Chapter 1
Interest Rates and Financial Derivatives

In this chapter we present the basic theory of interest rate instruments and the pricing of financial derivatives. The material we have chosen to present here is interesting and relevant in its own right but particularly so as the basis for the principles and methods considered in subsequent chapters.

The chapter consists of two sections. Section 1.1 presents the basic theory of interest rate instruments and focuses on the no-arbitrage valuation of cash flows. Section 1.2 presents the no-arbitrage principle for valuation of financial derivative contracts, contracts whose payoffs are functions of the value of another asset at a specified time in the future, and exemplifies the use of this principle. In a well-functioning market of derivative contracts, the derivative prices can be represented in terms of expected values of the payoffs, where the expectation is computed with respect to a probability distribution for the underlying asset value on which the contracts are written. If many derivative contracts are traded in the market, then we can say rather much about this probability distribution, and individual investors may compare it to their own subjective assessments of the underlying asset value and use the result of the comparison to make wise investment and risk management decisions.

1.1 Interest Rates and Deterministic Cash Flows

Consider a bank account that pays interest at the rate r per year. If yearly compounding is used, then one unit of currency on the bank account today has grown to $(1 + r)^n$ units after n years. Similarly, if monthly compounding is used, then one unit in the bank account today has grown to $(1 + r/12)^{12n}$ units after n years. Compounding can be done at any frequency. If a year is divided into m equally long time periods and if the interest rate r/m is paid at the end of each period, then one unit on the bank account today has grown to $(1 + r/m)^m$ units after 1 year. We say that the annual rate r is compounded at the frequency m. Note that $(1 + r/m)^m$ is increasing in m. In particular, a monthly rate r is better than a yearly

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rate r for the holder of a savings account. Continuous compounding means that we let m tend to infinity. Recall that $(1+1/m)^m \to e$ as $m \to \infty$, which implies that $(1+r/m)^m \to e^r$ as $m \to \infty$. Unless stated otherwise, interest rates in this book always refer to continuous compounding. That is, one unit deposited in a savings account with a 5% interest rate per year has grown to $e^{0.05t}$ units after t years. Note that the interest rate is just a means of expressing the rate of growth of cash. An investor cares about the rate of growth but not about which type of compounding is used to express this rate of growth.

In reality, things are certainly a bit more involved. The rate of interest on money deposited in a bank account differs from that for money borrowed from the bank. Moreover, the length of the time period also affects the interest rate. In most cases, the lender cannot ignore the risk that the borrower might be unable to live up to the borrower's obligations, and therefore the lender requires compensation in terms of a higher interest rate for accepting the risk of losing money.

1.1.1 Deterministic Cash Flows

Consider a set of times $0 = t_0 < t_1 < \cdots < t_n$, with $t_0 = 0$ being the present time. A deterministic cash flow is a set $\{(c_k, t_k); k = 0, 1, \ldots, n\}$ of pairs (c_k, t_k), where c_k and t_k are known numbers and where c_k represents the amount of cash received at time t_k by the owner of the cash flow. A negative value of c_k means that the owner of the cash flow must pay money at time t_k. Here we consider financial instruments that can be identified with deterministic cash flows. Any two parties can enter an agreement to exchange cash flows, but the contracted cash flow is not deterministic if there is a possibility that one party will fail to deliver the contracted cash flow.

An important instrument corresponding to a deterministic cash flow is the risk-free bond. The bonds issued by governments are typically good proxies. A risk-free bond issued at the present time corresponds to the cash flow

$$\{(-P_0, 0), (c, \Delta t), \ldots, (c, (n-1)\Delta t), (c + F, n\Delta t)\}, \tag{1.1}$$

where $P_0 > 0$ is the present bond price, $c \geq 0$ the periodic coupon amount paid to the bondholder, $F > 0$ the face value or principal of the bond, $\Delta t > 0$ the time between coupon payments, and $T = n\Delta t$ the time to maturity of the bond. Time is typically measured in years with $\Delta t = 0.5$ or $\Delta t = 1$. If $\Delta t = 0.5$, then the bond pays coupons semiannually and $2c$ is the annual coupon amount. If $c = 0$, then the bond is called a zero-coupon bond. Zero-coupon bonds often have less than 1 year to maturity. Buying a bond of the type given by (1.1) at time 0 that was issued at time $-u$, with $u \in (0, \Delta t)$, implies the cash flow

$$\{(-P_0, 0), (c, \Delta t - u), \ldots, (c, (n-1)\Delta t - u), (c + F, n\Delta t - u)\},$$

where P_0 is the price of the bond at time 0. Typically, $P_0 > P_{-u}$ since a buyer who purchases the bond at $-u$ would have to wait longer before receiving money.

Consider a market with an interest rate r per year that applies to all types of investment, loan and deposit (think of an ideal bank account without fees and restrictions on transactions). Then an amount A today is worth $e^{rt}A$ after t years. Similarly, an amount A received in t years from today is worth $e^{-rt}A$ today. We say that $e^{-rt}A$ is the present value of A at time t, and e^{-rt} is the discount factor for cash received at time t. The present value of a cash flow $\{(c_k, t_k); k = 0, \dots, n\}$ on this market is

$$P_0(r) = \sum_{k=0}^{n} c_k e^{-rt_k}.$$

The internal rate of return is the number r for which $P_0(r) = 0$. Note that the equation $P_0(r) = 0$ does not necessarily determine the internal rate of return uniquely for arbitrary deterministic cash flows. However, if $c_0 < 0$ and $c_k \geq 0$ for $k \geq 1$ with $c_k > 0$ for some k (e.g., the cash flow of a bond), then it is not difficult to verify that the internal rate of return is uniquely determined. For a bond, the internal rate of return is called the yield to maturity of the bond.

Consider a zero-coupon bond with current price $P_0 > 0$ that pays the amount $F > 0$ at t years from now, i.e., the cash flow $\{(-P_0, 0), (F, t)\}$. Clearly, there is a number r_t such that the relation $P_0 = e^{-r_t t}F$ holds. The number r_t is the t-year zero rate (or the t-year zero-coupon bond rate or spot rate), and the number $e^{-r_t t}$ is the discount factor for money received t years from now. Note that the discount factor $e^{-r_t t}$ is the current price for one unit received at time t. The graph of r_t viewed as a function of t is called the zero rate curve (or spot rate curve or yield curve). Market prices show that the zero rate curve is typically increasing and concave (the value of the second-order derivative with respect to t is negative). In particular, the assumption of a flat zero rate curve ($r_t = r$ for all t) is not consistent with market data.

The risk-free bonds discussed above are risk free in the sense that the buyer of such a bond will for sure receive the promised cash flow. However, a risk-free bond is risky if the holder sells the bond prior to maturity since the income from selling the bond is uncertain and depends on the market participants' demand for and valuation of the remaining cash flow. Moreover, the risk-free bond is risk free if held to maturity only in nominal terms. If, for instance, inflation is high, then the cash received at maturity may be worth little in the sense that you cannot buy much for the received amount. A bond is not risk free if it is possible that the issuer of the bond does not manage to pay the bondholder according to the specified cash flow of the bond. Such a bond is called risky or defaultable.

1.1.2 *Arbitrage-Free Cash Flows*

How are zero rates determined from prices of traded bonds or other cash flows? The simplest way would be to look up prices of zero-coupon bonds with the relevant maturity times. The problem with this approach is that such zero-coupon bond prices are typically not available. The cash flows priced by the market are typically more complicated cash flows such as coupon bonds. Moreover, the total number of cash flow times are often larger than the number of cash flows. Before addressing the question of how to determine zero rates from traded instruments, one must determine whether there exist any zero rates at all that are consistent with the observed prices.

Fix a set of times $0 = t_0 < \cdots < t_n$ and consider a market consisting of m cash flows:

$$\{(c_{k,0}, t_0), (c_{k,1}, t_1), \ldots, (c_{k,n}, t_n)\}, \quad k = 1, \ldots, m.$$

Since the times are held fixed, we represent the cash flows more compactly as m elements $\mathbf{c}_1, \ldots, \mathbf{c}_m$ in $\mathbb{R}^{n+1}$ (vectors with $n + 1$ real-valued components). It is assumed (although this is not entirely realistic) that you can buy and short-sell unlimited amounts of these contracts/cash flows. Short-selling a financial instrument should be interpreted as borrowing the instrument from a lender, then selling it at the current market price and at a later time purchasing an identical instrument at the prevailing market price and returning it to the lender. Here we ignore borrowing fees associated with short-selling. It is also assumed here (again not entirely realistically) that the market prices for buying and selling an instrument coincide and that there are no fees charged for buying and selling.

Under the imposed assumptions one can form linear portfolios of the original cash flows and thereby create new cash flows of the form $\mathbf{c} = \sum_{k=1}^{m} h_k \mathbf{c}_k$. The h_ks are any real numbers, and negative values correspond to short sales. The market therefore consists of arbitrary linear combinations of the original cash flows and can be represented as a linear subspace $\mathbb{C}$ of $\mathbb{R}^{n+1}$, spanned by the cash flows $\mathbf{c}_1, \ldots, \mathbf{c}_m$. We say that there exists an arbitrage opportunity if there exists a $\mathbf{c} \in \mathbb{C}$ such that $\mathbf{c} \neq \mathbf{0}$ ($c_k \neq 0$ for some k) and $\mathbf{c} \geq \mathbf{0}$ ($c_k \geq 0$ for all k). Such an element $\mathbf{c}$ corresponds to a contract that does not imply any initial or later costs and gives the buyer a positive amount of money. Such a contract cannot exist on a well-functioning market, at least not for long. If it did exist, some market participants would spot it and take advantage of it. Their actions would, in turn, drive the prices to the point where the arbitrage opportunity disappeared. The absence of arbitrage opportunities is equivalent to the existence of discount factors for the maturity times under consideration. This fact is a consequence of the following result from linear algebra.

Theorem 1.1. *Let $\mathbb{C}$ be a linear subspace of $\mathbb{R}^{n+1}$. Then the following statements are equivalent:*

(i) There exists no element $\mathbf{c} \in \mathbb{C}$ satisfying $\mathbf{c} \neq \mathbf{0}$ and $\mathbf{c} \geq \mathbf{0}$.
(ii) There exists an element $\mathbf{d} \in \mathbb{R}^{n+1}$ with $\mathbf{d} > \mathbf{0}$ satisfying $\mathbf{c}^{\mathrm{T}}\mathbf{d} = 0$ for all $\mathbf{c} \in \mathbb{C}$.

