely analogous to that of Theorem 3. Simply replace $S_n(z)$ with $\operatorname{Sgn}_n(z)$ and use (1.32) instead of (1.29).

1.4 Additional Remarks, Examples, and Corollaries

1.4.1 Specific Bounds

It should be evident from the above development that more specific bounds on the growth of the coefficients $\{a(n)\}$ will lead, via essentially the same calculations, to more specific bounds on the growth of the corresponding function (1.7) or (1.16).

For example, if in Theorem 2 we assume that $0 \leq p \leq 1$ and

$$\|\{a(n)\}\|_p = \sup_n \frac{|a(n)|}{(1+|n|)^p} < \infty,$$

then we may conclude that

$$|f(z)| \leq C\|\{a(n)\}\|_p\, e^{\pi|\operatorname{Im} z|}(1+|z|)^p \log(e+|z|),$$

where C is a constant that may depend on p but is otherwise independent of $\{a(n)\}$. Similar results are valid in all the other cases.

1.4.2 Some Special Functions

If

$$a(n) = (-1)^n = \cos \pi n,$$

then the statement in Theorem 2 concerning uniqueness can be applied to conclude that

$$\cos \pi z = \frac{\sin \pi z}{\pi} \left\{ \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{1}{(z^2 - k^2)} \right\}. \tag{1.33}$$

But the fact that the right-hand side of (1.33) is bounded when z is restricted to a strip about the real axis, $|\operatorname{Im} z| \leq \varepsilon < \infty$ does not follow from Theorem 3 since $|a(n+1) - a(n)| = 2$. On the other hand, unlike the partial sums of

$$1 = \frac{\sin \pi z}{\pi} \left\{ \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{(-1)^k}{(z^2 - k^2)} \right\} \tag{1.34}$$

and

$$\operatorname{Sgn}(z) = \frac{2 \sin \pi z}{\pi} \left\{ \sum_{k=1}^{\infty} \frac{(-1)^k k}{(z^2 - k^2)} \right\}, \tag{1.35}$$

the partial sums

$$\cos_n \pi z = \frac{\sin \pi z}{\pi} \left\{ \frac{1}{z} + 2z \sum_{k=1}^{n} \frac{1}{(z^2 - k^2)} \right\}$$

are not uniformly bounded.

In fact, choosing $z = n + 1/2$, we have for sufficiently large n

$$\begin{aligned} |\pi \cos_n \pi z| &= \frac{1}{n+1/2} + 2(n+1/2) \sum_{k=1}^{n} \frac{1}{n+k+1/2} \, \frac{1}{n-k+1/2} \\ &\geq 2(n+1/2) \sum_{k=1}^{n} \frac{1}{2(n+1/2)} \, \frac{1}{n-k+1/2} \\ &= \sum_{m=1}^{n} \frac{1}{m-1/2} \geq \log n, \end{aligned}$$

where the first inequality above follows from $\frac{1}{n+1/2} > 0$ and $n + k + 1/2 \leq 2(n+1/2)$. This implies that on the strips $|\operatorname{Im} z| \leq \varepsilon < \infty$ and for sufficiently large $|z|$, the uniform bound

$$|\cos_n \pi z| \leq C \log |z|$$

guaranteed by Theorem 2 cannot be improved.

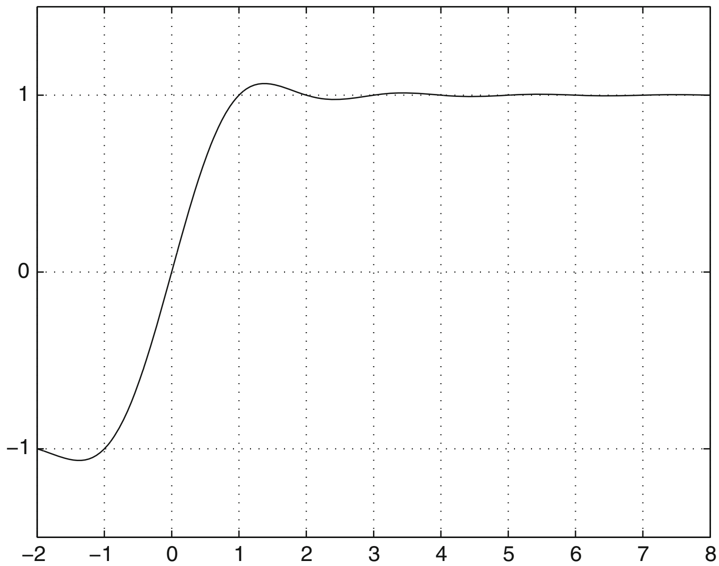

Fig. 1.1 Plot of $\mathrm{Sgn}(x)$ for $-2 \le x \le 8$

Formulas (1.33) and (1.34) are classical and well known, for example, see [1, formulas (11) and (13) on p 188]. But the fact that they also follow from Theorem 2, involving cardinal series expansions seems not to be so well known. We also bring attention to the elementary curiosity concerning the difference of behavior of their respective partial sums.

