7

Problems in Diophantine Approximation

7.1 Introduction

In this chapter, we examine three partial manuscripts on Diophantine approximation found in [269]. All are untitled and in rough form.

The first partial manuscript is on pages 262–265. At the top of page 262 are two appended notes. The first, possibly in the handwriting of G.H. Hardy's former research student, Gertrude Stanley, reads (in part) "Paper a little difficult to understand after the first page." The second, definitely in the handwriting of Hardy, surmises "Odd problem. I don't profess to know whether there's much to it."

On these four pages, Ramanujan considers the problem of finding the maximum value of a certain polynomial when the variable x is a rational number with prescribed denominator. We do not know what motivated Ramanujan to consider this particular problem, and it is natural to ask whether Ramanujan's analysis can be extended to other algebraic numbers. Probably, this is the case, but it appears to be complicated to state and prove a general theorem. Although this problem is outside the scope of contemporary research in Diophantine approximation, because only elementary number theory and elementary calculus are involved, we hope that readers will find Ramanujan's problem and its analysis to be appealing. We have decided that it would be unwise to dwell on every inaccuracy or vague statement in Ramanujan's manuscript. We emphasize that the principal ideas are due to Ramanujan, but that it took considerable effort to interpret and make them precise. The proofs are substantially due S. Kim and the second author [56].

The second manuscript is on pages 266 and 267 of [269]. This short manuscript is more precisely and clearly written. Ramanujan considers the Diophantine approximation of the exponential function $e^{2/a}$, where a is a nonzero integer. Remarkably, he obtains the best possible Diophantine approximation to $e^{2/a}$, a result that was first established in the literature by C.S. Davis [102] in 1978, probably about 60 years after Ramanujan had proved it. Our account of this manuscript is taken from a paper [61] that

G.E. Andrews and B.C. Berndt, *Ramanujan's Lost Notebook: Part IV*, DOI 10.1007/978-1-4614-4081-9_7,

Berndt coauthored with S. Kim and A. Zaharescu. This paper contains further results. In particular, the authors examine how often the convergents to the (simple) continued fraction of e coincide with partial sums of e. Moreover, they prove a conjecture of J. Sondow [292] asserting that only two partial sums of the Maclaurin series for e coincide with partial quotients of the simple continued fraction of e.

We have been unable to provide meaning to the third manuscript, which is on page 343. Its claims are wrong, and so it remains a challenge to determine whether something meaningful can be ascertained.

7.2 The First Manuscript

7.2.1 An Unusual Diophantine Problem

We begin by quoting Ramanujan at the beginning of his manuscript.

> Let us consider the maximum of
>
> $$\epsilon_m(1-\epsilon_m)(1-2\epsilon_m) \tag{7.2.1}$$
>
> when ϵ_m is a positive proper fraction and m and $m\epsilon_m$ are positive integers. Let v_m be the maximum of (7.2.1). If we do not assume that $m\epsilon_m$ is rational, we get that
>
> $$\epsilon_m = \frac{3-\sqrt{3}}{6}, \qquad v_m = \frac{1}{6\sqrt{3}}. \tag{7.2.2}$$

Here, as a *positive proper fraction*, Ramanujan intends ϵ_m to be a rational number (not necessarily in lowest terms) with denominator m. If

$$f(x) := x(1-x)(1-2x) = x - 3x^2 + 2x^3, \tag{7.2.3}$$

then it is easily seen that $x = (3-\sqrt{3})/6$ yields a local maximum of $f(x)$. Ramanujan desires to find the maximum value v_m of (7.2.3) when approximating $(3-\sqrt{3})/6$ by a rational number ϵ_m with denominator equal to m. He then claims that ϵ_m is either

$$g_m(\epsilon) := \frac{m\cdot\left(\dfrac{3-\sqrt{3}}{6}\right)-\epsilon}{m} \qquad \text{or} \qquad g_m(\epsilon-1) = \frac{m\cdot\left(\dfrac{3-\sqrt{3}}{6}\right)+1-\epsilon}{m}. \tag{7.2.4}$$

Here, we can see that ϵ is completely determined by m. We can assume that $0 < \epsilon < 1$, so that the two values in (7.2.4) give the two best rational approximations to $(3-\sqrt{3})/6$ with denominator m. In the first instance of (7.2.4), the approximation is from below, while in the second instance, the approximation is from above. Ramanujan then claims the following.

Proposition 7.2.1. *If*

$$\epsilon_m = \frac{m \cdot \left(\frac{3-\sqrt{3}}{6}\right) - \epsilon}{m}, \quad \textit{then} \quad v_m = \frac{1}{6\sqrt{3}} - \frac{\epsilon^2}{m^2}\sqrt{3} - 2\frac{\epsilon^3}{m^3}, \tag{7.2.5}$$

and if

$$\epsilon_m = \frac{m \cdot \left(\frac{3-\sqrt{3}}{6}\right) + 1 - \epsilon}{m}, \quad \textit{then} \quad v_m = \frac{1}{6\sqrt{3}} - \frac{(1-\epsilon)^2}{m^2}\sqrt{3} + 2\frac{(1-\epsilon)^3}{m^3}. \tag{7.2.6}$$

Proof. With the use of (7.2.3), both of these calculations are straightforward. □

We note that by replacing ϵ by $\epsilon - 1$ in the value of v_m in (7.2.5), we obtain the value of v_m in (7.2.6).

