3

Koshliakov's Formula and Guinand's Formula

3.1 Introduction

In his lecture at a conference to commemorate the centenary of Ramanujan's birth, held on June 1–5, 1987, at the University of Illinois at Urbana-Champaign, R. William Gosper remarked, "How can we pretend to love this man when he is forever reaching out from the grave to snatch away our neatest results?" In less colorful language, Gosper was asserting that it frequently happens that one proves an important theorem, only to discover later that it is ensconced somewhere in Ramanujan's writings. In other instances, we have learned that Ramanujan anticipated important later developments in his own inimitable way.

In this chapter, we examine two pages in Ramanujan's lost notebook [269, pp. 253–254], on one of which Gosper's observation is demonstrated once again. On page 253, Ramanujan states a version of a famous formula of A.P. Guinand, from which N.S. Koshliakov's equally famous formula follows as a corollary. On page 254, Ramanujan gives applications of Guinand's formula; these results are mostly new.

First, we discuss Koshliakov's formula. Koshliakov is chiefly remembered for one theorem, namely, *Koshliakov's formula* [188], which we now see was proved by Ramanujan about 10 years earlier. To state his formula, let $K_\nu(z)$ denote the modified Bessel function of order ν, defined in (2.1.3), and let $d(n)$ denote the number of positive divisors of the positive integer n. Then, if γ denotes Euler's constant and $a > 0$,

$$\gamma - \log\left(\frac{4\pi}{a}\right) + 4\sum_{n=1}^{\infty} d(n)K_0(2\pi an) = \frac{1}{a}\left(\gamma - \log(4\pi a) + 4\sum_{n=1}^{\infty} d(n)K_0\left(\frac{2\pi n}{a}\right)\right). \quad (3.1.1)$$

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Koshliakov's proof, as well as most subsequent proofs, depends upon Voronoï's summation formula [310]

$$\sum_{a\le n\le b}{}' d(n)f(n) = \int_a^b (\log x + 2\gamma)f(x)dx + \sum_{n=1}^{\infty} d(n)\int_a^b f(x)\left(4K_0(4\pi\sqrt{nx}) - 2\pi Y_0(4\pi\sqrt{nx})\right)dx, \quad (3.1.2)$$

where $Y_\nu(z)$ denotes the Weber–Bessel function of order ν, defined in (2.1.2). The prime $\prime$ on the summation sign on the left-hand side indicates that if a or b is an integer, then only $\frac{1}{2}f(a)$ or $\frac{1}{2}f(b)$, respectively, is counted. For conditions on $f(x)$ that ensure the validity of (3.1.2), see, for example, Berndt's paper [28].

A.L. Dixon and W.L. Ferrar [112] also proved (3.1.1) using the Voronoï summation formula. F. Oberhettinger and K.L. Soni [235] established a generalization of (3.1.1) using Voronoï's formula (3.1.2), and she derived further identities from Koshliakov's formula [295]. In contrast to the work of these authors, Ramanujan evidently did not appeal to Voronoï's formula.

Koshliakov's formula can be considered an analogue of the transformation formula for the classical theta function, namely,

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2/\tau} = \sqrt{\tau}\sum_{n=-\infty}^{\infty} e^{-\pi n^2\tau}, \quad \operatorname{Re}\tau > 0, \quad (3.1.3)$$

which, as is well known, is equivalent to the functional equation of the Riemann zeta function $\zeta(s)$ given by [306, p. 22]

$$\pi^{-s/2}\Gamma\left(\tfrac{1}{2}s\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\tfrac{1}{2}(1-s)\right)\zeta(1-s). \quad (3.1.4)$$

Ferrar [118] was evidently the first mathematician to prove indeed that (3.1.1) can be derived from the functional equation of $\zeta^2(s)$. Oberhettinger and Soni [235] showed that this functional equation and Koshliakov's formula are equivalent.

On page 253 in his lost notebook [269], Ramanujan states (3.1.1) as a corollary of a more general and especially beautiful formula at the top of the same page. This more general formula is stated in an equivalent formulation in Entry 3.1.1 below.

Entry 3.1.1 (p. 253). *Let $\sigma_k(n) = \sum_{d|n} d^k$, and let $\zeta(s)$ denote the Riemann zeta function. If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if s is any complex number, then*

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha)-\sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\beta)$$
$$=\frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\{\beta^{(1-s)/2}-\alpha^{(1-s)/2}\}+\frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\{\beta^{(1+s)/2}-\alpha^{(1+s)/2}\}. \tag{3.1.5}$$

The identity (3.1.5) is equivalent to a formula established by Guinand [136] in 1955. The series in Entry 3.1.1 are reminiscent of the Fourier expansion of nonanalytic Eisenstein series on $\mathbb{SL}(2,\mathbb{Z})$, or Maass wave forms [219], [226, pp. 230–232], [204, pp. 15–16], [304, pp. 208–209]. This Fourier series was published by H. Maass [219] in the language of Eisenstein series in the same year, 1949, that A. Selberg and S. Chowla [283], [282, pp. 367–378] published it in the similar vein of the Epstein zeta function, but with their proof not published until several years later [284], [282, pp. 521–545]. In the meanwhile, P.T. Bateman and E. Grosswald [24] published a proof. These Eisenstein series were shown by Maass [219] to satisfy a functional equation for automorphic forms. C.J. Moreno kindly informed the authors that he was easily able to derive Entry 3.1.1 from the aforementioned Fourier series expansion and functional equation. One may then regard (3.1.5) as an equivalent formulation of the functional equation for these nonholomorphic Eisenstein series or these particular Maass wave forms. The proof of Entry 3.1.1 that we give below is essentially the same as that of Guinand [136] and is completely independent of any considerations of nonanalytic Eisenstein series or their closely associated Epstein zeta functions. As is well known, Ramanujan made a large number of original contributions to Eisenstein series, many of which can be found in his lost notebook [13, Chaps. 11–16], [70].