Proof. The implication (ii) $\Rightarrow$ (i) in Theorem 1.1 is easily shown: if $\mathbf{d} > \mathbf{0}$ and $\mathbf{c}^{\mathrm{T}}\mathbf{d} = 0$ for all $\mathbf{c} \in \mathbb{C}$, then each nonzero $\mathbf{c} \in \mathbb{C}$ must have both a positive component and a negative component. The implication (i) $\Rightarrow$ (ii) is more difficult to show. Assume that (i) holds and let

$$K = \{\mathbf{k} = (k_0, \ldots, k_n)^{\mathrm{T}} \in \mathbb{R}^{n+1} \text{ such that } k_0 + \cdots + k_n = 1 \text{ and } k_i \geq 0 \text{ for all } i\}.$$

From (i) it follows that K and $\mathbb{C}$ have no common element. Let $\mathbf{d}$ be a vector in $\mathbb{R}^{n+1}$ of shortest length among all vectors in $\mathbb{R}^{n+1}$ of the form $\mathbf{k} - \mathbf{c}$ for $\mathbf{k} \in K$ and $\mathbf{c} \in \mathbb{C}$. The proof of the fact that such a vector $\mathbf{d}$ exists is postponed to Lemma 1.1 right after this proof. Take a representation $\mathbf{d} = \mathbf{k}^* - \mathbf{c}^*$, where $\mathbf{k}^* \in K$ and $\mathbf{c}^* \in \mathbb{C}$. For any $\lambda \in [0, 1]$, $\mathbf{k} \in K$, and $\mathbf{c} \in \mathbb{C}$ we notice that $\lambda \mathbf{k}^* + (1 - \lambda)\mathbf{k} \in K$ and $\lambda \mathbf{c}^* + (1 - \lambda)\mathbf{c} \in \mathbb{C}$. By the definition of $\mathbf{k}^*$ and $\mathbf{c}^*$, the function f defined on $[0, 1]$, given by

$$f(\lambda) = \big((\lambda \mathbf{k}^* + (1 - \lambda)\mathbf{k}) - (\lambda \mathbf{c}^* + (1 - \lambda)\mathbf{c})\big)^2$$

has a minimum at $\lambda = 1$. We may write

$$\begin{aligned} f(\lambda) &= (\lambda \mathbf{d} + (1 - \lambda)(\mathbf{k} - \mathbf{c}))^{\mathrm{T}}(\lambda \mathbf{d} + (1 - \lambda)(\mathbf{k} - \mathbf{c})) \\ &= \lambda^2 \mathbf{d}^{\mathrm{T}}\mathbf{d} + 2\lambda(1 - \lambda)\mathbf{d}^{\mathrm{T}}(\mathbf{k} - \mathbf{c}) + (1 - \lambda)^2(\mathbf{k} - \mathbf{c})^{\mathrm{T}}(\mathbf{k} - \mathbf{c}). \end{aligned}$$

The fact that f has a minimum at $\lambda = 1$ implies that

$$f'(1) = 2\Big(\mathbf{d}^{\mathrm{T}}\mathbf{d} - \mathbf{d}^{\mathrm{T}}(\mathbf{k} - \mathbf{c})\Big) \leq 0.$$

Equivalently, $\mathbf{d}^{\mathrm{T}}\mathbf{k} - \mathbf{d}^{\mathrm{T}}\mathbf{d} \geq \mathbf{d}^{\mathrm{T}}\mathbf{c}$ for any $\mathbf{k} \in K$ and $\mathbf{c} \in \mathbb{C}$. If $\mathbf{d}^{\mathrm{T}}\mathbf{c} \neq 0$ for some $\mathbf{c} \in \mathbb{C}$, then $\mathbf{d}^{\mathrm{T}}(t\mathbf{c}) \neq 0$ for $|t|$ arbitrarily large, which implies that $\mathbf{d}^{\mathrm{T}}\mathbf{k}$ is larger than any positive number for all $\mathbf{k} \in K$. This is clearly false, and we conclude that $\mathbf{d}^{\mathrm{T}}\mathbf{c} = 0$ for all $\mathbf{c} \in \mathbb{C}$, which implies that $\mathbf{d}^{\mathrm{T}}\mathbf{k} - \mathbf{d}^{\mathrm{T}}\mathbf{d} \geq 0$ for all $\mathbf{k} \in K$. It remains to show that the components of $\mathbf{d}$ are strictly positive. With $\mathbf{k} = (1, 0, \ldots, 0)^{\mathrm{T}}$ we get $d_0 \geq \mathbf{d}^{\mathrm{T}}\mathbf{d} > 0$, and similarly for the other components of $\mathbf{d}$ by choosing $\mathbf{k}$ among the standard basis vectors of $\mathbb{R}^{n+1}$. We conclude that the implication (i) $\Rightarrow$ (ii) holds. □

The following result from analysis is used in the proof of Theorem 1.1.

Lemma 1.1. *There exists a vector* $\mathbf{d}$ *of shortest length between* K *and* $\mathbb{C}$.

Proof. For $\mathbf{k}$ in K, let $\mathbf{v}$ be the corresponding vector of shortest length between $\mathbf{k}$ and $\mathbb{C}$. If $\mathbf{c}$ is the orthogonal projection of $\mathbf{k}$ onto $\mathbb{C}$, then $\mathbf{v} = \mathbf{k} - \mathbf{c}$. We will first show that the function f, given by $f(\mathbf{k}) = \mathbf{v}$, is continuous. For any $\mathbf{k}_1, \mathbf{k}_2$ in K, by orthogonality, the corresponding vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{c}_1, \mathbf{c}_2$ satisfy

$$|\mathbf{k}_2 - \mathbf{k}_1|^2 = |\mathbf{v}_2 - \mathbf{c}_2 - \mathbf{v}_1 + \mathbf{c}_1|^2 = |\mathbf{v}_2 - \mathbf{v}_1|^2 + |\mathbf{c}_2 - \mathbf{c}_1|^2.$$

In particular,

$$|f(\mathbf{k}_2) - f(\mathbf{k}_1)| = |\mathbf{v}_2 - \mathbf{v}_1| \le |\mathbf{k}_2 - \mathbf{k}_1|,$$

which proves the continuity of f. Since K is compact and f is continuous, $V = f(K)$ is compact, too. Vector $\mathbf{d}$ is a vector in V of minimal norm. Such a vector exists because it is a minimizer of a continuous function, the norm, over the compact set V. □

Consider statement (ii) of Theorem 1.1. Clearly the statement holds for some $\mathbf{d}$ if and only if it holds for $\mathbf{d}$ replaced by $t\mathbf{d}$ for any $t > 0$, in particular, for the choice $t = 1/d_0 > 0$. Therefore, Theorem 1.1 says that the market $\mathbb{C}$ has no arbitrage opportunities if and only if there exists a vector $\mathbf{d} = (1, d_1, \ldots, d_n)^{\mathrm{T}}$, $d_k > 0$ for all k, such that $\mathbf{c}^{\mathrm{T}}\mathbf{d} = 0$ for all $\mathbf{c} \in \mathbb{C}$. The components of such a vector $\mathbf{d}$ are the discount factors for the times $t_0, \ldots, t_n$. In particular, an arbitrage-free price of an instrument paying c_k at time t_k, for $k \ge 1$, is

$$P_0 = \sum_{k=1}^{n} c_k d_k. \tag{1.2}$$

Equivalently, $(-P_0, c_1, \ldots, c_n)^{\mathrm{T}}$ belongs to $\mathbb{C}$. There may exist a range of arbitrage-free prices p with each p satisfying (1.2) for some vector $\mathbf{d}$ with the property $\mathbf{c}^{\mathrm{T}}\mathbf{d} = 0$ for all $\mathbf{c} \in \mathbb{C}$. Note that the discount factors d_k, $k = 0, \ldots, n$, may be written $d_k = e^{-r_k t_k}$, where r_k is the zero rate corresponding to payment time t_k.

If there exists precisely one vector $\mathbf{d}$ of discount factors, then $\mathbb{C} = \{\mathbf{c}; \mathbf{c}^{\mathrm{T}}\mathbf{d} = 0\}$, and $\mathbb{C}$ is said to be complete. If $\mathbb{C}$ is complete, then any new cash flow (or contract) $\mathbf{c}$ introduced is either redundant (a linear combination of $\mathbf{c}_1, \ldots, \mathbf{c}_m$) or creates an arbitrage opportunity. Real-world markets are typically not complete: a new contract is not identical to a linear combination of existing contracts.

Suppose that the cash flow corresponds to bonds, i.e., for each $\mathbf{c}_k$ we have that $-c_{k,0}$ is the bond price today, $c_{k,n}$ is the face value plus a coupon, and the other $c_{k,j}$s ($j = 1, \ldots, n-1$) are coupons. Under the assumption that this bond market is complete and without arbitrage opportunities, the bond price $-c_{k,0}$ is given by

$$-c_{k,0} = \sum_{j=1}^{n} c_{k,j} e^{-t_j r_j},$$

where r_j are the (unique) zero rates.

Given a market consisting of the cash flows $\mathbf{c}_1, \ldots, \mathbf{c}_m$, it is not difficult to check if the market is arbitrage free and, if so, whether the market is complete

Table 1.1 Specifications of three bonds

Bond	A	B	C
Bond price	99.65	113.43	121.30
Maturity (days)	190	$32 + 2 \cdot 365$	$241 + 3 \cdot 365$
Annual coupon	0	5.5	6.75
Face value	100	100	100

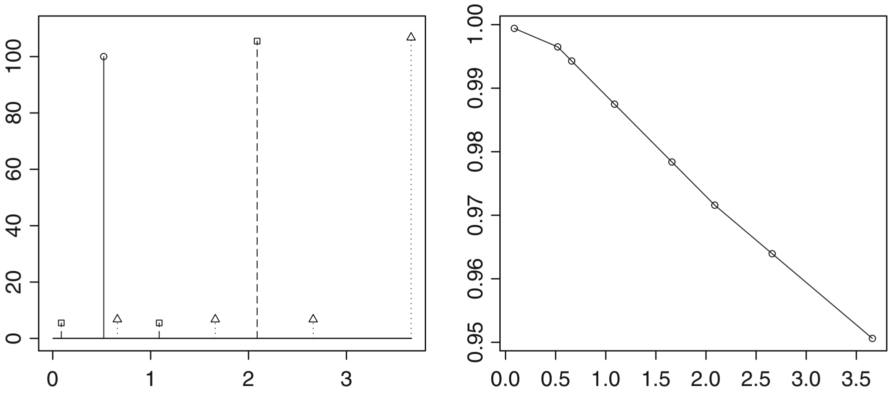

Fig. 1.1 *Left plot*: graphical illustration of cash flows for the three bonds; *right plot*: discount factors in Table 1.2. In the *left plot*, time is on the x-axis and the payment amounts on the y-axis. In the *right plot*, the time to maturity is on the x-axis and the value of the discount factors is on the y-axis

or not. An arbitrage-free (and complete) market is equivalent to the existence (and uniqueness) of a solution $\mathbf{d} = (d_1, \ldots, d_n)^{\mathrm{T}}$ to the matrix equation

$$\begin{pmatrix} -c_{1,0} \\ \vdots \\ -c_{m,0} \end{pmatrix} = \begin{pmatrix} c_{1,1} & \ldots & c_{1,n} \\ \vdots & \ldots & \vdots \\ c_{m,1} & \ldots & c_{m,n} \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}, \tag{1.3}$$

where $(c_{k,0}, \ldots, c_{k,n}) = \mathbf{c}_k^{\mathrm{T}}$. The analysis of solutions to matrix equation (1.3) is a standard problem in linear algebra.

