2

Double Series of Bessel Functions and the Circle and Divisor Problems

2.1 Introduction

In this chapter we establish identities that express certain finite trigonometric sums as double series of Bessel functions. These results, stated in Entries 2.1.1 and 2.1.2 below, are identities claimed by Ramanujan on page 335 in his lost notebook [269], for which no indications of proofs are given. (Technically, page 335 is *not* in Ramanujan's lost notebook; this page is a fragment published by Narosa *with* the original lost notebook.) As we shall see in the sequel, the identities are intimately connected with the famous *circle* and *divisor problems*, respectively. The first identity involves the ordinary Bessel function $J_1(z)$, where the more general ordinary Bessel function $J_\nu(z)$ is defined by

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{\nu+2n}, \qquad 0 < |z| < \infty, \qquad \nu \in \mathbb{C}. \quad (2.1.1)$$

The second identity involves the Bessel function of the second kind $Y_1(z)$ [314, p. 64, Eq. (1)], with $Y_\nu(z)$ more generally defined by

$$Y_\nu(z) := \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}, \quad (2.1.2)$$

and the modified Bessel function $K_1(z)$, with $K_\nu(z)$ [314, p. 78, Eq. (6)] defined, for $-\pi < \arg z < \frac{1}{2}\pi$, by

$$K_\nu(z) := \frac{\pi}{2} \frac{e^{\pi i\nu/2}J_{-\nu}(iz) - e^{-\pi i\nu/2}J_\nu(iz)}{\sin(\nu\pi)}. \quad (2.1.3)$$

If ν is an integer n, then it is understood that we define the functions by taking the limits as $\nu \to n$ in (2.1.2) and (2.1.3).

To state Ramanujan's claims, we need to next define

G.E. Andrews and B.C. Berndt, *Ramanujan's Lost Notebook: Part IV*, DOI 10.1007/978-1-4614-4081-9_2,

$$F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer,} \end{cases} \tag{2.1.4}$$

where, as customary, $[x]$ is the greatest integer less than or equal to x.

Entry 2.1.1 (p. 335). *Let $F(x)$ be defined by (2.1.4). If $0 < \theta < 1$ and $x > 0$, then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \sin(2\pi n\theta) = \pi x \left(\frac{1}{2} - \theta\right) - \frac{1}{4}\cot(\pi\theta) + \frac{1}{2}\sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}} \right\}. \tag{2.1.5}$$

Entry 2.1.2 (p. 335). *Let $F(x)$ be defined by (2.1.4). Then, for $x > 0$ and $0 < \theta < 1$,*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \cos(2\pi n\theta) = \frac{1}{4} - x\log(2\sin(\pi\theta)) + \frac{1}{2}\sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{I_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} + \frac{I_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}} \right\}, \tag{2.1.6}$$

where

$$I_\nu(z) := -Y_\nu(z) - \frac{2}{\pi}K_\nu(z). \tag{2.1.7}$$

Ramanujan's formulation of (2.1.5) is given in the form

$$\begin{aligned} &\left[\frac{x}{1}\right]\sin(2\pi\theta) + \left[\frac{x}{2}\right]\sin(4\pi\theta) + \left[\frac{x}{3}\right]\sin(6\pi\theta) + \left[\frac{x}{4}\right]\sin(8\pi\theta) + \cdots \\ &= \pi x\left(\frac{1}{2} - \theta\right) - \frac{1}{4}\cot(\pi\theta) + \frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\left\{\frac{J_1(4\pi\sqrt{m\theta x})}{\sqrt{m\theta}} - \frac{J_1(4\pi\sqrt{m(1-\theta)x})}{\sqrt{m(1-\theta)}}\right. \\ &\left. + \frac{J_1(4\pi\sqrt{m(1+\theta)x})}{\sqrt{m(1+\theta)}} - \frac{J_1(4\pi\sqrt{m(2-\theta)x})}{\sqrt{m(2-\theta)}} + \frac{J_1(4\pi\sqrt{m(2+\theta)x})}{\sqrt{m(2+\theta)}} - \cdots\right\}, \end{aligned} \tag{2.1.8}$$

"where $[x]$ denotes the greatest integer in x if x is not an integer and $x - \frac{1}{2}$ if x is an integer." His formulation of (2.1.6) is similar. Since Ramanujan employed the notation $[x]$ in a nonstandard fashion, we think it is advisable to introduce the alternative notation (2.1.4). As we shall see in the sequel, there is some evidence that Ramanujan did not intend the double sums to be interpreted as iterated sums, but as double sums in which the product mn of the summation indices tends to ∞.

Note that the series on the left-hand sides of (2.1.5) and (2.1.6) are finite, and discontinuous if x is an integer. To examine the right-hand side of (2.1.5), we recall [314, p. 199] that, as $x \to \infty$,

$$J_\nu(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right) + O\left(\frac{1}{x^{3/2}}\right). \tag{2.1.9}$$

Hence, as $m, n \to \infty$, the terms of the double series on the right-hand side of (2.1.5) are asymptotically equal to

$$\frac{1}{\pi\sqrt{2}x^{1/4}m^{3/4}} \left(\frac{\cos\left(4\pi\sqrt{m(n+\theta)x} - \frac{3}{4}\pi\right)}{(n+\theta)^{3/4}} - \frac{\cos\left(4\pi\sqrt{m(n+1-\theta)x} - \frac{3}{4}\pi\right)}{(n+1-\theta)^{3/4}} \right).$$

Thus, if indeed the double series on the right side of (2.1.5) does converge, it converges conditionally and not absolutely. A similar argument clearly pertains to (2.1.6).

We now discuss in detail Entry 2.1.1; our discourse will then be followed by a detailed account of Entry 2.1.2.

It is natural to ask what led Ramanujan to the double series on the right side of (2.1.5). Let $r_2(n)$ denote the number of representations of the positive integer n as a sum of two squares. Recall that the famous *circle problem* is to determine the precise order of magnitude, as $x \to \infty$, for the "error term" $P(x)$, defined by

$$\sideset{}{'}\sum_{0 \le n \le x} r_2(n) = \pi x + P(x), \tag{2.1.10}$$

where the prime $'$ on the summation sign on the left side indicates that if x is an integer, only $\frac{1}{2}r_2(x)$ is counted. Moreover, we define $r_2(0) = 1$. In [144], Hardy showed that $P(x) \neq O(x^{1/4})$, as x tends to ∞. (He actually showed a slightly stronger result.)

In 1906, W. Sierpiński [288] proved that $P(x) = O(x^{1/3})$, as $x \to \infty$. After Sierpiński's work, most efforts toward obtaining an upper bound for $P(x)$ have ultimately rested upon the identity

$$\sideset{}{'}\sum_{0 \le n \le x} r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n) \left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}), \tag{2.1.11}$$

(2.1.9), and methods of estimating the resulting trigonometric series. Here, the prime $'$ on the summation sign on the left side has the same meaning as above. The identity (2.1.11) was first published and proved in Hardy's paper [144], [150, pp. 243–263]. In a footnote, Hardy [150, p. 245] remarks, "The form of this equation was suggested to me by Mr. S. Ramanujan, to whom I had communicated the analogous formula for $d(1)+d(2)+\cdots+d(n)$, where $d(n)$ is the number of divisors of n." Thus, it is possible that Ramanujan was the first to prove (2.1.11), although we do not know anything about his derivation.

Observe that the summands in the series on the right side of (2.1.11) are similar to those on the right side of (2.1.5). Moreover, the sums on the left side in each formula are finite sums over $n \leq x$. Thus, it seems plausible that there is a connection between these two formulas, and as we shall see, indeed there is. Ramanujan might therefore have derived (2.1.5) in anticipation of applying it to the circle problem.

In his paper [144], Hardy relates a beautiful identity of Ramanujan connected with $r_2(n)$, namely, for $a, b > 0$, [144, p. 283], [150, p. 263],

$$\sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+a}} e^{-2\pi\sqrt{(n+a)b}} = \sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+b}} e^{-2\pi\sqrt{(n+b)a}},$$

which is not given elsewhere in any of Ramanujan's published or unpublished work. If we differentiate the identity above with respect to b, let $a \to 0$, replace $2\pi\sqrt{b}$ by s, and use analytic continuation, we find that for $\operatorname{Re} s > 0$,

$$\sum_{n=1}^{\infty} r_2(n) e^{-s\sqrt{n}} = \frac{2\pi}{s^2} - 1 + 2\pi s \sum_{n=1}^{\infty} \frac{r_2(n)}{(s^2 + 4\pi^2 n)^{3/2}},$$

which was the key identity in Hardy's proof that $P(x) \neq O(x^{1/4})$, as $x \to \infty$.

In summary, there is considerable evidence that while Ramanujan was at Cambridge, he and Hardy discussed the *circle problem*, and it is likely that Entry 2.1.1 was motivated by these discussions.

Note that if the factors $\sin(2\pi n\theta)$ were missing on the left side of (2.1.5), then this sum would coincide with the number of integral points (n, l) with $n, l \geq 1$ and $nl \leq x$, where the pairs (n, l) satisfying $nl = x$ are counted with weight $\frac{1}{2}$. Hence,

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) = \sideset{}{'}\sum_{1 \leq n \leq x} d(n), \tag{2.1.12}$$

where $d(n)$ denotes the number of divisors of n, and the prime $\prime$ on the summation sign indicates that if x is an integer, only $\frac{1}{2}d(x)$ is counted. Of course, similar remarks hold for the left side of (2.1.6). Therefore one may interpret the left sides of (2.1.5) and (2.1.6) as weighted divisor sums.

Berndt and A. Zaharescu [71] first proved Entry 2.1.1, but with the order of summation on the double sum *reversed* from that recorded by Ramanujan. The authors of [71] proved this emended version of Ramanujan's claim by first replacing Entry 2.1.1 with the following equivalent theorem.

Theorem 2.1.1. *For $0 < \theta < 1$ and $x > 0$,*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin(2\pi n\theta) - \pi x\left(\frac{1}{2}-\theta\right)$$
$$= \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{n+\theta}\sin^2\left(\frac{\pi(n+\theta)x}{m}\right) - \frac{1}{n+1-\theta}\sin^2\left(\frac{\pi(n+1-\theta)x}{m}\right)\right). \tag{2.1.13}$$

It should be emphasized that this reformulation fails to exist for Ramanujan's original formulation in Entry 2.1.1. After proving the aforementioned alternative version of Entry 2.1.1, the authors of [71] derived an identity involving the twisted character sums

$$d_\chi(n) = \sum_{k|n}\chi(k), \tag{2.1.14}$$

where χ is an odd primitive character modulo q. The following theorem on twisted character sums is proved in [71]; we have corrected the sign on the second expression on the right-hand side. The prime $'$ on the summation sign has the same meaning as it does in our discussions above, e.g., as in (2.1.10).

Theorem 2.1.2. *Let q be a positive integer, let χ be an odd primitive character modulo q, and let $d_\chi(n)$ be defined by (2.1.14). Then, for any $x > 0$,*

$$\sideset{}{'}\sum_{1\le n\le x} d_\chi(n) = L(1,\chi)x + \frac{i\tau(\chi)}{2\pi}L(1,\bar\chi) + \frac{i\sqrt{x}}{\tau(\bar\chi)}\sum_{1\le h<q/2}\bar\chi(h)$$
$$\times\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\frac{h}{q})x}\right)}{\sqrt{m(n+\frac{h}{q})}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\frac{h}{q})x}\right)}{\sqrt{m(n+1-\frac{h}{q})}}\right\}, \tag{2.1.15}$$

where $L(s,\chi)$ denotes the Dirichlet L-function associated with the character χ, and $\tau(\chi)$ denotes the Gauss sum

$$\tau(\chi) = \sum_{m=1}^{q}\chi(m)e^{2\pi i m/q}. \tag{2.1.16}$$

Using Theorem 2.1.2, Berndt and Zaharescu [71] derived a representation for $\sum'_{0\le n\le x} r_2(n)$.

Corollary 2.1.1. *For any $x > 0$,*

$$\sideset{}{'}\sum_{0\le n\le x} r_2(n) = \pi x$$
$$+ 2\sqrt{x}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\frac{1}{4})x}\right)}{\sqrt{m(n+\frac{1}{4})}} - \frac{J_1\left(4\pi\sqrt{m(n+\frac{3}{4})x}\right)}{\sqrt{m(n+\frac{3}{4})}}\right\}. \tag{2.1.17}$$

A possible advantage in using (2.1.17) in the circle problem is that $r_2(n)$ does not occur on the right side of (2.1.17), as in (2.1.11). On the other hand, the double series is likely to be more difficult to estimate than a single infinite series.

The summands in (2.1.17) have a remarkable resemblance to those in (2.1.11). It is therefore natural to ask whether the two identities are equivalent. We next show that (2.1.11) and (2.1.17) are *formally* equivalent. The key to this equivalence is a famous result of Jacobi. Let χ be the nonprincipal Dirichlet character modulo 4. Then Jacobi's formula [167], [44, p. 56, Theorem 3.2.1] is given by

$$r_2(n) = 4 \sum_{\substack{d|n \\ d\,\text{odd}}} (-1)^{(d-1)/2} =: 4d_\chi(n), \tag{2.1.18}$$

for all positive integers n. Therefore,

$$\begin{aligned}
&\sum_{k=1}^{\infty} r_2(k) \left(\frac{x}{k}\right)^{1/2} J_1(2\pi\sqrt{kx}) \\
&= 4 \sum_{k=1}^{\infty} \sum_{\substack{d|k \\ d \text{ odd}}} (-1)^{(d-1)/2} \left(\frac{x}{k}\right)^{1/2} J_1(2\pi\sqrt{kx}) \\
&= 4\sqrt{x} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left(\frac{J_1(2\pi\sqrt{m(4n+1)x})}{\sqrt{m(4n+1)}} - \frac{J_1(2\pi\sqrt{m(4n+3)x})}{\sqrt{m(4n+3)}} \right) \\
&= 2\sqrt{x} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left(\frac{J_1(4\pi\sqrt{m(n+\frac{1}{4})x})}{\sqrt{m(n+\frac{1}{4})}} - \frac{J_1(4\pi\sqrt{m(n+\frac{3}{4})x})}{\sqrt{m(n+\frac{3}{4})}} \right).
\end{aligned} \tag{2.1.19}$$

Hence, we have shown that (2.1.11) and (2.1.17) are versions of the same identity, *provided that the rearrangement of series in* (2.1.19) *is justified.* (J.L. Hafner [139] independently has also shown the formal equivalence of (2.1.11) and (2.1.17).)

In this chapter, we prove Entry 2.1.1 under two different interpretations, the first with the double series on the right-hand side summed in the order specified by Ramanujan, and the second with the double series on the right side interpreted as a double sum in which the product mn of the summation indices m and n tends to infinity. The former proof first appeared in a paper by Berndt, S. Kim, and Zaharescu [60], while the latter proof is taken from another paper [57] by the same trio of authors. We do not here give a proof of Entry 2.1.1 with the order of summation on the right-hand side of (2.1.5) reversed [71]. We emphasize that the three proofs of Entry 2.1.1 under different interpretations of the double sum on the right-hand side are entirely different; we are unable to use any portion or any idea of one proof in any of the other two proofs.

Having thoroughly discussed Entry 2.1.1, we turn our attention to Entry 2.1.2. Entry 2.1.2 was examined in detail in [48], where numerical calculations were extensively discussed with the conclusion that the entry might not be correct, because, in particular, the authors were not convinced that the double series of Bessel functions converges. Further evidence for the falsity of Entry 2.1.2 was also presented. Finding a proof of Entry 2.1.2, either in the form in which Ramanujan recorded it, or in the form in which the order of the double series is reversed, turned out to be more difficult than establishing a proof of Entry 2.1.2 in [71] for the following reasons: The Bessel functions $Y_1(z)$ and $K_1(z)$ have singularities at the origin. There is a lack of the "cancellation" in the pairs of Bessel functions on the right-hand side of (2.1.6) (where a plus sign separates the pairs of Bessel functions) that is evinced in (2.1.5) (where a minus sign separates the pairs of Bessel functions). We have a much less convenient intermediary theorem, Theorem 2.4.2, instead of Theorem 2.1.1, which replaces the proposed double Bessel series identity by a double trigonometric series identity. At this writing, we are unable to prove Entry 2.1.1 with the order of summation prescribed by Ramanujan. However, we can prove Entry 2.1.2 if we invert the order of summation or if we let the product of the indices of summation tend to infinity. Moreover, as we shall see in our proof, we need to make one further assumption in order to prove Entry 2.1.2 with the double series summed in reverse order.

As noted above, let $d(n)$ denote the number of positive divisors of the positive integer n. Define the "error term" $\Delta(x)$, for $x > 0$, by

$$\sum_{n\leq x}{}' d(n) = x\left(\log x + (2\gamma - 1)\right) + \frac{1}{4} + \Delta(x), \tag{2.1.20}$$

where γ denotes Euler's constant, and where the prime $'$ on the summation sign on the left side indicates that if x is an integer, then only $\frac{1}{2}d(x)$ is counted. The famous *Dirichlet divisor problem* asks for the correct order of magnitude of $\Delta(x)$ as $x \to \infty$. M.G. Voronoï [310] established a representation for $\Delta(x)$ in terms of Bessel functions with his famous formula

$$\sum_{n\leq x}{}' d(n) = x\left(\log x + (2\gamma - 1)\right) + \frac{1}{4} + \sum_{n=1}^{\infty} d(n)\left(\frac{x}{n}\right)^{1/2} I_1(4\pi\sqrt{nx}), \tag{2.1.21}$$

where $x > 0$ and $I_1(z)$ is defined by (2.1.7). Since the appearance of (2.1.21) in 1904, this identity has been the starting point for most attempts at finding an upper bound for $\Delta(x)$. Readers will note a remarkable similarity between the Bessel functions in (2.1.6) and those in (2.1.21), indicating that there must be a connection between these two formulas.

From the argument that we made in (2.1.19), it is reasonable to guess that Ramanujan might have regarded the double series in (2.1.5) symmetrically, i.e., that Ramanujan really was thinking of the double sum in the form $\lim_{N\to\infty}\sum_{mn\leq N}$. Thus, as with (2.1.5), we also prove (2.1.6) with the double series being interpreted symmetrically. Our proof uses (2.1.21) and twisted, or weighted, divisor sums. Our proofs of Entry 2.1.2 under the two interpretations that we have discussed first appeared in [57].

The identities in Entries 2.1.1 and 2.1.2, with the double series interpreted as iterated double series, might give researchers new tools in approaching the circle and divisor problems, respectively. The additional parameter θ in the two primary Bessel function identities might be useful in a yet unforeseen way.

In summary, there are three ways to interpret the double series in Entries 2.1.1 and 2.1.2. Our proofs in this volume cover both entries in two of the three possible interpretations.

Analogues of the problems of estimating the error terms $P(x)$ and $\Delta(x)$ exist for many other arithmetic functions $a(n)$ generated by Dirichlet series satisfying a functional equation involving the gamma function $\Gamma(s)$. See, for example, a paper by K. Chandrasekharan and R. Narasimhan [90]. As with the cases of $r_2(n)$ and $d(n)$, representations in terms of Bessel functions for $\sum_{n\leq x} a(n)$ and more generally for $\sum_{n\leq x} a(n)(x-n)^q$, which are occasionally called Riesz sums, play a critical role. See, for example, [26, 89], and [31]. A Bessel function identity for ${\sum}'_{n\leq x} a(n)(x-n)^q$ is, in fact, equivalent to the functional equation involving $\Gamma(s)$ of the corresponding Dirichlet series [89]. The second author, S. Kim, and Zaharescu [59] have established a Riesz sum identity for

$$\sum_{n\leq x}{}^{\prime}(x-n)^{\nu-1}\sum_{r|n}\sin(2\pi r\theta),$$

which in the special case $\nu = 1$ reduces to (2.1.5). One might also ask whether Ramanujan's identities in Entries 2.1.1 and 2.1.2 are isolated results, or whether they are forerunners of further theorems of this sort. To that end, the second author, S. Kim, and Zaharescu [58] have found three additional results akin to the aforementioned entries. We provide one example.