On page 254, Ramanujan recorded formulas similar to Koshliakov's formula (3.1.1) or to Guinand's formula (3.1.5). We show that each of the three main results on this page can be deduced from Ramanujan's (and Guinand's) beautiful generalization (3.1.5) of Koshliakov's formula.

We close this introduction by mentioning two recent papers by S. Kanemitsu, Y. Tanigawa, H. Tsukada, and M. Yoshimoto [168] and S. Kanemitsu, Y. Tanigawa, and M. Yoshimoto [171], in which the formulas of Koshliakov and Guinand are used or generalized.

The content of this chapter is taken from the second author's paper with Y. Lee and J. Sohn [62].

3.2 Preliminary Results

Throughout pages 253 and 254 of [269], Ramanujan expresses his theorems in terms of variants of the integral [126, p. 384, formula 3.471, no. 9]

$$\int_0^{\infty}x^{\nu-1}e^{-\beta/x-\gamma x}dx=2\left(\frac{\beta}{\gamma}\right)^{\nu/2}K_{\nu}(2\sqrt{\beta\gamma}), \tag{3.2.1}$$

where ν is any complex number and $\operatorname{Re}\beta > 0$, $\operatorname{Re}\gamma > 0$. Since the modified Bessel function $K_\nu(z)$ is such a well-known function and its notation is standard, it seems advisable to avoid Ramanujan's notation for variants of (3.2.1), which he calls ϕ, ψ, and χ. In summary, we have converted all of Ramanujan's theorems to identities involving the modified Bessel function K_ν.

We use the well-known fact [126, p. 978, formula 8.469, no. 3]

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}. \tag{3.2.2}$$

Necessary for us is the asymptotic behavior [314, p. 202]

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}, \qquad z \to \infty,$$

which we invoke to ensure the convergence of series and integrals and also to justify the interchange of integration and summation several times in the sequel. We need several integrals of Bessel functions beginning with [126, p. 705, formula 6.544, no. 8]

$$\int_0^\infty K_\nu\left(\frac{a}{x}\right)K_\nu(bx)\frac{dx}{x^2} = \frac{\pi}{a}K_{2\nu}(2\sqrt{ab}), \quad \operatorname{Re}a > 0, \operatorname{Re}b > 0. \tag{3.2.3}$$

We need the related pair [295, p. 544, Eq. (8)]

$$\int_0^\infty xK_0(ax)K_0(bx)dx = \frac{\log(a/b)}{a^2 - b^2}, \qquad a, b > 0, \tag{3.2.4}$$

and [126, p. 697, formula 6.521, no. 3]

$$\int_0^\infty xK_\nu(ax)K_\nu(bx)dx = \frac{\pi(ab)^{-\nu}(a^{2\nu} - b^{2\nu})}{2\sin(\pi\nu)(a^2 - b^2)}, \qquad |\operatorname{Re}\nu| < 1, \ \operatorname{Re}(a+b) > 0. \tag{3.2.5}$$

Lastly, we need the evaluation [126, p. 708, formula 6.561, no. 16], for $\operatorname{Re}a > 0$ and $\operatorname{Re}(\mu + 1 \pm \nu) > 0$,

$$\int_0^\infty x^\mu K_\nu(ax)dx = 2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right). \tag{3.2.6}$$

3.3 Guinand's Formula

We begin by restating Entry 3.1.1.

Entry 3.3.1 (p. 253). *As usual, let* $\sigma_k(n) = \sum_{d|n} d^k$, *and let* $\zeta(s)$ *denote the Riemann zeta function. If* α *and* β *are positive numbers such that* $\alpha\beta = \pi^2$, *and if* s *is any complex number, then*

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha) - \sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\beta)$$
$$= \frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\{\beta^{(1-s)/2}-\alpha^{(1-s)/2}\}+\frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\{\beta^{(1+s)/2}-\alpha^{(1+s)/2}\}. \tag{3.3.1}$$

To prove Entry 3.3.1, we need the following lemma.

Lemma 3.3.1. *Let $K_\nu(z)$ denote the modified Bessel function of order ν. If $x > 0$ and $\operatorname{Re}\nu > 0$, then*

$$\frac{1}{4}(\pi x)^{-\nu}\Gamma(\nu) + \sum_{n=1}^{\infty} n^{\nu}K_{\nu}(2\pi n x)$$
$$= \frac{1}{4}\sqrt{\pi}(\pi x)^{-\nu-1}\Gamma\left(\nu+\frac{1}{2}\right)+\frac{\sqrt{\pi}}{2x}\left(\frac{x}{\pi}\right)^{\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)\sum_{n=1}^{\infty}(n^2+x^2)^{-\nu-1/2}. \tag{3.3.2}$$

Lemma 3.3.1 is due to G.N. Watson [313], who proved it by using the Poisson summation formula. H. Kober [184] generalized Lemma 3.3.1 in two different directions. In one of them, the index n on the left-hand side of (3.3.2) was replaced by $n+\alpha$, $0<\alpha<1$, and in the other, $\cos(2\pi n\beta)$ was introduced into the summands on the left-hand side of (3.3.2). Berndt [32] generalized (3.3.2) by putting either an even or odd periodic sequence of coefficients in the infinite series of (3.3.2). The proof that we give below is essentially an elaboration of Guinand's proof [136].