Example 1.1 (Bootstrapping zero rates). Consider a market consisting of the bonds in Table 1.1. From Table 1.1 and Fig. 1.1 we see that there are in total eight nonzero cash flow times

$$(t_1, \ldots, t_8) \approx (0.09, 0.52, 0.66, 1.09, 1.66, 2.09, 2.66, 3.66),$$

where t_1 corresponds to 32 days from now and therefore $32/365 \approx 0.09$ years from now, etc. Therefore, there are also eight undetermined discount factors $d_1, \ldots, d_8$

Table 1.2 Cash flow times (years), discount factors, and zero rates (%) (discount factors obtained as in Example 1.1 by linear interpolation between discount factors)

Time	0.088	0.521	0.660	1.088	1.660	2.088	2.660	3.660
Discount factors	0.999	0.997	0.994	0.987	0.978	0.972	0.964	0.951
Zero rates	0.673	0.674	0.869	1.158	1.317	1.381	1.380	1.384

solving the matrix equation $\mathbf{Cd} = \mathbf{P}$ of the type in (1.3), where $\mathbf{d} = (d_1, \ldots, d_8)^{\mathrm{T}}$, $\mathbf{P} = (99.65, 113.43, 121.30)^{\mathrm{T}}$, and

$$\mathbf{C} = \begin{pmatrix} 0 & 100 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5.5 & 0 & 0 & 5.5 & 0 & 105.5 & 0 & 0 \\ 0 & 0 & 6.75 & 0 & 6.75 & 0 & 6.75 & 106.75 \end{pmatrix}.$$

There exist solutions to this matrix equation, so there are no arbitrage opportunities in this bond market. The problem here is that there is an infinite number of possibly very different solutions. One solution is obtained by setting the discount factors corresponding to coupon dates to one, $d_1 = d_3 = d_4 = d_5 = d_7 = 1$, which gives the equation system

$$\begin{pmatrix} 100 & 0 & 0 \\ 0 & 105.5 & 0 \\ 0 & 0 & 106.75 \end{pmatrix} \begin{pmatrix} d_2 \\ d_6 \\ d_8 \end{pmatrix} = \begin{pmatrix} 99.65 \\ 113.43 - 2 \cdot 5.5 \\ 121.30 - 3 \cdot 6.75 \end{pmatrix}$$

with solution $(d_2, d_6, d_8) \approx (0.9965, 0.9709, 0.9466)$. The corresponding zero rates are, in percentages, with two decimals, $r_1, \ldots, r_8 \approx 0, 0.67, 0, 0, 0, 1.41, 0, 1.50$. This is clearly a silly solution as it would imply that the price of a zero-coupon bond maturing 2.66 years from now with face value 100 is 100. Who would buy this bond?

Let us now take a step back and consider a better approach, which is often referred to as the bootstrap method (note: there are other methods referred to as bootstrap methods that have nothing to do with interest rates). The discount factor $d_2 = 0.9965$ corresponding to the zero-coupon bond is known. Also, the discount factor corresponding to cash flow today is clearly $d_0 = 1$. Therefore, it seems reasonable to assign a value to d_1 by interpolation between the two neighboring discount factors. Let us for simplicity use linear interpolation, which gives

$$d_1 = d_0 + \frac{d_2 - d_0}{t_2 - t_0}(t_1 - t_0) \approx 0.9994.$$

Now we have assigned values to the first two (nontrivial) discount factors, and we need an approach other than linear interpolation between known discount factors to assign values to the remaining ones. The second bond yields the equation

$$113.43 - 5.5d_1 = 5.5d_4 + 105.5d_6,$$

which is an equation with two unknowns. Assuming temporarily that the value of d_4 is given by linear interpolation between the last (in the sense of the order of the cash flow times) known discount factor d_2 and the unknown d_6 we get the equation

$$113.43 - 5.5d_1 = 5.5\left(d_2 + \frac{d_6 - d_2}{t_6 - t_2}(t_4 - t_2)\right) + 105.5d_6,$$

which can be solved for d_6, yielding $d_6 \approx 0.9716$. Now the discount factors d_3, d_4, d_5 are assigned values by linear interpolation between d_2 and d_6:

$$d_k = d_2 + \frac{d_6 - d_2}{t_6 - t_2}(t_k - t_2) \quad \text{for } k = 3, 4, 5.$$

This gives $(d_3, d_4, d_5) \approx (0.9943, 0.9875, 0.9784)$. The last two discount factors d_7 and d_8 are assigned values by repeating the foregoing procedure. This gives $(d_7, d_8) \approx (0.9639, 0.9506)$. The cash flow times, the discount factors, and the corresponding zero rates are given in Table 1.2.

Yield curves are not only derived from bond prices. The next example shows how a yield curve can be extracted from forward prices. In this example, the notion of present price and forward price of an asset is needed. Consider a contract for delivery of an asset at a future time $t > 0$. The forward price $G_0^{(t)}$ of the contract is the price, agreed upon at the current time 0, which will be paid at maturity, time t, of the contract. The present price $P_0^{(t)}$ of the contract is the price that is agreed upon and paid at the current time 0. In the sequel, when there is no risk of confusion about the maturity time, we will sometimes drop the superscript and write G_0 and P_0. The present price is the discounted forward price: $P_0^{(t)} = d_t G_0^{(t)}$, where d_t is the discount factor between 0 and t.

The present price of a share of a stock that does not pay dividends before time t must be identical to the spot price, S_0, for immediate delivery since there is no cost or benefit from holding the asset between time 0 and time t: the forward price must satisfy $d_t G_0^{(t)} = P_0^{(t)} = S_0$. The present price, for delivery at a future time t_2, of one share of a stock that pays a known dividend amount c at time $t_1 < t_2$ is determined by the relation

$$P_0^{(t_2)} = S_0 - d_{t_1} c.$$

The validity of the relation follows from the ensuing argument. Consider first the strategy of, at time 0, buying the share and short-selling a zero-coupon bond that matures at time t_1 with face value c, and at time t_2 selling the share. The initial cost of this strategy is $S_0 - d_{t_1} c$, and it gives the random payoff S_{t_2} at time t_2. On the other hand, consider a contract that delivers one share of the stock at time t_2. Since the contract and the foregoing strategy have identical future cash flows, their initial cash flows must coincide in order not to introduce arbitrage opportunities.

Table 1.3 Forward prices on April 8 for delivery of one share of H&M at different maturity times

Maturity	April 15	May 20	June 17	September 16	December 16	January 20	March 16
Forward price	218.64	209.52	209.92	211.29	212.85	213.50	214.59

Example 1.2 (Zero rates from forward prices). On April 8, the spot price S_0 for buying one share of H&M on the Nasdaq Nordic OMX exchange was 218.60 Swedish kronor. Table 1.3 shows forward prices on that same day for one share of the stock for delivery at different maturities. The company H&M announced that on May 6 it would pay a dividend of $c = 9.50$ kronor per share. This explains the large difference between the current forward prices for the maturity dates April 15 and May 20.

Consider the cash flow times $t_0, \dots, t_9$ given by

$$\begin{array}{lll} t_0 = 0 \quad \text{(Apr 8)}, & t_1 = 0.019 \text{ (Apr 15)}, & t_2 = 0.063 \text{ (May 6)}, \\ t_3 = 0.115 \text{ (May 20)}, & t_4 = 0.192 \text{ (Jun 17)}, & t_5 = 0.441 \text{ (Sep 16)}, \\ t_6 = 0.690 \text{ (Dec 16)}, & t_7 = 0.786 \text{ (Jan 20)}, & t_8 = 0.940 \text{ (Mar 16)}. \end{array}$$

The corresponding discount factors are denoted $d_0, \dots, d_8$. Since there is no dividend paid before t_1, the discount factor d_1 is derived from the relation $d_1 G_0^{(t_1)} = S_0$, where S_0 is the spot price and, hence, also the present price for delivery of one share of H&M at time t_1. The present price for delivery of one share of H&M at t_3 gives the relation $d_3 G_0^{(t_3)} = S_0 - cd_2$. Similarly, for the remaining maturities we have $d_k G_0^{(t_k)} = S_0 - cd_2$ for $k = 4, \dots, 8$. In all, we have seven equations and eight unknowns, which gives an underdetermined equation system with solution

$$d_1 = \frac{S_0}{G_0^{(t_1)}} \quad \text{and} \quad d_k = \frac{S_0}{G_0^{(t_k)}} - \frac{d_2 c}{G_0^{(t_k)}} \quad \text{for } k = 3, \dots, 8, \tag{1.4}$$

parameterized by d_2. To find a reasonable solution among all possible solutions, the bootstrapping procedure presented above suggests expressing d_2 by linear interpolation between d_1 and d_3. The equation

$$d_2 = d_1 + \frac{d_3 - d_1}{t_3 - t_1}(t_2 - t_1),$$

together with the equations for the maturity times t_1 and t_3, gives

$$d_2 = \left(1 + \frac{c}{G_0^{(t_3)}} \frac{t_2 - t_1}{t_3 - t_1}\right)^{-1} \left(\frac{S_0}{G_0^{(t_1)}}\left(1 - \frac{t_2 - t_1}{t_3 - t_1}\right) + \frac{S_0}{G_0^{(t_3)}} \frac{t_2 - t_1}{t_3 - t_1}\right),$$

from which the values of all discount factors can be computed from (1.4) (Fig. 1.2).

