6

Two Partial Manuscripts on Euler's Constant γ

6.1 Introduction

Like many mathematicians, Ramanujan was evidently fascinated with Euler's constant γ. He wrote only one paper on Euler's constant [264], [267, pp. 163–168], but published with his lost notebook [269, pp. 274–277] are two partial manuscripts devoted to γ.

First, on pages 274 and 275 in [269], there is the beginning of a manuscript that probably was to focus on integrals related to Euler's constant γ and $\psi(s) := \Gamma'(s)/\Gamma(s)$, and on integrals and series related to Frullani's integral theorem [37, p. 313, Eq. (2.15)], [142]. This fragment contains only two short sections, comprising one and a half pages. Afterward, Ramanujan wrote "3." to indicate the beginning of a third section, but the manuscript ends abruptly at this point.

The second partial manuscript is related to the first problem that Ramanujan submitted to the *Journal of the Indian Mathematical Society* [241], [267, p. 322] and to the first six entries of Chap. 2 in his second notebook [267], [37, pp. 25–35]. Moreover, the second partial manuscript gives Ramanujan's solution to another problem [243], [267, p. 325] that he submitted to the *Journal of the Indian Mathematical Society.* No solution to this problem was ever published in the *Journal of the Indian Mathematical Society.* The formula for γ in this problem was also recorded in Ramanujan's second notebook as Entry 16 of Chap. 8 [268], [37, p. 196]. In [37], we gave a solution based on material in Chap. 2 of Ramanujan's second notebook [268], [37, pp. 25–35], where he considers a more general series and derives several elegant theorems and examples. The solution that Ramanujan gives in his lost notebook is not fundamentally different from that given by the second author in [37], but since it is more self-contained and independent of our considerations in [37, pp. 25–35], for those readers not desiring to read the aforementioned material in Chap. 2 and only interested in a direct route to Ramanujan's formula for Euler's constant, we provide Ramanujan's solution in this chapter. We mildly

G.E. Andrews and B.C. Berndt, *Ramanujan's Lost Notebook: Part IV*, DOI 10.1007/978-1-4614-4081-9_6,

correct Ramanujan's claim and give his proof while providing a few additional details. Lastly, we employ Ramanujan's formula to numerically calculate γ.

The proofs in this chapter were first published in papers that Berndt wrote with D. Bowman [46] and T. Huber [55].

6.2 Theorems on γ and $\psi(s)$ in the First Manuscript

We first prove the primary theorem in the first section of the first-mentioned incomplete manuscript. Applications of this result have been made by H. Alzer and S. Koumandos [7] in deriving series representations for γ, Catalan's constant, $\zeta(3)$, π^2, and other familiar constants.

Entry 6.2.1 (p. 274). *Let p, q, and r be positive. Then*

$$\int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{rx^{q-1}}{1-x^r} \right) dx = \psi(q/r) - \psi(p) + \log r. \tag{6.2.1}$$

Proof. (Ramanujan) Using the continuity of the integrand on the right side below for $0 \leq x, s \leq 1$, a well-known integral representation for the beta function, the change of variable $t = x^r$ in the second part of the integrand, and L'Hospital's rule, we find that

$$\begin{aligned}
&\int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{rx^{q-1}}{1-x^r} \right) dx \\
&= \lim_{s \to 0+} \int_0^1 \left\{ x^{p-1}(1-x)^{s-1} - r^{1-s} x^{q-1}(1-x^r)^{s-1} \right\} dx \\
&= \lim_{s \to 0} \left\{ \frac{\Gamma(p)\Gamma(s)}{\Gamma(s+p)} - r^{-s} \frac{\Gamma(q/r)\Gamma(s)}{\Gamma(s+q/r)} \right\} \\
&= \lim_{s \to 0} \frac{\left\{ \dfrac{\Gamma(p)}{\Gamma(s+p)} - r^{-s} \dfrac{\Gamma(q/r)}{\Gamma(s+q/r)} \right\} \Gamma(s+1)}{s} \\
&= \lim_{s \to 0} \left\{ -\frac{\Gamma(p)\Gamma'(s+p)}{\Gamma^2(s+p)} + \Gamma(q/r) \left(\frac{r^{-s} \log r}{\Gamma(s+q/r)} + \frac{r^{-s}\Gamma'(s+q/r)}{\Gamma^2(s+q/r)} \right) \right\} \\
&= -\psi(p) + \log r + \psi(q/r),
\end{aligned}$$