Proof of Entry 3.3.1. Setting $n = kd$, we find that

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha) = \sqrt{\alpha}\sum_{n=1}^{\infty}\sum_{d|n}d^{-s}n^{s/2}K_{s/2}(2n\alpha)$$
$$= \sqrt{\alpha}\sum_{d=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{k}{d}\right)^{s/2}K_{s/2}(2dk\alpha). \tag{3.3.3}$$

We now invoke Lemma 3.3.1 on the right-hand side above to deduce that for $\operatorname{Re} s > 0$,

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha)$$
$$= \sqrt{\alpha}\sum_{d=1}^{\infty}\frac{1}{d^{s/2}}\left(-\frac{1}{4}(d\alpha)^{-s/2}\Gamma\left(\frac{s}{2}\right)+\frac{1}{4}\sqrt{\pi}(d\alpha)^{-s/2-1}\Gamma\left(\frac{s+1}{2}\right)\right.$$
$$\left.+\frac{\pi^{3/2}}{2d\alpha}\left(\frac{d\alpha}{\pi^2}\right)^{s/2+1}\Gamma\left(\frac{s+1}{2}\right)\sum_{n=1}^{\infty}\frac{1}{(n^2+(d\alpha/\pi)^2)^{(s+1)/2}}\right)$$

$$= -\frac{1}{4}\alpha^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) + \frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1)$$
$$+ \frac{1}{2}\alpha^{(s+1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(n^2\pi^2+d^2\alpha^2)^{(s+1)/2}} \tag{3.3.4}$$
$$= -\frac{1}{4}\alpha^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) + \frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1)$$
$$+ \frac{1}{2}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(n^2\beta^2/\pi^2+d^2)^{(s+1)/2}}$$
$$= -\frac{1}{4}\alpha^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) + \frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1)$$
$$+ \frac{1}{2}\beta^{(s+1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(n^2\beta^2+d^2\pi^2)^{(s+1)/2}}, \tag{3.3.5}$$

where we used the hypothesis $\alpha\beta = \pi^2$. By symmetry, from (3.3.4), for $\operatorname{Re} s > 0$,

$$\sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\beta)$$
$$= -\frac{1}{4}\beta^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) + \frac{1}{4}\beta^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1)$$
$$+ \frac{1}{2}\beta^{(s+1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(n^2\pi^2+d^2\beta^2)^{(s+1)/2}}. \tag{3.3.6}$$

Reversing the roles of the summation variables d and n in (3.3.6), subtracting (3.3.6) from (3.3.5), and rearranging slightly, we deduce that

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha) - \sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\beta)$$
$$= -\frac{1}{4}\alpha^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) + \frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1)$$
$$+ \frac{1}{4}\beta^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) - \frac{1}{4}\beta^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1). \tag{3.3.7}$$

On the other hand, using the functional equation (3.1.4) of $\zeta(s)$ and the fact that $\alpha\beta = \pi^2$, we find that

$$\frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1) = \frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\pi^{s+1/2}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)$$
$$= \frac{1}{4}\alpha^{(-s-1)/2}(\alpha\beta)^{(s+1)/2}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)$$
$$= \frac{1}{4}\beta^{(s+1)/2}\Gamma\left(-\frac{s}{2}\right)\zeta(-s). \tag{3.3.8}$$

Substituting (3.3.8) and its analogue with the roles of α and β reversed into (3.3.7), we find that

$$\begin{aligned}\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha)&-\sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\beta)\\ &=-\frac{1}{4}\alpha^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)+\frac{1}{4}\beta^{(s+1)/2}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\\ &\quad+\frac{1}{4}\beta^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)-\frac{1}{4}\alpha^{(s+1)/2}\Gamma\left(-\frac{s}{2}\right)\zeta(-s).\end{aligned}\tag{3.3.9}$$

The identity (3.3.9) is simply a rearrangement of (3.3.1), and so the proof of (3.3.1) is complete for $\operatorname{Re} s>0$. By analytic continuation, (3.3.1) is valid for all complex numbers s. □

Since $K_s(z)=K_{-s}(z)$ [314, p. 79, Eq. (8)], we see that (3.1.5) is invariant under the replacement of s by $-s$.

Ramanujan completes page 253 with two corollaries, which we now state and prove.

Entry 3.3.2 (p. 253). *Let α and β be positive numbers such that $\alpha\beta=\pi^2$. Then*

$$\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2n\alpha}-\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2n\beta}=\frac{\beta-\alpha}{12}+\frac{1}{4}\log\frac{\alpha}{\beta}.\tag{3.3.10}$$

Proof. Let $s=1$ in Entry 3.1.1. From (3.2.2),

$$\sqrt{\alpha n}K_{1/2}(2n\alpha)=\frac{1}{2}\sqrt{\pi}e^{-2n\alpha}.\tag{3.3.11}$$

Using (3.3.11), the values $\Gamma(-\frac{1}{2})=-2\Gamma(\frac{1}{2})=-2\sqrt{\pi}$ and $\zeta(-1)=-\frac{1}{12}$ [306, p. 19], and the Laurent expansion of $\zeta(s)$ about $s=1$ [306, p. 16, Eq. (2.1.16)] in (3.1.5), we find that

$$\begin{aligned}\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2n\alpha}&-\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2n\beta}-\frac{\beta-\alpha}{12}\\ &=\frac{1}{2\sqrt{\pi}}\lim_{s\to1}\Gamma\left(\frac{s}{2}\right)\zeta(s)\{\beta^{(1-s)/2}-\alpha^{(1-s)/2}\}\\ &=\frac{1}{2}\lim_{s\to1}\left(\frac{1}{s-1}+\gamma+\cdots\right)\\ &\qquad\times\left(\left\{1-\frac{s-1}{2}\log\beta+\cdots\right\}-\left\{1-\frac{s-1}{2}\log\alpha+\cdots\right\}\right)\\ &=\frac{1}{4}\log\frac{\alpha}{\beta}.\end{aligned}\tag{3.3.12}$$

We easily see that (3.3.12) is equivalent to (3.3.10), and so the proof is complete. □