Define, for Dirichlet characters χ_1 modulo p and χ_2 modulo q,

$$d_{\chi_1,\chi_2}(n) = \sum_{d|n}\chi_1(d)\chi_2(n/d).$$

Also, for arithmetic functions f and g, we define

$$\sum_{nm\leq x}{}^{\prime} f(n)g(m) = \begin{cases} \sum_{nm\leq x} f(n)g(m), & \text{if } x\notin\mathbb{Z},\\ \sum_{nm\leq x} f(n)g(m) - \frac{1}{2}\sum_{nm=x} f(n)g(m), & \text{if } x\in\mathbb{Z}. \end{cases}$$

Theorem 2.1.3. *Let $I_1(x)$ be defined by (2.1.7). If $0<\theta$, $\sigma<1$, and $x>0$, then*

$$\sum_{nm\le x}{}' \cos(2\pi n\theta)\cos(2\pi m\sigma)$$
$$= \frac{1}{4} + \frac{\sqrt{x}}{4}\sum_{n,m\ge 0}\left\{\frac{I_1(4\pi\sqrt{(n+\theta)(m+\sigma)x})}{\sqrt{(n+\theta)(m+\sigma)}} + \frac{I_1(4\pi\sqrt{(n+1-\theta)(m+\sigma)x})}{\sqrt{(n+1-\theta)(m+\sigma)}}\right.$$
$$\left. + \frac{I_1(4\pi\sqrt{(n+\theta)(m+1-\sigma)x})}{\sqrt{(n+\theta)(m+1-\sigma)}} + \frac{I_1(4\pi\sqrt{(n+1-\theta)(m+1-\sigma)x})}{\sqrt{(n+1-\theta)(m+1-\sigma)}}\right\}, \tag{2.1.22}$$

where in the double sum on the right-hand side of (2.1.22)*, the product mn of the two summation indices tends to infinity.*

The remaining two theorems in [58] involve sums of products of sines and sums of products of sines and cosines, respectively. The employment of sums of $d_{\chi_1,\chi_2}(n)$ is crucial in all of the proofs.

2.2 Proof of Ramanujan's First Bessel Function Identity (Original Form)

In this section we provide a proof of Entry 2.1.1 in the form given by Ramanujan. Our proof is a more detailed exposition of the proof given by the second author, S. Kim, and Zaharescu [60]. On the other hand, these authors actually prove a more general theorem. First, they introduce a family of Dirichlet series. For $x > 0$ and $0 < \theta < 1$, let

$$G(x,\theta,s) = \sum_{m=1}^{\infty}\frac{a(x,\theta,m)}{m^s}, \tag{2.2.1}$$

where the coefficients $a(x,\theta,m)$ are given by

$$a(x,\theta,m) = \sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{n+\theta}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{n+1-\theta}}\right\}. \tag{2.2.2}$$

For $x > 0$ and $0 < \theta < 1$, by (2.1.9), the series in (2.2.1) is absolutely convergent in the half-plane $\operatorname{Re} s > \frac{5}{4}$. Second, to prove Ramanujan's claim in Entry 2.1.1, we need to establish an analytic continuation of $G(x,\theta,s)$ to a larger region. In [60], the aforementioned authors prove the following theorem.

Theorem 2.2.1. *For $x > 0$ and $0 < \theta < 1$, $G(x,\theta,s)$ has an analytic continuation to the half-plane* $\operatorname{Re} s > \frac{8}{17}$*. For s in this half-plane, and $x > 0$, the series in* (2.2.1) *converges uniformly with respect to θ in every compact subinterval of* $(0,1)$*. If x is not an integer, these conclusions hold in the larger half-plane* $\operatorname{Re} s > \frac{1}{3}$.

Fourier analysis is then employed to recover the value of $G(x,\theta,s)$ at $s = \frac{1}{2}$, and in this way, Entry 2.1.1 is established.

2.2.1 Identifying the Source of the Poles

Fix $x > 0$, and define, for $0 < \theta < 1$,

$$g(\theta) := \frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}. \tag{2.2.3}$$

In order for Ramanujan's Entry 2.1.1 to be valid, the double series in (2.2.3) needs to converge, and the function $g(\theta)$ needs to be continuous on $(0,1)$. We prove this by showing that the double series converges uniformly with respect to θ in every compact subinterval of $(0,1)$. Also, in order for Entry 2.1.1 to hold, $g(\theta)$ needs to have simple poles at $\theta = 0$ and $\theta = 1$. We start by employing a heuristic argument, which allows us to identify that part of the double series that is responsible for these poles.

Setting $a = 4\pi\sqrt{x}$ and taking the terms from the right-hand side of (2.1.5) when $n = 0$, we are led to examine the series

$$T(\theta) := \sum_{m=1}^{\infty}\frac{J_1(a\sqrt{\theta m})}{\sqrt{m}}.$$

We consider the Mellin transform of $T(\theta)$, for σ sufficiently large, and make the change of variable $t^2 = a^2\theta m$ to find that

$$\begin{aligned}\int_0^{\infty} T(\theta)\theta^{s-1}d\theta &= \sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}\int_0^{\infty} J_1(a\sqrt{\theta m})\theta^{s-1}d\theta \\ &= \frac{2}{a^{2s}}\sum_{m=1}^{\infty}\frac{1}{m^{s+1/2}}\int_0^{\infty} J_1(t)t^{2s-1}dt \\ &= \frac{2}{a^{2s}}\zeta(s+\tfrac{1}{2})2^{2s-1}\frac{\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{3}{2}-s)},\end{aligned} \tag{2.2.4}$$

where we used a well-known Mellin transform for Bessel functions [126, p. 707, formula 6.561, no. 14], which is valid for $-\frac{1}{2} < \sigma < \frac{3}{4}$. Applying Mellin's inversion formula in (2.2.4), for $\frac{1}{2} < c < \frac{3}{4}$, we find that

$$T(\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta(s+\tfrac{1}{2})\frac{\Gamma(\frac{1}{2}+s)}{\Gamma(\frac{3}{2}-s)}\left(\frac{a^2\theta}{4}\right)^{-s}ds. \tag{2.2.5}$$

We would now like to shift the line of integration to the left of $\sigma = \frac{1}{2}$ by integrating over a rectangle with vertices $c \pm iT, b \pm iT$, where $T > 0$ and $0 < b < \frac{1}{2}$, and then letting $T \to \infty$. Thus, since the integrand has a simple pole at $s = \frac{1}{2}$ with residue

$$\left(\frac{a^2\theta}{4}\right)^{-1/2} = \frac{2}{a\sqrt{\theta}},$$

we find that

$$T(\theta) = \frac{2}{a\sqrt{\theta}} + \cdots .$$

We assume that the missing terms represented by $\cdots$ above are bounded as $\theta \to 0^+$. Returning to (2.1.5) and recalling the notation $a = 4\pi\sqrt{x}$, we find that the portion of (2.1.5) corresponding to the terms when $n = 0$ is asymptotically equal to, as $\theta \to 0^+$,

$$\frac{1}{2}\sqrt{\frac{x}{\theta}}\frac{2}{4\pi\sqrt{x\theta}} = \frac{1}{4\pi\theta}.$$

Since

$$-\frac{1}{4}\cot(\pi\theta) = -\frac{1}{4\pi\theta} + O(\theta),$$

as $\theta \to 0$, we see that the right-hand side of (2.1.5) is continuous at $\theta = 0$. A similar argument holds for $\theta = 1$.

By this heuristic argument, if we remove from the definition of $g(\theta)$ all the terms with $n = 0$, we should obtain a function that can be extended by continuity to $[0, 1]$. We prove that this is indeed the case, by showing that the sum of terms with $n \geq 1$ converges uniformly with respect to θ in $[0, 1]$. As for the terms with $n = 0$, we will show that their sum converges uniformly with respect to θ in every compact subinterval of $(0, 1)$, and that if each of these terms is multiplied by $\sin^2(\pi\theta)$, then their sum converges uniformly with respect to θ in $(0, 1)$ to a continuous function on $(0, 1)$, which tends to 0 as $\theta \to 0^+$ or $\theta \to 1^-$. If we assume that the aforementioned statements have been proved, it follows that the function $G(\theta)$ defined on $[0, 1]$ by $G(0) = 0$, $G(1) = 0$, and $G(\theta) = \sin^2(\pi\theta)g(\theta)$ is well-defined and continuous on $[0, 1]$. We return to the function $G(\theta)$ in Sect. 2.2.10. We now proceed to study the uniform convergence of the double series on the right side of Entry 2.1.1. In what follows, by "uniform convergence with respect to θ" of any series or double series below, we mean that one simultaneously has uniform convergence with respect to θ on every compact subinterval of $(0, 1)$ for the given series, and uniform convergence with respect to θ in $[0, 1]$ for the series obtained by removing the terms with $n = 0$ from the given series.

2.2.2 Large Values of *n*

Fix an $x > 0$, and set $a = \sqrt{4\pi x}$. With the use of (2.1.9), the problem of the uniform convergence with respect to θ of the double series on the right side of Entry 2.1.1 reduces to the study of the uniform convergence with respect to θ of the double series

$$S_1(a, \theta) := \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m^{3/4}}\left(\frac{\cos(a\sqrt{m(n+\theta)} - \frac{3\pi}{4})}{(n+\theta)^{3/4}} - \frac{\cos(a\sqrt{m(n+1-\theta)} - \frac{3\pi}{4})}{(n+1-\theta)^{3/4}}\right). \tag{2.2.6}$$

We first truncate the inner sum in order to further reduce the problem to one in which the summation over n is finite. Accordingly,

$$\begin{aligned}
&\left|\frac{\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})}{(n+\theta)^{3/4}}-\frac{\cos(a\sqrt{m(n+1-\theta)}-\frac{3\pi}{4})}{(n+1-\theta)^{3/4}}\right|\\
&\quad\le\frac{\left|\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})-\cos(a\sqrt{m(n+1-\theta)}-\frac{3\pi}{4})\right|}{(n+\theta)^{3/4}}\\
&\qquad+\left|\frac{1}{(n+\theta)^{3/4}}-\frac{1}{(n+1-\theta)^{3/4}}\right||\cos(a\sqrt{m(n+1-\theta)}-\tfrac{3\pi}{4})|\\
&\quad\le\frac{|a\sqrt{m(n+\theta)}-a\sqrt{m(n+1-\theta)}|}{(n+\theta)^{3/4}}+\left|\frac{1}{(n+\theta)^{3/4}}-\frac{1}{(n+1-\theta)^{3/4}}\right|.
\end{aligned}\tag{2.2.7}$$

For $n\ge 1$, uniformly with respect to $\theta\in[0,1]$,

$$\left|\sqrt{n+\theta}-\sqrt{n+1-\theta}\right|=O\left(\frac{1}{\sqrt{n}}\right)\tag{2.2.8}$$

and

$$\left|\frac{1}{(n+\theta)^{3/4}}-\frac{1}{(n+1-\theta)^{3/4}}\right|=O\left(\frac{1}{n^{7/4}}\right).\tag{2.2.9}$$

Thus, by (2.2.7)–(2.2.9),

$$\left|\frac{\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})}{(n+\theta)^{3/4}}-\frac{\cos(a\sqrt{m(n+1-\theta)}-\frac{3\pi}{4})}{(n+1-\theta)^{3/4}}\right|=O_a\left(\frac{\sqrt{m}}{n^{5/4}}\right).\tag{2.2.10}$$

(Here, and in what follows, if the constant implied by O is dependent on a parameter a, then we write O_a.) It follows that

$$\begin{aligned}
&\sum_{n\ge m^3\log^5 m}\frac{1}{m^{3/4}}\left|\frac{\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})}{(n+\theta)^{3/4}}-\frac{\cos(a\sqrt{m(n+1-\theta)}-\frac{3\pi}{4})}{(n+1-\theta)^{3/4}}\right|\\
&\quad=O_a\left(\frac{1}{m^{1/4}}\sum_{n\ge m^3\log^5 m}\frac{1}{n^{5/4}}\right)=O_a\left(\frac{1}{m\log^{5/4}m}\right),
\end{aligned}\tag{2.2.11}$$

which shows that the sum over m at the left-hand side of (2.2.11) is convergent. Therefore the double sum $S_1(a,\theta)$ is convergent, respectively uniformly convergent, if and only if the sum

$$S_2(a,\theta) := \sum_{m=1}^{\infty} \sum_{0 \le n < m^3 \log^5 m} \frac{1}{m^{3/4}} \left(\frac{\cos(a\sqrt{m(n+\theta)} - \frac{3\pi}{4})}{(n+\theta)^{3/4}} - \frac{\cos(a\sqrt{m(n+1-\theta)} - \frac{3\pi}{4})}{(n+1-\theta)^{3/4}} \right) \tag{2.2.12}$$

is convergent, respectively uniformly convergent.

2.2.3 Small Values of n

Our next goal is to remove from the sum those terms in which n is much smaller than m. To this end, let us consider a general sum of the form

$$S(\alpha, \beta, \mu, H_1, H_2) := \sum_{H_1 < m \le H_2} \frac{\cos(\alpha\sqrt{m+\mu} + \beta)}{(m+\mu)^{3/4}}, \tag{2.2.13}$$

where $\alpha > 0$, $\beta \in \mathbb{R}$, $\mu \in [0,1]$, and $H_1 < H_2$ are large positive integers. Define

$$f(y) := \frac{\cos(\alpha\sqrt{y+\mu} + \beta)}{(y+\mu)^{3/4}}. \tag{2.2.14}$$

We fix a small real number $\delta > 0$ and assume that H_1 and α satisfy the inequalities

$$c_1 \le \alpha \le c_2 H_1^{(1-\delta)/2}, \tag{2.2.15}$$

for some constants $c_1 > 0$, $c_2 > 0$ that depend only on a (which, in turn, depends only on x). Next, we fix a positive integer $k \ge 2$ such that

$$k\delta \ge 2. \tag{2.2.16}$$

So we may take $k = 1 + [2/\delta]$.

We apply the Euler–Maclaurin summation formula of order k in the form

$$S(\alpha, \beta, \mu, H_1, H_2) = \sum_{H_1 < m \le H_2} f(m) = \int_{H_1}^{H_2} \left(f(y) - \frac{(-1)^k}{k!} \psi_k(y) f^{(k)}(y) \right) dy + \sum_{\ell=1}^{k} \frac{(-1)^\ell}{\ell!} \left(f^{(\ell-1)}(H_2) - f^{(\ell-1)}(H_1) \right) B_\ell, \tag{2.2.17}$$

where $f(y)$ is defined in (2.2.14), B_ℓ, $\ell \ge 0$, is the ℓth Bernoulli number, and $\psi_k(y)$ is the kth Bernoulli function, defined by

$$\psi_k(y) := -k! \sum_{\substack{n=-\infty \\ n \ne 0}}^{\infty} (2\pi i n)^{-k} e(ny), \qquad k \ge 0, \tag{2.2.18}$$

where $e(x) = e^{2\pi i x}$. Since $k \geq 2$, the Fourier series on the right side of (2.2.18) converges absolutely.

Let us note that the integral of $f(y)$ on $[H_1, H_2]$ can be bounded via a change of variable followed by an integration by parts, namely,

$$\int_{H_1}^{H_2} f(y)dy = \int_{\sqrt{H_1+\mu}}^{\sqrt{H_2+\mu}} \frac{2\cos(\alpha t+\beta)}{\sqrt{t}}dt$$
$$= \frac{2\sin(\alpha t+\beta)}{\alpha\sqrt{t}}\Bigg|_{\sqrt{H_1+\mu}}^{\sqrt{H_2+\mu}} + \frac{1}{\alpha}\int_{\sqrt{H_1+\mu}}^{\sqrt{H_2+\mu}} \frac{\sin(\alpha t+\beta)}{t^{3/2}}dt = O\left(\frac{1}{H_1^{1/4}}\right), \tag{2.2.19}$$

uniformly with respect to β and μ.

Let us also observe that for each $\ell \in \{0, 1, \dots, k\}$, the derivative $f^{(\ell)}(y)$ can be expressed as a sum of the form

$$f^{(\ell)}(y) = \sum_{j=1}^{r_\ell} c_{\ell,j}\alpha^{a_{\ell,j}}(y+\mu)^{b_{\ell,j}} \sin(\alpha\sqrt{y+\mu}+\beta)$$
$$+ \sum_{j=1}^{r'_\ell} c'_{\ell,j}\alpha^{a'_{\ell,j}}(y+\mu)^{b'_{\ell,j}} \cos(\alpha\sqrt{y+\mu}+\beta), \tag{2.2.20}$$

where r_ℓ and r'_ℓ depend only on ℓ, the coefficients $c_{\ell,j}$ and the exponents $a_{\ell,j}$, $b_{\ell,j}$ depend only on ℓ and j, and similarly, $c'_{\ell,j}$ and the exponents $a'_{\ell,j}$, $b'_{\ell,j}$ depend only on ℓ and j. Consider the collection of all pairs $(a_{\ell,j}, b_{\ell,j})$, $1 \leq j \leq r_\ell$, and $(a'_{\ell,j}, b'_{\ell,j})$, $1 \leq j \leq r'_\ell$, and denote this collection by $\mathcal{C}_\ell$. Differentiating (2.2.3) with respect to y, and taking into account the possible cancellation of terms, we conclude that $\mathcal{C}_{\ell+1}$ is a subset of the set of all pairs of the form $(a, b-1)$ and $(a+1, b-\frac{1}{2})$, with $(a,b) \in \mathcal{C}_\ell$. Taking also into account that $\mathcal{C}_0$ consists of the single pair $(0, -\frac{3}{4})$ and using induction on ℓ, we see that each pair (a,b) in $\mathcal{C}_\ell$ satisfies $0 \leq a \leq \ell$ and $-\ell - \frac{3}{4} \leq b \leq -\frac{1}{2}\ell - \frac{3}{4}$. As a consequence, we derive that for each ℓ and for each $y \in [H_1, H_2]$,

$$f^{(\ell)}(y) = O_\ell\left(\frac{1}{y^{3/4}}\cdot\left(\frac{\alpha}{\sqrt{y}}\right)^\ell\right), \tag{2.2.21}$$

uniformly with respect to β and μ. Therefore, recalling (2.2.15), we find that

$$\sum_{\ell=1}^{k} \frac{(-1)^\ell}{\ell!}\left(f^{(\ell-1)}(H_2) - f^{(\ell-1)}(H_1)\right)B_\ell = O_k\left(\frac{1}{H_1^{3/4}}\right), \tag{2.2.22}$$

uniformly with respect to β and μ. Also,

$$\left|\int_{H_1}^{H_2} \frac{(-1)^k}{k!}\psi_k(y)f^{(k)}(y)dy\right| = O_k\left(\int_{H_1}^{H_2} |f^{(k)}(y)|dy\right)$$
$$= O_k\left(\int_{H_1}^{H_2} \frac{1}{y^{3/4}}\left(\frac{\alpha}{\sqrt{y}}\right)^k dy\right) = O_k\left(\frac{\alpha^k}{H_1^{\frac{1}{2}k-\frac{1}{4}}}\right) = O_{a,k}\left(\frac{1}{H_1^{\frac{1}{2}k\delta-\frac{1}{4}}}\right). \tag{2.2.23}$$

