5

Hypergeometric Series

5.1 Introduction

The purpose of this chapter is to discuss two entries on page 200 and two on page 327 in Ramanujan's lost notebook. All four entries fall under the purview of hypergeometric series. We begin with the two entries on page 200.

On page 200 of his lost notebook, Ramanujan offers two results on certain bilateral hypergeometric series. As we shall see, the second follows from a theorem of J. Dougall [113]. The first gives a formula for the derivative of a quotient of two particular bilateral hypergeometric series. Ramanujan's formula needs to be slightly corrected, but what is remarkable is that such a formula exists! This is one of those instances in which we can undauntedly claim that if Ramanujan had not discovered the formula, no one else, at least in the foreseeable future, would have done so. Our proofs of these two formulas first appeared in a paper by the second author and W. Chu [50].

We first state the second formula, which requires modest deciphering, because of Ramanujan's use of ellipses to denote missing terms. It will be used in the proof of Ramanujan's first formula on page 200.

Entry 5.1.1 (p. 200). *Let α, β, γ, δ, and ξ be complex numbers such that* $\mathrm{Re}(\alpha+\beta+\gamma+\delta)>3$. *Then*

$$\sum_{n=-\infty}^{\infty} \frac{\xi+2n}{\Gamma(\alpha+\xi+n)\Gamma(\beta-\xi-n)\Gamma(\gamma+\xi+n)\Gamma(\delta-\xi-n)\Gamma(\alpha-n)} \times \frac{1}{\Gamma(\beta+n)\Gamma(\gamma-n)\Gamma(\delta+n)} = \frac{\sin(\pi\xi)\,\Gamma(\alpha+\beta+\gamma+\delta-3)}{\pi\Gamma(\alpha+\gamma+\xi-1)\Gamma(\beta+\delta-\xi-1)\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)} \times \frac{1}{\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)}. \tag{5.1.1}$$

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We secondly state a corrected version of Ramanujan's more interesting formula, i.e., the first formula. At the end of Sect. 5.4, we indicate the mistakes in Ramanujan's original formula.

Entry 5.1.2 (Corrected, p. 200). *Define, for real numbers* s *and* θ, $0<\theta<2\pi$, *and for any complex numbers* α, β, γ, *and* δ *such that* $\mathrm{Re}(\alpha+\beta+\gamma+\delta)>4$,

$$\varphi_s(\theta) := \sum_{n=-\infty}^{\infty} \frac{e^{(n+s)i\theta}}{\Gamma(\alpha+s+n)\Gamma(\beta-s-n)\Gamma(\gamma+s+n)\Gamma(\delta-s-n)}. \tag{5.1.2}$$

Then

$$\frac{d}{d\theta}\frac{\varphi_s(\theta)}{\varphi_t(\theta)} = \frac{i\sin\{\pi(s-t)\}\left(2\sin\frac{\theta}{2}\right)^{\alpha+\beta+\gamma+\delta-4} e^{i(\pi-\theta)(\alpha-\beta+\gamma-\delta+2s+2t)/2}}{\pi\varphi_t^2(\theta)\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)} \tag{5.1.3}$$

On page 327 in his lost notebook [269], Ramanujan offers two beautiful continued fractions connected with hypergeometric polynomials, which we now offer.

Entry 5.1.3 (p. 327). *Let*

$$\varphi(a,x) := \frac{1}{\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+3}\right)^2\right\}\left\{1+\left(\frac{x}{a+5}\right)^2\right\}\cdots}. \tag{5.1.4}$$

Then, for $a+1>0$, $b+1>0$, *and* s *not purely imaginary,*

$$\int_0^\infty \varphi(a,x)\varphi(b,x)\frac{dx}{1+s^2x^2} = 2\sqrt{\pi}\,\frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma\left(1+\frac{b}{2}\right)\Gamma\left(1+\frac{a+b}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)\Gamma\left(\frac{1+a+b}{2}\right)} \times \frac{1}{a+b+1}\,{+}\,\frac{1(a+1)(b+1)(a+b+1)s^2}{a+b+3}\,{+}\,\frac{2(a+2)(b+2)(a+b+2)s^2}{a+b+5}\,{+}\cdots.$$

Entry 5.1.4 (p. 327). *If* $s=1$, *the continued fraction in Entry 5.1.3 can be written in the form*

$$\frac{1}{a+b+1}\,{+}\,\frac{1(a+1)(b+1)(a+b+1)}{a+b+3}\,{+}\,\frac{2(a+2)(b+2)(a+b+2)}{a+b+5}\,{+}\cdots = \frac{1}{a+b+1}(1-A_1+A_1A_2-A_1A_2A_3+\cdots), \tag{5.1.5}$$

where

$$A_t = \frac{(a+t)(b+t) - ab\cos^2\dfrac{\pi t}{2}}{(a+1+t)(b+1+t) - ab\cos^2\dfrac{\pi t}{2}}. \tag{5.1.6}$$

If we set $\alpha = (a+1)/2$ and $\beta = (b+1)/2$ and replace x with $2x$, then Entry 5.1.3 can be recast in the following form.

Entry 5.1.5 (p. 327). *Let*

$$\phi(\alpha, x) := \frac{1}{\left\{1+\left(\dfrac{x}{\alpha}\right)^2\right\}\left\{1+\left(\dfrac{x}{\alpha+1}\right)^2\right\}\left\{1+\left(\dfrac{x}{\alpha+2}\right)^2\right\}\cdots}. \tag{5.1.7}$$

Then, for $\alpha > 0$, $\beta > 0$, and s not purely imaginary,

$$\int_0^\infty \phi(\alpha,x)\phi(\beta,x)\frac{dx}{1+4s^2x^2} = \sqrt{\pi}\frac{\Gamma\left(\alpha+\frac12\right)\Gamma\left(\beta+\frac12\right)\Gamma\left(\alpha+\beta\right)}{\Gamma(\alpha)\Gamma(\beta)\Gamma\left(\alpha+\beta+\frac12\right)}\chi_1(s),$$

where

$$\chi_1(s) := \frac{1}{2+}\,\frac{2\cdot 1(2\alpha)(2\beta)s^2}{2\alpha+2\beta+1}\,\frac{}{+}\,\frac{2(2\alpha+1)(2\beta+1)(2\alpha+2\beta)s^2}{2\alpha+2\beta+3}$$
$$\frac{}{+}\,\frac{3(2\alpha+2)(2\beta+2)(2\alpha+2\beta+1)s^2}{2\alpha+2\beta+5}\,\frac{}{+\cdots}. \tag{5.1.8}$$

These continued fractions are connected with the continuous Hahn polynomials. In his Ph.D. thesis [318], J. Wilson found a remarkably general class of orthogonal hypergeometric polynomials, in which all of the classical and several additional polynomials can be expressed as special or limiting cases. In particular, certain ${}_3F_2$ polynomials with two free parameters, called the continuous symmetric Hahn polynomials, were found by R. Askey and Wilson [16]. They are defined for all nonnegative integers n by

$$P_n(x) := P_n(x;\alpha,\beta) := i^n\,{}_3F_2\left(\begin{matrix}-n, n+2\alpha+2\beta-1, \beta-ix \\ \alpha+\beta, 2\beta\end{matrix}; 1\right) \tag{5.1.9}$$

and are orthogonal with respect to the positive absolutely continuous weight function

$$W(x) := |\Gamma(\alpha+ix)\Gamma(\beta+ix)|^2, \tag{5.1.10}$$

where $-\infty < x < \infty$ and $\alpha, \beta > 0$ or $\alpha = \bar{\beta}$ and $\operatorname{Re}\alpha > 0$.

In Sects. 5.7 and 5.8, we provide two entirely different proofs of Entry 5.1.3, and in Sect. 5.9, we prove Entry 5.1.4. These proofs are due to S.-Y. Kang, S.-G. Lim, and J. Sohn [175]. The first proof of Entry 5.1.3 is instructive, because it relates Ramanujan's result to Hahn polynomials and the moment problem. The second proof is undoubtedly closer to Ramanujan's approach than the first, because it relies in the beginning stages on a theorem in Ramanujan's paper [255].