Proposition 7.2.2.

$$\textit{If} \quad \epsilon < \frac{1}{2} - \frac{m - \sqrt{m^2-1}}{2\sqrt{3}}, \quad \text{then } v_m \text{ in (7.2.5) is greater;} \tag{7.2.7}$$

$$\textit{if} \quad \epsilon > \frac{1}{2} - \frac{m - \sqrt{m^2-1}}{2\sqrt{3}}, \quad \text{then } v_m \text{ in (7.2.6) is greater;} \tag{7.2.8}$$

and

$$\textit{if} \ \epsilon = \frac{1}{2} - \frac{m - \sqrt{m^2-1}}{2\sqrt{3}}, \text{ then the values of } v_m \text{ in (7.2.5) and (7.2.6) are identical.} \tag{7.2.9}$$

Proof. An elementary calculation shows that

$$\frac{1}{6\sqrt{3}} - \frac{\epsilon^2}{m^2}\sqrt{3} - 2\frac{\epsilon^3}{m^3} > \frac{1}{6\sqrt{3}} - \frac{(1-\epsilon)^2}{m^2}\sqrt{3} + 2\frac{(1-\epsilon)^3}{m^3} \tag{7.2.10}$$

if and only if

$$6\epsilon^2 + (2m\sqrt{3} - 6)\epsilon + 2 - m\sqrt{3} < 0. \tag{7.2.11}$$

It is easily checked that the roots of $6\epsilon^2 + (2m\sqrt{3} - 6)\epsilon + 2 - m\sqrt{3} = 0$ are

$$r_1, r_2 := \frac{1}{2} + \frac{-m \pm \sqrt{m^2-1}}{2\sqrt{3}}, \qquad \text{with } r_2 < r_1.$$

Thus, (7.2.10) is true if and only if $r_2 < \epsilon < r_1$. Since the root that we seek is r_1, we see that the statements in Proposition 7.2.2 follow. □

Now, if

$$0 < \epsilon < \frac{1}{2} - \frac{m - \sqrt{m^2 - 1}}{2\sqrt{3}},$$

then

$$\frac{m-1}{2} - \sqrt{\frac{m^2-1}{12}} < m\epsilon_m < m\frac{3-\sqrt{3}}{6} < \frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}.$$

Also, if

$$\frac{1}{2} - \frac{m - \sqrt{m^2 - 1}}{2\sqrt{3}} < \epsilon < 1,$$

then

$$\frac{m-1}{2} - \sqrt{\frac{m^2-1}{12}} < m\frac{3-\sqrt{3}}{6} < m\epsilon_m < \frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}.$$

Thus, if

$$\epsilon \neq \frac{1}{2} - \frac{m - \sqrt{m^2 - 1}}{2\sqrt{3}},$$

we conclude that the maximum v_m occurs when

$$\epsilon_m = \frac{1}{m}\left[\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right]. \tag{7.2.12}$$

We also note that for those values of m for which

$$\epsilon = \frac{1}{2} - \frac{m - \sqrt{m^2 - 1}}{2\sqrt{3}}, \tag{7.2.13}$$

by (7.2.9), we can choose either expression from (7.2.4) for ϵ_m. Thus,

$$\epsilon_m = \frac{1}{m}\left(\frac{m-1}{2} - \sqrt{\frac{m^2-1}{12}}\right) \quad \text{or} \quad \frac{1}{m}\left(\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right). \tag{7.2.14}$$

We remark that by (7.2.4) and (7.2.13), we do not need greatest integer functions in (7.2.14). Hence, we have established the following proposition.

Proposition 7.2.3. *The formula for* ϵ_m *in* (7.2.12) *is valid for all values of* m, *and in the case of* (7.2.13), ϵ_m *can be determined by the alternative choices in* (7.2.14).

In conclusion, we use (7.2.12) to calculate ϵ_m. We then return to (7.2.3) to determine v_m.

In Table 7.1, we list the values of ϵ_m for each m, $1 \leq m \leq 10$, which were obtained from (7.2.12) or (7.2.14). We also add the corresponding values of ϵ in the table.

Ramanujan next discusses the *minimum order* and *maximum order* of v_m. He does not define these concepts, but in different words we relate what we think he intended.

m	ϵ_m	Value of ϵ	v_m
1	0, 1	$\frac{1}{2} - \frac{1}{2\sqrt{3}}$	0
2	$0, \frac{1}{2}$	$\frac{3-\sqrt{3}}{3}$	0
3	$\frac{1}{3}$	$\frac{3-\sqrt{3}}{2}$	$\frac{2}{27}$
4	$\frac{1}{4}$	$2 - \frac{2\sqrt{3}}{3}$	$\frac{3}{32}$
5	$\frac{1}{5}$	$\frac{3}{2} - \frac{5\sqrt{3}}{6}$	$\frac{12}{125}$
6	$\frac{1}{6}$	$2-\sqrt{3}$	$\frac{20}{6^3}$
7	$\frac{1}{7}, \frac{2}{7}$	$\frac{5}{2} - \frac{7\sqrt{3}}{6}$	$\frac{30}{7^3}$
8	$\frac{1}{4}$	$3 - \frac{4\sqrt{3}}{3}$	$\frac{3}{32}$
9	$\frac{2}{9}$	$\frac{7}{2} - \frac{3\sqrt{3}}{2}$	$\frac{70}{3^6}$
10	$\frac{1}{5}$	$3 - \frac{5\sqrt{3}}{3}$	$\frac{12}{5^3}$

Table 7.1. Table of values for v_m, $1 \le m \le 10$

Proposition 7.2.4. *For all values of* m,

$$v_m \ge \frac{m^2-4}{6m^3}\sqrt{\frac{m^2-1}{3}}, \tag{7.2.15}$$

with equality holding when

$$\epsilon = \frac{1}{2} - \frac{m-\sqrt{m^2-1}}{2\sqrt{3}}, \tag{7.2.16}$$

and the corresponding value of ϵ_m *is given by*

$$\epsilon_m = \frac{1}{m}\left(\frac{m-1}{2} - \sqrt{\frac{m^2-1}{12}}\right) \quad \text{or} \quad \frac{1}{m}\left(\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right). \tag{7.2.17}$$

Proof. From (7.2.12), we have

$$\frac{1}{m}\left(\frac{m-1}{2} - \sqrt{\frac{m^2-1}{12}}\right) \le \epsilon_m \le \frac{1}{m}\left(\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right).$$