which completes the proof. □

Entry 6.2.2 (p. 274). *Suppose that a, b, and c are positive with $b > 1$. Then*

$$\int_0^1 \left(\frac{x^{c-1}}{1-x} - \frac{bx^{bc-1}}{1-x^b} \right) \sum_{k=0}^{\infty} x^{ab^k} dx = \psi\left(\frac{a}{b} + c\right) - \log \frac{a}{b}.$$

Proof. By Entry 6.2.1 and the facts that $b > 1$ and $\psi(x) \sim \log x$, as x tends to ∞ (see (13.2.28)),

$$\begin{aligned}\int_0^1 \left(\frac{x^{c-1}}{1-x} - \frac{bx^{bc-1}}{1-x^b}\right)\sum_{k=0}^{n} x^{ab^k}\,dx &= \sum_{k=0}^{n}\int_0^1 \left(\frac{x^{c+ab^k-1}}{1-x} - \frac{bx^{bc+ab^k-1}}{1-x^b}\right)dx \\ &= \sum_{k=0}^{n}\left(\psi\left(ab^{k-1}+c\right) - \psi\left(ab^k+c\right) + \log b\right) \\ &= \psi\left(\frac{a}{b}+c\right) - \psi\left(ab^n+c\right) + (n+1)\log b \\ &= \psi\left(\frac{a}{b}+c\right) - \log\left(ab^n+c\right) + (n+1)\log b + o(1) \\ &= \psi\left(\frac{a}{b}+c\right) - n\log b - \log a + (n+1)\log b + o(1) \\ &= \psi\left(\frac{a}{b}+c\right) - \log\frac{a}{b} + o(1),\end{aligned}$$

as n tends to ∞. Letting $n \to \infty$, we complete the proof. □

Entry 6.2.3 (p. 275). *We have*

$$\int_0^1 \frac{1}{1+x}\sum_{k=1}^{\infty} x^{2^k}\,dx = 1-\gamma, \tag{6.2.2}$$

$$\int_0^1 \frac{1+2x}{1+x+x^2}\sum_{k=1}^{\infty} x^{3^k}\,dx = 1-\gamma, \tag{6.2.3}$$

$$\int_0^1 \frac{1+\frac{1}{2}\sqrt{x}}{(1+\sqrt{x})(1+\sqrt{x}+x)}\sum_{k=1}^{\infty} x^{(3/2)^k}\,dx = 1-\gamma. \tag{6.2.4}$$

Proof. In Entry 6.2.2, set, respectively, $c = 1$, $a = b = 2$; $c = 1$, $a = b = 3$; and $c = 1$, $a = b = 3/2$. Use the fact that [126, p. 954]

$$\psi(2) = 1-\gamma \tag{6.2.5}$$

to complete the proof. □

According to Bromwich [80, p. 526], (6.2.2) is due to E. Catalan. Parts (6.2.3) and (6.2.4) may be new. H. Alzer and S. Koumandos [8] have employed (6.2.2) in deriving further representations for γ; several references to the literature on γ can be found in [8].

Before discussing the very brief second section of Ramanujan's fragment, we offer some alternative proofs, references, and connections with further work of Ramanujan, as well as others.