For the record we also mention the following which follows from the development in Sect. 1.3.6.

Corollary 1. *The function* $\mathrm{Sgn}(z)$ *defined by* (1.35) *is a member of* E_π *that is odd and satisfies both*

$$\mathrm{Sgn}(n) = \mathrm{sgn}(n), \quad n = 0, \pm 1, \pm 2, \ldots$$

and

$$|\,\mathrm{Sgn}(z)| \le C\mathrm{e}^{\pi|\,\mathrm{Im}\, z|},$$

where

$$\mathrm{sgn}(z) = \begin{cases} z/|z| & \text{if } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$$

and C *is a constant independent of* z.

The above considerations suggest that reasonable candidates for a pair of odd functions in E_π that are analogous to the pair of even functions 1 and $\cos \pi z$ might be the pair $\mathrm{Sgn}(z)$ and an odd function $w(z)$ that satisfies (Figs. 1.1 and 1.2)

$$w(n) = (-1)^n \,\mathrm{sgn}(n), \quad n = 0, \pm 1, \pm 2 \ldots . \tag{1.36}$$

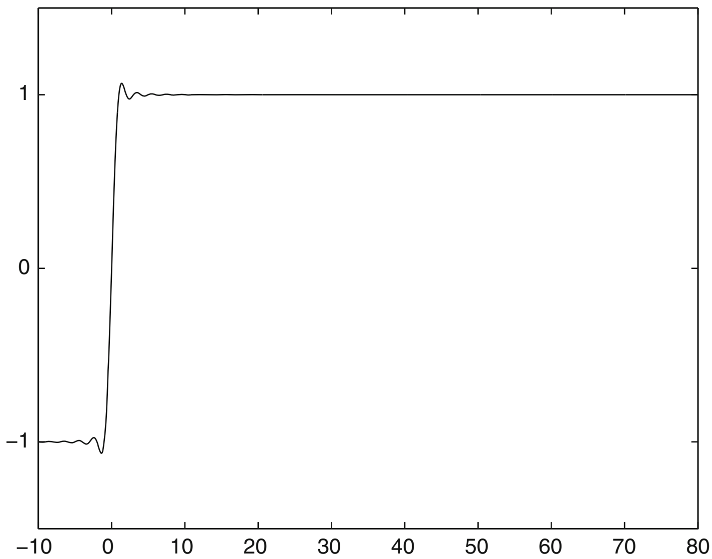

Fig. 1.2 Plot of $\mathrm{Sgn}(x)$ for $-10 \le x \le 80$

Note that in the case of the coefficients $\{w(n)\}$ given by (1.36)

$$\sum_{n=1}^{\infty}(-1)^n \frac{w(n)}{n} \qquad \text{does not converge}$$

so in view of second item in Theorem 4, such a function w, unlike the case of the cosine, cannot be represented by a cardinal series (1.1). Nevertheless, if we ignore the second term on the right-hand side of (1.30) and use the coefficients $a(n) = w(n)$ in the first term, we may write

$$w(z) = \frac{2\sin \pi z}{\pi}\left\{\sum_{n=1}^{\infty}\left(\frac{z^2}{n(z^2-n^2)}\right)\right\}, \tag{1.37}$$

where the series converges uniformly on compacta and defines an odd entire function in E_π that satisfies (1.36). A calculation analogous to the one used to obtain a lower bound on $|\cos_n(z)|$ shows that

$$|w(N+1/2)| \ge C\log(N) \quad \text{for sufficiently large } N$$

so that $w(z)$ is not bounded on the strips $|\operatorname{Im} z| \le \varepsilon < \infty$. But the function w defined by (1.37) does satisfy

$$|w(z)| \le C e^{\pi|\operatorname{Im} z|}\log|z| \quad \text{for sufficiently large } |z|$$

as can be verified by a calculation essentially identical to the one used to establish (1.17).