If the maximum v_m occurs at $\epsilon_m \le (3-\sqrt{3})/6$, then

$$v_m \ge f\left(\frac{1}{m}\left(\frac{m-1}{2} - \sqrt{\frac{m^2-1}{12}}\right)\right) = \frac{m^2-4}{6m^3}\sqrt{\frac{m^2-1}{3}},$$

since $f(x)$ is increasing when $x \le (3-\sqrt{3})/6$. On the other hand, $f(x)$ is decreasing when $(3-\sqrt{3})/6 \le x \le 1$. Thus, if the maximum v_m occurs at $\epsilon_m \ge (3-\sqrt{3})/6$, then

$$v_m \ge f\left(\frac{1}{m}\left(\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right)\right) = \frac{m^2-4}{6m^3}\sqrt{\frac{m^2-1}{3}},$$

which completes the proof. □

The previous proposition gives a lower bound for v_m. The next two propositions give upper bounds, with Proposition 7.2.5 due to Ramanujan; Proposition 7.2.6 was not given by Ramanujan in his partial manuscript.

Proposition 7.2.5. *If* $\epsilon_m = g(\epsilon)$, *then*

$$v_m \le \frac{m^2-1}{6m^3}\sqrt{\frac{m^2+2}{3}}, \tag{7.2.18}$$

with equality holding above when

$$\epsilon_m = \frac{1}{m}\left(\frac{m}{2} - \sqrt{\frac{m^2+2}{12}}\right). \tag{7.2.19}$$

Proof. We first note that

$$m\frac{3-\sqrt{3}}{6} = \frac{m}{2} - \sqrt{\frac{m^2}{12}} \quad \text{and} \quad \frac{m}{2} - \sqrt{\frac{m^2+1}{12}}$$

cannot be integers, whereas

$$\frac{m}{2} - \sqrt{\frac{m^2+2}{12}} \tag{7.2.20}$$

is an integer for $m = 1, 5, 19, \dots$. Also, it can easily be verified that

$$\left[\frac{m}{2} - \sqrt{\frac{m^2}{12}}\right] = \left[\frac{m}{2} - \sqrt{\frac{m^2+1}{12}}\right] = \left[\frac{m}{2} - \sqrt{\frac{m^2+2}{12}}\right] \le \frac{m}{2} - \sqrt{\frac{m^2+2}{12}}.$$

Thus, we obtain

$$v_m \le f\left(\frac{1}{m}\left(\frac{m}{2} - \sqrt{\frac{m^2+2}{12}}\right)\right) = \frac{m^2-1}{6m^3}\sqrt{\frac{m^2+2}{3}}.$$

□

Proposition 7.2.6. *If* $\epsilon_m = g(\epsilon - 1)$, *then*

$$v_m \leq \frac{m^2+2}{6m^3}\sqrt{\frac{m^2-4}{3}}, \tag{7.2.21}$$

with equality holding when

$$\epsilon_m = \frac{1}{m}\left(\frac{m}{2} - \sqrt{\frac{m^2-4}{12}}\right). \tag{7.2.22}$$

Proof. First, it can be easily verified that for $0 \leq i \leq 3$,

$$\frac{m}{2} - \sqrt{\frac{m^2-i}{12}}$$

does not take any integral values. So, we have

$$\left\lceil \frac{m}{2} - \sqrt{\frac{m^2}{12}} \right\rceil = \left\lceil \frac{m}{2} - \sqrt{\frac{m^2-4}{12}} \right\rceil \geq \frac{m}{2} - \sqrt{\frac{m^2-4}{12}}.$$

Thus, we obtain

$$v_m \leq f\left(\frac{1}{m}\left(\frac{m}{2} - \sqrt{\frac{m^2-4}{12}}\right)\right) = \frac{m^2+2}{6m^3}\sqrt{\frac{m^2-4}{3}}.$$

□

This concludes the first section of Ramanujan's partial manuscript.

7.2.2 The Periodicity of v_m

In the second and last section of his draft, Ramanujan considers the periodicity of v_m. To motivate the remainder of our paper, we move his table from the end of the manuscript to the beginning of this section (see Table 7.2).

We observe that there exist sequences of values that are periodic, e.g.,

$$v_5 = v_{10} = v_{15} = v_{20} = v_{25} = v_{30} = v_{35} = v_{40}. \tag{7.2.23}$$

Ramanujan then seeks to determine the maximum value of k such that

$$v_m = v_{2m} = v_{3m} = \cdots = v_{km}. \tag{7.2.24}$$

Theorem 7.2.1. *As in* (7.2.19), *consider only those values of* m *for which*

$$\epsilon_m = \frac{1}{m}\left(\frac{m}{2} - \sqrt{\frac{m^2+2}{12}}\right) \tag{7.2.25}$$

$v_1 = 0$	$v_{26} = 0.0955849$
$v_2 = 0$	$v_{27} = 0.0960219$
$v_3 = 0.0740741$	$v_{28} = 0.0962099$
$v_4 = 0.0937500$	$v_{29} = 0.0961909$
$v_5 = 0.0960000$	$v_{30} = 0.0960000$
$v_6 = 0.0925926$	$v_{31} = 0.0958679$
$v_7 = 0.0874436$	$v_{32} = 0.0961304$
$v_8 = 0.0937500$	$v_{33} = 0.0962239$
$v_9 = 0.0960219$	$v_{34} = 0.0961734$
$v_{10} = 0.0960000$	$v_{35} = 0.0960000$
$v_{11} = 0.0946657$	$v_{36} = 0.0960219$
$v_{12} = 0.0937500$	$v_{37} = 0.0961838$
$v_{13} = 0.0955849$	$v_{38} = 0.0962239$
$v_{14} = 0.0962099$	$v_{39} = 0.0961581$
$v_{15} = 0.0960000$	$v_{40} = 0.0960000$
$v_{16} = 0.0952148$	$v_{41} = 0.0961100$
$v_{17} = 0.0952575$	$v_{42} = 0.0962099$
$v_{18} = 0.0960219$	$v_{43} = 0.0962179$
$v_{19} = 0.0962239$	$v_{44} = 0.0961448$
$v_{20} = 0.0960000$	$v_{45} = 0.0960219$
$v_{21} = 0.0954541$	$v_{46} = 0.0961617$
$v_{22} = 0.0957926$	$v_{47} = 0.0962215$
$v_{23} = 0.0961617$	$v_{48} = 0.0962095$
$v_{24} = 0.0962095$	$v_{49} = 0.0961334$
$v_{25} = 0.0960000$	$v_{50} = 0.0960960$