5.2 Background on Bilateral Series

For every integer n, define

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{5.2.1}$$

The bilateral hypergeometric series ${}_pH_p$ is defined for complex parameters $a_1, a_2, \ldots, a_p$ and $b_1, b_2, \ldots, b_p$ by

$${}_pH_p \left[\begin{matrix} a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_p; \end{matrix} z\right] := \sum_{n=-\infty}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_p)_n} z^n.$$

With the use of D'Alembert's ratio test, it can be checked that ${}_pH_p$ converges only for $|z| = 1$, provided that [290, p. 181, Eq. (6.1.1.6)]

$$\operatorname{Re}(b_1 + b_2 + \cdots + b_p - a_1 - a_2 - \cdots - a_p) > 1. \tag{5.2.2}$$

The series ${}_pH_p$ is said to be well-poised if

$$a_1 + b_1 = a_2 + b_2 = \cdots = a_p + b_p.$$

In 1907, Dougall [113] proved that a well-poised series ${}_5H_5$ can be evaluated at $z = 1$. In order to state this evaluation, define

$$\Gamma \left[\begin{matrix} a_1, a_2, \ldots, a_m \\ b_1, b_2, \ldots, b_n \end{matrix}\right] := \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_m)}{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_n)}.$$

Then Dougall's formula [290, p. 182, Eq. (6.1.2.5)] is given by

$$\begin{aligned} &{}_5H_5 \left[\begin{matrix} 1+\frac{1}{2}a, & b, & c, & d, & e; \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e; \end{matrix} 1\right] \\ &\quad = \Gamma \left[\begin{matrix} 1-b, 1-c, 1-d, 1-e, 1+a-b, 1+a-c, 1+a-d, \\ 1+a, 1-a, 1+a-b-c, 1+a-b-d, 1+a-b-e, \end{matrix}\right. \\ &\qquad\qquad \left.\begin{matrix} 1+a-e, 1+2a-b-c-d-e \\ 1+a-c-d, 1+a-c-e, 1+a-d-e \end{matrix}\right], \end{aligned} \tag{5.2.3}$$

where for convergence, by (5.2.2),

$$1 + \operatorname{Re}(2a - b - c - d - e) > 0. \tag{5.2.4}$$

We need one further result, namely, the bilateral binomial theorem. If a and c are complex numbers with $\operatorname{Re}(c-a) > 1$ and if z is a complex number with $z = e^{i\theta}$, $0 < \theta < 2\pi$, then

$${}_1H_1 \left[\begin{matrix} a; \\ c; \end{matrix} z\right] = \frac{(1-z)^{c-a-1}}{(-z)^{c-1}} \frac{\Gamma(1-a)\Gamma(c)}{\Gamma(c-a)}. \tag{5.2.5}$$

It would seem that Ramanujan had discovered (5.2.5), but we are unaware of any mention of it by him in his papers or notebooks. We remark that the bilateral binomial theorem can also be recovered from another bilateral hypergeometric series identity [11, p. 110, Theorem 2.8.2] due to Dougall [113], namely,

$$ {}_2H_2\left[\begin{matrix} a,\ b; \\ c,\ d; \end{matrix}\ 1\right] = \frac{\Gamma(1-a)\Gamma(1-b)\Gamma(c)\Gamma(d)\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)}, \tag{5.2.6} $$

where $\mathrm{Re}(c+d-a-b) > 1$ for convergence. In fact, in the identity above, first replacing b by dz and second, letting $d \to +\infty$, we derive (5.2.5) in view of Stirling's asymptotic formula for the Γ-function.

The first appearance of (5.2.5) of which we are aware is in T.H. Koornwinder's paper [187, p. 91 (middle of the page)] in 1994. When the second author and W. Chu gave their proof of Entry 5.1.2 in [50], they used a formulation of (5.2.5) given by M.E. Horn [164] in 2003. His original formulation is incorrect, but it is corrected in the proof by J.M. Borwein, which follows the statement of the problem, and indeed the correct version (5.2.5) was used by Berndt and Chu in [50]. In addition to the proof accompanying the original problem, another proof published on the aforementioned website [164] is by G.C. Greubel.

In the sequel, we very often use the classical reflection formula

$$ \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{5.2.7} $$

5.3 Proof of Entry 5.1.1

We show that (5.2.3) leads to a proof of Entry 5.1.1.

Proof. Let S denote the series on the left-hand side of (5.1.1). Define

$$ \Omega := \frac{\sin\{\pi(\beta-\xi)\}\sin\{\pi(\delta-\xi)\}\sin\{\pi\alpha\}\sin\{\pi\gamma\}}{\pi^4}. \tag{5.3.1} $$

Using (5.2.7) and (5.3.1), we see that we can write S in the form

$$ \begin{aligned} S &= \Omega\xi \sum_{n=-\infty}^{\infty} \frac{(\xi+2n)\Gamma(1+\xi+n-\beta)\Gamma(1+\xi+n-\delta)\Gamma(1+n-\alpha)}{\xi\Gamma(\alpha+\xi+n)\Gamma(\gamma+\xi+n)\Gamma(\beta+n)\Gamma(\delta+n)} \\ &\quad \times \Gamma(1+n-\gamma) \\ &= \Omega\xi \frac{\Gamma(1+\xi-\beta)\Gamma(1+\xi-\delta)\Gamma(1-\alpha)\Gamma(1-\gamma)}{\Gamma(\alpha+\xi)\Gamma(\gamma+\xi)\Gamma(\beta)\Gamma(\delta)} \\ &\quad \times \sum_{n=-\infty}^{\infty} \frac{(1+\frac{1}{2}\xi)_n(1-\alpha)_n(1+\xi-\beta)_n(1-\gamma)_n(1+\xi-\delta)_n}{(\frac{1}{2}\xi)_n(\alpha+\xi)_n(\beta)_n(\gamma+\xi)_n(\delta)_n}. \end{aligned} \tag{5.3.2} $$

Note that the series (5.3.2) is well-poised, and so we can invoke (5.2.3) with $a = \xi$, $b = 1-\alpha$, $c = 1+\xi-\beta$, $d = 1-\gamma$, and $e = 1+\xi-\delta$. Thus, for $\mathrm{Re}(\alpha+\beta+\gamma+\delta) > 3$ for convergence, we deduce that

$$\begin{aligned} S &= \Omega\xi \frac{\Gamma(1-\alpha)\Gamma(1+\xi-\beta)\Gamma(1-\gamma)\Gamma(1+\xi-\delta)}{\Gamma(\alpha+\xi)\Gamma(\beta)\Gamma(\gamma+\xi)\Gamma(\delta)} \\ &\quad \times \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha+\gamma+\xi-1)\Gamma(\beta+\delta-\xi-1)} \\ &\quad \times \frac{\Gamma(\alpha+\xi)\Gamma(\beta-\xi)\Gamma(\gamma+\xi)\Gamma(\delta-\xi)\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(1+\xi)\Gamma(1-\xi)\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)} \\ &= \frac{\sin(\pi\xi)\Gamma(\alpha+\beta+\gamma+\delta-3)}{\pi\Gamma(\alpha+\gamma+\xi-1)\Gamma(\beta+\delta-\xi-1)\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)} \\ &\quad \times \frac{1}{\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)}, \end{aligned}$$

where we applied (5.2.7) five times, used the value of Ω from (5.3.1), and simplified. □

5.4 Proof of Entry 5.1.2

We first replace the functions in Entry 5.1.2 by another pair with which it is easier to work. With four applications of (5.2.7), we see that we can write $\varphi_s(\theta)$ in the form

$$\varphi_s(\theta) = \frac{e^{si\theta} H_s(\theta)}{\Gamma(\alpha+s)\Gamma(\beta-s)\Gamma(\gamma+s)\Gamma(\delta-s)}, \tag{5.4.1}$$

where

$$H_s(\theta) := {}_2H_2 \left[\begin{matrix} 1-\beta+s, & 1-\delta+s; \\ \alpha+s, & \gamma+s; \end{matrix} \, e^{i\theta} \right]. \tag{5.4.2}$$

Thus, we prove an analogue with φ_s and φ_t replaced by H_s and H_t, respectively. At the end of our proof, we convert our result to (5.1.2).