Entry 3.3.2 is equivalent to the identity

$$\sum_{m=1}^{\infty} \frac{1}{m(e^{2m\alpha}-1)} - \sum_{m=1}^{\infty} \frac{1}{m(e^{2m\beta}-1)} = \frac{\beta-\alpha}{12} + \frac{1}{4}\log\frac{\alpha}{\beta}. \tag{3.3.13}$$

To see this, expand the summands in (3.3.13) in geometric series and collect all terms with the same exponents in the resulting double series. The formula (3.3.13) (or (3.3.10)) is equivalent to the transformation formula for the logarithm of the Dedekind eta function. Ramanujan stated (3.3.13) twice in his second notebook [268], namely as Corollary (ii) in Sect. 8 of Chap. 14 [38, p. 256] and as Entry 27(iii) in Chap. 16 [39, p. 43]. He also recorded (3.3.13) in an unpublished manuscript on infinite series reproduced with Ramanujan's lost notebook [269]; in particular, see formula (29) on page 320 of [269]. See also Chap. 12 in this volume or [42, p. 65, Entry 3.5].

We next demonstrate that Koshliakov's formula (3.1.1) is a corollary of Entry 3.3.1. Our proof is a detailed explication of that of Guinand [136].

Entry 3.3.3 (p. 253). *Let α and β denote positive numbers such that $\alpha\beta = \pi^2$. Then, if γ denotes Euler's constant,*

$$\sqrt{\alpha}\left(\frac{1}{4}\gamma - \frac{1}{4}\log(4\beta) + \sum_{n=1}^{\infty} d(n)K_0(2n\alpha)\right) = \sqrt{\beta}\left(\frac{1}{4}\gamma - \frac{1}{4}\log(4\alpha) + \sum_{n=1}^{\infty} d(n)K_0(2n\beta)\right). \tag{3.3.14}$$

Proof. In order to let $s \to 0$ in Entry 3.1.1, we need the well-known Laurent expansions [126, p. 944, formula 8.321, no. 1]

$$\Gamma(s) = \frac{1}{s} - \gamma + \cdots \tag{3.3.15}$$

and [306, pp. 19–20, Eqs. (2.4.3) and (2.4.5)]

$$\zeta(s) = -\frac{1}{2} - \frac{1}{2}\log(2\pi)s + \cdots . \tag{3.3.16}$$

Hence, letting $s \to 0$ in (3.1.5) and using (3.3.15) and (3.3.16), we find that

$$\begin{aligned}
&\sqrt{\alpha}\sum_{n=1}^{\infty} d(n)K_0(2n\alpha) - \sqrt{\beta}\sum_{n=1}^{\infty} d(n)K_0(2n\beta) \\
&= \frac{1}{4}\lim_{s\to 0}\left(\left\{\left(\frac{1}{s/2} - \gamma + \cdots\right)\left(-\frac{1}{2} - \frac{1}{2}\log(2\pi)s + \cdots\right)\right.\right. \\
&\qquad \left.\left. \times \left(\sqrt{\beta}\left\{1 - \frac{1}{2}s\log\beta + \cdots\right\} - \sqrt{\alpha}\left\{1 - \frac{1}{2}s\log\alpha + \cdots\right\}\right)\right\}\right.
\end{aligned} \tag{3.3.17}$$

$$
\begin{aligned}
&+\left\{\left(\frac{1}{-s/2}-\gamma+\cdots\right)\left(-\frac{1}{2}+\frac{1}{2}\log(2\pi)s+\cdots\right)\right.\\
&\left.\left.\times\left(\sqrt{\beta}\left\{1+\frac{1}{2}s\log\beta+\cdots\right\}-\sqrt{\alpha}\left\{1+\frac{1}{2}s\log\alpha+\cdots\right\}\right)\right\}\right)\\
&=\frac{1}{4}\gamma(\sqrt{\beta}-\sqrt{\alpha})-\frac{1}{2}\log(2\pi)(\sqrt{\beta}-\sqrt{\alpha})+\frac{1}{4}(\sqrt{\beta}\log\beta-\sqrt{\alpha}\log\alpha)\\
&=\frac{1}{4}\gamma(\sqrt{\beta}-\sqrt{\alpha})-\frac{1}{4}\log(4\alpha\beta)(\sqrt{\beta}-\sqrt{\alpha})+\frac{1}{4}(\sqrt{\beta}\log\beta-\sqrt{\alpha}\log\alpha),
\end{aligned}
$$

where in the last step we used the equality $\alpha\beta=\pi^2$. A simplification and rearrangement of (3.3.17) yield (3.3.14) to complete the proof. □

3.4 Kindred Formulas on Page 254 of the Lost Notebook

Entry 3.4.1 (p. 254). *If* $a>0$,

$$
\begin{aligned}
\int_0^\infty \frac{dx}{x(e^{2\pi x}-1)(e^{2\pi a/x}-1)} &= 2\sum_{n=1}^\infty d(n)K_0(4\pi\sqrt{an})\\
&=\frac{a}{\pi^2}\sum_{n=1}^\infty\frac{d(n)\log(a/n)}{a^2-n^2}-\frac{1}{2}\gamma-\left(\frac{1}{4}+\frac{1}{4\pi^2 a}\right)\log a-\frac{\log(2\pi)}{2\pi^2 a}. \quad (3.4.1)
\end{aligned}
$$

Proof. Expanding the integrand in geometric series, we find that

$$
\begin{aligned}
\int_0^\infty \frac{dx}{x(e^{2\pi x}-1)(e^{2\pi a/x}-1)} &= \sum_{m=1}^\infty\sum_{k=1}^\infty\int_0^\infty\frac{1}{x}e^{-2\pi(mx+ak/x)}dx\\
&=\sum_{m=1}^\infty\sum_{k=1}^\infty\int_0^\infty\frac{1}{u}e^{-2\pi(u+akm/u)}du\\
&=\sum_{n=1}^\infty d(n)\int_0^\infty\frac{1}{u}e^{-2\pi(u+an/u)}du\\
&=2\sum_{n=1}^\infty d(n)K_0(4\pi\sqrt{an}),
\end{aligned}
$$

by (3.2.1), which proves the first part of (3.4.1).