Table 7.2. Table of values for v_m, $1 \leq m \leq 50$

is a rational number. Let k be the maximum value such that (7.2.24) holds. Then

$$k \not> \left[\frac{x}{m}\right] = \sqrt{3m^2+6} - 1, \tag{7.2.26}$$

where x is determined by

$$\frac{1}{x}\left(\frac{x-1}{2}-\sqrt{\frac{x^2-1}{12}}\right)=\frac{1}{m}\left(\frac{m}{2}-\sqrt{\frac{m^2+2}{12}}\right)=\epsilon_m. \tag{7.2.27}$$

Proof. From (7.2.12), recall that for every m

$$\epsilon_m=\frac{1}{m}\left[\frac{m+1}{2}-\sqrt{\frac{m^2-1}{12}}\right],$$

or in terms of the least integer function,

$$\epsilon_m=\frac{1}{m}\left\lceil\frac{m-1}{2}-\sqrt{\frac{m^2-1}{12}}\right\rceil.$$

With these values in mind, we first examine, for $x>1$, the two functions

$$f_1(x):=\frac{1}{x}\left(\frac{x+1}{2}-\sqrt{\frac{x^2-1}{12}}\right)\quad\text{and}\quad f_2(x):=\frac{1}{x}\left(\frac{x-1}{2}-\sqrt{\frac{x^2-1}{12}}\right).$$

An elementary calculation shows that

$$\begin{aligned}f_1'(x)&=-\frac{1}{2x^2}-\frac{1}{12x^2}\left(\frac{x^2-1}{12}\right)^{-1/2}<0,\\ f_2'(x)&=\frac{1}{2x^2}-\frac{1}{12x^2}\left(\frac{x^2-1}{12}\right)^{-1/2}>0,\end{aligned}$$

provided that $x>2/\sqrt{3}$. Thus, $f_1(x)$ is monotonically decreasing and $f_2(x)$ is monotonically increasing for $x>2/\sqrt{3}$. Also, we see that

$$f_2(x)=\frac{1}{2}-\frac{1}{2x}-\sqrt{\frac{1}{12}-\frac{1}{12x^2}}<\frac{3-\sqrt{3}}{6}<f_1(x)=\frac{1}{2}+\frac{1}{2x}-\sqrt{\frac{1}{12}-\frac{1}{12x^2}}. \tag{7.2.28}$$

Now we verify (7.2.26). Suppose that we have the sequence of equal values (7.2.24), which, in turn, implies that

$$\epsilon_m=\epsilon_{2m}=\epsilon_{3m}=\cdots=\epsilon_{km}.$$

Since $\epsilon_m=\epsilon_{km}$, by (7.2.12) and (7.2.27),

$$\begin{aligned}\frac{1}{x}\left(\frac{x-1}{2}-\sqrt{\frac{x^2-1}{12}}\right)&=\frac{1}{km}\left[\frac{km+1}{2}-\sqrt{\frac{k^2m^2-1}{12}}\right]\\&\geq\frac{1}{km}\left(\frac{km-1}{2}-\sqrt{\frac{k^2m^2-1}{12}}\right).\end{aligned}$$

Since $f_2(x)$ is monotonically increasing, it follows that $x\geq km$, which proves the first equality in (7.2.26).

Now we solve (7.2.27). Let

$$\alpha = \frac{1}{m}\left(\frac{m}{2} - \sqrt{\frac{m^2+2}{12}}\right).$$

Then, a straightforward calculation shows that

$$x = \frac{3(1-2\alpha) \pm \sqrt{1+12\alpha-12\alpha^2}}{3(1-2\alpha)^2-1}. \tag{7.2.29}$$

Since $\alpha = \frac{1}{2} - \frac{1}{m}\sqrt{\frac{m^2+2}{12}}$, we easily find that

$$\begin{aligned}
1+12\alpha-12\alpha^2 &= 4-12\Big(\frac{1}{2}-\alpha\Big)^2 = \frac{3m^2-2}{m^2},\\
3(1-2\alpha) &= \frac{\sqrt{3m^2+6}}{m},\\
3(1-2\alpha)^2-1 &= 2-12\alpha+12\alpha^2 = 12\Big(\alpha-\frac{1}{2}\Big)^2-1 = \frac{2}{m^2}.
\end{aligned}$$

Hence, by (7.2.29), we deduce that

$$\frac{x}{m} = \frac{\sqrt{3m^2+6}+\sqrt{3m^2-2}}{2}.$$

However, by (7.2.25), we see that $(m^2+2)/3$ is a perfect square, which is equivalent to $3m^2+6$ being a perfect square. Thus,

$$k \leq \left[\frac{x}{m}\right] = \sqrt{3m^2+6}-1, \tag{7.2.30}$$

which verifies the second equality in (7.2.26). □

In the next result, Ramanujan removes the restriction on (7.2.25) from Theorem 7.2.1 and claims a formula that is valid for *all* m.

Theorem 7.2.2. *Assume that* x *is chosen so that either*

$$\frac{1}{x}\left(\frac{x+1}{2} - \sqrt{\frac{x^2-1}{12}}\right) = \frac{1}{m}\left[\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right] > \frac{3-\sqrt{3}}{6} \tag{7.2.31}$$

or

$$\frac{1}{x}\left(\frac{x-1}{2} - \sqrt{\frac{x^2-1}{12}}\right) = \frac{1}{m}\left[\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right] < \frac{3-\sqrt{3}}{6}. \tag{7.2.32}$$

Then

$$k = \left[\frac{x}{m}\right]. \tag{7.2.33}$$

Moreover, if $3m^2 + 6$ *is a perfect square, then*

$$k = \sqrt{3m^2 + 6} - 1. \tag{7.2.34}$$

Proof. Observe that the last statement in Theorem 7.2.2 follows from (7.2.33) and (7.2.30).