Thus, by (2.2.16), (2.2.17), (2.2.19), (2.2.22), and (2.2.23), we find that, subject to (2.2.15) holding,

$$\sum_{H_1<m\leq H_2} \frac{\cos(\alpha\sqrt{m+\mu}+\beta)}{(m+\mu)^{3/4}} = O_{a,k}\left(\frac{1}{\alpha H_1^{1/4}}\right). \tag{2.2.24}$$

We now consider the sum

$$S_3(a,\theta,\delta) := \sum_{m=1}^{\infty} \sum_{0\leq n<m^{1-\delta}} \frac{1}{m^{3/4}} \left(\frac{\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})}{(n+\theta)^{3/4}} - \frac{\cos(a\sqrt{m(n+1-\theta)}-\frac{3\pi}{4})}{(n+1-\theta)^{3/4}}\right). \tag{2.2.25}$$

For each $M \geq 1$, we denote by $S_{3,M}(a,\theta,\delta)$ the corresponding restricted sum in $S_3(a,\theta,\delta)$, where the summation over m is restricted to $1 \leq m \leq M$. We intend to show that the sum $S_3(a,\theta,\delta)$ is convergent, and in order to do this, we apply Cauchy's criterion. Fix $\epsilon > 0$. We need to show that there exists an M_ϵ such that for every $M_1, M_2 > M_\epsilon$,

$$|S_{3,M_2}(a,\theta,\delta) - S_{3,M_1}(a,\theta,\delta)| < \epsilon. \tag{2.2.26}$$

Let $M_1 < M_2$ be large, and interchange the order of summation to rewrite $S_{3,M_2}(a,\theta,\delta) - S_{3,M_1}(a,\theta,\delta)$ in the form

$$\begin{aligned} S_{3,M_2}(a,\theta,\delta) - S_{3,M_1}(a,\theta,\delta) = \sum_{0\leq n\leq M_2^{1-\delta}} &\left(\sum_{\max\{n^{1/(1-\delta)},M_1\}<m\leq M_2}\right. \\ &\times \frac{1}{(n+\theta)^{3/4}} \frac{\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})}{m^{3/4}} \\ &\left. - \frac{1}{(n+1-\theta)^{3/4}} \frac{\cos(a\sqrt{m(n+1-\theta)}-\frac{3\pi}{4})}{m^{3/4}}\right). \end{aligned} \tag{2.2.27}$$

Using (2.2.24) with $\beta = -3\pi/4$, $\mu = 0$, $H_1 = \max\{n^{1/(1-\delta)}, M_1\}$, $H_2 = M_2$, and $\alpha = a\sqrt{n+\theta}, a\sqrt{n+1-\theta}$, respectively, and noting that (2.2.15) holds, we conclude from (2.2.24) that

$$\begin{aligned} &|S_{3,M_2}(a,\theta,\delta) - S_{3,M_1}(a,\theta,\delta)| \\ &= O_{a,\delta}\left(\sum_{0\leq n\leq M_2^{1-\delta}} \left\{\frac{1}{(n+\theta)^{3/4}} \cdot \frac{1}{\sqrt{n+\theta}\,(\max\{n^{1-\delta}, M_1\})^{1/4}}\right.\right. \\ &\qquad \left.\left. + \frac{1}{(n+1-\theta)^{3/4}} \cdot \frac{1}{\sqrt{n+1-\theta}\,\left(\max\{n^{1/(1-\delta)}, M_1\}\right)^{1/4}}\right\}\right) \end{aligned}$$

$$= O_{a,\delta}\left(\sum_{0\le n\le M_2^{1-\delta}} \frac{1}{n^{5/4}\max\{n^{1/(4(1-\delta))}, M_1^{1/4}\}}\right)$$

$$= O_{a,\delta}\left(\sum_{0\le n\le M_1^{1-\delta}} \frac{1}{n^{5/4}M_1^{1/4}}\right) + O_{a,\delta}\left(\sum_{M_1^{1-\delta}<n\le M_2^{1-\delta}} \frac{1}{n^{5/4+1/(4(1-\delta))}}\right)$$

$$= O_{a,\delta}\left(\frac{1}{M_1^{1/4}}\right) + O_{a,\delta}\left(\frac{1}{(M_1^{1-\delta})^{1/4+1/(4(1-\delta))}}\right)$$

$$= O_{a,\delta}\left(\frac{1}{M_1^{1/4}}\right) + O_{a,\delta}\left(\frac{1}{M_1^{(1-\delta)/4+1/4}}\right) = O_{a,\delta}\left(\frac{1}{M_1^{1/4}}\right). \qquad (2.2.28)$$

The foregoing analysis implies (2.2.26) for M_1 sufficiently large, and proves the convergence of $S_3(a,\theta,\delta)$. Therefore the convergence of $S_1(a,\theta,\delta)$ reduces to the convergence, respectively uniform convergence, of

$$S_4(a,\theta,\delta) := \sum_{m=1}^{\infty} \sum_{m^{1-\delta}\le n<m^3\log^5 m} \frac{1}{m^{3/4}}\left(\frac{\cos(a\sqrt{m(n+\theta)}-\frac{3\pi}{4})}{(n+\theta)^{3/4}} - \frac{\cos(a\sqrt{m(n+1-\theta)}-\frac{3\pi}{4})}{(n+1-\theta)^{3/4}}\right). \qquad (2.2.29)$$

2.2.4 Further Reductions

The remaining series under consideration, $S_4(a,\theta,\delta)$, does not contain any terms with $n = 0$. Therefore, in what follows, uniform convergence means uniform convergence with respect to θ in $[0,1]$. Next, we write

$$S_4(a,\theta,\delta) = S_5(a,\theta,\delta) + S_6(a,\theta,\delta), \qquad (2.2.30)$$

where $S_5(a,\theta,\delta)$ denotes the sum of those terms in $S_4(a,\theta,\delta)$ for which $n > m^{1+\delta}$, and $S_6(a,\theta,\delta)$ is the sum of terms with $n \le m^{1+\delta}$. The examination of $S_5(a,\theta,\delta)$ is like that for $S_3(a,\theta,\delta)$. In this case, we take the sum over n as the inner sum, and apply (2.2.24) with $\beta = -3\pi/4$, $\alpha = a\sqrt{m}$, and $\mu = \theta, 1-\theta$, respectively. We accordingly find that the sum $S_5(a,\theta,\delta)$ converges uniformly with respect to θ. It follows that the convergence of $S_1(a,\theta,\delta)$ reduces to that of $S_6(a,\theta,\delta)$.

Let us consider now the sum

$$S_7(a,\theta,\delta) := \sum_{m=1}^{\infty} \sum_{m^{1-\delta}\le n\le m^{1+\delta}} \frac{\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)}-\frac{3\pi}{4}\right)\sin\left(\frac{a(1-2\theta)}{4}\sqrt{\frac{m}{n}}\right)}{m^{3/4}n^{3/4}}. \qquad (2.2.31)$$

We claim that $S_6(a,\theta,\delta)$ is uniformly convergent if and only if $S_7(a,\theta,\delta)$ is. Indeed,

$$\frac{1}{(n+\theta)^{3/4}} = \frac{1}{n^{3/4}} + O\left(\frac{1}{n^{7/4}}\right), \qquad \frac{1}{(n+1-\theta)^{3/4}} = \frac{1}{n^{3/4}} + O\left(\frac{1}{n^{7/4}}\right), \tag{2.2.32}$$

and it is easily seen that the error terms in (2.2.32) are small enough so that the denominators $(n+\theta)^{3/4}$ and $(n+1-\theta)^{3/4}$ in $S_6(a,\theta,\delta)$ can both be replaced by $n^{3/4}$ without influencing the uniform convergence of the sum. Also,

$$\begin{aligned}\cos&\left(a\sqrt{m(n+\theta)} - \frac{3\pi}{4}\right) - \cos\left(a\sqrt{m(n+1-\theta)} - \frac{3\pi}{4}\right)\\ &= 2\sin\left(\frac{a\sqrt{m}\left(\sqrt{n+1-\theta}-\sqrt{n+\theta}\right)}{2}\right)\\ &\quad\times\sin\left(\frac{a\sqrt{m}\left(\sqrt{n+1-\theta}+\sqrt{n+\theta}\right)}{2} - \frac{3\pi}{4}\right).\end{aligned} \tag{2.2.33}$$

Here

$$\sqrt{n+1-\theta} - \sqrt{n+\theta} = \frac{1-2\theta}{2\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right) \tag{2.2.34}$$

and

$$\frac{\sqrt{n+1-\theta}+\sqrt{n+\theta}}{2} = \sqrt{n+\frac{1}{2}} + O\left(\frac{1}{n^{3/2}}\right). \tag{2.2.35}$$

By (2.2.33)–(2.2.35),

$$\begin{aligned}\cos&\left(a\sqrt{m(n+\theta)} - \frac{3\pi}{4}\right) - \cos\left(a\sqrt{m(n+1-\theta)} - \frac{3\pi}{4}\right)\\ &= 2\sin\left(\frac{a(1-2\theta)}{4}\sqrt{\frac{m}{n}}\right)\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3\pi}{4}\right) + O\left(\frac{\sqrt{m}}{n^{3/2}}\right).\end{aligned} \tag{2.2.36}$$

Again, it is easily checked that the error term in (2.2.36) is small enough so that the left side of (2.2.36) may be replaced by the main term from the right side of (2.2.36) in the modified version of $S_6(a,\theta,\delta)$ above without influencing the uniform convergence of the series. This proves our claim, and it remains to show the uniform convergence of $S_7(a,\theta,\delta)$.

We replace $S_7(a,\theta,\delta)$ by another series that has the same terms, but the double summation is performed over a union of rectangles. To be precise, for each positive integer r, we consider those m satisfying the inequalities $2^r \le m < 2^{r+1}$, and for each such m we replace the range of summation for n, which in $S_7(a,\theta,\delta)$ is $m^{1-\delta} \le n \le m^{1+\delta}$, with the somewhat larger range $2^{r(1-\delta)} \le n \le 2^{(r+1)(1+\delta)}$. This does not influence the uniform convergence of the series, because the extra terms added by this procedure are contained in the sums $S_3(a, \theta, \delta)$ and $S_5(a, \theta, \delta)$, which we have previously examined. More specifically, the extra terms arise from the ranges $2^{r(1-\delta)} \leq n < m^{1-\delta}$ and $m^{1+\delta} < n \leq 2^{(r+1)(1+\delta)}$. In both these ranges, either n is significantly smaller than m ($n < m^{1-\delta}$), or n is significantly larger than m ($n > m^{1+\delta}$), and so an appropriate use of (2.2.24) can be made in both cases. In conclusion, $S_7(a, \theta, \delta)$ is uniformly convergent if and only if the same is true for the sum

$$S_8(a, \theta, \delta) := \sum_{r=1}^{\infty} \sum_{2^r \leq m < 2^{r+1}} \sum_{2^{r(1-\delta)} \leq n \leq 2^{(r+1)(1+\delta)}} \times \frac{\sin\left(b\sqrt{\frac{m}{n}}\right) \sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3\pi}{4}\right)}{m^{3/4} n^{3/4}}, \tag{2.2.37}$$

where, henceforth, we define, for simplicity,

$$b = \frac{a(1-2\theta)}{4} = \pi\sqrt{x}(1-2\theta). \tag{2.2.38}$$

2.2.5 Refining the Range of Summation

In order to prove that $S_8(a, \theta, \delta)$ is uniformly convergent with respect to θ in $[0, 1]$, we need to show that the right side of (2.2.37) converges uniformly with respect to b in $[-\pi\sqrt{x}, \pi\sqrt{x}]$. To do this, we use Cauchy's criterion. Fix $\epsilon > 0$ and denote, as usual, for any $M > 1$, the partial sum in (2.2.37) corresponding to $1 \leq m \leq M$ by $S_{8,M}(a, \theta, \delta)$. Let $M_1 < M_2$ be large, and set $r_1 = [\log_2 M_1]$ and $r_2 = [\log_2 M_2]$. Then $S_{8,M_2}(a, \theta, \delta) - S_{8,M_1}(a, \theta, \delta)$ can be written as a sum over integral pairs (m, n) in the union of $r_2 - r_1 + 1$ rectangles, which we denote by $R_0, R_1, \ldots, R_{r_2-r_1}$, as follows. We let $R_0 = \left(M_1, 2^{r_1+1}\right) \times \left[2^{r_1(1-\delta)}, 2^{(r_1+1)(1+\delta)}\right]$, $R_j = [2^{r_1+j}, 2^{r_1+j+1}) \times \left[2^{(r_1+j)(1-\delta)}, 2^{(r_1+j+1)(1+\delta)}\right]$ for $1 \leq j \leq r_2 - r_1 - 1$, and $R_{r_2-r_1} = [2^{r_2}, M_2] \times \left[2^{r_2(1-\delta)}, 2^{r_2(1+\delta)}\right]$. Then

$$S_{8,M_2}(a, \theta, \delta) - S_{8,M_1}(a, \theta, \delta) = \sum_{j=0}^{r_2-r_1} \sum_{(m,n)\in R_j} \frac{\sin\left(b\sqrt{\frac{m}{n}}\right) \sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3\pi}{4}\right)}{m^{3/4} n^{3/4}}. \tag{2.2.39}$$

We now proceed to obtain bounds for the inner sum on the right side of (2.2.39) for each individual R_j. Fix such an R_j, and, to make a choice, assume that $1 \leq j \leq r_2 - r_1 - 1$. The cases $j = 0$ and $j = r_2 - r_1$ can be examined in a similar fashion. Also, for simplicity, we set $T = 2^{r_1+j}$. Then the corresponding inner sum on the right side of (2.2.39), which depends on a, b, δ, and T, and which we denote by $\sum_{a,b,\delta,T}$, or simply by $\sum$, has the form

$$\sum_{a,b,\delta,T} = \sum_{T \le m < 2T} \sum_{T^{1-\delta} \le n \le (2T)^{1+\delta}} \frac{\sin\left(b\sqrt{\frac{m}{n}}\right)\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)}-\frac{3\pi}{4}\right)}{m^{3/4}n^{3/4}}. \tag{2.2.40}$$

At this point, we fix a number λ, with $0 < \lambda < \frac{1}{2}$, whose precise value will be given later, and set $L = [T^\lambda]$. Then we subdivide the rectangle $[T, 2T) \times [T^{1-\delta}, (2T)^{1+\delta}]$ into squares of size $L \times L$. An explanation as to why we break the range of summation into such small squares of size $[T^\lambda] \times [T^\lambda]$, with $\lambda < \frac{1}{2}$, is in order. This choice may seem surprising, because for almost all exponential sums, the best one can hope to achieve is a square-root-type cancellation. And in our case, square-root cancellation over a square of size $[T^\lambda] \times [T^\lambda]$ means a savings over the trivial bound by a factor of T^λ. But this is not enough in our case, even if we achieve a square-root cancellation for each individual square of size $L \times L$, because the trivial bound for the entire sum $\sum_{a,b,\delta,T}$, even ignoring the small but strictly positive δ, is of order $O(T^{1/2})$. Thus we need cancellation in $\sum_{a,b,\delta,T}$ by a factor larger than $T^{1/2}$, and so a cancellation by a factor of T^λ with $\lambda < \frac{1}{2}$ will not suffice.

Our approach below, which proceeds via subdividing the range of summation into small squares of size $[T^\lambda] \times [T^\lambda]$, with $\lambda < \frac{1}{2}$, is based on two fundamental ideas. The first one is that on such small squares, the functions $(m, n) \mapsto m^{-3/4}n^{-3/4}$ and $(m, n) \mapsto b\sqrt{m/n}$ are almost constant, and the function $a\sqrt{m(n + \frac{1}{2})}$ is almost linear. This gives us a chance to approximate locally the corresponding sums on the right side of (2.2.40) by geometric series, for which we have better than square-root cancellation. The second idea is to approximate the function

$$(m, n) \mapsto \frac{\sin\left(b\sqrt{\frac{m}{n}}\right)\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)}-\frac{3\pi}{4}\right)}{m^{3/4}n^{3/4}}$$

by a short sum in which each term is a product of a function of m and a function of n. This, in turn, reduces the problem of bounding the right side of (2.2.40) to the problem of bounding certain sums that are products of a sum over m and a sum over n. This gives us the opportunity to combine the savings achieved due to cancellation in the sum over m with the savings achieved in the sum over n.

To proceed, we consider the set of integral points (m, n) in $[T, 2T) \times [T^{1-\delta}, (2T)^{1+\delta}]$ for which both m and n are divisible by L. We also consider all the squares of size $L \times L$ with vertices in the aforementioned set. These squares almost cover the rectangle above. We first examine the portion of the rectangle left uncovered, and bound its contribution on the right side of (2.2.40).

Let

$$T_1 := \left[\frac{T}{L}\right]+1, \quad T_2 := \left[\frac{2T}{L}\right]-1, \quad T_3 := \left[\frac{T^{1-\delta}}{L}\right]+1, \quad T_4 := \left[\frac{(2T)^{1+\delta}}{L}\right]-1. \tag{2.2.41}$$

For each $m_1 \in \{T_1, T_1+1, \ldots, T_2\}$ and $n_1 \in \{T_3, T_3+1, \ldots, T_4\}$, we consider the $L \times L$ square whose southwest corner has coordinates (Lm_1, Ln_1), and denote by $\sum_{m_1,n_1}$ its contribution on the right-hand side of (2.2.40). To be precise, we define

$$\sum_{m_1,n_1} := \sum_{Lm_1 \le m < L(m_1+1)} \sum_{Ln_1 \le n < L(n_1+1)} \times \frac{\sin\left(b\sqrt{\dfrac{m}{n}}\right) \sin\left(a\sqrt{m\left(n+\dfrac{1}{2}\right)} - \dfrac{3\pi}{4}\right)}{m^{3/4} n^{3/4}}. \tag{2.2.42}$$

Then we approximate the right side of (2.2.40) by the sum $\sum_{m_1,n_1}$ with (m_1, n_1) running over the pairs of integral points in the rectangle $[T_1, T_2] \times [T_3, T_4]$. The error made in this approximation is bounded as follows. Note that each integral point (m, n) in $[T, 2T) \times [T^{1-\delta}, (2T)^{1+\delta}]$ that does not belong to any of the $L \times L$ squares of the form $[Lm_1, L(m_1+1)) \times [Ln_1, L(n_1+1))$, with $T_1 \le m_1 \le T_2$, $T_3 \le n_1 \le T_4$, is at distance at most L from one of the four sides of the rectangle $[T, 2T) \times [T^{1-\delta}, (2T)^{1+\delta}]$. Therefore,

$$\begin{aligned}
&\left| \sum_{a,b,\delta,T} - \sum_{T_1 \le m_1 \le T_2} \sum_{T_3 \le n_1 \le T_4} \sum_{m_1,n_1} \right| \\
&= O\left(\sum_{\substack{|m-T| \le L \\ \text{or} \\ |m-2T| \le L}} \sum_{T^{1-\delta} \le n \le (2T)^{1+\delta}} \frac{1}{m^{3/4} n^{3/4}} \right) \\
&\quad + O\left(\sum_{\substack{|n-T^{1-\delta}| \le L \\ \text{or} \\ |n-(2T)^{1+\delta}| \le L}} \sum_{T \le m \le 2T} \frac{1}{m^{3/4} n^{3/4}} \right) \\
&= O\left(\frac{L}{T^{3/4}} \cdot T^{(1+\delta)/4} \right) + O\left(\frac{L}{T^{3(1-\delta)/4}} \cdot T^{1/4} \right) \\
&= O\left(\frac{L}{T^{\frac{1}{2}-\frac{3}{4}\delta}} \right) \\
&= O\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{3}{4}\delta}} \right).
\end{aligned} \tag{2.2.43}$$

In our approach, we first fix λ, and then we fix δ depending on λ. In particular, δ is chosen small enough so that $\frac{1}{2}-\lambda-\frac{3}{4}\delta>0$, which ensures that the far right side of (2.2.43) is negligible.