The second identity in (3.4.1) was actually first proved in print in 1966 by Soni [295]. Her proof is short, depends on Koshliakov's formula (3.1.1), and uses the integral evaluations (3.2.3) with $\nu=0$ and (3.2.4). We use her idea to prove the second major claim of Ramanujan on page 254. □

In contrast to the claims on the top and bottom thirds of page 254, the one claim in the middle of page 254 seems to be missing one element, and so we shall proceed as we think Ramanujan might have done. Proceeding as we did above and employing (3.2.1), we find that

$$\begin{aligned}\int_0^\infty \frac{dx}{\sqrt{x}(e^{2\pi x}-1)(e^{2\pi a/x}-1)} &= \sum_{m=1}^\infty \sum_{k=1}^\infty \frac{1}{\sqrt{m}} \int_0^\infty \frac{1}{\sqrt{u}} e^{-2\pi(u+akm/u)} du \\ &= \sum_{n=1}^\infty \sigma_{-1/2}(n) \int_0^\infty \frac{1}{\sqrt{u}} e^{-2\pi(u+an/u)} du \\ &= 2\sum_{n=1}^\infty \sigma_{-1/2}(n)(an)^{1/4} K_{1/2}(4\pi\sqrt{an}) \\ &= \frac{1}{\sqrt{2}} \sum_{n=1}^\infty \sigma_{-1/2}(n) e^{-4\pi\sqrt{an}}, \end{aligned} \tag{3.4.2}$$

where we have used (3.2.2). Ramanujan's next claim gives an identity for the last series above, with a replaced by $a/4$.

Entry 3.4.2 (p. 254). *For* $a > 0$,

$$\sum_{n=1}^\infty \sigma_{-1/2}(n) e^{-2\pi\sqrt{an}} = Ka \sum_{n=1}^\infty \frac{\sigma_{-1/2}(n)}{(n+a)(\sqrt{n}+\sqrt{a})} + \text{two trivial terms.} \tag{3.4.3}$$

Evidently, K on the right-hand side of (3.4.3) represents an unspecified constant. Ramanujan does not divulge the identities of the "two trivial terms." Our calculation in (3.4.2), showing a discrepancy with the series on the left-hand side of (3.4.3), actually provides a clue that this series in (3.4.3) *should be replaced by the series on the right-hand side of* (3.4.2). We next state a corrected version of Entry 3.4.2 providing the identities of the constant and the "trivial terms."

Entry 3.4.3 (p. 254). *If* $a > 0$, *then*

$$\begin{aligned}\sum_{n=1}^\infty \sigma_{-1/2}(n) e^{-4\pi\sqrt{an}} - \frac{a}{\pi} \sum_{n=1}^\infty \frac{\sigma_{-1/2}(n)}{(n+a)(\sqrt{n}+\sqrt{a})} \\ = \frac{1}{2}\zeta\left(\frac{1}{2}\right)\left(\frac{1}{\pi\sqrt{a}}-1\right) + \frac{1}{2}\zeta\left(-\frac{1}{2}\right)\left(4\pi\sqrt{a}-\frac{1}{\pi a}\right).\end{aligned} \tag{3.4.4}$$

Proof. In (3.1.5), set $s = \frac{1}{2}$ and $\alpha = x$, so that $\beta = \pi^2/x$. Then,

$$\begin{aligned}\sqrt{x}\sum_{n=1}^\infty \sigma_{-1/2}(n) n^{1/4} K_{1/4}(2nx) - \frac{\pi}{\sqrt{x}} \sum_{n=1}^\infty \sigma_{-1/2}(n) n^{1/4} K_{1/4}(2n\pi^2/x) \\ = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right)\left(\frac{\sqrt{\pi}}{x^{1/4}} - x^{1/4}\right) + \frac{1}{4}\Gamma\left(-\frac{1}{4}\right)\zeta\left(-\frac{1}{2}\right)\left(\frac{\pi^{3/2}}{x^{3/4}} - x^{3/4}\right).\end{aligned} \tag{3.4.5}$$

Multiply both sides of (3.4.5) by

$$\frac{1}{x^{5/2}} K_{1/4}(2a\pi^2/x)$$

and integrate over $(0,\infty)$. Inverting the order of summation and integration by absolute convergence, we find that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sigma_{-1/2}(n) n^{1/4} \int_0^{\infty} \frac{1}{x^2} K_{1/4}(2nx) K_{1/4}(2a\pi^2/x)dx \qquad (3.4.6)\\
&\quad - \pi \sum_{n=1}^{\infty} \sigma_{-1/2}(n) n^{1/4} \int_0^{\infty} \frac{1}{x^3} K_{1/4}(2n\pi^2/x) K_{1/4}(2a\pi^2/x)dx\\
&= \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right)\left(\sqrt{\pi}I_3 - I_1\right) + \frac{1}{4}\Gamma\left(-\frac{1}{4}\right)\zeta\left(-\frac{1}{2}\right)\left(\pi^{3/2}I_5 - I_{-1}\right),
\end{aligned}$$

where

$$I_j = \int_0^{\infty} u^{j/4} K_{1/4}(2a\pi^2 u)du, \qquad (3.4.7)$$

and where to obtain the four integrals on the right-hand side of (3.4.6), we made the change of variable $x = 1/u$ in each one.