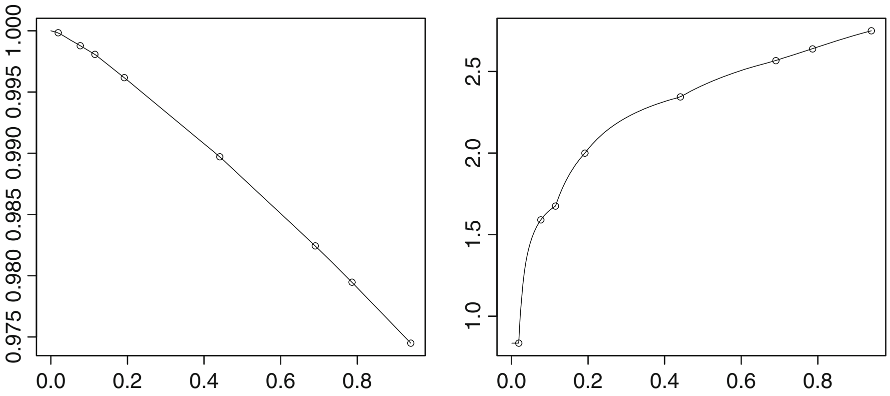

Fig. 1.2 *Left plot*: discount factors in Example 1.2. Time to maturity is on the x-axis; value of discount factors is on the y-axis. The *right plot* shows the zero rates (%) in Example 1.2 corresponding to the linearly interpolated discount factors

Example 1.3 (Interest rate swap). Let $0 = t_0 < t_1 < \cdots < t_n = T$ be a sequence of equally spaced times with $\Delta = t_k - t_{k-1} = T/n$, and let $d_1, \ldots, d_n$ be discount factors giving the value at time 0 of money at times $t_1, \ldots, t_n$.

An interest rate swap is an agreement at time 0 between two parties to exchange floating interest rate payments (a stochastic cash flow) for fixed interest rate payments (a deterministic cash flow) on a notional principal L (US \$100 million, say) until, and including, time t_n with zero initial cost for both parties.

The floating interest rate payments are paid at times $\Delta/m = \delta, 2\delta, \ldots, mn\delta = T$, where typically $m = 2$. The floating-rate payment due at time $k\delta$ is the interest earned between times $(k-1)\delta$ and $k\delta$ on the notional L, i.e., the random amount

$$L\Big(\frac{1}{d_{k-1,k}} - 1\Big),$$

where $d_{k-1,k}$ denotes the discount factor at time $(k-1)\delta$ between times $(k-1)\delta$ and $k\delta$. To determine the initial value of the floating-rate payments of the swap, we determine the value of a contract that pays the holder a never-ending stream of floating-rate payments at times $k\delta$, for $k = mn+1, mn+2, \ldots$, on principal L. The cash flow of the contract is obtained by investing at time $k\delta$ the amount L in zero-coupon bonds maturing at time $(k+1)\delta$ and at time $(k+1)\delta$, collecting the interest earned, and repeating the procedure with the remaining amount L. The value of this contract is therefore the value $d_n L$ of having the amount L at time $t_n = T$. Similarly, the value of a contract that pays the holder a never-ending stream of floating-rate payments at times $k\delta$, for $k = 1, 2, \ldots$, on principal L is L. Therefore, the initial value of the floating-rate payments of the swap is $L(1-d_n)$. Notice that the number δ does not show up, so the value of the floating-rate payments does not depend on the frequency of the floating-rate payments.

The initial value of the deterministic cash flow corresponding to payments cL at the times $t_1, \dots, t_n$ is simply the sum of the discounted payments: $cL(d_1+\cdots+d_n)$. Therefore, the fixed-rate payments of the swap have the initial value $cL(d_1+\cdots+d_n)$. The initial value of the swap is zero for both the floating-rate and fixed-rate receiver in the swap contract. Therefore, the number c must satisfy $cL(d_1+\cdots+d_n)=L(1-d_n)$, i.e., $c=(1-d_n)/(d_1+\cdots+d_n)$. The interest rate corresponding to the fixed-rate payment cL is called the swap rate. The swap rate can be seen as the yield to maturity of a bond with initial value L, maturing at t_n with face value L and coupons cL at times t_k. Notice that at time $t>0$ the discount factors will typically have changed and the value of the swap will be positive for one of the two parties and negative for the other.

The zero rates $r_k = -\log(d_k)/t_k$ corresponding to the discount factors $d_1, \dots, d_n$ are called swap zero rates. The discount factors $d_1, \dots, d_n$ and the corresponding swap zero rates are obtained from a set of swap contracts, with a corresponding set of contracted swap rates, by a bootstrap procedure similar to the one considered in Example 1.1.

There are many versions of interest rate swaps. The most common interest rate swap contract prescribes floating-rate payments every 6 months (3 months) and fixed-rate payments every 12 months (6 months), i.e., at half the frequency of the floating-rate payments. The floating interest rate is an interbank interest rate such as LIBOR (London Interbank Offered Rate) and not defined in terms of government bonds. A practical issue of some importance that we ignored previously is that different day count conventions typically apply to fixed rates and floating rates. When writing $r_k = -\log(d_k)/t_k$ one should specify if t_k equals the actual number of days divided by 360 or 365. Swap data show that two swap contracts with different values of δ, different frequencies of floating-rate payments, that are otherwise identical can have slightly different swap rates. This is at odds with the preceding swap valuation and shows that the credit risk borne by the floating-rate receiver from having to wait longer between the floating-rate payments is taken into account by the market in the valuation of the swap. Here credit risk refers to the risk of a failure to deliver the contracted cash flow.

1.2 Derivatives and No-Arbitrage Pricing

Consider the times 0 and $T>0$, with 0 being the present time, and let S_T be the spot price of some asset at time T. A contract with payoff $f(S_T)$ at time T for some function f is called a European derivative written on S_T. The derivative price π_f is the amount that is paid now in exchange for the payoff $f(S_T)$ at time T. A European call option on S_T with strike price K is a contract that gives the holder the right, but not the obligation, to purchase the underlying asset at time T for price K. Since this right is only exercised at time T if $S_T>K$, we see that the European call option is a derivative contract with payoff $f(S_T)=\max(S_T-K,0)$. A European put option on

S_T with strike price K is a derivative contract with payoff $f(S_T) = \max(K - S_T, 0)$. In this case, the holder has the right, but not the obligation, to sell the underlying asset at time T for price K.

We consider a market where m derivative contracts with current prices π_k and payoffs $f_k(S_T)$, for $k = 1, \dots, m$, and a risk-free zero-coupon bond maturing at time T with face value 1 and current price B_0 can be bought and sold. The bond saves us from difficulties in relating money at time 0 to money at time T. Here we assume that the market participants can buy and short-sell these contracts without paying any fees, and that for each contract the prices for buying and selling the contract coincide.

From the perspective of one of the market participants we want to understand how to assign a price to a new derivative contract in terms of the prices of the m existing derivative contracts and the bond. The market participants can form linear portfolios of the original derivative contracts, and such a portfolio will constitute a new derivative contract with payoff $f(S_T) = \sum_{k=1}^{m} h_k f_k(S_T)$ and price $\pi_f = \sum_{k=1}^{m} h_k \pi_k$. A contract of this type is called an arbitrage opportunity if $\pi_f = 0$, $\mathrm{P}(f(S_T) \geq 0) = 1$, and $\mathrm{P}(f(S_T) > 0) > 0$. An arbitrage opportunity is a contract that gives the holder a strictly positive probability of making a profit without taking any risk. The probability P is the subjective probability of the market participant under consideration. In particular, the existence of arbitrage opportunities depends on the subjective assessment of which events have probability zero.

Theorem 1.2. *The following statements are equivalent.*

1. *There are no arbitrage opportunities.*
2. *The prices π_f can be expressed as $\pi_f = B_0 \mathrm{E_Q}[f(S_T)]$, where the expectation is computed with respect to a probability* Q *that assigns zero probability to the same events as does the probability* P.

Remark 1.1. (i) The probability Q is called the forward probability. Note that $\mathrm{E_Q}[f(S_T)]$ is the forward price of the contract for delivery of $f(S_T)$ at time T.

(ii) There are examples of arbitrage opportunities that do not depend on the subjective probability P. Consider two derivative contracts with prices π_f and π_g and payoffs $f(S_T)$ and $g(S_T)$ satisfying $\pi_f < \pi_g$ and $f(S_T) \geq g(S_T)$ (for example, two European call options such that the one with the higher strike price costs more than the one with the lower strike price). A long position of size one in the cheaper derivative, a short position of size one in the expensive derivative, and a long position with initial value $\pi_g - \pi_f$ in the bond produces a contract with zero initial price and payoff $f(S_T) - g(S_T) + (\pi_g - \pi_f)/B_0 > 0$ at time T.

Proof. We begin by proving the implication (ii) $\Rightarrow$ (i). This implication is the easier one to prove and also probably the most relevant one since it means that as long as one comes up with a model for S_T that produces the observed prices, one can use this model for pricing new contracts without risking the introduction of arbitrage opportunities.

Suppose that (ii) holds, and consider a payoff $f(S_T)$ satisfying $\mathrm{P}(f(S_T) \geq 0) = 1$ and $\mathrm{P}(f(S_T) > 0) > 0$. We need to show that $\pi_f = B_0 \,\mathrm{E_Q}[f(S_T)] \neq 0$. By assumption, it also holds that $\mathrm{Q}(f(S_T) \geq 0) = 1$ and $\mathrm{Q}(f(S_T) > 0) > 0$. Since $\mathrm{Q}(f(S_T) \geq 0) = 1$, we may express $\mathrm{E_Q}[f(S_T)]$ as

$$\mathrm{E_Q}[f(S_T)] = \int_0^\infty \mathrm{Q}(f(S_T) > t)dt,$$

(see Remark 1.2), and since $\mathrm{Q}(f(S_T) > 0) > 0$, there exist $\varepsilon > 0$ and $\delta > 0$ such that $\mathrm{Q}(f(S_T) > \varepsilon) > \delta$. Therefore,

$$\frac{\pi_f}{B_0} = \mathrm{E_Q}[f(S_T)] = \int_0^\infty \mathrm{Q}(f(S_T) > t)dt \geq \int_0^\varepsilon \mathrm{Q}(f(S_T) > t)dt > \varepsilon\delta > 0,$$

which proves the claim, i.e., the implication (ii) $\Rightarrow$ (i).