For brevity, we introduce the notation

$$\langle s \rangle_n := \frac{(1-\beta+s)_n (1-\delta+s)_n}{(\alpha+s)_n (\gamma+s)_n}.$$

In particular, we can then write

$$H_s(\theta) = {}_2H_2 \left[\begin{matrix} 1-\beta+s, & 1-\delta+s; \\ \alpha+s, & \gamma+s; \end{matrix} \, e^{i\theta} \right] = \sum_{n=-\infty}^{\infty} \langle s \rangle_n e^{in\theta}.$$

Proof. By the quotient rule for derivatives,

$$\frac{d}{d\theta}\left\{\frac{H_s(\theta)}{H_t(\theta)}e^{(s-t)i\theta}\right\} = \frac{\Delta}{e^{2ti\theta}H_t^2(\theta)}, \tag{5.4.3}$$

where

$$\Delta = e^{ti\theta}H_t(\theta)\frac{d}{d\theta}\left\{e^{si\theta}H_s(\theta)\right\} - e^{si\theta}H_s(\theta)\frac{d}{d\theta}\left\{e^{ti\theta}H_t(\theta)\right\}. \tag{5.4.4}$$

Using the notation above and in the previous paragraph and setting $k = m + n$ in the second equality below, we find that

$$\begin{aligned}\Delta &= i\sum_{m,n=-\infty}^{\infty}(s-t+n-m)\,\langle s\rangle_n\,\langle t\rangle_m\,e^{(s+t+n+m)i\theta}\\ &= i\sum_{k,n=-\infty}^{\infty}(s-t-k+2n)\,\langle s\rangle_n\,\langle t\rangle_{k-n}\,e^{(s+t+k)i\theta}\\ &= i\sum_{k=-\infty}^{\infty}(s-t-k)\,\langle t\rangle_k\,e^{(s+t+k)i\theta}\sum_{n=-\infty}^{\infty}\frac{s-t-k+2n}{s-t-k}\langle s\rangle_n\langle k+t\rangle_{-n}.\end{aligned} \tag{5.4.5}$$

Observe that the inner sum above is a well-poised ${}_5H_5$, requiring that $\mathrm{Re}(\alpha+\beta+\gamma+\delta) > 3$ for convergence. Thus, we can use (5.2.3) to obtain the evaluation

$$\begin{aligned}&{}_5H_5\left[\begin{matrix}1+\frac{1}{2}(s-t-k),\ 1-\alpha-t-k,\ 1-\beta+s,\ 1-\gamma-t-k,\ 1-\delta+s;\\ \frac{1}{2}(s-t-k),\quad \alpha+s,\quad \beta-t-k,\quad \gamma+s,\quad \delta-t-k;\end{matrix}\,1\right]\\ &= \Gamma\left[\begin{matrix}\alpha+t+k,\quad \gamma+t+k,\quad \beta-t-k,\quad \delta-t-k\\ 1+s-t-k,\ 1-s+t+k,\ \alpha+\gamma+s+t+k-1,\ \beta+\delta-s-t-k-1\end{matrix}\right]\\ &\quad\times\ \Gamma\left[\begin{matrix}\alpha+s,\quad \beta-s,\quad \gamma+s,\quad \delta-s,\quad \alpha+\beta+\gamma+\delta-3\\ \alpha+\beta-1,\quad \beta+\gamma-1,\quad \gamma+\delta-1,\quad \delta+\alpha-1\end{matrix}\right].\end{aligned} \tag{5.4.6}$$

Using the evaluation (5.4.6) in (5.4.5) and simplifying the expressions involving gamma functions and rising factorials, we find that

$$\begin{aligned}\Delta = i\Gamma&\left[\begin{matrix}\alpha+t,\quad \beta-t,\quad \gamma+t,\quad \delta-t\\ s-t,\ 1-s+t,\ \alpha+\gamma+s+t-1,\ \beta+\delta-s-t-1\end{matrix}\right]\\ &\times\Gamma\left[\begin{matrix}\alpha+s,\ \beta-s,\ \gamma+s,\ \delta-s,\ \alpha+\beta+\gamma+\delta-3\\ \alpha+\beta-1,\ \beta+\gamma-1,\ \gamma+\delta-1,\ \delta+\alpha-1\end{matrix}\right]\\ &\times e^{i(s+t)\theta}\sum_{k=-\infty}^{\infty}\frac{(s+t-\beta-\delta+2)_k}{(s+t+\alpha+\gamma-1)_k}e^{ik\theta}.\end{aligned} \tag{5.4.7}$$

We next apply Koornwinder's bilateral binomial theorem (5.2.5) with $a = s+t-\beta-\delta+2$ and $b = s+t+\alpha+\gamma-1$, subject to the condition $\mathrm{Re}(\alpha+\beta+\gamma+\delta) > 4$. Thus,

$$\sum_{k=-\infty}^{\infty} \frac{(s+t-\beta-\delta+2)_k}{(s+t+\alpha+\gamma-1)_k} e^{ik\theta} = {}_1H_1 \left[\begin{matrix} s+t-\beta-\delta+2; \\ s+t+\alpha+\gamma-1; \end{matrix} \, e^{i\theta} \right]$$
$$= \left(-e^{i\theta}\right)^{2-\alpha-\gamma-s-t} (1-e^{i\theta})^{\alpha+\beta+\gamma+\delta-4}$$
$$\times \frac{\Gamma(\alpha+\gamma+s+t-1)\Gamma(\beta+\delta-s-t-1)}{\Gamma(\alpha+\beta+\gamma+\delta-3)}. \tag{5.4.8}$$

Now substitute (5.4.8) into (5.4.7), use (5.2.7), and cancel common gamma function factors to arrive at

$$\Delta = e^{(s+t)i\theta} \left(-e^{i\theta}\right)^{2-\alpha-\gamma-s-t} (1-e^{i\theta})^{\alpha+\beta+\gamma+\delta-4}$$
$$\times \; i\Gamma \left[\begin{matrix} \alpha+s,\ \beta-s,\ \gamma+s,\ \delta-s,\ \alpha+t,\ \beta-t,\ \gamma+t,\ \delta-t \\ s-t,\ 1-s+t,\ \alpha+\beta-1,\ \beta+\gamma-1,\ \gamma+\delta-1,\ \delta+\alpha-1 \end{matrix} \right]$$
$$= \frac{i}{\pi} \sin\{\pi(s-t)\} \left(2\sin\tfrac{\theta}{2}\right)^{\alpha+\beta+\gamma+\delta-4} e^{i(\pi-\theta)(\alpha-\beta+\gamma-\delta+2s+2t)/2}$$
$$\times \Gamma \left[\begin{matrix} \alpha+s,\ \beta-s,\ \gamma+s,\ \delta-s,\ \alpha+t,\ \beta-t,\ \gamma+t,\ \delta-t \\ \alpha+\beta-1,\ \beta+\gamma-1,\ \gamma+\delta-1,\ \delta+\alpha-1 \end{matrix} \right]. \tag{5.4.9}$$