Next, we proceed to bound each sum $\sum_{m_1,n_1}$. Fix $m_1\in\{T_1,T_1+1,\ldots,T_2\}$ and $n_1\in\{T_3,T_3+1,\ldots,T_4\}$. For each m and n, with $Lm_1\le m<L(m_1+1)$ and $Ln_1\le n<L(n_1+1)$, we find that, with several uses of (2.2.41) below,

$$\frac{1}{m^{3/4}}=\frac{1}{L^{3/4}m_1^{3/4}\left(1+O(1/m_1)\right)}=\frac{1}{L^{3/4}m_1^{3/4}}\left(1+O\left(\frac{1}{T^{1-\lambda}}\right)\right), \quad (2.2.44)$$

$$\frac{1}{n^{3/4}}=\frac{1}{L^{3/4}n_1^{3/4}\left(1+O(1/n_1)\right)}=\frac{1}{L^{3/4}n_1^{3/4}}\left(1+O\left(\frac{1}{T^{1-\lambda-\delta}}\right)\right), \quad (2.2.45)$$

$$\sqrt{\frac{m}{n}}=\frac{\sqrt{Lm_1}\cdot(1+O(1/m_1))}{\sqrt{Ln_1}\cdot(1+O(1/n_1))}=\sqrt{\frac{m_1}{n_1}}\left(1+O\left(\frac{1}{T^{1-\lambda-\delta}}\right)\right), \quad (2.2.46)$$

and, noting the definition of b given in (2.2.38), we further see that

$$\begin{aligned}\sin\left(b\sqrt{\frac{m}{n}}\right)&=\sin\left(b\sqrt{\frac{m_1}{n_1}}+O\left(\frac{|b|\sqrt{m_1}}{\sqrt{n_1}\,T^{1-\lambda-\delta}}\right)\right)\\&=\sin\left(b\sqrt{\frac{m_1}{n_1}}\right)+O_x\left(\frac{1}{T^{1-\lambda-\frac{3}{2}\delta}}\right), \quad (2.2.47)\end{aligned}$$

uniformly with respect to θ in $[0,1]$. Hence, by (2.2.42) and (2.2.44)–(2.2.47),

$$\begin{aligned}\sum_{m_1,n_1}&=\sum_{Lm_1\le m<L(m_1+1)}\sum_{Ln_1\le n<L(n_1+1)}\frac{1}{L^{3/2}m_1^{3/4}n_1^{3/4}}\left(1+O\left(\frac{1}{T^{1-\lambda-\delta}}\right)\right)\\&\quad\times\left(\sin\left(b\sqrt{\frac{m_1}{n_1}}\right)+O_x\left(\frac{1}{T^{1-\lambda-\frac{3}{2}\delta}}\right)\right)\cdot\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)}-\frac{3\pi}{4}\right)\\&=\frac{\sin\left(b\sqrt{m_1/n_1}\right)}{L^{3/2}m_1^{3/4}n_1^{3/4}}\sum_{Lm_1\le m<L(m_1+1)}\sum_{Ln_1\le n<L(n_1+1)}\\&\quad\times\sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)}-\frac{3\pi}{4}\right)+O_x\left(\frac{\sqrt{L}}{m_1^{3/4}n_1^{3/4}T^{1-\lambda-\frac{3}{2}\delta}}\right). \quad (2.2.48)\end{aligned}$$

Here,

$$m_1^{3/4}n_1^{3/4}\ge T_1^{3/4}T_3^{3/4}>\left(\frac{T}{L}\right)^{3/4}\left(\frac{T^{1-\delta}}{L}\right)^{3/4}\sim T^{\frac{3}{2}-\frac{3}{2}\lambda-\frac{3}{4}\delta}. \quad (2.2.49)$$

By (2.2.48) and (2.2.49),

$$\left|\sum_{m_1,n_1}\right| \ll \frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}} \left| \sum_{Lm_1 \le m < L(m_1+1)} \sum_{Ln_1 \le n < L(n_1+1)} \right. \\ \left. \times \sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3\pi}{4}\right)\right| + O_x\left(\frac{1}{T^{\frac{5}{2}-3\lambda-\frac{9}{4}\delta}}\right). \quad (2.2.50)$$

2.2.6 Short Exponential Sums

Consider now the exponential sum

$$E_{m_1,n_1} := \sum_{Lm_1 \le m < L(m_1+1)} \sum_{Ln_1 \le n < L(n_1+1)} e\left(2\sqrt{xm(n+\tfrac{1}{2})}\right), \quad (2.2.51)$$

where, as customary, $e(t) := e^{2\pi i t}$. Observe that

$$\sum_{Lm_1 \le m < L(m_1+1)} \sum_{Ln_1 \le n < L(n_1+1)} \sin\left(a\sqrt{m\left(n+\frac{1}{2}\right)} - \frac{3\pi}{4}\right) \\ = \operatorname{Im}\left(e\left(-\frac{3}{8}\right)E_{m_1,n_1}\right). \quad (2.2.52)$$

Since

$$\left|\operatorname{Im}\left(e\left(-\frac{3}{8}\right)E_{m_1,n_1}\right)\right| \le \left|e\left(-\frac{3}{8}\right)E_{m_1,n_1}\right| = |E_{m_1,n_1}|,$$

by (2.2.50), we see that

$$\left|\sum_{m_1,n_1}\right| = O\left(\frac{|E_{m_1,n_1}|}{T^{\frac{3}{2}-\frac{3}{4}\delta}}\right) + O\left(\frac{1}{T^{\frac{5}{2}-3\lambda-\frac{9}{4}\delta}}\right). \quad (2.2.53)$$

Adding the estimates (2.2.53) for all relevant values of m_1 and n_1 and using the bound

$$T_2T_4 = O\left(T^{2-2\lambda+\delta}\right),$$

we see that

$$\sum_{T_1 \le m_1 \le T_2} \sum_{T_3 \le n_1 \le T_4} \left|\sum_{m_1,n_1}\right| = O\left(\frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}} \sum_{T_1 \le m_1 \le T_2} \sum_{T_3 \le n_1 \le T_4} |E_{m_1,n_1}|\right) \\ + O\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right). \quad (2.2.54)$$

From (2.2.43) and (2.2.54), we deduce that

$$\left|\sum_{a,b,\delta,T}\right| = O\left(\frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}}\sum_{T_1\le m_1\le T_2}\sum_{T_3\le n_1\le T_4}|E_{m_1,n_1}|\right) + O\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right). \tag{2.2.55}$$

For fixed $\lambda < \frac{1}{2}$ and δ small enough so that $\frac{1}{2}-\lambda-\frac{13}{4}\delta > 0$, the second error term on the right-hand side of (2.2.55) is negligible. In order to estimate the first error term on the right side of (2.2.55), fix m_1 and n_1. We write each m and n with $Lm_1 \le m < L(m_1+1)$ and $Ln_1 \le n < L(n_1+1)$ in the forms

$$m = Lm_1 + m_2, \qquad n = Ln_1 + n_2, \qquad m_2, n_2 \in \{0, 1, \ldots, L-1\}. \tag{2.2.56}$$

Then,

$$\begin{aligned}\sqrt{m} &= \sqrt{Lm_1}\left(1+\frac{m_2}{Lm_1}\right)^{1/2}\\ &= \sqrt{Lm_1}\left(1+\frac{m_2}{2Lm_1}-\frac{m_2^2}{8L^2m_1^2}+O\left(\frac{m_2^3}{L^3m_1^3}\right)\right)\\ &= \sqrt{Lm_1}\left(1+\frac{m_2}{2Lm_1}-\frac{m_2^2}{8L^2m_1^2}+O\left(\frac{1}{T^{3-3\lambda}}\right)\right),\end{aligned} \tag{2.2.57}$$

$$\begin{aligned}\sqrt{n+\tfrac{1}{2}} &= \sqrt{Ln_1}\left(1+\frac{n_2+\frac{1}{2}}{Ln_1}\right)^{1/2}\\ &= \sqrt{Ln_1}\left(1+\frac{n_2+\frac{1}{2}}{2Ln_1}-\frac{(n_2+\frac{1}{2})^2}{8L^2n_1^2}+O\left(\frac{(n_2+\frac{1}{2})^3}{L^3n_1^3}\right)\right)\\ &= \sqrt{Ln_1}\left(1+\frac{n_2+\frac{1}{2}}{2Ln_1}-\frac{(n_2+\frac{1}{2})^2}{8L^2n_1^2}+O\left(\frac{1}{T^{3-3\lambda-3\delta}}\right)\right).\end{aligned} \tag{2.2.58}$$

Also, by (2.2.41) and (2.2.56),

$$\frac{m_2}{2Lm_1}\cdot\frac{(n_2+\frac{1}{2})^2}{8L^2n_1^2} = O\left(\frac{L}{T}\cdot\frac{L^2}{T^{2-2\delta}}\right) = O\left(\frac{1}{T^{3-3\lambda-2\delta}}\right), \tag{2.2.59}$$

$$\frac{m_2^2}{8L^2m_1^2}\cdot\frac{(n_2+\frac{1}{2})}{2Ln_1} = O\left(\frac{L^2}{T^2}\cdot\frac{L}{T^{1-\delta}}\right) = O\left(\frac{1}{T^{3-3\lambda-\delta}}\right), \tag{2.2.60}$$

$$\frac{m_2^2}{8L^2m_1^2}\cdot\frac{(n_2+\frac{1}{2})^2}{8L^2n_1^2} = O\left(\frac{L^2}{T^2}\cdot\frac{L^2}{T^{2-2\delta}}\right) = O\left(\frac{1}{T^{4-4\lambda-2\delta}}\right). \tag{2.2.61}$$

By (2.2.57)–(2.2.61),

$$\begin{aligned}\sqrt{m(n+\tfrac{1}{2})} = L\sqrt{m_1n_1}\Bigg(1&+\frac{m_2}{2Lm_1}+\frac{n_2+\frac{1}{2}}{2Ln_1}+\frac{m_2(n_2+\frac{1}{2})}{4L^2m_1n_1}-\frac{m_2^2}{8L^2m_1^2}\\ &-\frac{(n_2+\frac{1}{2})^2}{8L^2n_1^2}+O\left(\frac{1}{T^{3-3\lambda-3\delta}}\right)\Bigg).\end{aligned} \tag{2.2.62}$$

Next, again with the use of (2.2.41) and (2.2.56),

$$L\sqrt{m_1 n_1}\cdot\frac{1}{T^{3-3\lambda-3\delta}} = O\left(\frac{T^{1+\delta/2}}{T^{3-3\lambda-3\delta}}\right) = O\left(\frac{1}{T^{2-3\lambda-7\delta/2}}\right), \tag{2.2.63}$$

$$L\sqrt{m_1 n_1}\cdot\frac{\frac{1}{2}m_2}{4L^2 m_1 n_1} = O\left(\frac{T^{\lambda}}{L\sqrt{m_1 n_1}}\right) = O\left(\frac{1}{T^{1-\lambda-\delta/2}}\right), \tag{2.2.64}$$

$$L\sqrt{m_1 n_1}\cdot\frac{n_2+\frac{1}{4}}{8L^2 n_1^2} = O\left(\frac{\sqrt{m_1 n_1}}{n_1^2}\right) = O\left(\frac{1}{T^{1-\lambda-3\delta/2}}\right). \tag{2.2.65}$$

By (2.2.62)–(2.2.65), we see that

$$\begin{aligned}\sqrt{m(n+\tfrac{1}{2})} = L\sqrt{m_1 n_1}\left(1+\frac{1}{4Ln_1}\right) + \frac{1}{2}\sqrt{\frac{n_1}{m_1}}\cdot m_2 + \frac{1}{2}\sqrt{\frac{m_1}{n_1}}\cdot n_2 \\ - \frac{\sqrt{m_1 n_1}}{8L}\left(\frac{m_2}{m_1}-\frac{n_2}{n_1}\right)^2 + O\left(\frac{1}{T^{1-\lambda-3\delta/2}}\right) + O\left(\frac{1}{T^{2-3\lambda-7\delta/2}}\right).\end{aligned} \tag{2.2.66}$$

Note that for

$$3\delta < 1-2\lambda,$$

which we may assume in what follows, $T^{2-3\lambda-7\delta/2} > T^{1-\lambda-\delta/2}$. Therefore, by (2.2.66), we find that

$$\begin{aligned}&e\left(-2L\sqrt{xm_1 n_1}\left(1+\frac{1}{4Ln_1}\right)\right)e\left(2\sqrt{xm(n+\tfrac{1}{2})}\right) \\ &= e\left(\sqrt{\frac{xn_1}{m_1}}m_2\right)e\left(\sqrt{\frac{xm_1}{n_1}}n_2\right)e\left(-\frac{\sqrt{xm_1 n_1}}{4L}\left(\frac{m_2}{m_1}-\frac{n_2}{n_1}\right)^2\right) \\ &\quad + O\left(\frac{1}{T^{1-\lambda-3\delta/2}}\right).\end{aligned} \tag{2.2.67}$$

Summing up the relations (2.2.67) over m_2 and n_2 in their appropriate ranges, taking absolute values on both sides, and recalling (2.2.51), we find that

$$\begin{aligned}|E_{m_1,n_1}| &= \left|e\left(-2L\sqrt{xm_1 n_1}\left(1+\frac{1}{4Ln_1}\right)\right)\cdot E_{m_1,n_1}\right| \\ &= \left|\sum_{0\le m_2<L}\sum_{0\le n_2<L} e\left(\sqrt{\frac{xn_1}{m_1}}m_2\right)e\left(\sqrt{\frac{xm_1}{n_1}}n_2\right)\right. \\ &\qquad \left.\times\, e\left(-\frac{\sqrt{xm_1 n_1}}{4L}\left(\frac{m_2}{m_1}-\frac{n_2}{n_1}\right)^2\right)\right| \\ &\qquad + O\left(T^{3\lambda-1+3\delta/2}\right).\end{aligned} \tag{2.2.68}$$

We now use the Taylor expansion for

$$e\left(-\frac{\sqrt{xm_1n_1}}{4L}\left(\frac{m_2}{m_1}-\frac{n_2}{n_1}\right)^2\right).$$

Observe that

$$\begin{aligned}\frac{\sqrt{xm_1n_1}}{4L}\left(\frac{m_2}{m_1}-\frac{n_2}{n_1}\right)^2 &\leq \frac{\sqrt{xm_1n_1}}{4L}\max\left\{\frac{m_2^2}{m_1^2},\frac{n_2^2}{n_1^2}\right\}\\ &\leq \frac{\sqrt{xm_1n_1}}{4L}\max\left\{\frac{L^2}{m_1^2},\frac{L^2}{n_1^2}\right\}\\ &= \frac{L\sqrt{x}}{4}\max\left\{\frac{n_1^{1/2}}{m_1^{3/2}},\frac{m_1^{1/2}}{n_1^{3/2}}\right\}\\ &= O_x\left(T^{\lambda}\cdot\max\left\{\frac{T^{(1+\delta-\lambda)/2}}{T^{3(1-\lambda)/2}},\frac{T^{(1-\lambda)/2}}{T^{3(1-\delta-\lambda)/2}}\right\}\right)\\ &= O_x\left(\frac{1}{T^{1-2\lambda-3\delta/2}}\right). \end{aligned} \tag{2.2.69}$$

In what follows we fix a positive integer r, depending on λ only, such that $(r+1)(\frac{1}{2}-\lambda)\geq 1$. For example, we may take

$$r=\left[\frac{1}{\frac{1}{2}-\lambda}\right]. \tag{2.2.70}$$

We also assume that δ is small enough so that

$$3\delta<1-2\lambda.$$

Then $1-2\lambda-\frac{3}{2}\delta>\frac{1}{2}-\lambda$, and so by (2.2.69),

$$\frac{\sqrt{xm_1n_1}}{4L}\left(\frac{m_2}{m_1}-\frac{n_2}{n_1}\right)^2=O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda}}\right). \tag{2.2.71}$$

We may then truncate the Taylor series expansion mentioned above as

$$\begin{aligned}&e\left(-\frac{\sqrt{xm_1n_1}}{4L}\left(\frac{m_2}{m_1}-\frac{n_2}{n_1}\right)^2\right)\\ &=\sum_{j=0}^{r}\frac{(-1)^j(xm_1n_1)^{j/2}}{4^jL^jj!}\left(\frac{m_2}{m_1}-\frac{n_2}{n_1}\right)^{2j}+O_{x,\lambda,\delta}\left(\frac{1}{T^{(r+1)(\frac{1}{2}-\lambda)}}\right)\\ &=\sum_{j=0}^{r}\sum_{\ell=0}^{2j}\frac{(-1)^jx^{j/2}}{4^jj!}\binom{2j}{\ell}\frac{m_1^{\frac{1}{2}j-\ell}n_1^{\ell-\frac{3}{2}j}}{L^j}m_2^{\ell}n_2^{2j-\ell}+O_{x,\lambda,\delta}\left(\frac{1}{T}\right), \end{aligned} \tag{2.2.72}$$

by (2.2.71). Inserting (2.2.72) in (2.2.68), and noticing that the error term on the right side of (2.2.72) is small enough so that when inserted on the right side of (2.2.68) it can be subsumed in the existing error term from (2.2.68), we deduce that

$$|E_{m_1,n_1}| = \left|\sum_{j=0}^{r}\sum_{\ell=0}^{2j} A_{j,\ell}(m_1,n_1)V_{j,\ell}(m_1,n_1)\right| + O_{x,\lambda,\delta}\left(T^{3\lambda+\frac{3}{2}\delta-1}\right), \quad (2.2.73)$$

where we have defined

$$A_{j,\ell}(m_1,n_1) := \frac{(-1)^j x^{j/2}}{4^j j!}\binom{2j}{\ell}\frac{m_1^{\frac{1}{2}j-\ell}\, n_1^{\ell-\frac{3}{2}j}}{L^j} \quad (2.2.74)$$

and

$$V_{j,\ell}(m_1,n_1) := \sum_{0\le m_2<L}\sum_{0\le n_2<L} e\left(\sqrt{\frac{xn_1}{m_1}}m_2\right) e\left(\sqrt{\frac{xm_1}{n_1}}n_2\right) m_2^{\ell} n_2^{2j-\ell}. \quad (2.2.75)$$

In order to bound the coefficients $A_{j,\ell}(m_1,n_1)$, we distinguish two cases: $\ell \ge 3j/2$ and $\ell < 3j/2$. If $\ell \ge 3j/2$, in order to produce an upper bound for the right side of (2.2.74), we need an upper bound for n_1, which is $T^{1-\lambda+\delta}$. When $\ell < 3j/2$, we need a lower bound for n_1, which is $T^{1-\lambda-\delta}$. For m_1, both upper and lower bounds have the same size, $T^{1-\lambda}$. Combining the two cases, we find that

$$\begin{aligned}|A_{j,\ell}(m_1,n_1)| &= O_{x,\lambda,\delta}\left(\frac{T^{(1-\lambda)(\frac{1}{2}j-\ell)}\cdot T^{(1\pm\delta-\lambda)(\ell-\frac{3}{2}j)}}{T^{\lambda j}}\right)\\ &= O_{x,\lambda,\delta}\left(\frac{1}{T^{j-\frac{3}{2}\delta j}}\right), \quad (2.2.76)\end{aligned}$$

uniformly for $\ell \in \{0,1,\ldots,2j\}$.