In order to prove (7.2.33), we first need to show that $k \le [x/m]$ for arbitrary m. In the case of (7.2.32), we can use the same argument from the proof of (7.2.26). For the case of (7.2.31), if we assume $\epsilon_m = \epsilon_{2m} = \cdots = \epsilon_{km}$, then we have

$$\begin{aligned} \frac{1}{x}\left(\frac{x+1}{2} - \sqrt{\frac{x^2-1}{12}}\right) &= \frac{1}{km}\left[\frac{km+1}{2} - \sqrt{\frac{k^2m^2-1}{12}}\right] \\ &\le \frac{1}{km}\left(\frac{km+1}{2} - \sqrt{\frac{k^2m^2-1}{12}}\right). \end{aligned}$$

Since $f_1(x)$ is monotonically decreasing, we conclude that $km \le x$, or $k \le [x/m]$.

We now show that for every $1 \le t \le [x/m]$, $\epsilon_m = \epsilon_{tm}$, which proves (7.2.33). We first consider those values of m for which (7.2.31) holds. Since all the rational numbers with denominator tm include the rational numbers with denominator m, we have $v_{tm} \ge v_m$. Since $\epsilon_m > (3 - \sqrt{3})/6$ and the function $f(x) = x(1-x)(1-2x)$ is decreasing on the interval $[(3-\sqrt{3})/6, 1]$, we have $\epsilon_{tm} \le \epsilon_m$. On the other hand, since $f_1(x)$ is decreasing, by (7.2.31),

$$\epsilon_m = \frac{t}{tm}\left[\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right] \le \frac{1}{tm}\left(\frac{tm+1}{2} - \sqrt{\frac{t^2m^2-1}{12}}\right).$$

Thus,

$$t\left[\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right] \le \left[\frac{tm+1}{2} - \sqrt{\frac{t^2m^2-1}{12}}\right],$$

which implies that $\epsilon_m \le \epsilon_{tm}$, upon dividing both sides above by tm. Hence, the inequalities $\epsilon_m \ge \epsilon_{tm}$ and $\epsilon_m \le \epsilon_{tm}$ imply that $\epsilon_m = \epsilon_{tm}$ for all $1 \le t \le [x/m]$.

For those values of m that satisfy (7.2.32), we apply a similar argument. Since $v_{tm} \ge v_m$ and the function $f(x)$ is increasing on the interval $[0, (3-\sqrt{3})/6]$, we have $\epsilon_m \le \epsilon_{tm}$. Since $f_2(x)$ is increasing, by (7.2.32),

$$\epsilon_m = \frac{t}{tm}\left[\frac{m+1}{2} - \sqrt{\frac{m^2-1}{12}}\right] \ge \frac{1}{tm}\left(\frac{tm-1}{2} - \sqrt{\frac{t^2m^2-1}{12}}\right).$$

Thus,

$$t\left[\frac{m+1}{2}-\sqrt{\frac{m^2-1}{12}}\right] \geq \left[\frac{tm-1}{2}-\sqrt{\frac{t^2m^2-1}{12}}\right],$$

which implies that $\epsilon_{tm} \leq \epsilon_m$, upon dividing both sides above by tm. Thus, since we also had observed that $\epsilon_m \leq \epsilon_{tm}$, we conclude that $\epsilon_m = \epsilon_{tm}$, which completes the proof of (7.2.33). □

In summary, if $3m^2+6$ is a perfect square, then we use (7.2.34) to calculate the length k of the period. If $3m^2+6$ is not a perfect square, then we use (7.2.33), with x defined by (7.2.31) or (7.2.32), to calculate the period length k.

If $m=1$, then by (7.2.34), $k=2$. In our initial calculations above, we had observed that $v_1 = v_2 = 0$, but $v_3 \neq 0$, and so Ramanujan's periodic assertion is corroborated in this case. Ramanujan then gives seven periodic sequences corresponding to the values $m = 5, 9, 14, 19, 71, 265, 989$, with periods $8, 5, 12, 32, 122, 458, 1{,}712$, respectively, namely,

$$\begin{aligned}
&v_5 = v_{10} = v_{15} = \cdots = v_{40},\\
&v_9 = v_{18} = v_{27} = \cdots = v_{45},\\
&v_{14} = v_{28} = v_{42} = \cdots = v_{168},\\
&v_{19} = v_{38} = v_{57} = \cdots = v_{608},\\
&v_{71} = v_{142} = v_{213} = \cdots = v_{8{,}662},\\
&v_{265} = v_{530} = v_{795} = \cdots = v_{121{,}370},\\
&v_{989} = v_{1{,}978} = v_{2{,}967} = \cdots = v_{1{,}693{,}168}.
\end{aligned}$$

The first, fourth, fifth, sixth, and seventh sequences arise from (7.2.34), but for the second and third, we must use (7.2.33) and (7.2.31) to determine the values $k=5$ and $k=12$, respectively.

It is interesting to examine how often $3m^2+6$ is a perfect square. If we let $3m^2+6=n^2$ or $n^2-3m^2=6$, then $n+m\sqrt{3}$ is an element of $\mathbb{Z}[\sqrt{3}]$ with norm 6. Since $3+\sqrt{3}$ is such an element with positive smallest values of n and m, and $2+\sqrt{3}$ is the fundamental unit of $\mathbb{Z}[\sqrt{3}]$, all the values of n and m generated by $(3+\sqrt{3})(2+\sqrt{3})^r$ with $r \in \mathbb{Z}$ are solutions. In fact, we can also show that they are the only solutions, using the LMM algorithm as described by K. Matthews [221], for example. We remark that the values $m = 5, 19, 71, 265, 989$ are generated by $(3+\sqrt{3})(2+\sqrt{3})^r$ with $1 \leq r \leq 5$.