Lastly, substituting (5.4.9) into (5.4.3) and then reformulating the result according to the relation (5.4.1) between $\varphi_t(\theta)$ and $H_t(\theta)$, we derive the identity

$$\frac{d}{d\theta}\left\{\frac{\varphi_s(\theta)}{\varphi_t(\theta)}\right\} = \frac{i\ \sin\{\pi(s-t)\} \left(2\sin\frac{\theta}{2}\right)^{\alpha+\beta+\gamma+\delta-4} e^{i(\pi-\theta)(\alpha-\beta+\gamma-\delta+2s+2t)/2}}{\pi\varphi_t^2(\theta)\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)},$$

which is (5.1.3). The proof is thus complete. □

Let $\phi_t(\theta) = e^{-ti\theta}\varphi_t(\theta)$. We end this section with Ramanujan's rendition of Entry 5.1.2 given by

$$\frac{d}{d\theta}\left\{\frac{\phi_s(\theta)}{\phi_t(\theta)}\right\} \tag{5.4.10}$$
$$= \frac{i\sin\{\pi(s-t)\} \left|2\sin\frac{\theta}{2}\right|^{\alpha+\beta+\gamma+\delta-4} e^{i(\alpha-\beta+\gamma-\delta+2s-2t)\{(\pi-\theta)/2+\pi[\theta/(2\pi)]\}}}{\pi\phi_t^2(\theta)\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)}.$$

Note that Ramanujan's function $\phi_s(\theta)$ does not have the factor $e^{si\theta}$ in $\varphi_s(\theta)$, defined in (5.1.2). The second major difference between the two formulas is in the exponent of e on the right-hand sides. One would guess that $[x]$ in Ramanujan's exponent denotes the greatest integer less than or equal to x. The powers of $2\sin(\frac{1}{2}\theta)$ in both (5.1.3) and (5.4.10) are the same, except that Ramanujan has absolute values around $2\sin(\frac{1}{2}\theta)$. In conclusion, except for multiplicative expressions of absolute value equal to 1, the other parts of the formulas (5.4.10) and (5.1.3) are identical.

5.5 Background on Continued Fractions and Orthogonal Polynomials

Any set of polynomials $\{p_n(x)\}$ that is orthogonal with respect to a positive measure satisfies a three-term recurrence relation

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \tag{5.5.1}$$

where α_n, β_n, and γ_n are real and $\alpha_{n-1}\gamma_n > 0$, $n = 1, 2, \ldots$. Conversely, Farvard's theorem informs us that if a set of polynomials $\{p_n(x)\}$ satisfies (5.5.1) with α_n, β_n, and γ_n real and with $\alpha_{n-1}\gamma_n > 0$, $n = 1, 2, \ldots$, then there is a positive measure $d\psi(x)$ such that [10, 17]

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)\, d\psi(x) = \begin{cases} 0, & m \neq n, \\ \dfrac{\gamma_1 \cdots \gamma_n}{\alpha_0 \cdots \alpha_{n-1}} \displaystyle\int_{-\infty}^{\infty} d\psi(x), & m = n. \end{cases} \tag{5.5.2}$$

We next review some basic properties of continued fractions. For the continued fraction

$$b_0 + \frac{a_1}{b_1} {}_{+} \frac{a_2}{b_2} {}_{+} \cdots {}_{+} \frac{a_n}{b_n} {}_{+} \cdots,$$

the nth approximant f_n is given by

$$f_n = b_0 + \frac{a_1}{b_1} {}_{+} \frac{a_2}{b_2} {}_{+} \cdots {}_{+} \frac{a_n}{b_n} =: \frac{U_n}{V_n}.$$

We call U_n and V_n the nth numerator and denominator, respectively, of the continued fraction. If we define $U_{-1} = 1$, $V_{-1} = 0$, $U_0 = b_0$, and $V_0 = 1$, then, for $n = 1, 2, 3, \ldots$, the recurrence relations

$$b_n U_{n-1} + a_n U_{n-2} = U_n, \qquad b_n V_{n-1} + a_n V_{n-2} = V_n, \tag{5.5.3}$$

are valid [312, p. 15], [218, p. 8]. Using the recurrence relations in (5.5.3) and mathematical induction, one can easily deduce the following equivalence transformation [312, p. 19].

Proposition 5.5.1. *Let $c_0 = 1$ and $c_i \neq 0$ for $i > 0$. Then the two continued fractions*

$$b_0 + \frac{a_1}{b_1} {}_{+} \frac{a_2}{b_2} {}_{+} \frac{a_3}{b_3} {}_{+} \cdots$$

and

$$c_0 b_0 + \frac{c_0 c_1 a_1}{c_1 b_1} {}_{+} \frac{c_1 c_2 a_2}{c_2 b_2} {}_{+} \frac{c_2 c_3 a_3}{c_3 b_3} {}_{+} \cdots$$

have the same sequence of approximants.

The continuous symmetric Hahn polynomials $P_n(x)$, defined in (5.1.9), satisfy a three-term recurrence relation [16]

$$xP_n(x) = \alpha_n P_{n+1}(x) + \gamma_n P_{n-1}(x), \tag{5.5.4}$$

where

$$\alpha_n = \frac{(n+2\beta)(n+2\alpha+2\beta-1)}{2(2n+2\alpha+2\beta-1)} \quad \text{and} \quad \gamma_n = \frac{n(n+2\alpha-1)}{2(2n+2\alpha+2\beta-1)}. \tag{5.5.5}$$

Hence, by (5.5.3) and Proposition 5.5.1, the continued fraction corresponding to the orthogonal polynomials $P_n(x)$ with $\gamma_0 = -1$ is given by

$$\begin{aligned}\chi(x) := &\frac{1}{x} - \frac{\alpha_0\gamma_1}{x} - \frac{\alpha_1\gamma_2}{x} - \frac{\alpha_2\gamma_3}{x} - \cdots \\ = &\frac{1}{x} - \frac{1\cdot(2\alpha)(2\beta)}{4x(2\alpha+2\beta+1)} - \frac{2\cdot(2\alpha+1)(2\beta+1)(2\alpha+2\beta)}{x(2\alpha+2\beta+3)} \\ &- \frac{3\cdot(2\alpha+2)(2\beta+2)(2\alpha+2\beta+1)}{4x(2\alpha+2\beta+5)} - \cdots.\end{aligned} \tag{5.5.6}$$

In other words, $P_n(x)$ is the nth denominator of $\chi(x)$.

On the other hand, (5.5.2) along with (5.5.4) and (5.5.5) provides the L^2-norm of the continuous symmetric Hahn polynomials [16, p. 653], namely,

$$\int_{-\infty}^{\infty}[P_n(x;\alpha,\beta)]^2 W(x)\,dx = \frac{(1)_n(2\alpha)_n(\alpha+\beta-\frac{1}{2})_n}{(2\beta)_n(2\alpha+2\beta-1)_n(\alpha+\beta+\frac{1}{2})_n}W_I,$$

where

$$W_I = \int_{-\infty}^{\infty} W(x)\ dx = \sqrt{\pi}\frac{\Gamma(\alpha)\Gamma(\alpha+\frac{1}{2})\Gamma(\beta)\Gamma(\beta+\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+\frac{1}{2})}, \tag{5.5.7}$$

where $W(x)$ is defined by (5.1.10). The integral in (5.5.7) is a special case of Barnes's beta integral [22]. This particular evaluation was also established by Ramanujan [255], [267, pp. 53–58], and R. Roy [273] using Fourier transforms and Mellin transforms, respectively.

It follows from (5.5.7) that the normalized weight function of the continuous symmetric Hahn polynomials $P_n(x)$ is given by

$$W_{\mathbb{N}}(x) := \frac{\Gamma(\alpha+\beta+\frac{1}{2})|\Gamma(\alpha+ix)\Gamma(\beta+ix)|^2}{\sqrt{\pi}\,\Gamma(\alpha)\Gamma(\alpha+\frac{1}{2})\Gamma(\beta)\Gamma(\beta+\frac{1}{2})\Gamma(\alpha+\beta)}. \tag{5.5.8}$$

Since [255], [267, p. 54]

$$\phi(\alpha,x) = \frac{|\Gamma(\alpha+ix)|^2}{\Gamma^2(\alpha)}, \tag{5.5.9}$$

Entry 5.1.5 is equivalent to the following entry.