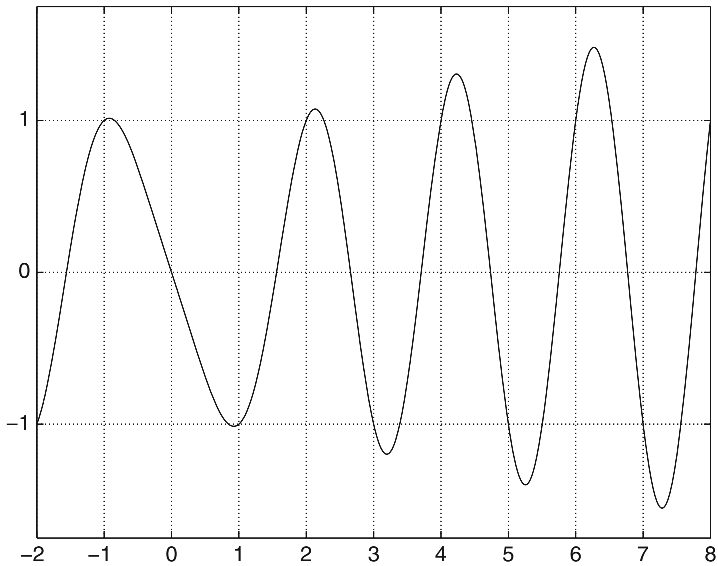

Fig. 1.3 Plot of $\mathrm{Cgn}(x)$ for $-2 \le x \le 8$

If w is the function defined by (1.37), then its derivative at $z = 0$ is 0, namely $w'(0) = 0$. This seems somewhat unnatural. A comparison with $\mathrm{Sgn}(z)$ suggests that the value of this derivative should be $-\mathrm{Sgn}'(0) = -\log 4$. This can be achieved without altering the values at the integers $z = 0, \pm 1, \pm 2, \dots$, by simply adding $-\frac{\log 4}{\pi} \sin \pi z$ to $w(z)$. Thus as an odd analogue of $\cos \pi z$ we propose the function (Figs. 1.3 and 1.4)

$$\mathrm{Cgn}(z) = w(z) - \frac{\log 4}{\pi} \sin \pi z.$$

1.4.3 Special Classes of Data

Here the term data is used to refer to the coefficients $\{a(n)\}$ in (1.1).

As mentioned in the introduction, the class E_π of entire functions $u(z)$ consists of Fourier transforms of distributions $\hat{u}$ with support in the interval $[-\pi, \pi]$. In other words, for every u in E_π there is a distribution $\hat{u}$ with support in the interval $[-\pi, \pi]$ such that $u(z)$ is the value of the distribution $\hat{u}$ evaluated at the test function $\varphi(\xi) = \frac{e^{iz\xi}}{2\pi}$ that, in the standard notation of linear functionals, can be expressed as

$$u(z) = \langle \varphi, \hat{u} \rangle.$$

In the case that $\hat{u}$ is an integrable function, the last identity can be re-expressed as

$$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\xi} \hat{u}(\xi) d\xi.$$

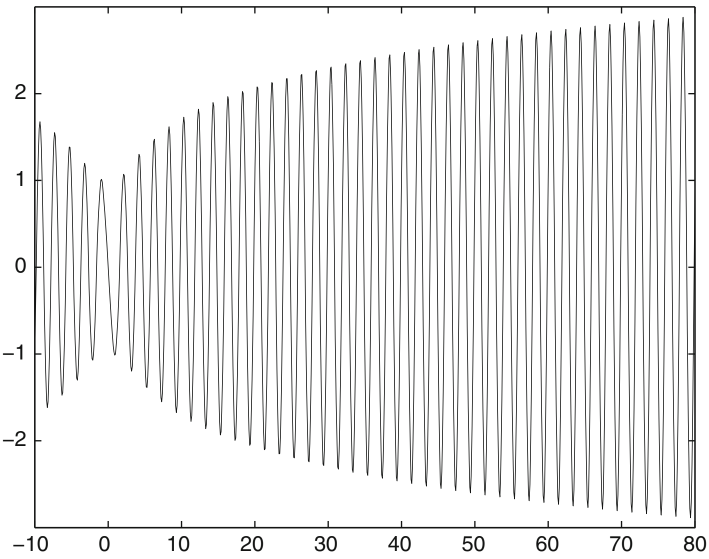

Fig. 1.4 Plot of Cgn(x) for $-10 \leq x \leq 80$

The Paley–Wiener class *PW* consists of those members u of E_π such that $\hat{u}$ is square integrable. This class is often referred to as the class of band-limited functions that plays a very prominent role in classical sampling theory. While it makes sense to refer to the members of E_π as being frequency band-limited, the term "band-limited" is so closely associated with the subclass *PW* in the literature that to avoid confusion, we have precluded its use in the wider sense.