The exponential sum on the right-hand side of (2.2.75), as hinted earlier, can be written as the product of two exponential sums, each in one variable,

$$V_{j,\ell}(m_1,n_1) = \left(\sum_{0\le m_2<L} e\left(\sqrt{\frac{xn_1}{m_1}}m_2\right) m_2^{\ell}\right)\left(\sum_{0\le n_2<L} e\left(\sqrt{\frac{xm_1}{n_1}}n_2\right) n_2^{2j-\ell}\right). \quad (2.2.77)$$

In the case $j=\ell=0$, the exponential sums above are geometric series, which can be accurately estimated. For any real number α, any integer M, and any positive integer H, we recall the well-known uniform upper bound

$$\left|\sum_{n=M+1}^{M+H} e(\alpha n)\right| = O\left(\min\left\{H, \frac{1}{\|\alpha\|}\right\}\right), \quad (2.2.78)$$

where $\|\alpha\|$ denotes the distance from α to the nearest integer. Using (2.2.78) in (2.2.77), we find that, for $j=\ell=0$,

$$|V_{0,0}(m_1,n_1)| = O\left(\min\left\{L, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\cdot\min\left\{L, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\right). \tag{2.2.79}$$

For general j and ℓ, a familiar argument based on (2.2.78) in combination with summation by parts for each of the two exponential sums on the right-hand side of (2.2.77) gives

$$|V_{j,\ell}(m_1,n_1)| = O_{j,\ell}\left(L^{2j}\min\left\{L, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\cdot\min\left\{L, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\right). \tag{2.2.80}$$

Using (2.2.76) and (2.2.80) for all $0 \le j \le r$, $0 \le \ell \le 2j$ and defining r by (2.2.70), we find from (2.2.73) that

$$\begin{aligned}|E_{m_1,n_1}| = O_{x,\lambda,\delta}\Bigg(&\sum_{0\le j\le[1/(\frac{1}{2}-\lambda)]}\sum_{\ell=0}^{2j}\frac{L^{2j}}{T^{j-\frac{3}{2}\delta j}}\cdot\min\left\{L, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\\ &\times\min\left\{L, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\Bigg) + O\left(T^{3\lambda+\frac{3}{2}\delta-1}\right).\end{aligned} \tag{2.2.81}$$

Here $L^2/T^{1-\frac{3}{2}\delta} < 1$ for $\delta < \frac{2}{3}(1-2\lambda)$, which we assume in the sequel, and so the maximum value of $L^{2j}/T^{j-\frac{3}{2}\delta j}$ is attained at $j=0$. Thus, by (2.2.81),

$$\begin{aligned}|E_{m_1,n_1}| = O_{x,\lambda,\delta}\Bigg(&\min\left\{L, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\\ &\times\min\left\{L, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\Bigg) + O\left(T^{3\lambda+\frac{3}{2}\delta-1}\right).\end{aligned} \tag{2.2.82}$$

Next, we employ (2.2.82) on the right side of (2.2.55). In doing so, note that the error term on the right side of (2.2.82) produces an error term on the right side of (2.2.55) that is bounded by

$$O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}}\cdot T_2\cdot T_4\cdot T^{3\lambda+\frac{3}{2}\delta-1}\right) = O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right).$$

This is smaller than the existing error term on the right side of (2.2.55), and we deduce that

$$\left|\sum_{a,b,\delta,T}\right| = O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}}\sum_{T_1\le m_1\le T_2}\sum_{T_3\le n_1\le T_4}\min\left\{T^\lambda, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\right.$$
$$\left.\times\min\left\{T^\lambda, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\right) + O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right). \quad (2.2.83)$$

2.2.7 Uniform Convergence When x Is Not an Integer

Our next idea is based on the observation that if for some m_1 and n_1, both $\|\sqrt{xn_1/m_1}\|$ and $\|\sqrt{xm_1/n_1}\|$ are simultaneously small, thus producing a large term on the right side of (2.2.83), then each of $\sqrt{xn_1/m_1}$ and $\sqrt{xm_1/n_1}$ is close to an integer, and hence their product is correspondingly close to an integer. But their product equals x, which is fixed throughout the proof, so this event cannot happen unless x is an integer. Fix an x that is not an integer. Then $\|x\| = \min\{|x-y| : y\in\mathbb{Z}\} > 0$. For each $m_1 \in \{T_1,\dots,T_2\}$ and $n_1 \in \{T_3,\dots,T_4\}$, let d_1 and d_2 be integers, depending on m_1 and n_1, such that

$$\left\|\sqrt{\frac{xn_1}{m_1}}\right\| = \left|d_1 - \sqrt{\frac{xn_1}{m_1}}\right| \quad (2.2.84)$$

and

$$\left\|\sqrt{\frac{xm_1}{n_1}}\right\| = \left|d_2 - \sqrt{\frac{xm_1}{n_1}}\right|. \quad (2.2.85)$$

Using (2.2.84) and (2.2.85) and the fact that d_1d_2 is an integer, we find that

$$\|x\| \le |x-d_1d_2| = \left|\sqrt{\frac{xn_1}{m_1}}\sqrt{\frac{xm_1}{n_1}} - d_1d_2\right| \le \left|\left(\sqrt{\frac{xn_1}{m_1}}-d_1\right)\sqrt{\frac{xm_1}{n_1}}\right|$$
$$+\left|d_1\left(\sqrt{\frac{xm_1}{n_1}}-d_2\right)\right| = \sqrt{\frac{xm_1}{n_1}}\left\|\sqrt{\frac{xn_1}{m_1}}\right\| + d_1\left\|\sqrt{\frac{xm_1}{n_1}}\right\|. \quad (2.2.86)$$

Here,

$$\sqrt{\frac{xm_1}{n_1}} = O_{x,\delta}(T^\delta) \quad (2.2.87)$$

and

$$d_1 \le \sqrt{\frac{xn_1}{m_1}} + \frac{1}{2} = O_{x,\delta}(T^\delta). \quad (2.2.88)$$

Thus, by (2.2.86)–(2.2.88),

$$\|x\| = O_{x,\delta}\left(T^\delta \max\left\{\left\|\sqrt{\frac{xn_1}{m_1}}\right\|, \left\|\sqrt{\frac{xm_1}{n_1}}\right\|\right\}\right). \quad (2.2.89)$$

It follows from (2.2.89) that, uniformly for $m_1 \in \{T_1, \ldots, T_2\}$ and $n_1 \in \{T_3, \ldots, T_4\}$,

$$\min\left\{\frac{1}{\|\sqrt{xn_1/m_1}\|}, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\} = O_{x,\delta}(T^{\delta}). \tag{2.2.90}$$

By (2.2.90), it follows that

$$\min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\} \cdot \min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}$$
$$= O_{x,\delta}\left(T^{\delta}\left(\min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\} + \min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\right)\right). \tag{2.2.91}$$

Inserting (2.2.91) into the right side of (2.2.83), we deduce that

$$\left|\sum_{a,b,\delta,T}\right| = O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{7}{4}\delta}} \sum_{T_1 \le m_1 \le T_2} \sum_{T_3 \le n_1 \le T_4} \min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\right.$$
$$\left. + \min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\right) + O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right). \tag{2.2.92}$$

The summation in the first of the two error terms on the right side of (2.2.92) yields two double sums. For the first, we keep the order of summation as in (2.2.92) and focus on the inner sum

$$F(x, \delta, \lambda, T, m_1) := \sum_{T_3 \le n_1 \le T_4} \min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}, \tag{2.2.93}$$

while for the second, we interchange the order of summation, so that the inner sum becomes

$$G(x, \delta, \lambda, T, n_1) := \sum_{T_1 \le m_1 \le T_2} \min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}. \tag{2.2.94}$$

We proceed to derive an upper bound for $F(x, \delta, \lambda, T, m_1)$. Each term in the sum on the right side of (2.2.93) lies in $[2, T^{\lambda}]$. We subdivide this interval into dyadic intervals $[2, 4), [4, 8), \ldots,$ $[2^s, T^{\lambda}]$, where $s = [\lambda \log_2 T]$. For each $j = 1, 2, \ldots, s$, set

$$B_{j,m_1} := \left\{n_1 \in \{T_3, \ldots, T_4\} : \left\|\sqrt{\frac{xn_1}{m_1}}\right\| \in \left[\frac{1}{2^{j+1}}, \frac{1}{2^j}\right]\right\}. \tag{2.2.95}$$

Then

$$F(x,\delta,\lambda,T,m_1) \\ \leq \sum_{j=1}^{s} 2^{j+1}\#\{B_{j,m_1}\} + T^{\lambda}\#\left\{T_3 \leq n_1 \leq T_4 : \left\|\sqrt{\frac{xn_1}{m_1}}\right\| \leq \frac{1}{T^{\lambda}}\right\}. \quad (2.2.96)$$

Now fix $j \in \{1,2,\dots,s\}$. We need an accurate upper bound for $\#\{B_{j,m_1}\}$. For each $n_1 \in B_{j,m_1}$, we let, as before, d_1 denote the closest integer to $\sqrt{xn_1/m_1}$. Then using (2.2.41), as we often have done and will continue to do, we see that, for T sufficiently large,

$$0 \leq d_1 \leq \sqrt{\frac{xT_4}{T_1}} + \frac{1}{2} \leq \sqrt{2^{1+\delta}xT^{\delta}} + \frac{1}{2} \leq \sqrt{3xT^{\delta}}. \quad (2.2.97)$$

By (2.2.95) and (2.2.97), it follows that

$$B_{j,m_1} \subseteq \bigcup_{d_1=0}^{[\sqrt{3xT^{\delta}}]} \{T_3 \leq n_1 \leq T_4 : \\ \sqrt{\frac{xn_1}{m_1}} \in \left[d_1 - \frac{1}{2^j}, d_1 - \frac{1}{2^{j+1}}\right] \cup \left[d_1 + \frac{1}{2^{j+1}}, d_1 + \frac{1}{2^j}\right]\right\} \\ \subseteq \bigcup_{d_1=0}^{[\sqrt{3xT^{\delta}}]} \left\{T_3 \leq n_1 \leq T_4 : n_1 \in \left[\frac{m_1}{x}\left(d_1 - \frac{1}{2^j}\right)^2, \frac{m_1}{x}\left(d_1 + \frac{1}{2^j}\right)^2\right]\right\}. \quad (2.2.98)$$

For each interval I of real numbers,

$$\#\{\mathbb{Z} \cap I\} \leq 1 + \text{length}(I). \quad (2.2.99)$$

From (2.2.98) and (2.2.99), we find that

$$\#\{B_{j,m_1}\} \leq \sum_{d_1=0}^{[\sqrt{3xT^{\delta}}]} \left(1 + \frac{m_1}{x}\left(d_1 + \frac{1}{2^j}\right)^2 - \frac{m_1}{x}\left(d_1 - \frac{1}{2^j}\right)^2\right) \\ = 1 + \left[\sqrt{3xT^{\delta}}\right] + \frac{m_1}{x}\sum_{d_1=0}^{[\sqrt{3xT^{\delta}}]} \frac{4d_1}{2^j} \\ = O_x\left(T^{\delta/2}\right) + O_x\left(\frac{m_1}{2^j} \cdot T^{\delta}\right). \quad (2.2.100)$$

Similarly,

$$\#\left\{T_3 \leq n_1 \leq T_4 : \left\|\sqrt{\frac{xn_1}{m_1}}\right\| \leq \frac{1}{T^{\lambda}}\right\} = O_x\left(T^{\delta/2}\right) + O_x\left(\frac{m_1}{T^{\lambda}} \cdot T^{\delta}\right). \quad (2.2.101)$$

Employing (2.2.100) and (2.2.101) on the right-hand side of (2.2.96), we deduce that

$$\begin{aligned} F(x,\delta,\lambda,T,m_1) &= O_x\left(\sum_{j=1}^{s} 2^{j+1}\left(T^{\delta/2}+\frac{m_1}{2^j}T^{\delta}\right)\right) \\ &\quad + O_x\left(T^{\lambda}\left(T^{\delta/2}+\frac{m_1}{T^{\lambda}}T^{\delta}\right)\right) \\ &= O_x(2^sT^{\delta/2}+sm_1T^{\delta})+O_x(T^{\lambda+\delta/2}+m_1T^{\delta}) \\ &= O_{x,\lambda,\delta}(T^{\lambda+\delta/2}+m_1T^{\delta}\log T), \end{aligned} \tag{2.2.102}$$

where we have recalled the definition $s=[\lambda\log_2 T]$. Reversing the roles of m_1 and n_1, using the same argument as above, appealing to (2.2.41), and invoking the inequalities

$$0 \le d_2 \le \sqrt{\frac{xm_1}{n_1}}+\frac{1}{2} \le \sqrt{\frac{2xT_2}{T_3}}+\frac{1}{2} = O_{x,\delta}(T^{\delta/2}) \tag{2.2.103}$$

in place of (2.2.97), we also deduce that

$$G(x,\delta,\lambda,T,n_1) = O_{x,\lambda,\delta}(T^{\lambda+\delta/2}+n_1T^{\delta}\log T). \tag{2.2.104}$$

Combining (2.2.92)–(2.2.94), (2.2.102), and (2.2.104), we find that

$$\begin{aligned} \left|\sum_{a,b,\delta,T}\right| &= O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{7}{4}\delta}}\sum_{T_1\le m_1\le T_2}\left(T^{\lambda+\delta/2}+m_1T^{\delta}\log T\right)\right) \\ &\quad + O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{7}{4}\delta}}\sum_{T_3\le n_1\le T_4}\left(T^{\lambda+\delta/2}+n_1T^{\delta}\log T\right)\right) \\ &\quad + O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right) \\ &= O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{7}{4}\delta}}\left(T^{1+\delta/2}+T^{2-2\lambda+\delta}\log T\right)\right) \\ &\quad + O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{7}{4}\delta}}\left(T^{1+3\delta/2}+T^{2-2\lambda+3\delta}\log T\right)\right) \\ &\quad + O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right) \\ &= O_{x,\lambda,\delta}\left(\frac{\log T}{T^{2\lambda-\frac{1}{2}-\frac{19}{4}\delta}}\right)+O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right). \end{aligned} \tag{2.2.105}$$

So far, the only condition we have put on λ is that $\lambda\in(0,\frac{1}{2})$. We now see that in order for the argument above to work, we also need $\lambda>\frac{1}{4}$. Then the first error term on the far right side of (2.2.105) will be small enough. In order to balance the exponents not involving δ in the two error terms on the far right side of (2.2.105), we now choose $\lambda = \frac{1}{3}$. Then by (2.2.105) and the fact that $\log T$ is smaller than $T^{\delta/4}$ for fixed $\delta > 0$ and T sufficiently large,

$$\left| \sum_{a,b,\delta,T} \right| = O_{x,\delta}\left(\frac{1}{T^{\frac{1}{6}-5\delta}} \right). \tag{2.2.106}$$

Let us recall that $\sum_{a,b,\delta,T}$ is one of the inner sums on the right-hand side of (2.2.40), with, from the discourse prior to (2.2.40), $T = 2^{r_1+j}$. Using (2.2.106) for each of these sums and recalling the definition $r_1 = [\log_2 M_1]$, we find from (2.2.39) that

$$\begin{aligned} |S_{8,M_2}(a,\theta,\delta) - S_{8,M_1}(a,\theta,\delta)| &= O_{x,\delta}\left(\sum_{j=0}^{r_2-r_1} \frac{1}{2^{(r_1+j)(\frac{1}{6}-5\delta)}} \right) \\ &= O_{x,\delta}\left(\frac{1}{2^{r_1(\frac{1}{6}-5\delta)}} \sum_{j=0}^{\infty} \frac{1}{2^{j(\frac{1}{6}-5\delta)}} \right) \\ &= O_{x,\delta}\left(\frac{1}{2^{r_1(\frac{1}{6}-5\delta)}} \right) \\ &= O_{x,\delta}\left(\frac{1}{M_1^{(\frac{1}{6}-5\delta)}} \right), \end{aligned} \tag{2.2.107}$$

uniformly with respect to θ in $[0,1]$. This completes the proof that the sum $S_8(a,\theta,\delta)$ converges uniformly with respect to θ in $[0,1]$, which in turn implies a corresponding statement for the initial double sum $S_1(a,\theta)$.

2.2.8 The Case That x Is an Integer

We now proceed to examine the case that x is an integer. In the case above, in which x is not an integer, the relations (2.2.106) and consequently (2.2.107) were stronger than needed, in the sense that a weaker savings, where the exponent $\frac{1}{6}$ is replaced by any smaller strictly positive constant, would have sufficed. The fact that we had some room to spare in the proof above naturally leads us to expect that exactly the same argument as above would cover as well, at least partially, the case that x is an integer. With this in mind, we subdivide the sum $\sum_{a,b,\delta,T}$ into two sums, one for which the argument above applies, to be examined first, and the second, to be examined later.