Proving the implication (i) $\Rightarrow$ (ii) in a general setting is rather difficult. It becomes much less difficult if we assume that S_T takes values in a finite (but arbitrarily large) set. This is not at all an unrealistic assumption; S_T will take values with finitely many decimals, and it is plausible that $\mathrm{P}(S_T > s) = 0$ for all s greater than some sufficiently large number. Let $\{s_1, \dots, s_n\}$, with $\mathrm{P}(S_T = s_k) > 0$ and $\mathrm{P}(S_T = s_1) + \cdots + \mathrm{P}(S_T = s_n) = 1$, be the set of possible outcomes for S_T. Then every contract can be represented as a vector $\mathbf{x} = (x_0, x_1, \dots, x_n)^{\mathrm{T}}$ in $\mathbb{R}^{n+1}$. The contract with payoff $f(S_T)$ and price π_f can be represented as the vector $\mathbf{x} = (-\pi_f, f(s_1), \dots, f(s_n))^{\mathrm{T}}$. Therefore, the set of all contracts constructed from the original m derivative contracts forms a linear subspace of $\mathbb{R}^{n+1}$. Let us denote this linear space by $\mathbb{X}$. We see that $\mathbf{x} \in \mathbb{X}$ is an arbitrage opportunity if $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{x} \geq \mathbf{0}$. Theorem 1.1 says that there are no arbitrage opportunities if and only if there exists a vector $\mathbf{y} \in \mathbb{R}^{n+1}$ with $\mathbf{y} > \mathbf{0}$ such that $\mathbf{x}^{\mathrm{T}}\mathbf{y} = 0$ for all $\mathbf{x} \in \mathbb{X}$. Of course, the same result holds if $\mathbf{y}$ is replaced by $y_0^{-1}\mathbf{y}$. The bond corresponds to the vector $\mathbf{x} = (-B_0, 1, \dots, 1)^{\mathrm{T}}$. Since $\mathbf{x}^{\mathrm{T}}(y_0^{-1}\mathbf{y}) = 0$, we have $\sum_{k=1}^n y_0^{-1} y_k = B_0$. For $k = 1, \dots, n$ set $q_k = (B_0 y_0)^{-1} y_k$ and note that $q_k > 0$ and $\sum_{k=1}^n q_k = 1$. In particular, the q_k constitute a probability distribution on the set $\{s_1, \dots, s_n\}$ of possible outcomes for S_T. With $\mathbf{x} = (-\pi_f, f(s_1), \dots, f(s_n))^{\mathrm{T}}$ we see that $\mathbf{x}^{\mathrm{T}}(B_0 y_0)^{-1}\mathbf{y} = 0$ is equivalent to $\pi_f = B_0 \sum_{k=1}^n f(s_k) q_k$, which is precisely what Theorem 1.2 says. □

Remark 1.2. The representation of the expected value of a nonnegative random variable as an integral of its tail probabilities is not difficult to justify. Consider a random variable $X \geq 0$ with distribution function F, and set $\overline{F} = 1 - F$. If F has a density f, then

$$\int_0^\infty \overline{F}(t)dt = \int_0^\infty \Big[\int_t^\infty f(u)du\Big]dt = \int_0^\infty f(u)\Big[\int_0^u dt\Big]du = \int_0^\infty u f(u)du,$$

where we have simply changed the order of integration. The existence of a density f is actually not needed for the result to hold, but it simplifies the presentation.

Theorem 1.2 tells us how to price a new contract with payoff $g(S_T)$ such that no arbitrage opportunity is introduced: simply assign the price $\pi_g = B_0 \, \mathrm{E_Q}[g(S_T)]$ to the derivative contract. The expected value $\mathrm{E_Q}[g(S_T)]$ is the expected value of the random variable $g(S_T)$ computed with respect to the probability Q. Theorem 1.2 does not say that this price π_g is the unique arbitrage-free price of the new contract. There are typically many possible representations of the existing prices as discounted expected values, and the different representations are likely to give different prices to new contracts. More precisely: suppose that you assign a probability distribution to S_T with more than m parameters and that there is more than one solution (a set of parameters) to the nonlinear system of equations $\pi_k = B_0 \, \mathrm{E_Q}[f_k(S_T)]$, $k = 1, \ldots, m$, where the left-hand side is the market price of the kth original derivative and the right-hand side is the discounted expected payoff according to your chosen parametric model. Then there are probably several solutions, and the different solutions are likely to give different prices $B_0 \, \mathrm{E_Q}[g(S_T)]$ to a new derivative contract with payoff $g(S_T)$.

Example 1.4 (Rolling dice). Let S_T be the value of a six-sided die. The die is not necessarily fair. Suppose for now that there are two derivative contracts on S_T available on the market, a bet on even numbers (contract A) and a bet on odd numbers (contract B). Both contracts pay 1 if the bet turns up right and 0 otherwise, and the market prices of both contracts are $1/2$. There are no arbitrage opportunities on this market if the subjective probabilities $\mathrm{P}(S_T = 1), \ldots, \mathrm{P}(S_T = 6)$ are strictly positive. There are infinitely many choices of strictly positive probabilities $\mathrm{Q}(S_T = 1), \ldots, \mathrm{Q}(S_T = 6)$ such that (ii) of Theorem 1.2 holds. One such choice is given by

$$\mathrm{Q}(S_T = 1) = \cdots = \mathrm{Q}(S_T = 6) = 1/6.$$

Depending on the subjective view of the probabilities $\mathrm{P}(S_T = 1), \ldots, \mathrm{P}(S_T = 6)$, there may be opportunities for good deals: portfolios whose expected payoffs are greater than their prices. Consider an agent whose subjective view of the probabilities are such that

$$\mathrm{P}(S_T = k) = 0 \quad \text{for } k = 4, 5, 6.$$

To this agent the set of possible outcomes is reduced to $\{1, 2, 3\}$. Note that the observed prices are still consistent with no arbitrage. Suppose a new contract C is introduced paying 1 if the outcome of S_T is 1 or 2, and that the market price of this contract is $1/3$. The original market is still free of arbitrage (the same Q still works). However, on the reduced set of outcomes $\{1, 2, 3\}$ it is not possible to find a probability Q that reproduces the market prices. To the agent who believes in the reduced set of possible outcomes there seems to be an arbitrage opportunity. A portfolio consisting of a long position in C and a short position in A of the same size has a strictly negative price equal to $-1/6$ (you get money now) and has a nonnegative payoff with P-probability 1. The agent now has two choices: try to capitalize on the perceived arbitrage opportunity by going long in C and short in A, or revise the subjective probabilities. This example illustrates that there may be portfolios that are perceived as arbitrage opportunities because the subjective model used to assign probabilities to future events is too simplistic.

Example 1.5 (Calls and digitals). Consider a derivative with payoff $I\{S_T > K\}$ (meaning the value 1 if the event occurs and 0 otherwise) at time T, referred to as a digital or binary option, with current price $D_0(K)$. Consider also two call options with payoffs $\max(S_T - K, 0)$ and $\max(S_T - (K-1), 0)$ at time T and current prices $C_0(K)$ and $C_0(K-1)$. Let $x_+ = \max(x, 0)$, and notice that

$$(S_T - K + 1)_+ - (S_T - K)_+ = \begin{cases} 0 & \text{if } S_T < K - 1, \\ S_T - K + 1 & \text{if } S_T \in [K-1, K], \\ 1 & \text{if } S_T > K. \end{cases}$$

In particular, $(S_T - K + 1)_+ - (S_T - K)_+ \geq I\{S_T > K\}$.

If $C_0(K-1) - C_0(K) < D_0(K)$, then there are arbitrage opportunities. Buying the call option with strike $K-1$ and short-selling the call and the digital option with the strike K gives a strictly positive cash flow at time 0, which can be used to buy zero coupon bonds maturing at time T. Moreover, the cash flow from the payoffs of the options at time T is nonnegative. We have thus constructed a contract with zero initial cash flow that gives a strictly positive cash flow at time T. This is an arbitrage opportunity regardless of the probability distribution assigned to S_T.

If $C_0(K-1) - C_0(K) = D_0(K)$, then there may be arbitrage opportunities. Buying the call option with strike $K-1$ and short-selling the call and the digital option with the strike K gives zero initial cash flow and a cash flow $(S_T - K + 1)I\{S_T \in [K-1, K]\} \geq 0$ at time T. If $\mathrm{P}(S_T \in [K-1, K]) > 0$, then this is an arbitrage opportunity.

Example 1.6 (Put–call parity). Suppose there is a risk-free zero-coupon bond maturing at time T with face value 1, a call option with strike price K on the value S_T at time T, and a put option with the same strike price K on S_T. Write B_0, C_0, and P_0 for the current prices of the bond, call option, and put option, respectively. Suppose further that there is a forward contract on S_T with forward price G_0, the amount agreed upon today that is paid at time T in exchange for the random amount S_T.

A position of size $G_0 - K$ in the bond (long or short depending on the sign of $G_0 - K$) and a long position in the forward contract give the price $B_0(G_0 - K)$ for the derivative contract with payoff $S_T - K$. However, the same payoff can be produced by taking positions in the options. A long position in the call option and a short position in the put option correspond to a long position in a derivative contract with price $C_0 - P_0$ and the payoff

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K$$

at time T. In an arbitrage-free market, the prices of two derivative contracts with the same payoffs must coincide. Otherwise a risk-free profit is made by buying the cheaper of the two and short-selling the more expensive one. Therefore,

$$C_0 - P_0 = B_0(G_0 - K).$$

This relation between bond, forward, call option, and put option prices is called the put–call parity.

Example 1.7 (Parametric forward distribution). Suppose you want to use the parametric density function $q_{\boldsymbol{\theta}}$, whose argument is a real number and whose parameter vector $\boldsymbol{\theta}$ is multidimensional, as a model for the forward probability. Suppose further that the nonlinear system of equations in $\boldsymbol{\theta}$

$$\pi_k = B_0 \int f_k(s) q_{\boldsymbol{\theta}}(s) ds, \quad k = 1, \ldots, m$$

has a solution $\boldsymbol{\theta}^*$. Theorem 1.2 tells us that the market is arbitrage free if for any interval (a, b) it holds that

$$\int_a^b q_{\boldsymbol{\theta}^*}(s) ds = 0 \quad \text{if and only if} \quad \int_a^b p(s) ds = 0,$$

where p is your subjective probability density for the future spot price S_T. In this case you may assign the arbitrage-free price

$$\pi_g = B_0 \int g(s) q_{\boldsymbol{\theta}^*}(s) ds$$

to a derivative contract with payoff $g(S_T)$.