Lemma 6.2.1. *For $x > 0$, $x \neq 1$, and any integer $n > 1$,*

$$\frac{1}{\log x} + \frac{1}{1-x} = \sum_{k=1}^{\infty} \frac{(n-1) + (n-2)x^{1/n^k} + (n-3)x^{2/n^k} + \cdots + x^{(n-2)/n^k}}{n^k\left(1 + x^{1/n^k} + x^{2/n^k} + \cdots + x^{(n-1)/n^k}\right)}. \tag{6.2.6}$$

Proof. It is easy to verify that

$$\frac{1}{1-x^n} = \frac{1}{n}\left(\frac{(n-1)+(n-2)x+(n-3)x^2+\cdots+x^{n-2}}{1+x+x^2+\cdots+x^{n-1}} + \frac{1}{1-x}\right). \tag{6.2.7}$$

Replacing x by $x^{1/n}$ and iterating m times, we find that

$$\frac{1}{1-x} = \sum_{k=1}^{m} \frac{(n-1)+(n-2)x^{1/n^k}+(n-3)x^{2/n^k}+\cdots+x^{(n-2)/n^k}}{n^k(1+x^{1/n^k}+x^{2/n^k}+\cdots+x^{(n-1)/n^k})} + \frac{1}{n^m(1-x^{1/n^m})}.$$

If we now let m tend to ∞ and apply L'Hospital's rule, we complete the proof. □

The special cases $n = 2, 3$ of Lemma 6.2.1 can be found in Ramanujan's third notebook [268, p. 364], and proofs can be found in Berndt's book [40, pp. 399–400]. Our proof here generalizes these proofs.

Lemma 6.2.2. *For every integer $n > 1$,*

$$\gamma = \int_0^1 \left(\frac{n}{1-x^n} - \frac{1}{1-x}\right) \sum_{k=1}^{\infty} x^{n^k-1} dx. \tag{6.2.8}$$

Proof. Integrate (6.2.6) over $0 \le x \le 1$ and employ the well-known integral representation [80, p. 507], [126, p. 955]

$$\gamma = \int_0^1 \left(\frac{1}{\log x} + \frac{1}{1-x}\right) dx.$$

Accordingly, replacing x by x^{n^k}, we find that

$$\begin{aligned}
\gamma &= \int_0^1 \sum_{k=1}^{\infty} \frac{(n-1)+(n-2)x^{1/n^k}+(n-3)x^{2/n^k}+\cdots+x^{(n-2)/n^k}}{n^k(1+x^{1/n^k}+x^{2/n^k}+\cdots+x^{(n-1)/n^k})} dx \\
&= \sum_{k=1}^{\infty} \int_0^1 \frac{1}{n^k} \frac{(n-1)+(n-2)x^{1/n^k}+(n-3)x^{2/n^k}+\cdots+x^{(n-2)/n^k}}{1+x^{1/n^k}+x^{2/n^k}+\cdots+x^{(n-1)/n^k}} dx \\
&= \sum_{k=1}^{\infty} \int_0^1 \frac{(n-1)+(n-2)x+(n-3)x^2+\cdots+x^{n-2}}{1+x+x^2+\cdots+x^{n-1}} x^{n^k-1} dx \\
&= \int_0^1 \left(\frac{n}{1-x^n} - \frac{1}{1-x}\right) \sum_{k=1}^{\infty} x^{n^k-1} dx,
\end{aligned}$$

by (6.2.7). This completes the proof. □

Lemma 6.2.2 is equivalent to Entry 6.2.2 in the case $c = 1, a = b = n$. To see this, first make these substitutions in Entry 6.2.2 and use (6.2.5) to deduce that

$$1 - \gamma = \int_0^1 \left(\frac{1}{1-x} - \frac{nx^{n-1}}{1-x^n} \right) \sum_{k=1}^{\infty} x^{n^k} dx. \tag{6.2.9}$$

Adding (6.2.8) and (6.2.9) and simplifying, we readily find that

$$1 = (n-1) \int_0^1 \sum_{k=1}^{\infty} x^{n^k - 1} dx,$$

which is trivially verified by termwise integration.

The arguments in the proof of Lemma 6.2.2 lead to another formula for γ. A proof of this formula can be found in the paper by Berndt and Bowman [46] and in the Master's Thesis of C.S. Haley [140].

Theorem 6.2.1. *If b is an integer exceeding 1, let*

$$\epsilon_r = \begin{cases} b-1, & \text{if } b \mid r, \\ -1, & \text{if } b \nmid r. \end{cases} \tag{6.2.10}$$

Then

$$\gamma = \sum_{r=1}^{\infty} \frac{\epsilon_r}{r} \left[\frac{\log r}{\log b} \right],$$

where $[x]$ denotes the greatest integer $\leq x$.