We begin by fixing a positive integer x. Next, we fix an arbitrary small real number $\eta > 0$. With η fixed, we then choose a real number $\lambda < \frac{1}{2}$, depending on η. The exact dependence of λ on η will be clarified later, with the crux of the matter being that λ is chosen such that $\frac{1}{2}-\lambda$ is much smaller than η. With η and λ fixed, we then choose $\delta > 0$, depending on η and λ. The dependence of δ on η and λ will be made explicit later, with the goal being that δ will be chosen to be much smaller than $\frac{1}{2} - \lambda$. Once η, λ, and δ are fixed, we start by following the same reduction procedure from the foregoing beginning of the proof, which reduces the convergence, respectively uniform convergence, of $S_1(a, \theta)$ to that of $S_8(a, \theta, \delta)$. In order to investigate the convergence of $S_8(a, \theta, \delta)$, we again employ Cauchy's criterion, and arrive at (2.2.40). We need to show that the right side of (2.2.40) is in absolute value less than ϵ, for an arbitrary fixed $\epsilon > 0$. We again bound each of the inner sums on the right-hand side of (2.2.40) separately. As before, we fix j, with $1 \le j \le r_2 - r_1 - 1$, set $T = 2^{r_1+j}$, and consider the sum $\sum_{a,b,\delta,T}$, defined in (2.2.42). At this point, we divide the sum $\sum_{a,b,\delta,T}$ into two parts, depending on η, as follows. Consider in $\mathbb{R}^2$ the rectangle

$$D(\delta, T) := [T, 2T) \times [T^{1-\delta}, (2T)^{1+\delta}].$$

For each divisor d of x, draw the ray from the origin with slope d^2/x. Around this ray, consider the thin trapezoidal region, say $V(x, d, \eta, \delta, T)$, that consists of all the points in $D(\delta, T)$ for which the slope of the line from the origin through the point lies in the interval

$$\left[\frac{d^2}{x} - \frac{1}{T^{\frac{1}{2}-\eta}}, \frac{d^2}{x} + \frac{1}{T^{\frac{1}{2}-\eta}}\right]. \tag{2.2.108}$$

Set

$$U_1(a, b, \delta, T, \eta) := \sum_{(m,n)\in D(\delta,T)\setminus\cup_{d|x} V(x,d,\eta,\delta,T)} \frac{\sin\left(b\sqrt{\dfrac{m}{n}}\right)\sin\left(a\sqrt{m\left(n+\dfrac{1}{2}\right)} - \dfrac{3\pi}{4}\right)}{m^{3/4}n^{3/4}} \tag{2.2.109}$$

and

$$U_2(a, b, \delta, T, \eta) := \sum_{d|x}\ \sum_{(m,n)\in V(x,d,\eta,\delta,T)} \frac{\sin\left(b\sqrt{\dfrac{m}{n}}\right)\sin\left(a\sqrt{m\left(n+\dfrac{1}{2}\right)} - \dfrac{3\pi}{4}\right)}{m^{3/4}n^{3/4}}. \tag{2.2.110}$$

Thus,

$$\sum_{a,b,\delta,T} = U_1(a, b, \delta, T, \eta) + U_2(a, b, \delta, T, \eta). \tag{2.2.111}$$

Before proceeding further, we observe that although the trapezoids $V(x,d,\eta,\delta,T)$ are very thin, we cannot afford to trivially estimate $U_2(a,b,\delta,T,\eta)$, as we did in (2.2.45). Indeed, $V(x,d,\eta,\delta,T)$ is a trapezoid with (horizontal) height T lying inside an angle of measure roughly $1/T^{\frac{1}{2}-\eta}$, and so the two bases have size of the order of magnitude of $T^{\frac{1}{2}+\eta}$. Therefore the area of $V(x,d,\eta,\delta,T)$ is of order $T^{\frac{3}{2}+\eta}$. The number of integral points (m,n) in $V(x,d,\eta,\delta,T)$ is asymptotic to this area, since the perimeter of the trapezoid is of smaller order, $O(T)$. On the other hand, the denominator $m^{3/4}n^{3/4}$ on the right side of (2.2.110) is of precise order of magnitude $T^{3/2}$. To see this, note that n/m lies between $1/x$ and x. For other points in the trapezoid $V(x,d,\eta,\delta,T)$, for T sufficiently large, n/m lies between $1/2x$ and $2x$, say. Since $T<m<2T$, this implies that $T/2x<n<4xT$. Therefore, if we estimate the sum on the right side of (2.2.110) trivially, we obtain

$$|U_2(a,b,\delta,T,\eta)| = O_{x,\eta,\delta}(T^{\eta}), \tag{2.2.112}$$

which is not sufficient for our purposes. This discussion also shows that any cancellation on the right side of (2.2.110) allowing us to save a factor of T^{c_0}, for some constant $c_0>0$ independent of η, would suffice (by taking η smaller than c_0).

Taking into account the shape of these trapezoids, we see that it does not appear appropriate to consider subdividing them into small squares as before. Instead, it is more natural to try to achieve cancellation on large exponential sums taken along parallel lines of corresponding slope d^2/x, which is what we will do later.

We first bound $U_1(a,b,\delta,T,\eta)$. Subdivide $D(\delta,T)\setminus\cup_{d|x}V(x,d,\eta,\delta,T)$ into squares of size $L\times L$, where, as before, $L=[T^{\lambda}]$. Let T_1,T_2,T_3, and T_4 be as defined in (2.2.41). For each $m_1\in\{T_1,\dots,T_2\}$ and $n_1\in\{T_3,\dots,T_4\}$, we define $\sum_{m_1,n_1}$ by (2.2.42). We consider all those squares $[Lm_1,L(m_1+1))\times[Ln_1,L(n_1+1))$ for which the lower left corner does not belong to any of the trapezoids $V(x,d,\eta,\delta,T)$. Since the slope of the ray from the origin to this lower left corner equals n_1/m_1, the condition above can be stated as

$$\frac{n_1}{m_1}\notin\cup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{\frac{1}{2}-\eta}},\frac{d^2}{x}+\frac{1}{T^{\frac{1}{2}-\eta}}\right]. \tag{2.2.113}$$

Note that all the integral points (m,n) in $D(\delta,T)\setminus\cup_{d|x}V(x,d,\eta,\delta,T)$ that do not belong to the union of squares $[Lm_1,L(m_1+1))\times[Ln_1,L(n_1+1))$, $m_1\in\{T_1,\dots,T_2\}$ and $n_1\in\{T_3,\dots,T_4\}$, and that satisfy (2.2.113) are at a distance $O(L)$ from the boundary of $D(\delta,T)\setminus\cup_{d|x}V(x,d,\eta,\delta,T)$. We bound the contribution of these points (m,n) on the right side of (2.2.109) as follows. The contribution of those points (m,n) that are at a distance $O(L)$ from the four edges of the rectangle $D(\delta,T)$ was estimated in (2.2.43), and it was found to be

$$O\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{3}{4}\delta}}\right).$$

The remaining points, namely, those (m, n) lying inside the rectangle $D(\delta, T)$ that are at a distance $O(L)$ from the union over $d \mid x$ of the rays from the origin of slopes

$$\frac{d^2}{x} - \frac{1}{T^{\frac{1}{2}-\eta}} \quad \text{and} \quad \frac{d^2}{x} + \frac{1}{T^{\frac{1}{2}-\eta}}$$

can be bounded in a similar manner. One then finds that their contribution to the right side of (2.2.109) is

$$O_x\left(\frac{1}{T^{\frac{1}{2}-\lambda}}\right).$$

Combining all these bounds, we deduce that

$$\left| U_1(a, b, \delta, T, \eta) - \sum_{\substack{T_1 \le m_1 \le T_2 \\ T_3 \le n_1 \le T_4 \\ \frac{n_1}{m_1} \notin \cup_{d|x}\left[\frac{d^2}{x} - \frac{1}{T^{\frac{1}{2}-\eta}}, \frac{d^2}{x} + \frac{1}{T^{\frac{1}{2}-\eta}}\right]}} \sum_{m_1, n_1} \right| = O_x\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{3}{4}\delta}}\right). \tag{2.2.114}$$

Next, we apply (2.2.53) to each $\sum_{m_1,n_1}$ in (2.2.114), and obtain a relation analogous to (2.2.55), namely,

$$|U_1(a, b, \delta, T, \eta)| = O\left(\frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}} \sum_{\substack{T_1 \le m_1 \le T_2 \\ T_3 \le n_1 \le T_4 \\ \frac{n_1}{m_1} \notin \cup_{d|x}\left[\frac{d^2}{x} - \frac{1}{T^{\frac{1}{2}-\eta}}, \frac{d^2}{x} + \frac{1}{T^{\frac{1}{2}-\eta}}\right]}} |E_{m_1,n_1}|\right) + O_x\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right), \tag{2.2.115}$$

where E_{m_1,n_1} is defined in (2.2.51). The exponential sums E_{m_1,n_1} were bounded in (2.2.82). Employing those bounds on the right-hand side of (2.2.115), we derive a relation analogous to (2.2.83), namely,

$$|U_1(a,b,\delta,T,\eta)| = O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}} \sum_{\substack{T_1\le m_1\le T_2\\ T_3\le n_1\le T_4\\ \frac{n_1}{m_1}\notin\cup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{\frac{1}{2}-\eta}},\frac{d^2}{x}+\frac{1}{T^{\frac{1}{2}-\eta}}\right]}} \min\left\{T^\lambda, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\cdot\min\left\{T^\lambda, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\right) + O_x\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right). \tag{2.2.116}$$

Unlike the previous case, in which x was not an integer and $\|\sqrt{xn_1/m_1}\|$ and $\|\sqrt{xm_1/n_1}\|$ cannot be simultaneously small, in the present case in which x is an integer, $\|\sqrt{xn_1/m_1}\|$ and $\|\sqrt{xm_1/n_1}\|$ can be small simultaneously. This can happen only if n_1/m_1 is close to a number of the form d^2/x with $d|x$. Conversely, if n_1/m_1 is close to d^2/x for some divisor d of x, then automatically m_1/n_1 is close to ${d'}^2/x$, where $dd' = x$, and $\|\sqrt{xn_1/m_1}\|$ and $\|\sqrt{xm_1/n_1}\|$ are simultaneously small. The extra condition on n_1/m_1 in the summation on the right side of (2.2.116) assures us that $\|\sqrt{xn_1/m_1}\|$ and $\|\sqrt{xm_1/n_1}\|$ cannot be simultaneously small. This does not prevent the possibility that one of $\|\sqrt{xn_1/m_1}\|$ and $\|\sqrt{xm_1/n_1}\|$ is much smaller than the other, of course. But in that case, the other is larger than $1/T^\delta$, by (2.2.41), and so the term corresponding to the pair (m_1, n_1) on the right side of (2.2.116) is harmless, as we have seen before.

With this in mind, we proceed as follows. Consider the sets of integral points (m_1, n_1) defined by

$$\mathcal{B}_1(x,\eta,\lambda,\delta,T) := \Bigg\{(m_1,n_1) : T_1\le m_1\le T_2, T_3\le n_1\le T_4, \frac{n_1}{m_1}\notin\cup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{\frac{1}{2}-\eta}},\frac{d^2}{x}+\frac{1}{T^{\frac{1}{2}-\eta}}\right], \max\left\{\left\|\sqrt{\frac{xn_1}{m_1}}\right\|,\left\|\sqrt{\frac{xm_1}{n_1}}\right\|\right\} > \frac{1}{T^\delta}\Bigg\} \tag{2.2.117}$$

and

$$\mathcal{B}_2(x,\eta,\lambda,\delta,T) := \Bigg\{(m_1,n_1) : T_1\le m_1\le T_2, T_3\le n_1\le T_4, \frac{n_1}{m_1}\notin\cup_{d|x}\left[\frac{d^2}{x}-\frac{1}{T^{\frac{1}{2}-\eta}},\frac{d^2}{x}+\frac{1}{T^{\frac{1}{2}-\eta}}\right], \left\|\sqrt{\frac{xn_1}{m_1}}\right\|\le\frac{1}{T^\delta}, \left\|\sqrt{\frac{xm_1}{n_1}}\right\|\le\frac{1}{T^\delta}\Bigg\}. \tag{2.2.118}$$

The last condition in the definition of $\mathcal{B}_1(x,\eta,\lambda,\delta,T)$ is equivalent to

$$\min\left\{\frac{1}{\|\sqrt{xn_1/m_1}\|},\frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}<T^{\delta},\tag{2.2.119}$$

which is analogous to (2.2.90). Therefore the contribution of $\mathcal{B}_1(x,\eta,\lambda,\delta,T)$ on the right side of (2.2.116) can be estimated as in the previous case when x was not an integer. In the present case, we arrive at (2.2.91) and proceed similarly as in the proof that previously led to (2.2.105), but now there remains the estimate of the summation over (m_1,n_1) in $\mathcal{B}_2(x,\eta,\lambda,\delta,T)$. Accordingly, up to this point, we obtain the bounds

$$\begin{aligned}
&|U_1(a,b,\delta,T,\eta)|\\
&=O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}}\sum_{(m_1,n_1)\in\mathcal{B}_2(x,\eta,\lambda,\delta,T)}\min\left\{T^{\lambda},\frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\right.\\
&\quad\left.\times\min\left\{T^{\lambda},\frac{1}{\|\sqrt{xm_1/n_1}\|}\right\}\right)+O_{x,\lambda,\delta}\left(\frac{\log T}{T^{2\lambda-\frac{1}{2}-\frac{19}{4}\delta}}\right)+O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right).
\end{aligned}\tag{2.2.120}$$

Next, let us observe that for each $(m_1,n_1)\in\mathcal{B}_2(x,\eta,\lambda,\delta,T)$, if we denote by d_1 and d_2 the closest integers to $\sqrt{xn_1/m_1}$ and $\sqrt{xm_1/n_1}$, respectively, then

$$\begin{aligned}
|d_1d_2-x|&=\left|\left(d_1-\sqrt{\frac{xn_1}{m_1}}\right)d_2+\sqrt{\frac{xn_1}{m_1}}\left(d_2-\sqrt{\frac{xm_1}{n_1}}\right)\right|\\
&=\left\|\sqrt{\frac{xn_1}{m_1}}\right\||d_2+\sqrt{\frac{xn_1}{m_1}}\left\|\sqrt{\frac{xm_1}{n_1}}\right\|.
\end{aligned}\tag{2.2.121}$$

Here, by (2.2.41),

$$\sqrt{\frac{xn_1}{m_1}}=O_x(T^{\delta/2})\qquad\text{and}\qquad d_2=\sqrt{\frac{xm_1}{n_1}}+O(1)=O_x(T^{\delta/2}),$$

while

$$\left\|\sqrt{\frac{xn_1}{m_1}}\right\|\leq\frac{1}{T^{\delta}}\qquad\text{and}\qquad\left\|\sqrt{\frac{xm_1}{n_1}}\right\|\leq\frac{1}{T^{\delta}},$$

by (2.2.118). On using the foregoing estimates in (2.2.121), we find that

$$|d_1d_2-x|=O_x\left(\frac{1}{T^{\delta/2}}\right),\tag{2.2.122}$$

and since d_1, d_2, and x are integers, (2.2.122) implies that $d_1d_2=x$. Let us further observe that for $(m_1,n_1)\in\mathbb{B}_2(x,\eta,\lambda,\delta,T)$, the quantities $1/\|\sqrt{xn_1/m_1}\|$

and $1/\|\sqrt{xm_1/n_1}\|$, which are both larger than T^δ by (2.2.118), have the same order of magnitude. Indeed,

$$\frac{\left\|\sqrt{\frac{xn_1}{m_1}}\right\|}{\left\|\sqrt{\frac{xm_1}{n_1}}\right\|}=\frac{\left|d_1-\sqrt{\frac{xn_1}{m_1}}\right|}{\left|d_2-\sqrt{\frac{xm_1}{n_1}}\right|}=\frac{\left|d_1^2-\frac{xn_1}{m_1}\right|\left(d_2+\sqrt{\frac{xm_1}{n_1}}\right)}{\left|d_2^2-\frac{xm_1}{n_1}\right|\left(d_1+\sqrt{\frac{xn_1}{m_1}}\right)}. \tag{2.2.123}$$

Here, by (2.2.41),

$$d_2+\sqrt{\frac{xm_1}{n_1}}=2d_2+O\left(\frac{1}{T^\delta}\right)=2d_2\left(1+O_x\left(\frac{1}{T^\delta}\right)\right), \tag{2.2.124}$$

$$d_1+\sqrt{\frac{xn_1}{m_1}}=2d_1\left(1+O_x\left(\frac{1}{T^\delta}\right)\right), \tag{2.2.125}$$

$$\left|d_1^2-\frac{xn_1}{m_1}\right|=\frac{1}{m_1}\left|d_1^2m_1-d_1d_2n_1\right|=\frac{d_1}{m_1}|d_1m_1-d_2n_1|, \tag{2.2.126}$$

and

$$\left|d_2^2-\frac{xm_1}{n_1}\right|=\frac{d_2}{n_1}|d_2n_1-d_1m_1|. \tag{2.2.127}$$

By (2.2.123)–(2.2.127), we see that unless $d_2n_1=d_1m_1$,

$$\frac{\left\|\sqrt{\frac{xn_1}{m_1}}\right\|}{\left\|\sqrt{\frac{xm_1}{n_1}}\right\|}=\frac{n_1}{m_1}\left(1+O_x\left(\frac{1}{T^\delta}\right)\right). \tag{2.2.128}$$

But

$$\frac{n_1}{m_1}=\frac{\sqrt{\frac{xn_1}{m_1}}}{\sqrt{\frac{xm_1}{n_1}}}=\frac{d_1+O_x\left(\frac{1}{T^\delta}\right)}{d_2+O_x\left(\frac{1}{T^\delta}\right)}=\frac{d_1}{d_2}\left(1+O_x\left(\frac{1}{T^\delta}\right)\right). \tag{2.2.129}$$

By (2.2.128) and (2.2.129), it follows that

$$\frac{\left\|\sqrt{\frac{xn_1}{m_1}}\right\|}{\left\|\sqrt{\frac{xm_1}{n_1}}\right\|}=\frac{d_1}{d_2}\left(1+O_x\left(\frac{1}{T^\delta}\right)\right), \tag{2.2.130}$$

unless $d_2n_1=d_1m_1$, in which case both quantities $\|\sqrt{xn_1/m_1}\|$ and $\|\sqrt{xm_1/n_1}\|$ are equal to zero. In both cases, we can conclude that

$$\min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xm_1/n_1}\|}\right\} = O_x\left(\min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\right). \tag{2.2.131}$$

Inserting (2.2.131) into the right-hand side of (2.2.120), we find that

$$\begin{aligned}
&|U_1(a,b,\delta,T,\eta)| \\
&= O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{3}{2}-\frac{3}{4}\delta}} \sum_{(m_1,n_1)\in\mathcal{B}_2(x,\eta,\lambda,\delta,T)} \left(\min\left\{T^{\lambda}, \frac{1}{\|\sqrt{xn_1/m_1}\|}\right\}\right)^2\right) \\
&\quad + O_{x,\lambda,\delta}\left(\frac{\log T}{T^{2\lambda-\frac{1}{2}-\frac{19}{4}\delta}}\right) + O_{x,\lambda,\delta}\left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}}\right).
\end{aligned} \tag{2.2.132}$$

We proceed to estimate the sum in the first error term on the right-hand side of (2.2.132). Recall that for any $(m_1, n_1) \in \mathbb{B}_2(x, \eta, \lambda, \delta, T)$, on the one hand, $\|\sqrt{xn_1/m_1}\| \leq 1/T^{\delta}$, and on the other hand,

$$\frac{n_1}{m_1} \notin \cup_{d|x}\left[\frac{d^2}{x} - \frac{1}{T^{\frac{1}{2}-\eta}}, \frac{d^2}{x} + \frac{1}{T^{\frac{1}{2}-\eta}}\right],$$

and so, in particular,

$$\left|\frac{n_1}{m_1} - \frac{d_1^2}{x}\right| \geq \frac{1}{T^{\frac{1}{2}-\eta}}, \tag{2.2.133}$$

where as before, d_1 is the closest integer to $\sqrt{xn_1/m_1}$. By (2.2.133) and (2.2.125),

$$\begin{aligned}
\left\|\sqrt{\frac{xn_1}{m_1}}\right\| &= \left|d_1 - \sqrt{\frac{xn_1}{m_1}}\right| = \frac{x\left|\frac{d_1^2}{x} - \frac{n_1}{m_1}\right|}{d_1 + \sqrt{\frac{xn_1}{m_1}}} \geq \frac{x}{T^{\frac{1}{2}-\eta}\left(d_1 + \sqrt{\frac{xn_1}{m_1}}\right)} \\
&= \frac{x}{2d_1 T^{\frac{1}{2}-\eta}}\left(1 + O_x\left(\frac{1}{T^{\delta}}\right)\right) > \frac{1}{4T^{\frac{1}{2}-\eta}},
\end{aligned} \tag{2.2.134}$$

for sufficiently large T.