We complete our discussion of this first manuscript by adding an explanation for those readers who are reading this chapter in conjunction with Ramanujan's original manuscript. In fact, instead of (7.2.27) in Theorem 7.2.1, Ramanujan had written

$$\frac{1}{x}\left(\frac{x+1}{2}-\sqrt{\frac{x^2-1}{12}}\right) = \frac{1}{m}\left(\frac{m}{2}-\sqrt{\frac{m^2+2}{12}}\right). \tag{7.2.35}$$

Now, the right-hand side of (7.2.35) is

$$\frac{1}{m}\left(\frac{m}{2}-\sqrt{\frac{m^2+2}{12}}\right)=\frac{1}{2}-\sqrt{\frac{1}{12}+\frac{1}{6m^2}}<\frac{3-\sqrt{3}}{6}, \tag{7.2.36}$$

while the left-hand side of (7.2.35), by (7.2.28), is equal to

$$\frac{1}{2}+\frac{1}{2x}-\sqrt{\frac{1}{12}-\frac{1}{12x^2}}=f_1(x)>\frac{3-\sqrt{3}}{6}. \tag{7.2.37}$$

Clearly, (7.2.36) and (7.2.37) are incompatible. This mistake caused confusion for the writer of the first note appended to Ramanujan's manuscript. She (or he) writes, "I don't see where eqn (7.2.26) (the second equality) comes from, e.g., $m = 5$, $k = 8$ does not come from the value of k given $[x/m]$, as x is negative."

7.3 A Manuscript on the Diophantine Approximation of $e^{2/a}$

In this section, we discuss the partial manuscript on pages 266–267 of [269], in which Ramanujan examines the Diophantine approximation of $e^{2/a}$ when a is a nonzero integer. At the top of page 266 is a note, "See Q. 784(ii) in volume. This goes further," which is in G.H. Hardy's handwriting. Question 784 is a problem on the Diophantine approximation submitted by Ramanujan to the *Journal of the Indian Mathematical Society* [261] [267, p. 334]; "volume" evidently refers to Ramanujan's *Collected Papers* [267]. It took more than a decade before A.A. Krishnaswami Aiyangar [203] published a partial solution and T. Vijayaraghavan and G.N. Watson [309] published a complete solution to Question 784. In Question 784, Ramanujan improved upon the classical approximation. But in the partial manuscript on pages 266 and 267, Ramanujan made a further improvement and moreover derived the best possible Diophantine approximation for $e^{2/a}$. As remarked in the introduction, such a theorem was first proved in print by C.S. Davis [102] in 1978, approximately 60 years after Ramanujan discovered it. Of course, Davis was unaware that his theorem was ensconced in Ramanujan's lost notebook. As we indicate in the sequel, Ramanujan's proof is different, and considerably more elementary, than Davis's proof. Thus, Hardy's remark is on the mark. Using methods similar to those of Ramanujan (but of course, without knowledge of Ramanujan's work), B.G. Tasoev [300] established a general result, for which Davis's theorem is a special case. In regard to Ramanujan's original problem, readers might find a letter from S.D. Chowla to S.S. Pillai, written on August 25, 1929, of interest [20, p. 612].

7.3.1 Ramanujan's Claims

Ramanujan established three different, but related, results, which we relate in a moderately more contemporary style. As customary, $[x]$ denotes the greatest integer in x.

Entry 7.3.1 (p. 266). *Let $\epsilon > 0$ be given. If a is any nonzero integer, then there exist infinitely many positive integers N such that*

$$Ne^{2/a} - [Ne^{2/a}] < \frac{(1+\epsilon)\log\log N}{|a|N\log N}. \tag{7.3.1}$$

Moreover, for all sufficiently large positive integers N,

$$Ne^{2/a} - [Ne^{2/a}] > \frac{(1-\epsilon)\log\log N}{|a|N\log N}. \tag{7.3.2}$$

Entry 7.3.1 might be compared with a theorem of P. Bundschuh established in 1971 [84]. If t is a nonzero integer, then there exist positive constants c_1 and infinitely many rational numbers p/q such that

$$\left|e^{1/t} - \frac{p}{q}\right| < c_1\frac{\log\log q}{q^2\log q};$$

and there exists a positive constant c_2 such that for all rational numbers p/q,

$$\left|e^{1/t} - \frac{p}{q}\right| > c_2\frac{\log\log q}{q^2\log q}.$$

In his next theorem, Ramanujan considers two cases, – a even and a odd. His result for a even is identical to that for Entry 7.3.1, except that he formulates his conclusion in terms of $1 + [Ne^{2/a}] - Ne^{2/a}$. We therefore state Ramanujan's claim only in the case that a is odd.

Entry 7.3.2 (p. 266). *If a is any odd integer and $\epsilon > 0$ is given, then there exist infinitely many positive integers N such that*

$$1 + [Ne^{2/a}] - Ne^{2/a} < \frac{(1+\epsilon)\log\log N}{4|a|N\log N}. \tag{7.3.3}$$

Furthermore, given $\epsilon > 0$, for all positive integers N sufficiently large,

$$1 + [Ne^{2/a}] - Ne^{2/a} > \frac{(1-\epsilon)\log\log N}{4|a|N\log N}. \tag{7.3.4}$$

It will be seen, from the proofs of these entries below, that the constants multiplying

$$\frac{\log\log N}{N\log N}$$

on the right-hand sides of (7.3.1)–(7.3.4) are optimal.

We now provide a precise statement of Davis's theorem [102, Theorem 2], which readers will immediately see is equivalent to Ramanujan's Entries 7.3.1 and 7.3.2. In his paper, Davis, in fact, proves his theorem only in the special case of e, indicating that the proof of the more general result follows along the same lines. Although both the proofs of Davis and Ramanujan employ continued fractions, they are quite different. Davis uses, for example, integrals, hypergeometric functions, and Tannery's theorem. On the other hand, Ramanujan utilizes only elementary properties of continued fractions.