Corollary 6.2.1. *We have*

$$\gamma = \sum_{r=1}^{\infty} \frac{(-1)^r}{r} \left[\frac{\log r}{\log 2} \right]. \tag{6.2.11}$$

Proof. Let $b = 2$ in Theorem 6.2.1. □

The representation for γ given in (6.2.11) was discovered in 1909 by G. Vacca [307] and is known as Dr. Vacca's series for γ. Corollary 6.2.1 was rediscovered by H.F. Sandham, who submitted it as a problem [274]. M. Koecher [185] obtained a generalization of (6.2.11) that includes a formula for γ submitted by Ramanujan as a problem [243], [267, p. 325] to the *Journal of the Indian Mathematical Society*, and found in his notebooks [268], [37, p. 196]. Further series in the spirit of those of Ramanujan and Koecher were found by F.L. Bauer [25]. A result similar to that of Bauer was found by A.W. Addison [2], with a simpler version later established by I. Gerst [121]. For alternative versions of Vacca's series for γ, for generalizations, and for approximations to γ, see papers by J. Sondow [293], Sondow and W. Zudilin [294], and Kh. Hessami Pilehrood and T. Hessami Pilehrood [154–156].

J.W.L. Glaisher [123] generalized Theorem 6.2.1. We offer a theorem that is equivalent to his theorem. For a proof, we refer to the paper by Berndt and Bowman [46]. Another proof has been found by Haley [140].

Theorem 6.2.2. *Let a and b be positive integers with $b > 1$, and let ϵ_r be defined by* (6.2.10). *Then*

$$\log a + \gamma - \sum_{n=1}^{a-1} \frac{1}{n} = \sum_{r=a}^{\infty} \frac{\epsilon_r}{r} \left[\frac{\log(r/a)}{\log b} \right].$$

We complete this section with a remark about Entry 6.2.1. After replacing x by e^{-x} in (6.2.1), we obtain an integral of Frullani type. In his third quarterly report, Ramanujan found a beautiful generalization of Frullani's theorem. In particular, the formula

$$\int_0^\infty \frac{(1+ax)^{-p} - (1+bx)^{-q}}{x} dx = \psi(q) - \psi(p) + \log \frac{b}{a}, \qquad (6.2.12)$$

where $a, b, p, q > 0$, is a special instance of Ramanujan's theorem [37, p. 314]. In view of the right sides of (6.2.1) and (6.2.12), one might surmise that (6.2.1) can be derived from (6.2.12), or Ramanujan's generalization of Frullani's theorem, and this was accomplished by J.-P. Allouche [3].

6.3 Integral Representations of $\log x$

Section 2 in Ramanujan's first unpublished fragment is devoted solely to the statements of the following theorem and (6.3.1) below.

Entry 6.3.1 (p. 275). *If a, b, and c are positive with $b > 1$, then*

$$\int_0^1 \frac{x^{c-1} - x^{bc-1}}{\log x} \sum_{k=0}^{\infty} x^{ab^k} dx = -\log\left(1 + \frac{bc}{a}\right).$$

Proof. As indicated by Ramanujan, we begin with the equality [126, p. 575]

$$\int_0^1 \frac{x^{p-1} - x^{q-1}}{\log x} dx = -\log \frac{q}{p}, \qquad (6.3.1)$$

where $p, q > 0$. Thus, since $b > 1$,

$$\begin{aligned}
-\int_0^1 \frac{x^{c-1} - x^{bc-1}}{\log x} \sum_{k=0}^{n} x^{ab^k} dx &= \sum_{k=0}^{n} \int_0^1 \frac{x^{c+ab^k-1} - x^{bc+ab^k-1}}{\log x} dx \\
&= \sum_{k=0}^{n} \log \frac{bc + ab^k}{c + ab^k} \\
&= \sum_{k=0}^{n} \left(\log b + \log(c + ab^{k-1}) - \log(c + ab^k) \right) \\
&= (n+1)\log b + \log(c + a/b) - \log(c + ab^n) \\
&= (n+1)\log b + \log(c + a/b) - n\log b - \log a + o(1) \\
&= \log(1 + bc/a) + o(1),
\end{aligned}$$