We next subdivide the interval

$$\left[\frac{1}{4T^{\frac{1}{2}-\eta}}, \frac{1}{T^{\delta}}\right]$$

into dyadic intervals of the form

$$\left[\frac{1}{2^{j+1}}, \frac{1}{2^j}\right],$$

and, for each j, bound the contribution to the first O-term on the right side of (2.2.132) of those pairs (m_1, n_1) for which

$$\left\| \sqrt{\frac{xn_1}{m_1}} \right\| \in \left[\frac{1}{2^{j+1}}, \frac{1}{2^j} \right].$$

Recall that $\frac{1}{2} - \lambda$ is smaller than η, and so $1/T^\lambda < 1/(4T^{\frac{1}{2}-\eta})$. Hence,

$$\min \left\{ T^\lambda, \frac{1}{\|\sqrt{xn_1/m_1}\|} \right\} = \frac{1}{\|\sqrt{xn_1/m_1}\|} \tag{2.2.135}$$

for all $(m_1, n_1) \in \mathbb{B}_2(x, \eta, \lambda, \delta, T)$. In conclusion, if we set $s_1 := [\delta \log_2 T]$ and $s_2 := 2 + \left[\left(\frac{1}{2} - \eta \right) \log_2 T \right]$, then, with the use of (2.2.99) below,

$$\begin{aligned}
&\sum_{(m_1,n_1)\in\mathbb{B}_2(x,\eta,\lambda,\delta,T)} \left(\min \left\{ T^\lambda, \frac{1}{||\sqrt{xn_1/m_1}||} \right\} \right)^2 \\
&\le \sum_{j=s_1}^{s_2} 2^{2j+2} \# \left\{ (m_1, n_1) \in \mathbb{B}_2(x, \eta, \lambda, \delta, T) : \left\| \sqrt{\frac{xn_1}{m_1}} \right\| \in \left[\frac{1}{2^{j+1}}, \frac{1}{2^j} \right] \right\} \\
&\le \sum_{j=s_1}^{s_2} 2^{2j+2} \sum_{d|x} \sum_{T_1 \le m_1 \le T_2} \# \left\{ n_1 : \left| \sqrt{\frac{xn_1}{m_1}} - d \right| \in \left[\frac{1}{2^{j+1}}, \frac{1}{2^j} \right] \right\} \\
&\le \sum_{j=s_1}^{s_2} 2^{2j+2} \sum_{d|x} \sum_{T_1 \le m_1 \le T_2} \# \left\{ \mathbb{Z} \cap \left[\frac{m_1}{x} \left(d - \frac{1}{2^j} \right)^2, \frac{m_1}{x} \left(d + \frac{1}{2^j} \right)^2 \right] \right\} \\
&\le \sum_{j=s_1}^{s_2} 2^{2j+2} \sum_{d|x} \sum_{T_1 \le m_1 \le T_2} \left(1 + \frac{dm_1}{x2^{j-2}} \right) \\
&= O_x \left(\sum_{j=s_1}^{s_2} 2^{2j} T_2 \right) + O_x \left(\sum_{j=s_1}^{s_2} 2^j \sum_{T_1 \le m_1 \le T_2} m_1 \right) \\
&= O_{x,\eta,\delta,\lambda} \left(2^{2s_2} T^{1-\lambda} \right) + O_{x,\eta,\delta,\lambda} \left(2^{s_2} T^{2-2\lambda} \right) \\
&= O_{x,\eta,\delta,\lambda}(T^{2-2\eta-\lambda}) + O_{x,\eta,\delta,\lambda}(T^{\frac{5}{2}-\eta-2\lambda}).
\end{aligned} \tag{2.2.136}$$

Combining (2.2.136) and (2.2.132), we finally deduce that

$$\begin{aligned}
|U_1(a, b, \delta, T, \eta)| = O_{x,\lambda,\delta,\eta} \left(\frac{1}{T^{\lambda+2\eta-\frac{1}{2}-\frac{3}{4}\delta}} \right) + O_{x,\lambda,\delta,\eta} \left(\frac{1}{T^{2\lambda+\eta-1-\frac{3}{4}\delta}} \right) \\
+ O_{x,\lambda,\delta} \left(\frac{\log T}{T^{2\lambda-\frac{1}{2}-\frac{19}{4}\delta}} \right) + O_{x,\lambda,\delta} \left(\frac{1}{T^{\frac{1}{2}-\lambda-\frac{13}{4}\delta}} \right).
\end{aligned} \tag{2.2.137}$$

We now see that for any fixed $\eta > 0$, we can make all the O-terms on the right side of (2.2.137) sufficiently small by choosing λ close to $\frac{1}{2}$ and then choosing $\delta > 0$ small enough. To be precise, we fix a small $\eta > 0$, and then let $\lambda = \frac{1}{2} - \frac{1}{3}\eta$. Thus, (2.2.137) takes the shape

$$\begin{aligned} |U_1(a,b,\delta,T,\eta)| &= O_{x,\delta,\eta}\left(\frac{1}{T^{\frac{5}{3}\eta-\frac{3}{4}\delta}}\right) + O_{x,\delta,\eta}\left(\frac{1}{T^{\frac{1}{3}\eta-\frac{3}{4}\delta}}\right) \\ &\quad + O_{x,\delta,\eta}\left(\frac{\log T}{T^{\frac{1}{2}-\frac{2}{3}\eta-\frac{19}{4}\delta}}\right) + O_{x,\delta,\eta}\left(\frac{1}{T^{\frac{1}{3}\eta-\frac{13}{4}\delta}}\right) \\ &= O_{x,\delta,\eta}\left(\frac{1}{T^{\frac{1}{3}\eta-\frac{13}{4}\delta}}\right). \end{aligned} \tag{2.2.138}$$

We now let $\delta = \eta/39$, and so from (2.2.137) we can now deduce that

$$|U_1(a,b,T,\eta)| = O_{x,\eta}\left(\frac{1}{T^{\eta/4}}\right), \tag{2.2.139}$$

where, for simplicity, we deleted the symbol δ on the left-hand side of (2.2.139), because δ is a function of η.

There remains the problem of obtaining a suitable bound for the sum $U_2(a,b,\delta,T,\eta)$. As above, we delete δ from the notations $V(x,d,\eta,\delta,T)$ and $U_2(a,b,\delta,T,\eta)$, which we now proceed to estimate.

2.2.9 Estimating $U_2(a,b,T,\eta)$

In order to bound $U_2(a,b,T,\eta)$, we estimate, for each divisor d of x, the inner sum on the right side of (2.2.110). For each $(m,n) \in V(x,d,\eta,T)$, by (2.2.108),

$$\left|\frac{n}{m} - \frac{d^2}{x}\right| \leq \frac{1}{T^{\frac{1}{2}-\eta}}. \tag{2.2.140}$$

By (2.2.140),

$$\sin\left(b\sqrt{\frac{m}{n}}\right) = \sin\left(\frac{b\sqrt{x}}{d}\right) + O_x\left(\frac{1}{T^{\frac{1}{2}-\eta}}\right) \tag{2.2.141}$$

and

$$\frac{1}{m^{3/4}n^{3/4}} = \frac{x^{3/4}}{d^{3/2}m^{3/2}}\left(1 + O_x\left(\frac{1}{T^{\frac{1}{2}-\eta}}\right)\right). \tag{2.2.142}$$

Hence, by (2.2.110),

$$U_2(a,b,T,\eta) = x^{3/4} \sum_{d|x} \frac{\sin\left(\frac{b\sqrt{x}}{d}\right)}{d^{3/2}} \sum_{(m,n)\in V(x,d,\eta,T)} \frac{\sin\left(a\sqrt{m(n+\frac{1}{2})} - \frac{3\pi}{4}\right)}{m^{3/2}}$$
$$+ O_x\left(\frac{1}{T^{\frac{1}{2}-\eta}} \sum_{d|x} \sum_{(m,n)\in V(x,d,\eta,T)} \frac{1}{m^{3/2}}\right). \tag{2.2.143}$$

Recall from the reasoning leading to (2.2.112) that the number of integral pairs (m,n) in each $V(x,d,\eta,T)$ is of the order of $T^{\frac{3}{2}+\eta}$. Thus, using this estimate in the O-term above and recalling that $T \le m < 2T$, we find that (2.2.143) reduces to

$$U_2(a,b,T,\eta) = x^{3/4} \sum_{d|x} \frac{\sin\left(\frac{b\sqrt{x}}{d}\right)}{d^{3/2}} \sum_{(m,n)\in V(x,d,\eta,T)} \frac{\sin\left(a\sqrt{m(n+\frac{1}{2})} - \frac{3\pi}{4}\right)}{m^{3/2}}$$
$$+ O_x\left(\frac{1}{T^{\frac{1}{2}-2\eta}}\right). \tag{2.2.144}$$

From the inequalities $T \le m < 2T$ combined with (2.2.140), it follows that

$$\left| n - \frac{d^2 m}{x} \right| \le 2T^{\frac{1}{2}+\eta} \tag{2.2.145}$$

and

$$\begin{aligned}
\sqrt{m(n+\tfrac{1}{2})} &= \sqrt{m\left(\frac{d^2m}{x} + \frac{1}{2} + n - \frac{d^2m}{x}\right)} \\
&= \frac{dm}{\sqrt{x}}\left(1 + \frac{x\left(\frac{1}{2}+n-\frac{d^2m}{x}\right)}{d^2m}\right)^{1/2} \\
&= \frac{dm}{\sqrt{x}}\left(1 + \frac{x\left(\frac{1}{2}+n-\frac{d^2m}{x}\right)}{2d^2m} - \frac{x^2\left(\frac{1}{2}+n-\frac{d^2m}{x}\right)^2}{8d^4m^2}\right. \\
&\qquad \left. + O_x\left(\frac{\left|\frac{1}{2}+n-\frac{d^2m}{x}\right|^3}{m^3}\right)\right) \\
&= \frac{dm}{\sqrt{x}} + \frac{\sqrt{x}}{4d} + \frac{\sqrt{x}\,n}{2d} - \frac{dm}{2\sqrt{x}} - \frac{x^{3/2}\left(\frac{1}{2}+n-\frac{d^2m}{x}\right)^2}{8d^3m} + O_x\left(\frac{1}{T^{\frac{1}{2}-3\eta}}\right).
\end{aligned} \tag{2.2.146}$$

Recall that $a = 4\pi\sqrt{x}$. Therefore,

$$\begin{aligned}
&\sin\left(a\sqrt{m(n+\tfrac{1}{2})} - \frac{3\pi}{4}\right) \\
&\quad = \sin\left(2\pi dm + \frac{\pi x}{d} + \frac{2\pi xn}{d} - \frac{\pi x^2\left(\frac{1}{2}+n-\frac{d^2m}{x}\right)^2}{2d^3m} - \frac{3\pi}{4}\right) \\
&\qquad + O_x\left(\frac{1}{T^{\frac{1}{2}-3\eta}}\right).
\end{aligned} \tag{2.2.147}$$

Here, $2\pi dm + 2\pi xn/d$ is an integral multiple of 2π, and $\pi x/d$ is an integral multiple of π, which is a multiple of 2π if and only if x/d is even. It follows from (2.2.147) and (2.2.144) that

$$\begin{aligned}
&U_2(a,b,T,\eta) \\
&\quad = x^{3/4}\sum_{d|x}\frac{1}{d^{3/2}}\sin\left(\frac{b\sqrt{x}}{d}\right) \\
&\qquad \times \sum_{(m,n)\in V(x,d,\eta,T)}\frac{(-1)^{x/d}}{m^{3/2}}\sin\left(-\frac{\pi x^2\left(\frac{1}{2}+n-\frac{d^2m}{x}\right)^2}{2d^3m} - \frac{3\pi}{4}\right) \\
&\qquad + O_x\left(\frac{1}{T^{\frac{1}{2}-3\eta}}\sum_{d|x}\sum_{(m,n)\in V(x,d,\eta,T)}\frac{1}{m^{3/2}}\right) + O_x\left(\frac{1}{T^{\frac{1}{2}-2\eta}}\right) \\
&\quad = x^{3/4}\sum_{d|x}\frac{(-1)^{x/d+1}}{d^{3/2}}\sin\left(\frac{b\sqrt{x}}{d}\right) \\
&\qquad \times \sum_{(m,n)\in V(x,d,\eta,T)}\frac{1}{m^{3/2}}\sin\left(\frac{\pi x^2\left(\frac{1}{2}+n-\frac{d^2m}{x}\right)^2}{2d^3m} + \frac{3\pi}{4}\right) \\
&\qquad + O_x\left(\frac{1}{T^{\frac{1}{2}-4\eta}}\right).
\end{aligned} \tag{2.2.148}$$

Furthermore, by (2.2.145) and the inequalities $T \leq m < 2T$,

$$\frac{\pi x^2\left(\frac{1}{2}+n-\frac{d^2m}{x}\right)^2}{2d^3m} = \frac{\pi(xn-d^2m)^2}{2d^3m} + O_x\left(\frac{1}{T^{\frac{1}{2}-\eta}}\right), \tag{2.2.149}$$

which, when inserted in (2.2.148), gives

$$U_2(a,b,T,\eta) = x^{3/4} \sum_{d|x} \frac{(-1)^{x/d+1}}{d^{3/2}} \sin\left(\frac{b\sqrt{x}}{d}\right) \times \sum_{(m,n)\in V(x,d,\eta,T)} \frac{1}{m^{3/2}} \sin\left(\frac{\pi(xn-d^2m)^2}{2d^3m} + \frac{3\pi}{4}\right) + O_x\left(\frac{1}{T^{\frac{1}{2}-4\eta}}\right). \tag{2.2.150}$$

Next, for each divisor d of x, consider the function $H_d(u,v)$ of two real variables defined on $[T,2T) \times [T^{1-\delta},(2T)^{1+\delta}]$ by

$$H_d(u,v) := \frac{1}{u^{3/2}} \sin\left(\frac{\pi(xv-d^2u)^2}{2d^3u} + \frac{3\pi}{4}\right). \tag{2.2.151}$$

Note that on $V(x,d,\eta,T)$, $|xv-d^2u| \le 2xT^{\frac{1}{2}+\eta}$, by (2.2.145), and so

$$\left|\frac{\partial H_d}{\partial v}\right| = \left|\frac{1}{u^{3/2}} \cos\left(\frac{\pi(xv-d^2u)^2}{2d^3u} + \frac{3\pi}{4}\right) \cdot \frac{\pi x}{d^3u}(xv-d^2u)\right| = O_x\left(\frac{1}{T^{2-\eta}}\right) \tag{2.2.152}$$

and

$$\left|\frac{\partial H_d}{\partial u}\right| \le \left|\frac{3}{2u^{5/2}} \sin\left(\frac{\pi(xv-d^2u)^2}{2d^3u} + \frac{3\pi}{4}\right)\right| + \left|\frac{1}{u^{3/2}} \cos\left(\frac{\pi(xv-d^2u)^2}{2d^3u} + \frac{3\pi}{4}\right)\right| \frac{\pi}{2d^3} \frac{|2(d^2u-xv)d^2u-(d^2u-xv)^2|}{u^2} = O_x\left(\frac{1}{T^{2-\eta}}\right). \tag{2.2.153}$$

Using (2.2.152) and (2.2.153), we may replace each sum on the right side of (2.2.150) by a double integral. More precisely, for each $(m,n) \in V(x,d,\eta,T)$,

$$\left| \frac{\sin\left(\frac{\pi(xn-d^2m)^2}{2d^3m} + \frac{3\pi}{4}\right)}{m^{3/2}} - \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} H_d(u,v)\,dv\,du \right|$$

$$= \left| \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \{H_d(u,v) - H_d(m,n)\}\,dv\,du \right|$$

$$= O\left(\sup_{\substack{u\in[m-\frac{1}{2},m+\frac{1}{2}] \\ v\in[n-\frac{1}{2},n+\frac{1}{2}]}} \left\{ \left|\frac{\partial H_d}{\partial u}(u,v)\right| + \left|\frac{\partial H_d}{\partial v}(u,v)\right| \right\} \right)$$

$$= O_x\left(\frac{1}{T^{2-\eta}}\right). \tag{2.2.154}$$

Adding relations (2.2.154) for all $(m,n) \in V(x,d,\eta,T)$, we see that

$$\sum_{(m,n)\in V(x,d,\eta,T)} \frac{1}{m^{3/2}} \sin\left(\frac{\pi(xn-d^2m)^2}{2d^3m}+\frac{3\pi}{4}\right)$$
$$= \sum_{(m,n)\in V(x,d,\eta,T)} \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} H_d(u,v)dv\,du + O_x\left(\frac{1}{T^{\frac{1}{2}-2\eta}}\right). \quad (2.2.155)$$

Let us observe that if we define

$$V^*(x,d,\eta,T) = \cup_{(m,n)\in V(x,d,\eta,T)}[m-\tfrac{1}{2}, m+\tfrac{1}{2}] \times [n-\tfrac{1}{2}, n+\tfrac{1}{2}], \quad (2.2.156)$$

then

$$\text{Area}\,(V(x,d,\eta,T)\backslash V^*(x,d,\eta,T)) \cup (V^*(x,d,\eta,T)\backslash V(x,d,\eta,T)) = O_x(T), \quad (2.2.157)$$

because the perimeter of the trapezoid defining $V(x,d,\eta,T)$ is $O(T)$. Since

$$|H_d(u,v)| = O\left(\frac{1}{T^{3/2}}\right)$$

on $V(x,d,\eta,T) \cup V^*(x,d,\eta,T)$, by (2.2.157), it follows that

$$\sum_{(m,n)\in V(x,d,\eta,T)} \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} H_d(u,v)dv\,du$$
$$= \int\int_{V(x,d,\eta,T)} H_d(u,v)dv\,du + O_x\left(\frac{1}{T^{1/2}}\right). \quad (2.2.158)$$

Combining (2.2.150) with (2.2.155) and (2.2.158), we find that

$$U_2(a,b,T,\eta) = x^{3/4} \sum_{d|x} \frac{(-1)^{x/d+1}}{d^{3/2}} \sin\left(\frac{b\sqrt{x}}{d}\right) \int\int_{V(x,d,\eta,T)} H_d(u,v)dv\,du$$
$$+ O_x\left(\frac{1}{T^{\frac{1}{2}-4\eta}}\right). \quad (2.2.159)$$

To evaluate the double integrals on the right side of (2.2.159), we perform the change of variable $v = u(w + d^2/x)$ to deduce that

$$\iint_{V(x,d,\eta,T)} H_d(u,v)dvdu = \int_T^{2T} \int_{-1/T^{\frac{1}{2}-\eta}}^{1/T^{\frac{1}{2}-\eta}} H_d(u, u(w+d^2/x))u\,dw\,du$$
$$= \int_T^{2T} \int_{-1/T^{\frac{1}{2}-\eta}}^{1/T^{\frac{1}{2}-\eta}} \frac{\sin\left(\frac{\pi x^2 uw^2}{2d^3}+\frac{3\pi}{4}\right)}{\sqrt{u}} dw\,du. \quad (2.2.160)$$

Next, we make a second change of variable to balance the shape of the region of integration by setting $u = Tt$ and $w = z/\sqrt{T}$. Then (2.2.160) reduces to

$$\iint_{V(x,d,\eta,T)} H_d(u,v)dv\,du = \int_1^2 \frac{1}{\sqrt{t}} \int_{-T^\eta}^{T^\eta} \sin\left(\frac{\pi x^2 tz^2}{2d^3} + \frac{3\pi}{4}\right) dz\,dt. \tag{2.2.161}$$

In the inner integral we make a further change of variable, $z = d^{3/2}y/(xt^{1/2})$, so that (2.2.161) now takes the form

$$\iint_{V(x,d,\eta,T)} H_d(u,v)dv\,du = \frac{d^{3/2}}{x} \int_1^2 \frac{1}{t} \int_{-T^\eta xt^{1/2}d^{-3/2}}^{T^\eta xt^{1/2}d^{-3/2}} \sin\left(\frac{\pi}{2}y^2 + \frac{3\pi}{4}\right) dy\,dt. \tag{2.2.162}$$