Theorem 7.3.1. *Let $a = \pm 2/t$, where t is a positive integer, and set*

$$c = \begin{cases} 1/t, & \text{if } t \text{ is even,} \\ 1/(4t), & \text{if } t \text{ is odd.} \end{cases}$$

Then, for each $\epsilon > 0$, the inequality

$$\left| e^a - \frac{p}{q} \right| < (c+\epsilon) \frac{\log\log q}{q^2 \log q}$$

has an infinity of solutions in integers p, q. Furthermore, there exists a number q', depending only on ϵ and t, such that

$$\left| e^a - \frac{p}{q} \right| > (c-\epsilon) \frac{\log\log q}{q^2 \log q}$$

for all integers p, q, with $q \geq q'$.

7.3.2 Proofs of Ramanujan's Claims on Page 266

Proof. We begin with the continued fraction

$$\tanh x = \frac{x}{1} \underset{+}{} \frac{x^2}{3} \underset{+}{} \frac{x^2}{5} \underset{+}{} \frac{x^2}{7} \underset{+}{} \cdots, \qquad x \in \mathbb{C}, \tag{7.3.5}$$

first established by J.H. Lambert, and rediscovered by Ramanujan, who recorded it in his second notebook [268, Chap. 12, Sect. 18], [38, p. 133, Corollary 3]. Write

$$\tanh x = 1 - \frac{2}{e^{2x}+1}$$

in (7.3.5), solve for $2/(e^{2x}+1)$, take the reciprocal of both sides, and set $x = 1/a$, where a is any nonzero integer. Hence,

$$\frac{1}{2}\left(e^{2/a}+1\right) = \frac{1}{1} \underset{-}{} \frac{1}{a} \underset{+}{} \frac{1}{3a} \underset{+}{} \frac{1}{5a} \underset{+}{} \frac{1}{7a} \underset{+}{} \cdots. \tag{7.3.6}$$

Now consider the nth approximant u_n/v_n of (7.3.6) [218, pp. 8–9], [38, p. 105, Entry 1], i.e., for $n \geq 3$,

$$\frac{1}{1} - \frac{1}{a} + \frac{1}{3a} + \frac{1}{5a} + \frac{1}{7a} + \cdots + \frac{1}{(2n-3)a} = \frac{u_n}{v_n}.$$

Then, provided that $|a| \geq 2$,

$$u_1 = 1, \quad v_1 = 1; \qquad u_2 = |a|, \quad v_2 = |a-1|. \tag{7.3.7}$$

Also, from standard recurrence relations [218, pp. 8–9],

$$u_{n+1} - u_{n-1} = (2n-1)|a|u_n; \qquad v_{n+1} - v_{n-1} = (2n-1)|a|v_n. \tag{7.3.8}$$

From the second equality in (7.3.8), we can deduce that

$$v_{n+1} \sim 2|a|nv_n \quad \text{and} \quad \log v_n \sim n \log n, \tag{7.3.9}$$

as $n \to \infty$.

Now in general, if we define $v_0 = 1$, then [38, p. 105, Entry 1] [312, p. 18]

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} =: a_1 \frac{u_n}{v_n} = \sum_{k=1}^{n} \frac{(-1)^{k+1} a_1 a_2 \cdots a_k}{v_{k-1} v_k}.$$

If we use the formula above in (7.3.6), we easily find that

$$\frac{1}{2}\left(e^{2/a} + 1\right) = \frac{u_n}{v_n} + (-1)^n \left(\frac{1}{v_n v_{n+1}} - \frac{1}{v_{n+1} v_{n+2}} + \cdots\right). \tag{7.3.10}$$

It follows from (7.3.9) and (7.3.10) that as n tends to ∞,

$$e^{2/a} + 1 - \frac{2u_n}{v_n} \sim \frac{(-1)^n}{|a|nv_n^2}. \tag{7.3.11}$$

We now subdivide our examination of (7.3.11) into two cases. First, suppose that a is even. Then, using the fact that v_1 and v_2 in (7.3.7) are odd, the recurrence relation for v_n in (7.3.8), and induction, we easily find that v_n is odd for all $n \geq 1$. Now choose $N = v_n$. By (7.3.9), we see that $n \sim \log N / \log\log N$, as $N \to \infty$. Hence, by (7.3.11), as $N \to \infty$,

$$N(e^{2/a} + 1) - 2u_n \sim \frac{(-1)^n \log\log N}{|a| N \log N}. \tag{7.3.12}$$

Second, suppose that a is odd. Ramanujan then claims that if n is odd, then v_n is odd, while if n is even, then v_n is even. However, these claims are incorrect. By (7.3.7), (7.3.8), and induction, we find, instead, that

$$v_{3m} \text{ and } v_{3m+1} \text{ are odd}; \quad v_{3m+2} \text{ is even}.$$

Thus, choose $N = v_n$, when $n = 3m$ or $n = 3m + 1$. In these cases, as in (7.3.12), we conclude that

$$N(e^{2/a} + 1) - 2u_n \sim \frac{(-1)^n \log\log N}{|a| N \log N}. \tag{7.3.13}$$

However, if $n = 3m + 2$, we can choose $N = \frac{1}{2} v_{3m+2}$. Hence, in this case,

$$N(e^{2/a} + 1) - u_n \sim \frac{(-1)^m \log\log N}{4|a| N \log N}. \tag{7.3.14}$$

Turning to Ramanujan's claims in Entries 7.3.1 and 7.3.2, from the asymptotic formulas (7.3.12) and (7.3.14), we see that all of Ramanujan's claims in these entries readily follow. This completes the proof. □