Example 1.8 (Online sports betting). Suppose you are visiting the Web site of an online sports betting agent, the bookmaker, with the intent of betting on a Premier League game, Chelsea vs. Liverpool. The odds offered by the bookmaker are "Chelsea": 2.50, "draw": 3.25, and "Liverpool": 2.70. The corresponding outcome of the game are denoted by 1, X, and 2, and for each of the outcomes it is assumed that you do not assign zero probability to the occurrence of that outcome. This game may be viewed as a market with three digital derivatives with prices $q_1 = 1/2.50$, $q_X = 1/3.25$, and $q_2 = 1/2.70$ and payoffs X_1, X_X, and X_2, where $X_1 = 1$ if the outcome of the game is "Chelsea" and 0 otherwise, and similarly for the other payoffs. Notice that

$$q_1 + q_X + q_2 = \frac{1}{2.50} + \frac{1}{3.25} + \frac{1}{2.70} \approx 1.078.$$

Since the prices do not sum up to one, they cannot be interpreted as probabilities. Equivalently, they cannot be expressed as (discounted) expected payoffs. A natural question, in light of Theorem 1.2, is therefore: is there an arbitrage opportunity? The answer is no. The reason is that you cannot sell the contracts short on this market (the bookmaker is not willing to switch sides with you). To see that there is no arbitrage, one could argue as follows. Consider dividing the initial capital 1 into bets on "Chelsea," "draw," and "Liverpool," where $w_1, w_X, w_2 \geq 0$, with $w_1 + w_X + w_2 = 1$, are the amounts placed on the respective possible outcomes. The portfolio (w_1, w_X, w_2) is an arbitrage opportunity if its post game value

$$\frac{w_1}{q_1} X_1 + \frac{w_X}{q_X} X_X + \frac{w_2}{q_2} X_2$$

is greater than or equal to one for sure and strictly greater than one with a strictly positive probability. Suppose that (w_1, w_X, w_2) is an arbitrage opportunity. For the postgame portfolio value to be greater than or equal to one it is necessary that $w_1/q_1 \geq 1$, $w_X/q_X \geq 1$, and $w_2/q_2 \geq 1$. Therefore,

$$w_1 + w_X + w_2 \geq q_1 + q_X + q_2 > 1,$$

which is a contradiction. We conclude that there are no arbitrage opportunities. The key to arriving at this conclusion is, of course, that the sum of the reciprocal odds is greater than one. The excess $1.078 - 1 = 0.078$ can be interpreted as the margin the bookmaker takes as a profit.

Occasionally, when examining the odds of many different sports betting agents, you may find better odds. If the best available odds happen to be 2.75 on "Chelsea," 3.50 on "Draw," and 2.95 on "Liverpool," then there is an arbitrage opportunity. In the analogy with the digital derivative market, here the sum of the digital derivative prices sum up to a number less than one. Therefore, a portfolio can be formed whose initial value is less than one and whose postgame value is one, from which an arbitrage portfolio can be formed.

1.2.1 The Lognormal Model

Suppose that there exist a risk-free zero-coupon bond with price B_0 that pays the amount 1 at time T and a forward contract on S_T with current forward price G_0. A long position in the bond of size G_0 together with a long position of size one in the forward contract produces a European derivative contract with price $B_0 G_0$ and payoff S_T at time T. Therefore, we are in the setting of Theorem 1.2 [with $m = 1$ and $f_1(s) = s$].

Here we will choose a lognormal distribution for S_T in the representation $B_0 G_0 = B_0 \operatorname{E}_{\mathrm{Q}}[S_T]$ and derive arbitrage-free pricing formulas for European derivatives. Note that S_T has a lognormal distribution if $\log S_T$ has a normal distribution. If we choose μT and $\sigma^2 T$ to be the mean and variance of the normal distribution for $\log S_T$, then we may write $\log S_T = \mu T + \sigma\sqrt{T} Z$ for a standard normally distributed random variable Z. Since

$$G_0 = \mathrm{E_Q}[S_T] = \int_{-\infty}^{\infty} e^{\mu T + \sigma\sqrt{T} z} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz = e^{\mu T + \sigma^2 T/2} \int_{-\infty}^{\infty} \frac{e^{-(z-\sigma\sqrt{T})^2/2}}{\sqrt{2\pi}} dz$$
$$= e^{\mu T + \sigma^2 T/2},$$

we see that $\mu T = \log G_0 - \sigma^2 T/2$ and $\log S_T$ is $\mathrm{N}(\log G_0 - \sigma^2 T/2, \sigma^2 T)$-distributed. In particular, we may write

$$S_T = G_0 e^{\sigma\sqrt{T} Z - \sigma^2 T/2}$$

with Z standard normally distributed, and therefore the price of a derivative on S_T with payoff $g(S_T)$ may be expressed as

$$\pi_g = B_0 \,\mathrm{E_Q}[g(S_T)] = B_0 \int_{-\infty}^{\infty} g\Big(G_0 e^{\sigma\sqrt{T} z - \sigma^2 T/2}\Big) \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz. \qquad (1.5)$$

This representation of the derivative price is known as Black's formula (Fisher Black). For call (and put) options, Black's formula turns into a very nice explicit expression. The price C_0 of a call option on S_T with strike price K can be expressed as

$$\begin{aligned} C_0 &= B_0 \,\mathrm{E_Q}[\max(S_T - K, 0)] \\ &= B_0 \,\mathrm{E_Q}[(S_T - K) I\{S_T > K\}] \\ &= B_0 \,\mathrm{E_Q}[(G_0 e^{-\sigma^2 T/2 + \sigma\sqrt{T} Z} - K) I\{Z > \gamma\}] \\ &= B_0 G_0 e^{-\sigma^2 T/2} \,\mathrm{E_Q}[e^{\sigma\sqrt{T} Z} I\{Z > \gamma\}] - K B_0 \,\mathrm{E_Q}[I\{Z > \gamma\}], \end{aligned}$$

where

$$\gamma = \frac{\log(K/G_0)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}.$$

Therefore, with Φ denoting the standard normal distribution function,

$$\begin{aligned} C_0 &= B_0 G_0 e^{-\sigma^2 T/2} \int_{\gamma}^{\infty} e^{\sigma\sqrt{T} z} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz - B_0 K (1 - \Phi(\gamma)) \\ &= B_0 G_0 \int_{\gamma}^{\infty} \frac{e^{-(z-\sigma\sqrt{T})^2/2}}{\sqrt{2\pi}} dz - B_0 K \Phi(-\gamma) \end{aligned}$$

$$= B_0 G_0 \int_{\gamma-\sigma\sqrt{T}}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz - B_0 K \Phi(-\gamma)$$
$$= B_0 G_0 \Phi(\sigma\sqrt{T} - \gamma) - B_0 K \Phi(-\gamma).$$

This expression for the call option price, called Black's formula for call options, is typically written as

$$C_0^{\mathrm{B}} = B_0(G_0 \Phi(d_1) - K \Phi(d_2)), \tag{1.6}$$

where

$$d_1 = \frac{\log(G_0/K)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

If the underlying asset is a pure investment asset (holding the asset brings neither benefits nor costs), then a buyer of the underlying asset at time 0 does not care whether the asset is delivered at that time or at the later time T. This implies that the spot price S_0 for immediate delivery at time 0 must coincide with the derivative price $B_0 G_0$ for delivery of the asset at time T. If the underlying asset is a pure investment asset, then Black's formula for call option prices is called the Black–Scholes, or the Black–Merton–Scholes formula for call option prices, and reads

$$C_0 = S_0 \Phi(d_1) - B_0 K \Phi(d_2), \tag{1.7}$$

where

$$d_1 = \frac{\log(S_0/(B_0 K))}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

If the market provides us with the prices C_0 and G_0, or with C_0, S_0, and B_0 if the underlying asset is a pure investment asset, then the model parameter σ is obtained as the solution to a nonlinear equation in one variable [(1.6) or (1.7)] and is called the implied volatility (implied by the market prices). For a given underlying asset and maturity time, an option price is often quoted in volatility rather than in monetary units. The implied volatilities for two call options on S_T with different strike prices typically do not coincide. Therefore, the lognormal model is inconsistent with price data. However, the very simple lognormal model is still surprisingly accurate and is used as a benchmark model with the modification that the volatility parameter σ is viewed as a function of the strike price K (thereby violating the assumption of the lognormal model). The graph of the function $\sigma(K)$ is called the volatility smile or volatility skew.

Table 1.4 Current prices of options maturing in 35 days

Strike	980	990	1,000	1,020	1,040
Call price	63.625	56.625	50.000	37.625	27.250
Strike	1,060	1,080	1,100	1,120	1,140
Call price	18.500	12.000	7.125	3.825	1.875
Strike	980	990	1,000	1,020	1,040
Put price	23.875	26.875	30.375	38.125	47.625

1.2.2 Implied Forward Probabilities

Consider n call option prices $C_0(K_1), \dots, C_0(K_n)$ on S_T, the forward price G_0 of S_T, and the price B_0 of a zero-coupon bond maturing at time T with face value 1. It is assumed that the set of prices do not give rise to arbitrage opportunities. From Black's formula (1.6) the implied volatilities $\sigma(K_1), \dots, \sigma(K_n)$ are obtained, and by interpolation and extrapolation among the implied volatilities a volatility smile can be created that can be used together with Black's formula to price any European derivative on S_T. For call options, write $C_0(K) = C_0^{\mathrm{B}}(K, \sigma(K))$, where C_0^{B} denotes Black's formula and $\sigma(K)$ is the volatility smile evaluated at K. The produced prices are arbitrage free if and only if there is a probability distribution for S_T so that $C_0(K) = B_0 \,\mathrm{E_Q}[\max(S_T - K, 0)]$ for all K. We may write

$$\begin{aligned} C_0(K) &= B_0 \,\mathrm{E_Q}[\max(S_T - K, 0)] \\ &= B_0 \int_0^\infty \mathrm{Q}(\max(S_T - K, 0) > t)dt \\ &= B_0 \int_K^\infty \mathrm{Q}(S_T > t)dt. \end{aligned}$$

In particular, the prices are arbitrage free if and only if there exists a distribution function Q, the forward probability distribution function, such that

$$\frac{dC_0}{dK}(k) = -B_0(1 - Q(k)) \quad \text{for all } k \geq 0.$$

Moreover, we see that if $C_0(K)$ is twice differentiable, then the prices are arbitrage free if and only if there exists a density function q such that

$$\frac{d^2C_0}{dK^2}(k) = B_0 q(k) \quad \text{for all } k \geq 0.$$

Example 1.9 (Implied volatilities). Consider the option prices specified in Table 1.4. The options were the actively traded European call and put options that day on the value of a stock market index 35 trading days later (7 weeks later). For simplicity, the prices in the table are computed as mid prices; the mid price is

Table 1.5 Zero rates derived from put–call parity

Strike	980	990	1,000	1,020	1,040
Zero rate (%)	0.632	0.626	0.529	0.431	0.509

Table 1.6 Implied volatilities using Black's formula

Strike	980	990	1,000	1,020	1,040
Implied vol.	0.274	0.268	0.263	0.250	0.239
Strike	1,060	1,080	1,100	1,120	1,140
Implied vol.	0.227	0.218	0.208	0.198	0.190

the average of the bid price (the highest price at which a buyer is willing to buy) and the ask price (the lowest price at which a seller is willing to sell). The index level at the time, here called the spot, was $S_0 = 1{,}018.89$.