An issue of interest in sampling theory are requirements on u or $\hat{u}$ that guarantee that the cardinal series $f(z)$ with coefficients $a(n) = u(n)$, $n = 0, \pm 1, \pm 2, \dots$ exists and satisfies the property that $f = u$. In what follows we give several such conditions that are consequences of the results in Sect. 1.2 and are somewhat less restrictive than those associated with classical sampling theory.

As an immediate consequence of the uniqueness statement in Theorem 2 we have

Corollary 2. *Suppose u is an even entire function in E_π such that for some value of $p < 1$*

$$u(x) = O(|x|^p) \quad as \quad x \to \pm\infty$$

on the real axis. Then the symmetric partial sums (1.3) of the cardinal series with coefficients $a(n) = u(n)$, $n = 0, \pm 1, \pm 2, \dots$ converge uniformly on compacta to $u(z)$.

If $\hat{u}$ is integrable or, more generally, a finite measure, then $u(z)$ is bounded on the real axis. Hence Corollary 2 can be applied in this case to get

Corollary 3. *Suppose u is an even entire function in E_π such that $\hat{u}$ is an integrable function or, more generally, a finite measure. Then the symmetric partial sums (1.3)* *of the cardinal series with coefficients* $a(n) = u(n)$*,* $n = 0, \pm 1, \pm 2, \ldots$ *converge uniformly on compacta to* $u(z)$*.*

If k is a nonnegative integer, then PW^k denotes the class of those entire functions u whose derivative of order k, $u^{(k)}$, is in the Paley–Wiener class PW. In other words, $PW^k = \{u : u^{(k)} \in PW\}$. The class PW^k is endowed with the natural semi-norm

$$\|u\|_{PW^k} = \left\{ \int_{-\infty}^{\infty} |u^{(k)}(x)|^2 \mathrm{d}x \right\}^{1/2}.$$

Note that $PW^0 = PW$ and $PW^k \subset PW^{k+1}$ where the containment is proper.

The standard sampling theorem for PW does not apply to PW^k when $k \geq 1$. Nevertheless, it was shown in [14] that members u of PW^k can be recovered from their samples $\{u(n)\}$ via the spline summability method. Additional properties of PW^k can be found in [12].

The following facts concerning PW^k will be useful in what follows:

If u is in PW^k, $k \geq 1$, then

$$u(x) = O\left(|x|^{k-1/2}\right) \quad \text{as} \quad x \to \pm\infty \tag{1.38}$$

on the real axis, and the samples $\{u(n)\}$ enjoy

$$\sum_{n=-\infty}^{\infty} |\Delta^k u(n)|^2 \leq C\|u\|^2_{PW^k}, \tag{1.39}$$

where $\Delta^k u(n)$ are the forward differences of order k of $\{u(n)\}$ that can be defined recursively as

$$\Delta^1 u(n) = \Delta u(n) = u(n+1) - u(n), \quad \Delta^{k+1} u(n) = \Delta^k u(n+1) - \Delta^k u(n).$$

In view of (1.38) Corollary 2 implies the following.

Corollary 4. *Suppose* u *is an even function in* PW^1*. Then the symmetric partial sums* (1.3) *of the cardinal series with coefficients* $a(n) = u(n)$*,* $n = 0, \pm 1, \pm 2, \ldots$ *converge uniformly on compacta to* $u(z)$*.*

The corresponding results for odd functions are not quite so transparent. Nevertheless Theorem 4 can be used to show that the following is true.

Corollary 5. *Suppose* u *is an odd entire function in* E_π *such that* $\hat{u}$ *is an integrable function. Then the symmetric partial sums* (1.3) *of the cardinal series with coefficients* $a(n) = u(n)$*,* $n = 0, \pm 1, \pm 2, \ldots$ *converge uniformly on compacta to* $u(z)$*.*

To see this, note that $u(n)$ are the Fourier coefficients of $\hat{u}(\xi)$ while $\{(-1)^n u(n)\}$ are the Fourier coefficients of $\hat{u}(\xi - \pi)$. In other words

$$\hat{u}(\xi) \sim \sum_{n=-\infty}^{\infty} u(n) e^{-in\xi} = -2i \sum_{n=1}^{\infty} u(n) \sin n\xi, \quad -\pi \le \xi \le \pi$$

with partial sums

$$\hat{u}_N(\xi) = -2i \sum_{n=1}^{N} u(n) \sin n\xi, \quad -\pi \le \xi \le \pi$$

and

$$\hat{u}(\xi - \pi) \sim -2i \sum_{n=1}^{\infty} (-1)^n u(n) \sin n\xi, \quad -\pi \le \xi \le \pi.$$

Since both $\hat{u}(\xi)$ and $\hat{u}(\xi - \pi)$ are integrable functions it follows that both

$$\sum_{n=1}^{\infty} \frac{u(n)}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^n \frac{u(n)}{n} \quad \text{converge,}$$

see, for example, [21, Theorem 8.7 and the remarks that follow on p 59]. In view of Theorem 4 the convergence of the second series implies that the symmetric partial sums (1.3) of the cardinal series with coefficients $a(n) = u(n)$, $n = 0, \pm 1, \pm 2, \ldots$ converge uniformly on compacta to an entire function $f(z)$.