7.4 The Third Manuscript

Page 343 in the volume [269] containing Ramanujan's lost notebook is devoted to an unusual kind of approximation to certain algebraic numbers. Ramanujan's claims are surprising, and, indeed they do not appear to be valid. We copy page 343 verbatim below, and afterward we briefly discuss Ramanujan's claims:

ℓ, m, n are any integers including 0.

$$\theta = \sqrt[5]{2}.$$

$$a = \frac{1}{\sqrt[5]{2} - 1}, \qquad b = \frac{\sqrt{5}}{(1 + \sqrt[5]{4})^{5/2}}$$

$$a^m b^n \theta = p_{m,n} + \epsilon_{m,n}$$

where $-\frac{1}{2} < \epsilon_{m,n} < \frac{1}{2}$ and $p_{m,n}$ is an integer. Then

$$\epsilon_{m,n} = O\left(\frac{5^{n/2}}{(\sqrt[5]{4} - 2\sqrt[5]{2}\cos\frac{2\pi s}{5} + 1)^{m/2} (\sqrt[5]{16} + 2\sqrt[5]{4}\cos\frac{4\pi s}{5} + 1)^{5n/4}} \right) \tag{7.4.1}$$

where s is the most unfavorable of the integers 1, 2, 3, 4.

$$\theta = \sqrt[7]{2}$$

$$a = \frac{1}{\sqrt[7]{2} - 1}, \qquad b = \frac{7}{(\sqrt[7]{8} - 1)^7}, \qquad c = \frac{\sqrt[7]{2} + 1}{\sqrt[7]{4} - \sqrt[7]{2} + 1},$$

$$a^\ell b^m c^n \theta = p_{\ell,m,n} + \epsilon_{\ell,m,n}$$

$$\epsilon_{\ell,m,n} = O\left(\frac{7^m(\sqrt[7]{4}+2\sqrt[7]{2}\cos\frac{2\pi s}{5}+1)^{2n}}{(\sqrt[7]{64}-2\sqrt[7]{8}\cos\frac{2\pi s}{7}+1)^{\ell/2}} \times \frac{1}{(\sqrt[7]{64}-2\sqrt[7]{8}\cos\frac{6\pi s}{7}+1)^{7m/2}(\sqrt[7]{64}+2\sqrt[7]{8}\cos\frac{6\pi s}{7}+1)^{n/2}}\right), \tag{7.4.2}$$

where s is the most unfavorable of the integers 1, 2, 3, 4, 5, 6.

We do not know for certain what Ramanujan meant by the term "unfavorable." We think that Ramanujan was indicating that we should choose that value of s that makes the displayed error term the largest. It is unclear why Ramanujan listed $s = 1, 2, 3, 4$ below (7.4.1) instead of just writing $s = 1, 2$, because $\cos\frac{2\pi s}{5} = \cos\frac{8\pi s}{5}$ and $\cos\frac{4\pi s}{5} = \cos\frac{6\pi s}{5}$. Of course, a similar remark holds for the corresponding phrase below (7.4.2). It is also unclear what roles θ play in Ramanujan's thinking.

In order for Ramanujan's claims to have some validity, the numbers $a^m b^n \theta$ and $a^\ell b^m c^n \theta$ would need to become close to integers as ℓ, m, and n become large. It would be astounding if such were the case. Table 7.3 provides some calculations of $p_{m,n}$, $\epsilon_{m,n}$, and the error terms for $s = 1, 2$. We first notice that with increasing m and n, the remainders $\epsilon_{m,n}$ do not appear to be tending to 0, but, as we might expect, appear to be randomly distributing themselves in the interval $[-\frac{1}{2}, \frac{1}{2}]$. Also, note that if we set $m = 0$ and choose $s = 1$, then the error terms in these apparently "unfavorable" instances actually tend to infinity as n tends to infinity. In other words, in order to obtain a meaningful claim in the case $s = 1$, both m and n would both need to tend to infinity. Thus, Ramanujan's claim is meaningless in these cases. Moreover, if we set $m = 0$, then $p_{0,n} \equiv 0$ and $\epsilon_{0,n} \to 0$. Thus, for another reason, to obtain a meaningful claim, both m and n would need to tend to infinity.

If Ramanujan's assertions were correct, then ℓ, m, and n would need to tend to infinity on very special sequences. However, it is doubtful that such sequences exist.

m,n	$a^m b^n \theta$	$p_{m,n}$	$\epsilon_{m,n}$	Error, $s=1$	Error, $s=2$
$1,0$	7.725	8	-0.27	0.7882	0.4892
$2,0$	51.951	52	-0.05	0.6213	0.2393
$3,0$	349.372	349	$+0.37$	0.4897	0.1171
$4,0$	2,349.532	$2,350$	-0.47	0.3860	0.0573
$5,0$	15,800.658	$15,801$	-0.34	0.3042	0.0280
$6,0$	106,259.805	$106,260$	-0.19		
$7,0$	714,599.734	$714,600$	-0.27		
$8,0$	4,805,700.336	$4,805,700$	$+0.34$		
$9,0$	32,318,449.897	$32,318,450$	-0.10		
$10,0$	217,342,349.872	$217,342,350$	-0.13		
$0,1$	0.2729	0	$+0.27$	4.1813	0.4578
$0,2$	0.0745	0	$+0.07$	17.4833	0.2096
$0,3$	0.0203	0	$+0.02$	73.1028	0.0960
$0,4$	0.0055	0	$+0.01$		
$1,1$	2.108	2	$+0.11$	3.2958	0.2240
$2,2$	3.869	4	-0.13	10.8621	0.0502
$3,3$	7.100	7	$+0.10$	35.7988	0.0112
$4,4$	13.031	13	$+0.03$	117.9842	0.0025
$5,5$	23.914	24	-0.09		
$6,6$	23.887	24	-0.11		
$7,7$	80.543	81	-0.46		
$8,8$	147.815	148	-0.18		
$9,9$	271.274	271	$+0.27$		
$10,10$	497.849	498	-0.15		

Table 7.3. Values of $p_{m,n}$ and $\epsilon_{m,n}$