From the put–call parity in Example 1.6 we see that the put and call prices can be combined to get prices of the derivative that pays one unit of the index at maturity (we ignore commissions and trading costs). The index does not pay dividends, and therefore the spot S_0 equals $B_0 G_0$, where B_0 is the price of a zero-coupon bond that matures at the same time as the options and G_0 is the forward price of the index. Therefore, the put–call parity reads

$$C_0 - P_0 = S_0 - B_0 K.$$

From this relation we can derive B_0 and the zero rate $r = -\log(B_0)/T$, where $T = 35/252$ is the time to maturity (assuming 252 trading days per year). As we have prices on calls and puts for several strikes, each pair will give a possibly different value of r. The extracted zero rates r are presented in Table 1.5. The zero rates are not identical over the range of strikes, but we make a rough approximation and assume the zero rate is equal to 0.5%.

Now we can compute the implied volatilities using Black's formula (1.6). The implied volatilities are presented in Table 1.6. They are also shown in the left-hand plot in Fig. 1.3. The implied volatilities often have a convex looking shape and are therefore often referred to as the volatility smile.

We now turn to the question of how implied volatilities for strikes $K_1, \dots, K_n$ should be used to price a derivative that is not actively traded on a market. For instance, a digital option with payoff $I\{S_T \geq K\}$, where $K_i < K < K_{i+1}$. The arbitrage-free price of the digital option is given by

$$D_0(K) = B_0 \, \mathrm{E_Q}[I\{S_T \geq K\}] = B_0 \, \mathrm{Q}(S_T \geq K) = B_0(1 - Q(K)),$$

where Q is a choice of pricing probability, satisfying the conditions in Theorem 1.2, and Q is the corresponding distribution function for S_T. If we use the lognormal model, then

$$\mathrm{E_Q}[I\{S_T \geq K\}] = \mathrm{Q}(S_T \geq K) = \Phi(d_2),$$

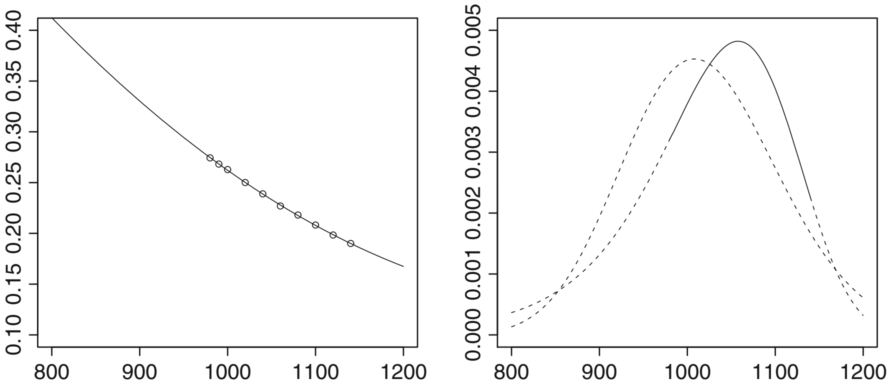

Fig. 1.3 *Left plot*: implied volatilities and graph of fitted second-degree polynomial. The strike price is on the x-axis, and volatility on the y-axis. *Right plot*: graph of implied forward density corresponding to fitted volatility smile, drawn by a *solid curve* within the range of strikes and by a *dashed curve* outside the range of the strikes. The *dashed curve* shows the graph of the density corresponding to the lognormal model with the volatility parameter chosen as the average of the implied volatilities

where

$$d_2 = \frac{\log(G_0/K)}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2},$$

but it is far from clear what volatility σ we should use.

A common practice is to use Black's model together with a suitable implied volatility smile $\sigma(k)$ and express the price of a call option with an arbitrary strike price k as $C_0(k) = C_0^{\mathrm{B}}(k, \sigma(k))$. Recall that these prices are arbitrage free if there exists a forward distribution function Q such that

$$Q(k) = 1 + \frac{1}{B_0}\frac{dC_0}{dK}(k). \tag{1.8}$$

If Black's model together, with a suitable implied volatility smile $\sigma(k)$, is used, then $C_0(k) = C_0^{\mathrm{B}}(k, \sigma(k))$ and

$$\begin{aligned}\frac{dC_0}{dK}(k) &= \frac{\partial C_0^{\mathrm{B}}}{\partial K}(k, \sigma(k)) + \frac{\partial C_0^{\mathrm{B}}}{\partial \sigma}(k, \sigma(k))\frac{d\sigma}{dK}(k)\\ &= -B_0\Phi(d_2) + B_0 G_0 \phi(d_1)\sqrt{T}\frac{d\sigma}{dK}(k).\end{aligned} \tag{1.9}$$

The last equality is not obvious and requires an explanation. Recall that the standard normal density is given by $\phi(z) = \exp\{-z^2/2\}/\sqrt{2\pi}$, and notice that

$$d_1^2 = \Big(\frac{\log(G_0/K)}{\sigma\sqrt{T}}\Big)^2 + \frac{\sigma^2 T}{4} + \log(G_0/K),$$

$$(d_1 - \sigma\sqrt{T})^2 = \Big(\frac{\log(G_0/K)}{\sigma\sqrt{T}}\Big)^2 + \frac{\sigma^2 T}{4} - \log(G_0/K)$$

$$= d_1^2 - 2\log(G_0/K).$$

Therefore,

$$\frac{\partial C_0^{\mathrm{B}}}{\partial K} = -B_0\Phi(d_2) + B_0\left(G_0\phi(d_1)\frac{\partial d_1}{\partial K} - K\phi\left(d_1 - \sigma\sqrt{T}\right)\frac{\partial d_2}{\partial K}\right)$$

$$= -B_0\Phi(d_2)$$

since $\partial d_1/\partial K = \partial d_2/\partial K$, and

$$\frac{\partial C_0^{\mathrm{B}}}{\partial \sigma} = B_0\left(G_0\phi(d_1)\frac{\partial d_1}{\partial \sigma} - K\phi\left(d_1 - \sigma\sqrt{T}\right)\frac{\partial d_2}{\partial \sigma}\right)$$

$$= B_0 G_0\phi(d_1)\sqrt{T}$$

since $\partial d_1/\partial\sigma = \partial d_2/\partial\sigma + \sqrt{T}$.

With Black's model with a volatility smile $\sigma(k)$ the price of the digital option with payoff $I\{S_T \geq K\}$ follows from (1.8) and (1.9) and is given by

$$D_0(K) = B_0(1 - Q(K)) = B_0\Phi(d_2) - B_0 G_0\phi(d_1)\sqrt{T}\frac{d\sigma}{dK}(K).$$

Notice that the expression in (1.9) must be nondecreasing in k (recall also that d_1 and d_2 depend on k) and takes values in the interval $[-B_0, 0]$ for the function Q in (1.8) to be a distribution function. In particular, there are conditions that a volatility smile $\sigma(k)$ must satisfy to give rise to arbitrage-free derivative prices.

A natural approach to constructing a volatility smile is to use some interpolation method to interpolate between the implied volatilities. Linear interpolation is one choice, and then $\sigma(k)$, for $k \in [K_i, K_{i+1}]$, is given by

$$\sigma(k) = \sigma(K_i) + \frac{\sigma(K_{i+1}) - \sigma(K_i)}{K_{i+1} - K_i}(k - K_i).$$

However, linear interpolation may lead to a model that admits arbitrage. We now show this claim. For this model to be free of arbitrage it is necessary that the slope of $\sigma(k)$ be nondecreasing, i.e., that the linearly interpolated volatility smile $\sigma(k)$ be a convex function. Suppose, on the contrary, that the slope between K_{i-1} and K_i is larger than the slope between K_i and K_{i+1}. In mathematical terms, suppose that

$$\text{slope}_{i+1} = \frac{\sigma(K_{i+1}) - \sigma(K_i)}{K_{i+1} - K_i} < \frac{\sigma(K_i) - \sigma(K_{i-1})}{K_i - K_{i-1}} = \text{slope}_i .$$

As a consequence,

$$\begin{aligned}
\lim_{k \uparrow K_i} Q(k) &= 1 + \frac{1}{B_0} \lim_{k \uparrow K_i} \frac{dC_0}{dK}(k) \\
&= 1 + \frac{1}{B_0} \lim_{k \uparrow K_i} \left(\frac{\partial C_0^{\mathrm{B}}}{\partial K}(k, \sigma(k)) + \frac{\partial C_0^{\mathrm{B}}}{\partial \sigma}(k, \sigma(k)) \frac{d\sigma}{dk}(k) \right) \\
&= 1 + \frac{1}{B_0} \left(\frac{\partial C_0^{\mathrm{B}}}{\partial K}(K_i, \sigma(K_i)) + \frac{\partial C_0^{\mathrm{B}}}{\partial \sigma}(K_i, \sigma(K_i)) \text{slope}_i \right) \\
&> 1 + \frac{1}{B_0} \left(\frac{\partial C_0^{\mathrm{B}}}{\partial K}(K_i, \sigma(K_i)) + \frac{\partial C_0^{\mathrm{B}}}{\partial \sigma}(K_i, \sigma(K_i)) \text{slope}_{i+1} \right) \\
&= 1 + \frac{1}{B_0} \lim_{k \downarrow K_i} \left(\frac{\partial C_0^{\mathrm{B}}}{\partial K}(k, \sigma(k)) + \frac{\partial C_0^{\mathrm{B}}}{\partial \sigma}(k, \sigma(k)) \frac{d\sigma}{dk}(k) \right) \\
&= \lim_{k \downarrow K_i} Q(k).
\end{aligned}$$

We find that the function Q has a negative jump at K_i, and therefore it cannot be a distribution function. We conclude that it is necessary that $\text{slope}_{i+1} \geq \text{slope}_i$, i.e., that

$$\frac{\sigma(K_{i+1}) - \sigma(K_i)}{K_{i+1} - K_i} \geq \frac{\sigma(K_i) - \sigma(K_{i-1})}{K_i - K_{i-1}}$$

for a pricing model with linearly interpolated implied volatilities to be free of arbitrage.