as n tends to ∞. Letting n tend to ∞, we complete the proof. □

Entry 6.3.2 (p. 275). *We have*

$$\int_0^1 \frac{1-x}{\log x} \sum_{k=1}^{\infty} x^{2^k}\, dx = -\log 2.$$

Proof. Set $c = 1$ and $a = b = 2$ in Entry 6.3.1. □

Observe that if x is replaced by e^{-x} in (6.3.1), we obtain an example of Frullani's integral theorem. Ramanujan's ideas can be extended to other examples of Frullani-type integrals found by, among others, Ramanujan in his quarterly reports [37] and Hardy [142], [151, pp. 195–226]. For example, we note the integral [142, Eq. (29)], [267, p. 200]

$$\int_0^{\infty} \frac{e^{-ax}\cos(\alpha x) - e^{-bx}\cos(\beta x)}{x}\, dx = -\frac{1}{2}\log\frac{a^2+\alpha^2}{b^2+\beta^2}, \tag{6.3.2}$$

where $a, b, \alpha, \beta > 0$.

6.4 A Formula for γ in the Second Manuscript

At the top of page 276 in [269], Ramanujan writes

$$\gamma = \log 2 - \frac{2}{3^3-3} - 4\left(\frac{1}{6^3-6} + \frac{1}{9^3-9} + \frac{1}{12^3-12}\right) - 6\left(\frac{1}{15^3-15} + \frac{1}{18^3-18} + \cdots + \frac{1}{39^3-39}\right) - \cdots,$$

$$\text{the last term of the } n\text{th group being } \frac{1}{\left(\frac{3^n+3}{2}\right)^3} - \frac{1}{\frac{3^n+3}{2}}. \tag{6.4.1}$$

Ramanujan's assertion (6.4.1) needs to be slightly corrected. The *first*, not the last, term of the nth group is $\frac{1}{\left(\frac{3^n+3}{2}\right)^3} - \frac{1}{\frac{3^n+3}{2}}$. We give a more precise statement of Ramanujan's claim.

Entry 6.4.1 (p. 276).

$$\gamma = \log 2 - \sum_{n=1}^{\infty} 2n \sum_{k=\frac{3^{n-1}+1}{2}}^{\frac{3^n-1}{2}} \frac{1}{(3k)^3 - 3k}. \tag{6.4.2}$$

Proof. It is easily checked that for each positive integer k,

$$\frac{1}{3k-1} + \frac{1}{3k} + \frac{1}{3k+1} = \frac{1}{k} + \frac{2}{(3k)^3 - 3k}. \tag{6.4.3}$$

Set $k = 1, 2, \ldots, n$ in (6.4.3) and add the n equalities to find that

$$\sum_{k=2}^{3n+1} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} \frac{2}{(3k)^3 - 3k},$$

i.e.,

$$\sum_{k=1}^{2m+1} \frac{1}{m+k} = 1 + \sum_{k=1}^{m} \frac{2}{(3k)^3 - 3k}. \tag{6.4.4}$$

The first three cases, $m = 1, 2, 3$, of (6.4.4) are, respectively,

$$\begin{aligned}
\frac{1}{2} + \frac{1}{3} + \frac{1}{4} &= 1 + \frac{2}{3^3 - 3}, \\
\frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{13} &= 1 + \frac{2}{3^3 - 3} + \frac{2}{6^3 - 6} + \frac{2}{9^3 - 9} + \frac{2}{12^3 - 12}, \\
\frac{1}{4} + \frac{1}{15} + \cdots + \frac{1}{40} &= 1 + \frac{2}{3^3 - 3} + \cdots + \frac{2}{39^3 - 39}.
\end{aligned}$$