We examine each of the six integrals in (3.4.6) in turn. First, using (3.2.3) and (3.2.2), we find that

$$\begin{aligned}
\int_0^{\infty} \frac{1}{x^2} K_{1/4}(2nx) K_{1/4}(2a\pi^2/x)dx &= \frac{1}{2a\pi} K_{1/2}(4\pi\sqrt{an})\\
&= \frac{1}{4\sqrt{2}a^{5/4}n^{1/4}\pi} e^{-4\pi\sqrt{an}}. \qquad (3.4.8)
\end{aligned}$$

Second, making the change of variable $u = \pi^2/x$ and using (3.2.5), we deduce that

$$\begin{aligned}
\int_0^{\infty} \frac{1}{x^3} K_{1/4}(2n\pi^2/x) K_{1/4}(2a\pi^2/x)dx &= \frac{1}{\pi^4}\int_0^{\infty} u K_{1/4}(2nu) K_{1/4}(2au)du\\
&= \frac{1}{\pi^4}\frac{\pi(4na)^{-1/4}(\sqrt{2n}-\sqrt{2a})}{2\sin(\pi/4)(4n^2-4a^2)}\\
&= \frac{\sqrt{2}(an)^{-1/4}}{8\pi^3(n+a)(\sqrt{n}+\sqrt{a})}. \qquad (3.4.9)
\end{aligned}$$

In our calculations of I_j, $j = 3, 1, 5, -1$, we employ (3.2.6). Thus,

$$I_3 = 2^{-1/4}(2a\pi^2)^{-7/4}\Gamma(1)\Gamma\left(\frac{3}{4}\right) = \frac{1}{4a^{7/4}\pi^{7/2}}\Gamma\left(\frac{3}{4}\right), \qquad (3.4.10)$$

$$I_1 = 2^{-3/4}(2a\pi^2)^{-5/4}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right) = \frac{1}{4a^{5/4}\pi^2}\Gamma\left(\frac{3}{4}\right), \qquad (3.4.11)$$

$$I_5 = 2^{1/4}(2a\pi^2)^{-9/4}\Gamma\left(\frac{5}{4}\right)\Gamma(1) = \frac{1}{4a^{9/4}\pi^{9/2}}\Gamma\left(\frac{5}{4}\right), \tag{3.4.12}$$

$$I_{-1} = 2^{-5/4}(2a\pi^2)^{-3/4}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right) = \frac{1}{4a^{3/4}\pi}\Gamma\left(\frac{1}{4}\right). \tag{3.4.13}$$

Using (3.4.8)–(3.4.13) in (3.4.6) and making frequent use of the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

we deduce that

$$\frac{1}{4\sqrt{2}a^{5/4}\pi}\sum_{n=1}^{\infty}\sigma_{-1/2}(n)e^{-4\pi\sqrt{an}} - \frac{1}{4\sqrt{2}a^{1/4}\pi^2}\sum_{n=1}^{\infty}\frac{\sigma_{-1/2}(n)}{(n+a)(\sqrt{n}+\sqrt{a})}$$
$$= \frac{\sqrt{2}}{16}\zeta\left(\frac{1}{2}\right)\left(\frac{1}{a^{7/4}\pi^2} - \frac{1}{a^{5/4}\pi}\right) + \frac{\sqrt{2}}{16}\zeta\left(-\frac{1}{2}\right)\left(-\frac{1}{a^{9/4}\pi^2} + \frac{4}{a^{3/4}}\right). \tag{3.4.14}$$

If we multiply both sides of (3.4.14) by $4\sqrt{2}a^{5/4}\pi$ and rearrange slightly, we obtain (3.4.4) to complete the proof. □

We record the last two results on page 254 as Ramanujan wrote them, except that we express the results in terms of Bessel functions. The constant K and the "two trivial terms" are not the same as they are in Entry 3.4.2.

Entry 3.4.4 (p. 254). *If $a > 0$, then*

$$\int_0^{\infty}\frac{dx}{(e^{2\pi x}-1)(e^{2\pi a/x}-1)} = 2\sqrt{a}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an}) \tag{3.4.15}$$

$$= Ka^2\sum_{n=1}^{\infty}\frac{\sigma_{-1}(n)}{n(n+a)} + \text{two trivial terms}. \tag{3.4.16}$$

Proof. We prove (3.4.15). Expanding the integrand in geometric series, setting $mx = u$, and invoking (3.2.1), we find that

$$\int_0^{\infty}\frac{dx}{(e^{2\pi x}-1)(e^{2\pi a/x}-1)} = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{m}\int_0^{\infty}e^{-2\pi(u+akm/u)}du$$
$$= \sum_{n=1}^{\infty}\sigma_{-1}(n)\int_0^{\infty}e^{-2\pi(u+an/u)}du$$
$$= 2\sqrt{a}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an}).$$

□

Lastly, we provide and prove a more precise version of (3.4.16) giving the identities of the missing terms.

Entry 3.4.5 (p. 254). *If* $a > 0$ *and* γ *denotes Euler's constant, then*

$$2\sqrt{a}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an})$$
$$= -\frac{a^2}{2\pi}\sum_{n=1}^{\infty}\frac{\sigma_{-1}(n)}{n(n+a)}+\frac{a}{2\pi}\left((\log a+\gamma)\zeta(2)+\zeta'(2)\right)+\frac{1}{4\pi}(\log 2a\pi+\gamma)+\frac{1}{48a\pi}. \tag{3.4.17}$$

Proof. In (3.3.10), set $\alpha = x$, so that $\beta = \pi^2/x$. Recalling (3.2.2), we find that

$$\begin{aligned}\frac{2}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{nx}K_{1/2}(2nx)\\ &= \left(\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2n\pi^2/x}-\frac{x}{12}\right)+\frac{1}{2}\log\frac{x}{\pi}+\frac{\pi^2}{12x}\\ &=: I_1+I_2+I_3.\end{aligned} \tag{3.4.18}$$

Next, multiply both sides of (3.4.18) by

$$\frac{1}{x^{5/2}}K_{1/2}(2a\pi^2/x)$$

and integrate over $(0,\infty)$.