An alternative to linear interpolation between implied volatilities, although still rather ad hoc, is to fit a second-degree polynomial $\sigma(k) = c_0 + c_1 k + c_2 k^2$ to the implied volatilities using least squares. The least-squares-fitted volatilities $\sigma(K_i)$ will not coincide with the original implied volatilities. However, typically the second degree polynomial gives a good enough fit so that the resulting model prices for the call and put options lie between the observed bid and ask prices.

Let us illustrate the procedure on the option data in Example 1.9. The resulting second-degree polynomial and the implied volatilities are shown in the left-hand plot in Fig. 1.3. Here we observe a very good fit to the implied volatilities. Note that in the left-hand plot in Fig. 1.3 the graph of the function $\sigma(k)$ is plotted also outside the range of the strikes. However, to the left of the smallest strike (980) and to the right of the highest strike (1,140) we do not have information on what the volatility smile looks like. Extrapolating outside the range of the data is nothing but a crude guess.

The resulting implied forward distribution Q can now be computed from (1.8), and the corresponding implied density q is given by

$$
\begin{aligned}
q(k) &= \frac{1}{B_0}\frac{d^2}{dk^2}C_0^{\mathrm{B}}(k,\sigma(k)) \\
&= \frac{1}{B_0}\left(\frac{\partial^2 C_0^{\mathrm{B}}}{\partial K^2}(k,\sigma(k)) + 2\frac{\partial^2 C_0^{\mathrm{B}}}{\partial K\partial\sigma}(k,\sigma(k))\frac{d\sigma}{dK}(k)\right. \\
&\qquad \left. + \frac{\partial C_0^{\mathrm{B}}}{\partial\sigma}(k,\sigma(k))\frac{d^2\sigma}{dK^2}(k) + \frac{\partial^2 C_0^{\mathrm{B}}}{\partial\sigma^2}(k,\sigma(k))\left(\frac{d\sigma}{dK}(k)\right)^2\right).
\end{aligned}
$$

Computing the second-order partial derivatives of $C_0^{\mathrm{B}}(K,\sigma)$ from Black's formula is straightforward—but tedious. Therefore, we simply state them and leave it to the reader as an exercise to verify them. The second-order derivatives are given by

$$
\begin{aligned}
\frac{\partial^2 C_0^{\mathrm{B}}}{\partial K^2}(K,\sigma) &= \frac{B_0}{K\sigma\sqrt{T}}\phi(-d_2), \\
\frac{\partial^2 C_0^{\mathrm{B}}}{\partial\sigma\partial K}(K,\sigma) &= \frac{B_0 G_0}{K\sigma}d_1\phi(d_1), \\
\frac{\partial^2 C_0^{\mathrm{B}}}{\partial\sigma^2}(K,\sigma) &= \frac{B_0 G_0\sqrt{T}}{\sigma}d_1 d_2\phi(d_1).
\end{aligned}
$$

The resulting density $q(k)$ is shown in the right-hand plot in Fig. 1.3. The solid part of the curve indicates the range between the smallest and largest strikes (the interval on which we have information from the price data). The dashed part of the curve is the extrapolation outside the range of the strikes of the option data. We compare the density implied by the volatility smile and Black's call option price formula to the density for the lognormal model with the volatility parameter chosen as the average of the implied volatilities. This lognormal density is shown as the dashed curve in the plot to the right in Fig. 1.3. We observe that the effect of the volatility smile, compared to a constant volatility, is that the implied forward density is left-skewed and has more probability mass in the left tail.

1.3 Notes and Comments

To prove the no-arbitrage theorem for deterministic cash flows, Theorem 1.1, we used a proof we learned from Harald Lang. The same idea was also used to prove Theorem 1.2 under the assumption that the spot price S_T takes values in a finite set. Theorem 1.2 is called the First Fundamental Theorem of Asset Pricing and appears here in its simplest form. A more general version of the theorem, without the assumption of a finite set of possible outcomes for S_T and with multiple time periods instead of one, was proved by Robert Dalang, Andrew Morton, and Walter Willinger in [10]. Since then, many alternative proofs of their theorem have appeared.

The material in Sect. 1.2 on no-arbitrage pricing and the lognormal model is a very brief summary of selected topics from the enormous amount of literature written on no-arbitrage pricing of derivatives contracts since the seminal work in [6, 32], and [5] of Fisher Black, Myron Scholes, and Robert Merton in the early 1970s. A natural motivation for the use of lognormal models can be found in the work [42] of Paul Samuelson, which predates that of Fisher Black, Myron Scholes, and Robert Merton.

The reader who seeks more information about financial markets and contracts, including the topics presented here, is recommended to consult the popular textbooks of John Hull, for instance [21].

1.4 Exercises

In the exercises below, it is assumed, wherever applicable, that you can take positions corresponding to fractions of assets.

Exercise 1.1 (Arbitrage in bond prices). (a) Consider a market consisting of the five risk-free bonds shown in Table 1.7. Show that the market is free of arbitrage and determine the zero rates, or construct an arbitrage portfolio.

(b) Consider a market consisting of the three bonds denoted A, D, and E in Table 1.7. Show that the market is free of arbitrage and use the bootstrapping procedure to determine the zero rates, or construct an arbitrage portfolio.

Exercise 1.2 (Put–call parity). (a) Consider a European derivative contract, called a collar, with payoff function f given by

$$f(x) = \begin{cases} K_1 & \text{if } x < K_1, \\ x & \text{if } x \in [K_1, K_2], \\ K_2 & \text{if } x > K_2, \end{cases}$$

where $K_1 < K_2$. Express the forward price of a collar in terms of the forward prices of appropriate European call and put options and the forward price of the underlying asset.

Table 1.7 Bond specifications

Bond	A	B	C	D	E
Bond price ($)	98.51	100.71	188.03	111.55	198.96
Maturity (years)	0.5	1	1.5	1.5	2
Annual coupon ($)	0	4	0	12	8
Face value ($)	100	100	200	100	200

Half of the annual coupon is paid every 6 months from today and including the time of maturity; the first coupon payment is in 6 months

Table 1.8 Odds offered by seven bookmakers

Bookmaker	1	2	3	4	5	6	7
Everton	4.30	4.55	4.35	4.30	4.55	4.60	4.70
Draw	3.50	3.55	3.35	3.70	3.30	3.45	3.55
Manchester City	1.85	1.80	1.95	1.80	1.85	1.85	1.75

Table 1.9 Bond specifications

Bond	A	B	C	D
Bond price ($)	98	104	93	98
Maturity (years)	1	2	1	2
Annual coupon ($)	0	5	0	10
Face value ($)	100	100	100	100

The annual coupon is paid every 12 months starting from today and including the time of maturity. The first coupon payment is in 12 months

(b) A risk reversal is a position made up of a long position in an out-of-the-money (worthless if it were to expire today) European call option and a short position of the same size in an out-of-the-money European put option; both options have the same maturity and are written on the same underlying asset. Express the forward price of a risk reversal in terms of the forward prices of the underlying asset and a collar on this asset.

Exercise 1.3 (Sports betting). Consider the odds shown in Table 1.8 of seven bookmakers on the outcome of a football game between Everton and Manchester City. Is it possible to create an arbitrage opportunity by making bets corresponding to long positions?

Exercise 1.4 (Lognormal model). (a) Let Z have a standard normal distribution. For any $a > 0$ and $b \in \mathbb{R}$ compute $\mathrm{E}[e^{aZ} I\{Z > b\}]$.

(b) Let R have the lognormal distribution $\mathrm{LN}(\mu, \sigma^2)$. For an arbitrary number c, compute

$$\mathrm{E}[(R-c)_+],\ \mathrm{Var}((R-c)_+), \mathrm{Cov}(R, (R-c)_+), \mathrm{Cov}((R-c)_+, (R-d)_+),$$

where $d > c$.

(c) What happens to the preceding quantities if R is replaced by $S = S_0 R$ for a constant $S_0 > 0$?

Exercise 1.5 (Risky bonds). Consider investments in long positions in the four bonds shown in Table 1.9. Bonds A and B are issued by the United States government, and their cash flows are considered risk free. Bonds C and D are issued by a bank in the USA experiencing serious financial difficulties. If the bank were to default, bonds C and D would be worthless.

(a) Determine the current 1- and 2-year US Treasury zero rates and the 1- and 2-year credit spreads for the bonds issued by the bank. Determine the probabilities of default within 1 and within 2 years implied by the bond prices.
(b) An investor is certain that the bank is considered by the government to be too important to the financial system to be allowed to default. However, the investor believes that the market will continue to believe that the bank may default on its bonds. The investor believes that the 1-year Treasury zero rate in 1 year is $\mathrm{N}(0.03, 0.01^2)$-distributed and that the 1-year credit spread in 1 year is $\mathrm{N}(0.13, 0.03^2)$-distributed. Determine the investment strategy in terms of \$10,000 invested in long positions in the bonds and a strategy for how to reinvest any cash flow received in 1 year that maximizes the expected value of the cash flow in 2 years.
(c) Suppose that the investor is wrong and the market's view, corresponding to the current bond prices, is right. Determine the distribution function of the investor's (perceived) optimal cash flow in 2 years in (b).

Project 1 (Implied forward distribution). Find prices of traded European put and call options of the future value of a stock market index. Consider prices of the options that are traded in sufficiently large volumes so that the prices contain relevant information about the future index value.

Use the material in Sect. 1.2.2 to estimate the density function of the future index value implied by the option prices. Make the plots correspond to Fig. 1.3. Make sure that the method used for interpolation and extrapolation of the implied Black's model volatilities does not lead to arbitrage in the pricing model you suggest.