Entry 5.5.1 (p. 327). *For $\alpha > 0$ and $\beta > 0$,*

$$\int_0^\infty \frac{W_{\mathbb{N}}(x)\,dx}{1+4s^2x^2} = \frac{1}{2} + \frac{2(2\alpha)(2\beta)s^2}{2\alpha+2\beta+1} + \frac{2(2\alpha+1)(2\beta+1)(2\alpha+2\beta)s^2}{2\alpha+2\beta+3} + \cdots.$$

Entry 5.5.1 gives a representation for the Stieltjes transform of the weight function of the continuous symmetric Hahn polynomials in terms of a continued fraction. A more general continued fraction with five free parameters was found by M.E.H. Ismail, J. Letessier, G. Valent, and J. Wimp [165]. Using contiguous relations for generalized hypergeometric functions of the type ${}_7F_6$, they derived explicit representations for the associated Wilson polynomials and computed the corresponding continued fraction.

5.6 Background on the Hamburger Moment Problem

Let $\{\mu_n\}$, $0 \le n < \infty$, be a sequence of real numbers. The Hamburger moment problem seeks to find a bounded, nondecreasing function $\psi(x)$ on the interval $(-\infty,\infty)$ satisfying the equations

$$\mu_n = \int_{-\infty}^{\infty} x^n \; d\psi(x), \qquad n = 0,1,2,\ldots. \tag{5.6.1}$$

Throughout this section, it is assumed that a solution $\psi(x)$ of the Hamburger moment problem (5.6.1) is increasing on an infinite number of points. If this solution is unique, the moment problem is said to be determinate; otherwise, it is indeterminate.

For any solution $\psi(x)$ of the moment problem (5.6.1), let

$$I(z,\psi) := \int_{-\infty}^{\infty} \frac{d\psi(x)}{z-x}, \tag{5.6.2}$$

where $z \in \mathbb{H} := \{z : \operatorname{Im} z > 0\}$. The following two lemmas show that there is a one-to-one correspondence between the elements of a certain class of functions to which $I(z,\psi)$ belongs and those in the class of solutions $\psi(x)$ of the moment problem (5.6.1).

Lemma 5.6.1. [286, Theorem 2.1] *The function $I(z,\psi)$ is analytic,* $\operatorname{Im} I(z,\psi) \le 0$ *on $\mathbb{H}$, and*

$$I(z,\psi) \sim \sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}}, \qquad 0 < \epsilon \le \arg z \le \pi - \epsilon, \quad 0 < \epsilon < \pi/2, \tag{5.6.3}$$

where μ_n, $n \ge 0$, is defined by (5.6.1).

Lemma 5.6.2. [286, Theorem 2.1] *If* $F(z)$ *is analytic,* $\operatorname{Im} F(z) \le 0$ *on* $\mathbb{H}$, *and*

$$F(z) \sim \sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}}, \qquad 0 < \epsilon \le \arg z \le \pi - \epsilon, \quad 0 < \epsilon < \pi/2, \tag{5.6.4}$$

where μ_n, $n \ge 0$, *is defined by* (5.6.1), *then there exists a unique solution* $\psi(x)$ *of the moment problem* (5.6.1) *such that* $F(z) = I(z, \psi)$.

The integral $I(z, \psi)$ is also closely related to a continued fraction.

Lemma 5.6.3. [286, Theorem 2.4] *There exists a function* $F(z)$ *such that* $F(z)$ *is analytic,* $\operatorname{Im} F(z) \le 0$, *and* $F(z)$ *has a representation* (5.6.4) *if and only if there exists an associated continued fraction*

$$b_0 + \frac{a_0}{b_1 + z} - \frac{a_1}{b_2 + z} - \frac{a_2}{b_3 + z} - \cdots \tag{5.6.5}$$

such that $a_n > 0$, $n \ge 0$, $b_n \in \mathbb{R}$ *for* $n \ge 0$, *and*

$$F(z) = b_0 + \frac{a_0}{b_1 + z} - \frac{a_1}{b_2 + z} - \cdots - \frac{a_n}{F_{n+1}(z) + z},$$

where $F_{n+1}(z)$ *is an arbitrary analytic function,* $\operatorname{Im} F_{n+1}(z) \le 0$, *and* $F_{n+1}(z) = o(z)$ *as* $z \to \infty$ *on* $\mathbb{H}$.

In fact, the nth approximant, say $q_n(z)/p_n(z)$, of the continued fraction (5.6.5) can be expanded in the form [286, p. 35]

$$\frac{q_n(z)}{p_n(z)} = \frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \cdots + \frac{\mu_{2n-1}}{z^{2n}} + \frac{\mu'_{2n}}{z^{2n+1}} + \frac{\mu'_{2n+1}}{z^{2n+2}} + \cdots, \tag{5.6.6}$$

where μ_j, $0 \le j \le 2n-1$, is defined in (5.6.1). (The definitions of μ'_n can be found in [286, p. 35]. Because their definitions are somewhat complicated and are not important in the present context, we do not give them here.) As we have seen in Sect. 5.5, the denominators $p_n(z)$ comprise a set of orthogonal polynomials of degree n by (5.5.3), and by (5.5.1) and (5.5.2) in Farvard's theorem. Moreover, the orthogonality relation

$$\int_{-\infty}^{\infty} p_n(x) p_m(x)\, d\psi(x) = \begin{cases} 0, & m \ne n, \\ h_n, & m = n, \end{cases} \tag{5.6.7}$$

is satisfied by the solution $\psi(x)$ of the moment problem (5.6.1), [286, p. 35].

Next, we state two lemmas that provide, respectively, a sufficient and a necessary condition for a unique solution to the moment problem (5.6.1).

Lemma 5.6.4. [286, Theorem 2.9] *The moment problem* (5.6.1) *is determinate if*

$$\sum_{n=0}^{\infty} |p_n(z)|^2$$

diverges at a nonreal number z.

Lemma 5.6.5. [286, Theorem 2.10] *If the moment problem* (5.6.1) *is determinate, then the associated continued fraction* (5.6.5) *converges for all complex numbers z.*

5.7 The First Proof of Entry 5.1.5

Using the lemmas in Sect. 5.6, we prove the following theorem.

Theorem 5.7.1. *Let $\alpha > 0$, $\beta > 0$, and let z be nonreal. Then*

$$\int_{-\infty}^{\infty} \frac{W_{\mathbb{N}}(x)\,dx}{z-x} = \frac{1}{z} - \frac{1(2\alpha)(2\beta)}{4z(2\alpha+2\beta+1)} - \frac{2(2\alpha+1)(2\beta+1)(2\alpha+2\beta)}{z(2\alpha+2\beta+3)} - \cdots,$$

where $W_{\mathbb{N}}(x)$ is defined in (5.5.8).

Since

$$\int_{-\infty}^{\infty} \frac{W_{\mathbb{N}}(x)}{z-x}\,dx = 2z\int_{0}^{\infty} \frac{W_{\mathbb{N}}(x)}{z^2-x^2}\,dx,$$

Theorem 5.7.1 is equivalent to

$$\int_{0}^{\infty} \frac{W_{\mathbb{N}}(x)}{z^2-x^2}dx = \frac{1}{2z^2} - \frac{2(2\alpha)(2\beta)}{4(2\alpha+2\beta+1)} - \frac{2\cdot(2\alpha+1)(2\beta+1)(2\alpha+2\beta)}{z^2(2\alpha+2\beta+3)} - \frac{3\cdot(2\alpha+2)(2\beta+2)(2\alpha+2\beta+1)}{4(2\alpha+2\beta+5)} - \cdots, \tag{5.7.1}$$

from which Entry 5.1.5 or Entry 5.5.1 immediately follows after replacing z by $i/2s$. Therefore, Theorem 5.7.1 implies that Theorem 5.1.5 holds for every complex number s except when s is purely imaginary.

In order to complete the proof of Theorem 5.7.1, we need a lemma of Stieltjes that gives the power series representation of a continued fraction of the type in (5.6.5).