Consider first the series arising on the left-hand side of (3.4.18). Inverting the order of summation and integration on the left-hand side by absolute convergence, we arrive at

$$\begin{aligned}\frac{2}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{n}\int_0^{\infty}\frac{1}{x^2}K_{1/2}(2nx)K_{1/2}(2a\pi^2/x)dx\\ &= \frac{1}{a\pi^{3/2}}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an}),\end{aligned} \tag{3.4.19}$$

where we have employed (3.2.3).

Second, the contribution from I_3 in (3.4.18) is given by

$$\frac{\pi^2}{12}\int_0^{\infty}x^{-7/2}K_{1/2}(2a\pi^2/x)dx = \frac{\pi^2}{12}\int_0^{\infty}u^{3/2}K_{1/2}(2a\pi^2u)du = \frac{1}{96a^{5/2}\pi^{5/2}}, \tag{3.4.20}$$

where we used (3.2.6) in the last step with $\mu = \frac{3}{2}$, $\nu = \frac{1}{2}$, and a replaced by $2a\pi^2$.

Third, using (3.2.2), we find that the contribution from I_2 in (3.4.18) is equal to

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty x^{-5/2}\log(x/\pi)K_{1/2}(2a\pi^2/x)dx \\
&\quad= \frac{1}{4\sqrt{a\pi}}\int_0^\infty x^{-2}\log(x/\pi)e^{-2a\pi^2/x}dx \\
&\quad= \frac{1}{8a^{3/2}\pi^{5/2}}\int_0^\infty \log(2a\pi/u)e^{-u}du \\
&\quad= \frac{1}{8a^{3/2}\pi^{5/2}}\left\{\int_0^\infty e^{-u}\log(2a\pi)du - \int_0^\infty e^{-u}\log u\, du\right\} \\
&\quad= \frac{1}{8a^{3/2}\pi^{5/2}}\left\{\log(2a\pi) - \int_0^\infty e^{-u}\log u\, du\right\} \\
&\quad= \frac{1}{8a^{3/2}\pi^{5/2}}\left\{\log(2a\pi)+\gamma\right\}, \qquad (3.4.21)
\end{aligned}$$

since [126, p. 602, formula 4.331, no. 1]

$$\gamma = -\int_0^\infty e^{-u}\log u\, du.$$

Finally, the contribution from I_1 in (3.4.18) is given by

$$J := \int_0^\infty \left(\sum_{n=1}^\infty \sigma_{-1}(n)e^{-2n\pi^2/x} - \frac{1}{12}x\right)x^{-5/2}K_{1/2}(2a\pi^2/x)dx. \qquad (3.4.22)$$

Recall that $\zeta(2) = \pi^2/6$. Thus, we can write

$$\begin{aligned}
\sum_{n=1}^\infty \sigma_{-1}(n)e^{-2n\pi^2/x} - \frac{1}{12}x &= \sum_{n=1}^\infty\sum_{d|n}\frac{1}{d}e^{-2n\pi^2/x} - \frac{1}{12}x \\
&= \sum_{d=1}^\infty\sum_{m=1}^\infty \frac{1}{d}e^{-2md\pi^2/x} - \frac{1}{12}x \\
&= \sum_{d=1}^\infty \frac{1}{d}\frac{1}{e^{2d\pi^2/x}-1} - \left(\sum_{n=1}^\infty\frac{1}{n^2}\right)\frac{x}{2\pi^2}. \qquad (3.4.23)
\end{aligned}$$

Using (3.4.23) and (3.2.2) in (3.4.22), we see that

$$\begin{aligned}
J &= \int_0^\infty \left(\sum_{n=1}^\infty \frac{1}{n}\frac{1}{e^{2n\pi^2/x}-1} - \left(\sum_{n=1}^\infty\frac{1}{n^2}\right)\frac{x}{2\pi^2}\right)\frac{1}{2\sqrt{a\pi}}e^{-2a\pi^2/x}\frac{dx}{x^2} \\
&= \frac{1}{2\sqrt{a\pi}}\int_0^\infty \sum_{n=1}^\infty \frac{1}{n}\left(\frac{1}{e^{2n\pi^2/x}-1} - \frac{1}{2n\pi^2/x}\right)e^{-2a\pi^2/x}\frac{dx}{x^2}. \qquad (3.4.24)
\end{aligned}$$

Since for $z > 0$,

$$\frac{1}{e^z - 1} - \frac{1}{z} < 0,$$

we can change the order of summation and integration by the monotone convergence theorem. Hence,

$$\begin{aligned} J &= \frac{1}{2\sqrt{a\pi}} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} \left(\frac{1}{e^{2n\pi^2/x} - 1} - \frac{1}{2n\pi^2/x} \right) e^{-2a\pi^2/x} \frac{dx}{x^2} \\ &= \frac{1}{4\sqrt{a}\pi^{5/2}} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{\infty} \left(\frac{1}{e^u - 1} - \frac{1}{u} \right) e^{-au/n} du. \end{aligned} \tag{3.4.25}$$

Consider now two different expressions for the logarithmic derivative of the gamma function, namely [126, p. 952, formula 8.362, no. 1; formula 8.361, no. 8],

$$\begin{aligned} \frac{\Gamma'(z)}{\Gamma(z)} &= -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(n+z)} \\ &= \log z - \frac{1}{z} - \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) e^{-tz} dt, \end{aligned}$$

where $\operatorname{Re} z > 0$. Hence,

$$\int_0^{\infty} \left(\frac{1}{e^u - 1} - \frac{1}{u} \right) e^{-au/n} du = \log(a/n) + \gamma - \sum_{m=1}^{\infty} \frac{a}{m(mn+a)}. \tag{3.4.26}$$