More generally, taking $m = 1, 2, \ldots, n$ in (6.4.4) and adding the n equalities, we find that

$$\begin{aligned}
\sum_{k=1}^{\frac{3^n-1}{2}} \frac{1}{k} &= n + (n-1)\frac{2}{3^3 - 3} + (n-2)\left(\frac{2}{6^3 - 6} + \frac{2}{9^3 - 9} + \frac{2}{12^3 - 12}\right) \\
&\quad + (n-3)\left(\frac{2}{15^3 - 15} + \frac{2}{18^3 - 18} + \cdots + \frac{2}{39^3 - 39}\right),
\end{aligned} \tag{6.4.5}$$

where there are n expressions on the right-hand side of (6.4.5). Now, from the standard definition of Euler's constant, as $n \to \infty$,

$$\sum_{k=1}^{\frac{3^n-1}{2}} \frac{1}{k} = \log\left(\frac{3^n - 1}{2}\right) + \gamma + o(1) = n \log 3 - \log 2 + \gamma + o(1). \tag{6.4.6}$$

If we use (6.4.6) in (6.4.5), divide both sides of the resulting equality by n, and then let $n \to \infty$, we deduce that

$$\log 3 = 1 + \sum_{k=1}^{\infty} \frac{2}{(3k)^3 - 3k}. \tag{6.4.7}$$

(The identity (6.4.7) is also found in Sect. 2 of Chap. 2 in Ramanujan's second notebook [268]; see also [37, p. 27].) Lastly, using (6.4.6) in (6.4.5), letting $n \to \infty$ while invoking (6.4.7), and rearranging, we readily arrive at (6.4.2) to complete the proof. □

6.5 Numerical Calculations

Define

$$S_j := \sum_{n=1}^{j} 2n \sum_{k=\frac{3^{n-1}+1}{2}}^{\frac{3^n-1}{2}} \frac{1}{(3k)^3 - 3k}. \tag{6.5.1}$$

The first 14 values of $-\gamma + \log 2 - S_j$ are given in the following table.

j	$-\gamma + \log 2 - S_j$	j	$-\gamma + \log 2 - S_j$
1	3.25982×10^{-2}	8	3.14043×10^{-8}
2	5.66401×10^{-3}	9	3.87176×10^{-9}
3	8.37419×10^{-4}	10	4.72684×10^{-10}
4	1.15710×10^{-4}	11	5.72414×10^{-11}
5	1.53668×10^{-5}	12	6.88472×10^{-12}
6	1.98621×10^{-6}	13	8.23230×10^{-13}
7	2.51665×10^{-7}	14	6.05812×10^{-14}

These calculations were carried out using Mathematica 5.2. The partial sums in (6.5.1) are taken with respect to the index n of the outer sum. Thus, (6.4.2) converges quite rapidly, with only 14 terms needed to determine γ up to an error of order 10^{-14}. If we regard (6.5.1), or (6.4.2), as a single sum, i.e., each partial sum contains only one additional term from the inner sum, then the computations take much longer.

Ramanujan's series for γ converges much more rapidly than the standard series definition for γ, namely,

$$\gamma = \lim_{n\to\infty} C_n, \qquad C_n := \left(\sum_{j=1}^{n} \frac{1}{j} - \log n \right). \tag{6.5.2}$$

To compare the use of (6.5.2) with that of (6.5.1), which we used in computing the previous table, we list the first 14 values of $C_n - \gamma$ in the following table.

n	$C_n - \gamma$	n	$C_n - \gamma$
1	0.42278	8	0.061200
2	0.22964	9	0.054528
3	0.15751	10	0.049167
4	0.11982	11	0.044766
5	0.09668	12	0.041088
6	0.081025	13	0.037969
7	0.069731	14	0.035289

For several years, the most effective algorithm for computing γ has been that of R.P. Brent and E.M. McMillan [77]. The current world record, at the writing of this book, for calculating the digits of

$$\gamma = 0.57721566490153286060651209008240243104215933593992\ldots$$

is held by Alexander J. Yee and R. Chan [320], who calculated 29,844,489,545 digits.

Another representation for γ can be found in Entry 44 of Chap. 12 in Ramanujan's second notebook [268], [38, p. 167]. Asymptotic expansions for γ are located in Corollaries 1 and 2 in Sect. 9 of Chap. 4 in his second notebook [268], [37, p. 98]. An extension of these results along with an interesting discussion of them has been given by R.P. Brent [75, 76].