Lemma 5.7.1. [312, Theorem 53.1] *The coefficients in the J-fraction*

$$\frac{1}{b_1+z} - \frac{a_1}{b_2+z} - \frac{a_2}{b_3+z} - \cdots$$

and its power series expansion

$$\sum_{n=0}^{\infty} \frac{(-1)^n c_n}{z^{n+1}}$$

are connected by the relations

$$c_{p+q} = k_{0,p}k_{0,q} + a_1k_{1,p}k_{1,q} + a_1a_2k_{2,p}k_{2,q} + \cdots,$$

where $k_{0,0} = 1$*,* $k_{r,s} = 0$ *if* $r > s$*, and* $k_{r,s}$*, for* $s \geq r$*, is recursively given by the matrix equations*

$$\begin{pmatrix} k_{00} & 0 & 0 & 0 & \cdots \\ k_{01} & k_{11} & 0 & 0 & \cdots \\ k_{02} & k_{12} & k_{22} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} \cdot \begin{pmatrix} b_1 & 1 & 0 & 0 & \cdots \\ a_1 & b_2 & 1 & 0 & \cdots \\ 0 & a_2 & b_3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} = \begin{pmatrix} k_{01} & k_{11} & 0 & 0 & \cdots \\ k_{02} & k_{12} & k_{22} & 0 & \cdots \\ k_{03} & k_{13} & k_{23} & k_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}.$$

Proof of Theorem 5.7.1. Note that the continued fraction in Theorem 5.7.1 is $\chi(z)$ in (5.5.6), the continued fraction corresponding to the continuous symmetric Hahn polynomials $P_n(z)$. In the case of $\chi(z)$, $a_n = \alpha_{n-1}\gamma_n > 0$ and $b_n = 0$ for $n \geq 1$. It is therefore easy to see that $k_{ij} = 0$ when $i+j$ is odd, and thus $c_{2n+1} = 0$ and $c_{2n} > 0$. Let $Q_n(z)$ be the nth numerator of $\chi(z)$. Then for positive real numbers c_{2n} obtained from Lemma 5.7.1,

$$\frac{Q_n(z)}{P_n(z)} = \frac{1}{z} + \frac{c_2}{z^3} + \cdots + \frac{c_{2n-2}}{z^{2n-1}} + \frac{c_{2n}}{z^{2n+1}} \cdots .$$

Consider the moment problem

$$c_n = \int_{-\infty}^{\infty} x^n \, d\psi(x) \quad (n = 0, 1, 2, \dots) \tag{5.7.2}$$

for the sequence of real numbers c_n given above.

Observe that $P_0(\beta i) = 1$ and that more generally, by the Chu–Vandermonde theorem [11, p. 67, Corollary 2.2.3], $P_{4n}(\beta i) = 1$ for $n \geq 0$. Hence,

$$\sum_{n=0}^{\infty} |P_{4n}(\beta i)|^2$$

diverges, and thus the moment problem (5.7.2) has a unique solution $W_{\mathbb{N}}(x)$ by (5.6.7) and Lemma 5.6.4. It now follows from Lemmas 5.6.1–5.6.3, Lemma 5.6.5, and (5.6.6) that the continued fraction $\chi(z)$ converges to $I(z, \psi)$, for every nonreal number z, where $d\psi(x) = W_{\mathbb{N}}(x)dx$. □

5.8 The Second Proof of Entry 5.1.5

Recalling the definition of $\phi(\alpha, x)$ from either (5.1.7) or (5.5.9), set, for $t > 0$,

$$\Phi(\alpha, \beta, t) := \int_0^{\infty} \phi(\alpha, x)\phi(\beta, x) \cos(tx) \, dx. \tag{5.8.1}$$

Then, with the use of the elementary evaluation, for $x > 0$ and $s > 0$,

$$\int_0^{\infty} \cos(xt) e^{-st} \, dt = \frac{s}{s^2 + x^2},$$

the integral in Entry 5.1.5 can be rewritten in the form

$$\begin{aligned}\mathbb{I} := &\int_0^\infty \phi(\alpha,x)\phi(\beta,x)\left(\frac{1}{2s}\int_0^\infty e^{-t/(2s)}\cos(xt)\,dt\right)\,dx\\ &= \frac{1}{2s}\int_0^\infty e^{-t/(2s)}\left(\int_0^\infty \phi(\alpha,x)\phi(\beta,x)\cos(tx)\,dx\right)\,dt\\ &= \frac{1}{2s}\int_0^\infty e^{-t/(2s)}\Phi(\alpha,\beta,t)\,dt, \end{aligned} \tag{5.8.2}$$

where we inverted the order of integration by absolute convergence.

Ramanujan [255], [267, p. 53] showed that by integrating termwise the partial fraction decomposition of the integrand,

$$\int_0^\infty \phi(\alpha,x)\cos(yx)\,dx = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(\alpha+\frac12)}{\Gamma(\alpha)}\operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right), \qquad y>0.$$

Hence, from the theory of Fourier cosine transforms,

$$\int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right)\cos(xy)\,dy = \sqrt{\pi}\,\frac{\Gamma(\alpha)}{\Gamma(\alpha+\frac12)}\phi(\alpha,x), \qquad x>0. \tag{5.8.3}$$

Consequently,

$$\phi(\alpha,x) = \frac{1}{\sqrt{\pi}}\,\frac{\Gamma(\alpha+\frac12)}{\Gamma(\alpha)}\int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right)\cos(xy)\,dy. \tag{5.8.4}$$

Applying (5.8.4) to (5.8.1), we deduce that

$$\Phi(\alpha,\beta,t) = \frac{1}{\pi}\,\frac{\Gamma(\alpha+\frac12)}{\Gamma(\alpha)}\,\frac{\Gamma(\beta+\frac12)}{\Gamma(\beta)}\,\mathcal{T}, \tag{5.8.5}$$

where $\mathcal{T}$ is the triple integral

$$\mathcal{T} := \int_0^\infty\int_0^\infty\int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right)\operatorname{sech}^{2\beta}\left(\frac{z}{2}\right)\cos(xy)\cos(xz)\cos(tx)\,dz\,dy\,dx. \tag{5.8.6}$$

Using the elementary trigonometric identity $2\cos(xy)\cos(xz) = \cos(y+z)x + \cos(y-z)x$, replacing $-z$ by z in the integral involving $\cos(y-z)x$, setting $u=y+z$ in the second equality, inverting the order of integration with respect to x and y, and then replacing $-u$ by u in the integral over $-\infty<u\le 0$, we find that

$$\begin{aligned}\mathcal{T} &= \frac12\int_0^\infty\int_0^\infty\int_{-\infty}^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right)\operatorname{sech}^{2\beta}\left(\frac{z}{2}\right)\cos((y+z)x)\cos(tx)\,dz\,dy\,dx\\ &= \frac12\int_0^\infty\int_0^\infty\int_{-\infty}^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right)\operatorname{sech}^{2\beta}\left(\frac{y-u}{2}\right)\cos(ux)\cos(tx)\,du\,dy\,dx\\ &= \frac12\int_0^\infty\int_0^\infty\int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right)\left(\operatorname{sech}^{2\beta}\left(\frac{y+u}{2}\right)\right.\\ &\qquad \left.+\operatorname{sech}^{2\beta}\left(\frac{y-u}{2}\right)\right)\cos(ux)\cos(tx)\,du\,dx\,dy. \end{aligned} \tag{5.8.7}$$

Utilize the Fourier integral formula [305, p. 2]

$$\int_0^\infty \cos(nx)\,dx \int_0^\infty f(u)\cos(ux)\,du = \frac{\pi}{2} f(n)$$

in (5.8.7) to deduce that

$$\mathcal{T} = \frac{\pi}{4}\int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right)\left(\operatorname{sech}^{2\beta}\left(\frac{y+t}{2}\right) + \operatorname{sech}^{2\beta}\left(\frac{y-t}{2}\right)\right) dy. \quad (5.8.8)$$

In summary, so far, we have shown from (5.8.1), (5.8.5), and (5.8.8) that

$$\begin{aligned} \Phi(\alpha,\beta,t) = & \frac{1}{4}\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)}\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta)} \quad (5.8.9) \\ & \times \int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right)\left(\operatorname{sech}^{2\beta}\left(\frac{y+t}{2}\right) + \operatorname{sech}^{2\beta}\left(\frac{y-t}{2}\right)\right) dy. \end{aligned}$$

The equality (5.8.9), which is a generalization of the integral of $W(x)$ in (5.5.7), was established also in [38, p. 226] as a consequence of Parseval's theorem, (5.8.3) above, and Legendre's duplication formula. In [38, p. 226], it was mentioned that M.L. Glasser [124] evaluated integrals like that on the right side in (5.8.9). Glasser used contour integration, but we use the binomial theorem and Euler's beta integral below.