Putting (3.4.26) in (3.4.25), we find that

$$\begin{aligned} J &= \frac{1}{4a^{1/2}\pi^{5/2}} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\log(a/n) + \gamma - \sum_{m=1}^{\infty} \frac{a}{m(mn+a)} \right) \\ &= \frac{1}{4a^{1/2}\pi^{5/2}} \left((\log a + \gamma)\zeta(2) - \sum_{n=1}^{\infty} \frac{\log n}{n^2} - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a}{n^2 m(mn+a)} \right) \\ &= \frac{1}{4a^{1/2}\pi^{5/2}} \left((\log a + \gamma)\zeta(2) + \zeta'(2) - a \sum_{n=1}^{\infty} \frac{\sigma_{-1}(n)}{n(n+a)} \right) \\ &= -\frac{a^{1/2}}{4\pi^{5/2}} \sum_{n=1}^{\infty} \frac{\sigma_{-1}(n)}{n(n+a)} + \frac{1}{4a^{1/2}\pi^{5/2}} ((\log a + \gamma)\zeta(2) + \zeta'(2)). \end{aligned} \tag{3.4.27}$$

We now combine all our calculations that arose from (3.4.18), namely, (3.4.19)–(3.4.22), and (3.4.27), to deduce that

$$\begin{aligned} \frac{1}{a\pi^{3/2}} \sum_{n=1}^{\infty} \sigma_{-1}(n) \sqrt{n} K_1(4\pi\sqrt{an}) &= \frac{1}{96a^{5/2}\pi^{5/2}} + \frac{1}{8a^{3/2}\pi^{5/2}} \{\log(2a\pi) + \gamma\} \\ &\quad - \frac{a^{1/2}}{4\pi^{5/2}} \sum_{n=1}^{\infty} \frac{\sigma_{-1}(n)}{n(n+a)} + \frac{1}{4a^{1/2}\pi^{5/2}} ((\log a + \gamma)\zeta(2) + \zeta'(2)). \end{aligned} \tag{3.4.28}$$

Finally multiply both sides of (3.4.28) by $2\pi^{3/2}a^{3/2}$ to deduce (3.4.17) and complete the proof. □

Analogues of Guinand's formula in Entry 3.3.1 and Watson's lemma (Lemma 3.3.1) have been derived by Berndt [27]. These analogues are also discussed in the paper [62] on which this chapter is based. Analogues of Guinand's and Koshliakov's formulas with characters in the summands have been derived by Berndt, A. Dixit, and Sohn [52]. A different character analogue of Koshliakov's formula along with a connection to integrals of Dirichlet L-functions that are analogues of Ramanujan's famous integrals involving Riemann's Ξ-function [257] has been derived by Dixit [110]. H. Cohen [98] has continued the line of investigation represented by Entry 3.4.5 and has derived several interesting formulas of the same sort.

Dixit [107] has derived a beautiful extension of Koshliakov's formula. Recall that Riemann's ξ-function is defined by

$$\xi(s) := (s-1)\pi^{-s/2}\Gamma(1+\tfrac{1}{2}s)\zeta(s), \tag{3.4.29}$$

and that his Ξ-function is defined by

$$\Xi(t) := \xi(\tfrac{1}{2}+it). \tag{3.4.30}$$

We now state Dixit's extension [107].

Theorem 3.4.1 (Extended version of Koshliakov's formula). *Let $\Xi(t)$ be defined by (3.4.30). If α and β are positive numbers such that $\alpha\beta = 1$, then*

$$\begin{aligned}
\sqrt{\alpha}&\left(\frac{\gamma-\log(4\pi\alpha)}{\alpha} - 4\sum_{n=1}^{\infty} d(n)K_0(2\pi n\alpha)\right)\\
&= \sqrt{\beta}\left(\frac{\gamma-\log(4\pi\beta)}{\beta} - 4\sum_{n=1}^{\infty} d(n)K_0(2\pi n\beta)\right)\\
&= -\frac{32}{\pi}\int_0^\infty \frac{\left(\Xi\left(\tfrac{1}{2}t\right)\right)^2\cos\left(\tfrac{1}{2}t\log\alpha\right)\,dt}{(1+t^2)^2}.
\end{aligned} \tag{3.4.31}$$

Dixit first showed that the far left side of (3.4.31) is equal to the integral on the far right-hand side. Next observe that if we put $\alpha = 1/\beta$ in this equality, then the first equality in (3.4.31) easily follows. Koshliakov [191] derived a formula similar to (3.4.31). Essentially, his formula arises from taking the Fourier cosine transform of both sides of (3.4.31).

Dixit [109] has also extended Guinand's formula.

Theorem 3.4.2 (Extended version of Guinand's formula). *If α and β are positive numbers such that $\alpha\beta = 1$, then for $-1 < \operatorname{Re} z < 1$,*

$$\begin{aligned}
&\sqrt{\alpha}\left(\alpha^{z/2-1}\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z)+\alpha^{-z/2-1}\pi^{z/2}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\right.\\
&\qquad\left.-4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{z/2}\left(2n\pi\alpha\right)\right)\\
&=\sqrt{\beta}\left(\beta^{z/2-1}\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z)+\beta^{-z/2-1}\pi^{z/2}\Gamma\left(-\frac{z}{2}\right)\zeta(-z)\right.\\
&\qquad\left.-4\sum_{n=1}^{\infty}\sigma_{-z}(n)n^{z/2}K_{z/2}\left(2n\pi\beta\right)\right)\\
&=-\frac{32}{\pi}\int_0^{\infty}\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\,dt.
\end{aligned}\tag{3.4.32}$$

As with Dixit's extension of Koshliakov's formula, suppose that we can show that the far left side of (3.4.32) is equal to the far right side above. Then if we set $\alpha = 1/\beta$ in this equality, the first equality of (3.4.32) follows. Dixit [109] has obtained a companion theorem to Theorem 3.4.2 for $|\operatorname{Re} z| > 1$.