To see that $f(z) = u(z)$ we argue as follows: the arithmetic means of the partial sums $\hat{u}_n(\xi)$ converge to $\hat{u}(\xi)$ in L^1, for example, see [21, Theorem 5.5(ii) on p 144]. That is

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \left| \hat{u}(\xi) - \frac{1}{N} \sum_{n=1}^{N} \hat{u}_n(\xi) \right| d\xi = 0.$$

Hence the arithmetic means of the partial sums f_N of the corresponding cardinal series converge to $u(z)$ uniformly on strips, namely

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_n(z) = u(z)$$

uniformly on the strips $|\operatorname{Im} z| \le \varepsilon < \infty$. Since the arithmetic means of a sequence converge to the same limit as the original sequence we may conclude that $f(z) = u(z)$.

Corollaries 3 and 5 can be combined to obtain

Corollary 6. *Suppose u is an entire function in E_π such that $\hat{u}$ is an integrable function. Then the symmetric partial sums (1.3) of the cardinal series with coefficients $a(n) = u(n)$, $n = 0, \pm 1, \pm 2, \ldots$ converge uniformly on compacta to $u(z)$.*

Versions of the statement of Corollary 6 have been recorded in [3, Theorem 1 and the cited references] and [7, Theorem 3 on p 70]. For alternate proofs see [2, p 124] and [13, p 499].

There is an analogue of Corollary 4 for odd functions $u(z)$, but its proof is significantly more complicated. For example, to see that (1.39) implies that (1.15) is valid for the coefficients $a(n) = u(n)$ we argue as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^n \frac{a(n)}{n} &= \sum_{k=1}^{\infty}\left\{\frac{a(2k)}{2k} - \frac{a(2k-1)}{2k-1}\right\} \\
&= \sum_{k=1}^{\infty}\left\{\frac{a(2k)}{2k} - \frac{a(2k-1)}{2k}\right\} + \sum_{k=1}^{\infty}\left\{\frac{1}{2k} - \frac{1}{2k-1}\right\} a(2k-1) \\
&= \sum_{k=1}^{\infty}\frac{a(2k)-a(2k-1)}{2k} - \sum_{k=1}^{\infty}\frac{a(2k-1)}{2k(2k-1)}.
\end{aligned}$$

Now, the Schwarz inequality and (1.39) yield

$$\Big|\sum_{k=1}^{\infty}\frac{a(2k)-a(2k-1)}{2k}\Big|^2 \le \left\{\sum_{k=1}^{\infty}\frac{1}{(2k)^2}\right\}\sum_{k=1}^{\infty}|a(2k)-a(2k-1)|^2 \le C\|u\|_{PW^1}^2$$

while (1.38) yields

$$\sum_{k=1}^{\infty}\frac{|a(2k-1)|}{2k(2k-1)} < \infty.$$

Altogether the above identity and inequalities imply (1.15).

It now follows from Theorem 4 that if $a(n) = u(n)$ and u is in PW^1, then the symmetric partial sums for the cardinal series (1.3) converge to the entire function $f(z)$ given by (1.16). An argument analogous to the one used to prove the uniqueness portion of Theorem 3 shows that $f(z) = u(z) + c\sin\pi z$ where c is a constant. But our argument for the fact that the constant c is indeed 0 involves more intricate properties of PW^k and is too complicated to be included here.

However, let us bring attention to the fact that a variant of the above argument used to show that the coefficients $a(n) = u(n)$ satisfy (1.15) when $u(z)$ is an odd function in PW^1 can be used to show that such coefficients satisfy (1.6) when $u(z)$ is an even function in PW^2.

We summarize these observations as follows:

Corollary 7. *Suppose u is an odd function in PW^1. Then the symmetric partial sums (1.3) of the cardinal series with coefficients $a(n) = u(n)$, $n = 0, \pm 1, \pm 2, \ldots$ converge uniformly on compacta to $u(z) + c\sin\pi z$. If u is an even function in PW^2, then the symmetric partial sums (1.3) of the cardinal series with coefficients $a(n) = u(n)$, $n = 0, \pm 1, \pm 2, \ldots$ converge uniformly on compacta to $u(z) + cz\sin\pi z$.*

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