Using the elementary identity

$$\begin{aligned} \operatorname{sech}^{2\beta}\left(\frac{y+t}{2}\right) + \operatorname{sech}^{2\beta}\left(\frac{y-t}{2}\right) &= \operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\operatorname{sech}^{2\beta}\left(\frac{y}{2}\right) \\ \times \left\{\left(\frac{1}{1+\tanh\left(\frac{1}{2}y\right)\tanh\left(\frac{1}{2}y\right)}\right)^{2\beta} \right. &+ \left. \left(\frac{1}{1-\tanh\left(\frac{1}{2}y\right)\tanh\left(\frac{1}{2}y\right)}\right)^{2\beta}\right\} \end{aligned}$$

and the binomial theorem in (5.8.8), we find that

$$\mathcal{T} = \frac{\pi}{2}\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\int_0^\infty \operatorname{sech}^{2\alpha+2\beta}\left(\frac{y}{2}\right)\sum_{n=0}^\infty \frac{(2\beta)_{2n}}{(2n)!}\tanh^{2n}\left(\frac{t}{2}\right)\tanh^{2n}\left(\frac{y}{2}\right) dy. \quad (5.8.10)$$

Setting $v = \tanh^2(\frac{1}{2}y)$ in (5.8.10), we arrive at

$$\mathcal{T} = \frac{\pi}{2}\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\sum_{n=0}^\infty \frac{(2\beta)_{2n}}{(2n)!}\tanh^{2n}\left(\frac{t}{2}\right)\int_0^1 (1-v)^{\alpha+\beta-1}\, v^{n-1/2}\,dv. \quad (5.8.11)$$

Using Euler's beta integral $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$, we deduce that

$$\mathcal{T} = \frac{\pi}{2}\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\sum_{n=0}^{\infty}\frac{(2\beta)_{2n}}{(2n)!}\tanh^{2n}\left(\frac{t}{2}\right)\frac{\Gamma(n+\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n+\frac{1}{2})} \tag{5.8.12}$$

$$= \frac{\pi}{2}\frac{\Gamma(\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+\frac{1}{2})}\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\sum_{n=0}^{\infty}\frac{(2\beta)_{2n}(\frac{1}{2})_n}{(2n)!(\alpha+\beta+\frac{1}{2})_n}\tanh^{2n}\left(\frac{t}{2}\right)$$

$$= \frac{\pi}{2}\frac{\Gamma(\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+\frac{1}{2})}\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right){}_2F_1\left(\beta,\beta+\frac{1}{2};\alpha+\beta+\frac{1}{2};\tanh^2\left(\frac{t}{2}\right)\right).$$

Set $w = \tanh\left(\frac{1}{4}t\right)$, so that

$$F(t) := \operatorname{sech}^{2\beta}\left(\frac{t}{2}\right){}_2F_1\left(\beta,\beta+\frac{1}{2};\alpha+\beta+\frac{1}{2};\tanh^2\left(\frac{t}{2}\right)\right)$$

$$= \left(\frac{1-w^2}{1+w^2}\right)^{2\beta}{}_2F_1\left(\beta,\beta+\frac{1}{2};\alpha+\beta+\frac{1}{2};\frac{4w^2}{(1+w^2)^2}\right). \tag{5.8.13}$$

Using the quadratic transformation [11, p. 128, Eq. (3.1.9)]

$${}_2F_1(a,b;a-b+1;z) = (1+z)^{-a}{}_2F_1\left(\frac{a}{2},\frac{a+1}{2};a-b+1;\frac{4z}{(1+z)^2}\right)$$

with $z = w^2$, $a = 2\beta$, and $b = \beta-\alpha+\frac{1}{2}$, we find that

$$F(t) = (1-w^2)^{2\beta}{}_2F_1\left(-\alpha+\beta+\frac{1}{2},2\beta;\alpha+\beta+\frac{1}{2};w^2\right),$$

and using Pfaff's transformation formula [11, p. 68, Theorem 2.2.5]

$$(1-z)^a{}_2F_1(a,b;c;z) = {}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right)$$

with $a = 2\beta$, $b = -\alpha+\beta+\frac{1}{2}$, $c = \alpha+\beta+\frac{1}{2}$, and $z = w^2$, we find that

$$F(t) = {}_2F_1\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};\frac{w^2}{w^2-1}\right). \tag{5.8.14}$$

Therefore, by (5.8.12)–(5.8.14),

$$\mathcal{T} = \frac{\pi}{2}\frac{\Gamma(\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+\frac{1}{2})}{}_2F_1\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};-\sinh^2\frac{t}{4}\right). \tag{5.8.15}$$

From (5.8.5) and (5.8.15), we now see that

$$\Phi(\alpha,\beta,t) = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+\frac{1}{2})}$$

$$\times\,{}_2F_1\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};-\sinh^2\frac{t}{4}\right). \tag{5.8.16}$$

Then, it follows from (5.8.2) and (5.8.16) that

$$\mathbb{I} = \frac{\sqrt{\pi}}{4s} \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+\frac{1}{2})} \times \int_0^\infty {}_2F_1\left(2\alpha, 2\beta; \alpha+\beta+\frac{1}{2}; -\sinh^2\frac{t}{4}\right) e^{-t/(2s)}\, dt. \tag{5.8.17}$$

Recall the continued fraction expansion of Stieltjes [297], [298, pp. 282–291],

$$\int_0^\infty {}_2F_1\left(a, b; \frac{a+b+1}{2}; -\sinh^2 t\right) e^{-tz}\, dt = \frac{1}{z}+ \frac{1\cdot ab\cdot 4}{(a+b+1)z}+ \frac{2\cdot(a+1)(b+1)(a+b)\cdot 4}{(a+b+3)z} + \frac{3\cdot(a+2)(b+2)(a+b+1)\cdot 4}{(a+b+5)z} + \cdots, \qquad \operatorname{Re} z > 0, \tag{5.8.18}$$

from which, upon replacing t by $4t$ and setting $a = 2\alpha$ and $b = 2\beta$, we deduce that, for $s > 0$,

$$\int_0^\infty {}_2F_1\left(2\alpha, 2\beta; \alpha+\beta+\frac{1}{2}; -\sinh^2\frac{t}{4}\right) e^{-t/(2s)}\, dt = \frac{4}{2/s}+ \frac{1\cdot(2\alpha)(2\beta)\cdot 4}{(2\alpha+2\beta+1)2/s}+ \frac{2\cdot(2\alpha+1)(2\beta+1)(2\alpha+2\beta)\cdot 4}{(2\alpha+2\beta+3)2/s} + \cdots. \tag{5.8.19}$$

By (5.8.17) and (5.8.19), we finally deduce that

$$\mathbb{I} = \sqrt{\pi}\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)}\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+\frac{1}{2})} \times \left(\frac{1}{2}+ \frac{2\cdot 1\cdot(2\alpha)(2\beta)s^2}{(2\alpha+2\beta+1)}+ \frac{2\cdot(2\alpha+1)(2\beta+1)(2\alpha+2\beta)s^2}{(2\alpha+2\beta+3)} + \cdots\right),$$

which completes the proof of Entry 5.1.5 for $s > 0$. By (5.5.7), Entry 5.1.5 holds for $s = 0$. Since both sides of Theorem 5.1.5 are even functions of s, Theorem 5.1.5 is valid for all real s.

5.9 Proof of Entry 5.1.2

We use the recurrence relation (5.5.3) and induction to prove that for all integers $n \geq 1$,

$$f_n = \frac{1}{a+b+1}+ \frac{1\cdot(a+1)(b+1)(a+b+1)}{a+b+3} + \cdots + \frac{n\cdot(a+n)(b+n)(a+b+n)}{a+b+(2n+1)} = \frac{1}{a+b+1}(1 - A_1 + A_1A_2 - A_1A_2A_3 + \cdots + (-1)^n A_1A_2\cdots A_n) := R_n, \tag{5.9.1}$$

where A_n is defined in (5.1.6). Entry 5.1.2 then readily follows from (5.9.1).

First, observe from (5.1.6) that

$$A_t = \begin{cases} \dfrac{(a+t)(b+t)}{(a+1+t)(b+1+t)}, & \text{if } t \text{ is odd,} \\ \dfrac{t(a+b+t)}{(a+b+t+1)(t+1)}, & \text{if } t \text{ is even.} \end{cases} \tag{5.9.2}$$

From the recurrence relations (5.5.3), we find that

$$\begin{aligned} f_1 &= \frac{U_1}{V_1} = \frac{1}{a+b+1} = R_1, \\ f_2 &= \frac{U_2}{V_2} = \frac{a+b+3}{(a+b+1)(a+2)(b+2)} = \frac{1}{a+b+1}(1-A_1) = R_2, \\ f_3 &= \frac{U_3}{V_3} = \frac{3(a+b+3)^2 + 2(a+1)(b+1)(a+b+2)}{3(a+b+1)(a+b+3)(a+2)(b+2)} \\ &= \frac{1}{a+b+1}(1 - A_1 + A_1A_2) = R_3, \\ f_4 &= \frac{U_4}{V_4} = \frac{3(a+b+3)^2(a+4)(b+4) + 2(a+1)(b+1)(a+b+2)(a+b+7)}{3(a+b+1)(a+b+3)(a+2)(b+2)(a+4)(b+4)} \\ &= \frac{1}{1+a+b}(1 - A_1 + A_1A_2 - A_1A_2A_3) = R_4. \end{aligned}$$

Assume that (5.9.1) holds up to k. Then by (5.5.3),

$$f_{k+1} = \frac{U_{k+1}}{V_{k+1}} = \frac{(a+b+(2k+1))U_k + k(a+k)(b+k)(a+b+k)U_{k-1}}{(a+b+(2k+1))V_k + k(a+k)(b+k)(a+b+k)V_{k-1}}.$$

By the induction hypothesis, the numerator above equals

$$\begin{aligned} &\frac{a+b+(2k+1)}{a+b+1}(1 - A_1 + A_1A_2 + \cdots + (-1)^{k-1}A_1A_2\cdots A_{k-1})V_k \\ &+ \frac{k(a+k)(b+k)(a+b+k)}{a+b+1} \\ &\times (1 - A_1 + A_1A_2 + \cdots + (-1)^{k-2}A_1A_2\cdots A_{k-2})V_{k-1}. \end{aligned}$$

Hence, we may write

$$\begin{aligned} f_{k+1} = &\frac{1}{a+b+1}(1 - A_1 + A_1A_2 + \cdots + (-1)^{k-1}A_1A_2\cdots A_{k-1}) \\ &- \frac{(-1)^{k-1}A_1A_2\cdots A_{k-1}}{a+b+1} \\ &\times \frac{k(a+k)(b+k)(a+b+k)V_{k-1}}{(a+b+(2k+1))V_k + k(a+k)(b+k)(a+b+k)V_{k-1}}. \end{aligned}$$

It therefore suffices to prove that

$$\frac{k(a+k)(b+k)(a+b+k)V_{k-1}}{(a+b+(2k+1))V_k + k(a+k)(b+k)(a+b+k)V_{k-1}} = A_k. \tag{5.9.3}$$

We claim that

$$V_k = \begin{cases} V_{k-1}(a+b+2k-1+(k-1)(a+b+k-1)), & \text{if } k \text{ is odd,} \\ V_{k-1}(a+b+2k-1+(a+k-1)(b+k-1)), & \text{if } k \text{ is even.} \end{cases} \tag{5.9.4}$$

We shall defer the proof of the claim above until the end of the proof (5.9.1).

Assuming the truth of (5.9.4) for the moment, let k be odd. Then the left side of (5.9.3) is equal to

$$\begin{aligned} &\frac{k(a+k)(b+k)(a+b+k)V_{k-1}}{\begin{pmatrix}(a+b+2k+1)V_{k-1}\{a+b+2k-1+(k-1)(a+b+k-1)\} \\ +k(a+k)(b+k)(a+b+k)V_{k-1}\end{pmatrix}} \\ &\quad= \frac{k(a+k)(b+k)(a+b+k)}{(a+b+2k+1)(ak+bk+k^2)+k(a+k)(b+k)(a+b+k)} \\ &\quad= \frac{(a+k)(b+k)}{a+b+2k+1+(a+k)(b+k)} = \frac{(a+k)(b+k)}{(a+1+k)(b+1+k)} = A_k, \end{aligned}$$

as desired. When k is even, the left-hand side of (5.9.3) takes the shape

$$\begin{aligned} &\frac{k(a+k)(b+k)(a+b+k)V_{k-1}}{\begin{pmatrix}(a+b+2k+1)V_{k-1}\{a+b+2k-1+(a+k-1)(b+k-1)\} \\ +k(a+k)(b+k)(a+b+k)V_{k-1}\end{pmatrix}} \\ &\quad= \frac{k(a+k)(b+k)(a+b+k)}{(a+b+2k+1)(a+k)(b+k)+k(a+k)(b+k)(a+b+k)} \\ &\quad= \frac{k(a+b+k)}{(a+b+k+1)(k+1)} = A_k, \end{aligned}$$

which again is what we wanted to prove. It remains to prove the claim.

We can recast (5.9.4) in the equivalent form

$$\frac{V_k}{V_{k-1}} = \begin{cases} (a+k)(b+k), & \text{if } k \text{ is even,} \\ k(a+b+k), & \text{if } k \text{ is odd.} \end{cases} \tag{5.9.5}$$

We now prove (5.9.5). The first few instances of (5.9.5) are

$$\begin{aligned} \frac{V_2}{V_1} &= \frac{(a+2)(b+2)(a+b+1)}{a+b+1} = (a+2)(b+2), \\ \frac{V_3}{V_2} &= \frac{3(a+2)(b+2)(a+b+1)(a+b+3)}{(a+2)(b+2)(a+b+1)} = 3(a+b+3), \\ \frac{V_4}{V_3} &= (a+4)(b+4). \end{aligned}$$

Assume that (5.9.5) is true up to $2k$. Then, by (5.5.3) and the induction hypothesis,

$$\begin{aligned}\frac{V_{2k+1}}{V_{2k}} &= \frac{(a+b+4k+1)V_{2k} + 2k(a+2k)(b+2k)(a+b+2k)V_{2k-1}}{V_{2k}} \\ &= (a+b+4k+1) + 2k(a+2k)(b+2k)(a+b+2k) \cdot \frac{1}{(a+2k)(b+2k)} \\ &= (2k+1)(a+b+2k+1),\end{aligned}$$

which is in agreement with (5.9.5). Assuming that (5.9.5) is valid up to $2k+1$ and using (5.5.3) again, we find, upon the use of the induction hypothesis, that

$$\begin{aligned}&\frac{V_{2k+2}}{V_{2k+1}} \\ &= (a+b+4k+3)V_{2k+1} + (2k+1)(a+2k+1)(b+2k+1)(a+b+2k+1) \\ &\quad \times \frac{V_{2k}}{V_{2k+1}} \\ &= (a+b+4k+3) + (2k+1)(a+2k+1)(b+2k+1)(a+b+2k+1) \\ &\quad \times \frac{1}{(2k+1)(a+b+2k+1)} \\ &= (a+2k+2)(b+2k+2),\end{aligned}$$

which again is in harmony with (5.9.5). This then completes the proof of Ramanujan's assertion in (5.9.1) and Entry 5.1.2. As mentioned in the introduction, this proof is due to S.-Y. Kang.