We approximate the inner integral by

$$c_0 := \int_{-\infty}^{\infty} \sin\left(\frac{\pi}{2}y^2 + \frac{3\pi}{4}\right) dy. \tag{2.2.163}$$

A change of variables followed by an integration by parts yields

$$\begin{aligned}\int_{T^\eta xt^{1/2}d^{-3/2}}^{\infty} \sin\left(\frac{\pi}{2}y^2 + \frac{3\pi}{4}\right) dy &= \frac{1}{2}\int_{T^{2\eta}x^2td^{-3}}^{\infty} \frac{\sin\left(\frac{\pi}{2}\rho + \frac{3\pi}{4}\right)}{\rho^{1/2}} d\rho \\ &= -\left.\frac{\cos\left(\frac{\pi}{2}\rho + \frac{3\pi}{4}\right)}{\pi\rho^{1/2}}\right|_{T^{2\eta}x^2td^{-3}}^{\infty} - \frac{1}{2\pi}\int_{T^{2\eta}x^2td^{-3}}^{\infty} \frac{\cos\left(\frac{\pi}{2}\rho + \frac{3\pi}{4}\right)}{\rho^{3/2}} d\rho.\end{aligned} \tag{2.2.164}$$

By (2.2.164), it follows that, uniformly for $1 \le t \le 2$,

$$\left|\int_{T^\eta xt^{1/2}d^{-3/2}}^{\infty} \sin\left(\frac{\pi}{2}y^2 + \frac{3\pi}{4}\right) dy\right| = O_x\left(\frac{1}{T^\eta}\right). \tag{2.2.165}$$

It is clear that the same bound as in (2.2.165) also holds for the integral from $-\infty$ to $-T^\eta xt^{1/2}d^{-3/2}$. Using these relations in combination with (2.2.162), we deduce that

$$\iint_{V(x,d,\eta,T)} H_d(u,v)dv\,du = \frac{d^{3/2}c_0\log 2}{x} + O_x\left(\frac{1}{T^\eta}\right). \tag{2.2.166}$$

We now insert (2.2.166) into the right-hand side of (2.2.159) to deduce that

$$\begin{aligned}U_2(a,b,T,\eta) = x^{-1/4}c_0\log 2\sum_{d|x}(-1)^{x/d+1}\sin\left(\frac{b\sqrt{x}}{d}\right) \\ + O_x\left(\frac{1}{T^\eta}\right) + O_x\left(\frac{1}{T^{\frac{1}{2}-4\eta}}\right).\end{aligned} \tag{2.2.167}$$

Recall that $b = \pi\sqrt{x}(1-2\theta)$. Therefore, the series over d on the right-hand side of (2.2.167) cannot cancel for general θ. Thus, in order for the convergence of our initial series $S_1(a,\theta)$ to hold for general θ, it is necessary that c_0 be equal to 0, and indeed it is. To that end [126, p. 435, formula 3.691, no. 1],

$$\begin{aligned} c_0 &= \int_{-\infty}^{\infty} \sin\left(\frac{\pi}{2}y^2 + \frac{3\pi}{4}\right) dy \\ &= -\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} \sin\left(\frac{\pi}{2}y^2\right) dy + \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} \cos\left(\frac{\pi}{2}y^2\right) dy \\ &= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0. \end{aligned}$$

In particular, we note that the term $\frac{3\pi}{4}$ in the argument of the sine on the right side of (2.2.163) *is essential* in order to have $c_0 = 0$. We conclude from (2.2.167) that

$$|U_2(a,b,T,\eta)| = O_x\left(\frac{1}{T^{\eta}}\right) + O_x\left(\frac{1}{T^{\frac{1}{2}-4\eta}}\right). \tag{2.2.168}$$

By (2.2.168) and (2.2.139),

$$|U_1(a,b,T,\eta)| + |U_2(a,b,T,\eta)| = O_x\left(\frac{1}{T^{\eta/4}}\right) + O_x\left(\frac{1}{T^{\frac{1}{2}-4\eta}}\right). \tag{2.2.169}$$

We now let $\eta = \frac{2}{17}$. Then both O-terms on the right-hand side of (2.2.169) are $O_x(1/T^{1/34})$, and so by (2.2.111),

$$\left|\sum_{a,b,T}\right| = O_x\left(\frac{1}{T^{1/34}}\right), \tag{2.2.170}$$

uniformly for $\theta \in [0,1]$, where on the left side of (2.2.170) we deleted δ, which is fixed (recall that $\delta = \eta/39 = 2/663$). With (2.2.170) in hand, the proof of the uniform convergence of the initial series $S_1(a,\theta)$ can immediately be completed, as in the previous case when x was not an integer.

2.2.10 Completion of the Proof of Entry 2.1.1

We return to the function $G(\theta)$ defined in Sect. 2.2.1, which we now know is well-defined and continuous on $[0,1]$. We want to prove that

$$\begin{aligned} &\sin^2(\pi\theta)\left\{\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin(2\pi n\theta) - \pi x\left(\frac{1}{2}-\theta\right) + \frac{1}{4}\cot(\pi\theta)\right\} \\ &= \frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}\sin^2(\pi\theta) \\ &= G(\theta). \end{aligned} \tag{2.2.171}$$

The identity $G(\theta) = -G(1-\theta)$ is also satisfied. We find the Fourier sine series of $G(\theta)$ on $(0, \frac{1}{2})$, and so write

$$G(\theta) = \sum_{j=1}^{\infty} b_j \sin(2\pi j\theta). \tag{2.2.172}$$

For $j \geq 1$, interchanging the order of integration and double summation by the uniform convergence and continuity established in the foregoing sections, we find that

$$\begin{aligned}
b_j &= 2\sqrt{x} \int_0^{1/2} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}} \right\} \\
&\qquad \times \sin^2(\pi\theta)\sin(2\pi j\theta)d\theta \\
&= \sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \int_0^{1/2} \left\{ \frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}} \right\} \\
&\qquad \times \Big(\sin(2\pi j\theta) - \frac{1}{2}\sin(2\pi\theta(j+1)) - \frac{1}{2}\sin(2\pi\theta(j-1)) \Big) d\theta.
\end{aligned} \tag{2.2.173}$$

In the first set of integrals of the series on the far right-hand side of (2.2.173), set

$$u = 4\pi\sqrt{m(n+\theta)x}, \quad \text{so that} \quad \frac{d\theta}{\sqrt{m(n+\theta)}} = \frac{du}{2\pi m\sqrt{x}},$$

and in the second set of integrals of the series, set

$$u = 4\pi\sqrt{m(n+1-\theta)x}, \qquad \text{so that} \qquad \frac{d\theta}{\sqrt{m(n+1-\theta)}} = -\frac{du}{2\pi m\sqrt{x}}.$$

Thus, we find that for each $j \geq 1$,

$$\begin{aligned}
&\sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \int_0^{1/2} \left\{ \frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}} \right\} \\
&\qquad\qquad \times \sin(2\pi j\theta)d\theta \\
&= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2\pi m} \left\{ \int_{4\pi\sqrt{mnx}}^{4\pi\sqrt{m(n+1/2)x}} J_1(u)\sin\left(2\pi j\left(\frac{u^2}{16\pi^2 mx} - n\right)\right) du \right. \\
&\qquad \left. + \int_{4\pi\sqrt{m(n+1)x}}^{4\pi\sqrt{m(n+1/2)x}} J_1(u)\sin\left(2\pi j\left(n+1-\frac{u^2}{16\pi^2 mx}\right)\right) du \right\} \\
&= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2\pi m} \left\{ \int_{4\pi\sqrt{mnx}}^{4\pi\sqrt{m(n+1/2)x}} J_1(u)\sin\left(\frac{u^2 j}{8\pi mx}\right) du \right.
\end{aligned}$$

$$- \int_{4\pi\sqrt{m(n+1)x}}^{4\pi\sqrt{m(n+1/2)x}} J_1(u) \sin\left(\frac{u^2 j}{8\pi m x}\right) du \Bigg\}$$

$$= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2\pi m} \int_{4\pi\sqrt{mnx}}^{4\pi\sqrt{m(n+1)x}} J_1(u) \sin\left(\frac{u^2 j}{8\pi m x}\right) du$$

$$= \sum_{m=1}^{\infty} \frac{1}{2\pi m} \int_0^{\infty} J_1(u) \sin\left(\frac{u^2 j}{8\pi m x}\right) du. \tag{2.2.174}$$

Similar calculations hold for the integrals involving $j+1$ and $j-1$ in (2.2.173). Thus, for each $j \geq 1$,

$$b_j = \sum_{m=1}^{\infty} \frac{1}{2\pi m} \int_0^{\infty} J_1(u) \left\{ \sin\left(\frac{u^2 j}{8\pi m x}\right) - \frac{1}{2} \sin\left(\frac{u^2 (j+1)}{8\pi m x}\right) - \frac{1}{2} \sin\left(\frac{u^2 (j-1)}{8\pi m x}\right) \right\} du.$$

For $a, b > 0$, recall the formula [126, p. 759, formula 6.686, no. 5]

$$\int_0^{\infty} \sin(au^2) J_1(bu) du = \frac{1}{b} \sin\left(\frac{b^2}{4a}\right).$$

Thus,

$$b_j = \sum_{m=1}^{\infty} \frac{1}{2\pi m} \left\{ \sin\left(\frac{2\pi m x}{j}\right) - \frac{1}{2} \sin\left(\frac{2\pi m x}{j+1}\right) - \frac{1}{2} \sin\left(\frac{2\pi m x}{j-1}\right) \right\}, \tag{2.2.175}$$

where the last term is not present if $j = 1$. From the fact that for any real number y,

$$- \sum_{m=1}^{\infty} \frac{\sin(2\pi m y)}{\pi m} = \begin{cases} 0, & \text{if } y \text{ is an integer}, \\ y - [y] - \frac{1}{2}, & \text{if } y \text{ is not an integer}, \end{cases} \tag{2.2.176}$$

we deduce that

$$\sum_{m=1}^{\infty} \frac{\sin(2\pi m x/j)}{\pi m} = \begin{cases} 0, & \text{if } x/j \text{ is an integer}, \\ -\dfrac{x}{j} + \left[\dfrac{x}{j}\right] + \dfrac{1}{2}, & \text{if } x/j \text{ is not an integer}, \end{cases}$$
$$= F\left(\frac{x}{j}\right) - \frac{x}{j} + \frac{1}{2}. \tag{2.2.177}$$

Hence, from (2.2.175) and (2.2.177), we find that

$$b_j = \frac{1}{2} \left\{ F\left(\frac{x}{j}\right) - \frac{x}{j} + \frac{1}{2} - \frac{1}{2}\left(F\left(\frac{x}{j+1}\right) - \frac{x}{j+1} + \frac{1}{2} \right) - \frac{1}{2}\left(F\left(\frac{x}{j-1}\right) - \frac{x}{j-1} + \frac{1}{2} \right) \right\},$$

where the last term is not present if $j = 1$. Thus,

$$b_1 = \frac{1}{8} - \frac{3x}{8} + \frac{1}{2}F(x) - \frac{1}{4}F\left(\frac{x}{2}\right), \tag{2.2.178}$$

and for $j \geq 2$,

$$b_j = \frac{1}{2}F\left(\frac{x}{j}\right) - \frac{1}{4}F\left(\frac{x}{j+1}\right) - \frac{1}{4}F\left(\frac{x}{j-1}\right) + \frac{x}{2j(j^2-1)}. \tag{2.2.179}$$

Next, we find the Fourier sine series on $(0, \frac{1}{2})$ of the left-hand side of (2.2.171). We have

$$\begin{aligned} &F\left(\frac{x}{n}\right)\sin(2\pi n\theta)\sin^2(\pi\theta) \\ &\quad = \frac{1}{2}F\left(\frac{x}{n}\right)\left\{\sin(2\pi n\theta) - \frac{1}{2}\sin(2\pi\theta(n+1)) - \frac{1}{2}\sin(2\pi\theta(n-1))\right\} \end{aligned}$$

and

$$\cot(\pi\theta)\sin^2(\pi\theta) = \cos(\pi\theta)\sin(\pi\theta) = \frac{1}{2}\sin(2\pi\theta).$$

Also, since $0 < \theta < 1$, by (2.2.176),

$$\begin{aligned} \sin^2(\pi\theta)\Big(\frac{1}{2} - \theta\Big) &= \frac{1}{2}\Big(1 - \cos(2\pi\theta)\Big)\sum_{m=1}^{\infty}\frac{\sin(2\pi m\theta)}{\pi m} \\ = \frac{1}{2}\sum_{m=1}^{\infty}\frac{\sin(2\pi m\theta)}{\pi m} &- \frac{1}{4}\sum_{m=1}^{\infty}\frac{\sin(2\pi\theta(m+1))}{\pi m} - \frac{1}{4}\sum_{m=1}^{\infty}\frac{\sin(2\pi\theta(m-1))}{\pi m}. \end{aligned}$$

Thus, if the Fourier sine series of the left-hand side of (2.2.171) is

$$\sum_{j=1}^{\infty} c_j \sin(2\pi j\theta),$$

then

$$c_1 = \frac{1}{8} - \frac{3x}{8} + \frac{1}{2}F(x) - \frac{1}{4}F\left(\frac{x}{2}\right) = b_1,$$

by (2.2.178), and for $j \geq 2$,

$$c_j = \frac{x}{2j(j^2-1)} + \frac{1}{2}F\left(\frac{x}{j}\right) - \frac{1}{4}F\left(\frac{x}{j+1}\right) - \frac{1}{4}F\left(\frac{x}{j-1}\right) = b_j,$$

by (2.2.179), which completes the proof of (2.1.5).

2.3 Proof of Ramanujan's First Bessel Function Identity (Symmetric Form)

We prove Ramanujan's first Bessel function identity (2.1.5), emphasizing that the double sum on the right-hand side of (2.1.5) is being interpreted symmetrically, i.e., the product mn of the summation indices m and n tends to infinity. A slight modification of the analysis from [26, pp. 354–356], in particular, Lemma 14 of [26], shows that the series on the right-hand side of (2.1.5) converges uniformly with respect to θ on any interval $0 < \theta_1 \le \theta \le \theta_2 < 1$. (There is a misprint in (3.5) of Theorem 4 in [26]; read $b(n)/\mu_n^{\sigma-1/2m}$ for $b(n)\mu_n^{\sigma-1/2m}$.) By continuity, it therefore suffices to prove Entry 2.1.1 for rational $\theta = a/q$, where q is prime and $0 < a < q$.

First define

$$\begin{aligned} &H(a,q,x)\\ &:= \frac{\sqrt{x}}{2}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+a/q)x}\right)}{\sqrt{m(n+a/q)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-a/q)x}\right)}{\sqrt{m(n+1-a/q)}}\right\}\\ &= \frac{\sqrt{qx}}{2}\left\{\sum_{m=1}^{\infty}\sum_{\substack{r=1\\ r\equiv a \bmod q}}^{\infty}\frac{J_1\left(4\pi\sqrt{mrx/q}\right)}{\sqrt{mr}} - \sum_{m=1}^{\infty}\sum_{\substack{r=1\\ r\equiv -a \bmod q}}^{\infty}\frac{J_1\left(4\pi\sqrt{mrx/q}\right)}{\sqrt{mr}}\right\}. \end{aligned}$$

With the restriction $\theta = a/q$ and with the notation above, we now reformulate Entry 2.1.1.

Theorem 2.3.1. *If q is prime and $0 < a < q$, then*

$$H(a,q,x) = \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin\left(\frac{2\pi na}{q}\right) - \pi x\left(\frac{1}{2}-\frac{a}{q}\right) + \frac{1}{4}\cot\left(\frac{a\pi}{q}\right) =: P(a,q,x).$$

In the analysis that follows, we demonstrate that in order to prove Theorem 2.3.1, it suffices to prove the next theorem.

Theorem 2.3.2. *Let q be a positive integer, and let χ be an odd primitive character modulo q. Then, for any $x > 0$,*

$$\sum_{n\le x}{}' d_\chi(n) = L(1,\chi)x + \frac{i\tau(\chi)}{2\pi}L(1,\overline{\chi}) + \frac{i\sqrt{q}}{\tau(\overline{\chi})}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\sqrt{\frac{x}{n}}J_1\left(4\pi\sqrt{nx/q}\right). \tag{2.3.1}$$

Proof. Suppose that χ is a primitive nonprincipal odd character modulo q. Then [101, p. 71]

$$\left(\frac{\pi}{q}\right)^{-(2s+1)/2}\Gamma\left(s+\frac{1}{2}\right)L(2s,\chi) = -\frac{i\tau(\chi)}{\sqrt{q}}\left(\frac{\pi}{q}\right)^{-(1-s)}\Gamma(1-s)L(1-2s,\overline{\chi}). \tag{2.3.2}$$

Recall again the functional equation of $\zeta(s)$, namely,

$$\pi^{-s}\Gamma(s)\zeta(2s) = \pi^{-(1/2-s)}\Gamma\left(\frac{1}{2}-s\right)\zeta(1-2s). \tag{2.3.3}$$

Multiply (2.3.2) and (2.3.3) to deduce that

$$\frac{\pi^{-2s-1/2}}{q^{-s-1/2}}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)\zeta(2s)L(2s,\chi) = -\frac{i\tau(\chi)}{\sqrt{q}}\frac{\pi^{-3/2+2s}}{q^{-1+s}}\Gamma(1-s)\Gamma\left(\frac{1}{2}-s\right)L(1-2s,\overline{\chi})\zeta(1-2s). \tag{2.3.4}$$

If we invoke the duplication formula for the gamma function,

$$\Gamma(2s)\sqrt{\pi} = 2^{2s-1}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right),$$

then (2.3.4) can be written as

$$\begin{aligned}\frac{\pi^{-2s-1/2}}{q^{-s-1/2}}&\frac{\sqrt{\pi}\Gamma(2s)}{2^{2s-1}}\zeta(2s)L(2s,\chi)\\ &= -\frac{i\tau(\chi)}{\sqrt{q}}\frac{\pi^{-3/2+2s}}{q^{-1+s}}\frac{\Gamma\Big(2(1/2-s)\Big)\sqrt{\pi}}{2^{2(1/2-s)-1}}L(1-2s,\overline{\chi})\zeta(1-2s)\\ &= -\frac{i\tau(\chi)}{\sqrt{q}}\frac{\pi^{-1+2s}}{q^{-1+s}}\frac{\Gamma(1-2s)}{2^{-2s}}L(1-2s,\overline{\chi})\zeta(1-2s).\end{aligned}$$

Thus,

$$\left(\frac{2\pi}{\sqrt{q}}\right)^{-2s}\Gamma(2s)L(2s,\chi)\zeta(2s) = -\frac{i\tau(\chi)}{\sqrt{q}}\left(\frac{2\pi}{\sqrt{q}}\right)^{2s-1}\Gamma(1-2s)L(1-2s,\overline{\chi})\zeta(1-2s).$$

Replacing s by $s/2$, we have

$$\left(\frac{2\pi}{\sqrt{q}}\right)^{-s}\Gamma(s)L(s,\chi)\zeta(s) = -\frac{i\tau(\chi)}{\sqrt{q}}\left(\frac{2\pi}{\sqrt{q}}\right)^{s-1}\Gamma(1-s)L(1-s,\overline{\chi})\zeta(1-s).$$

In the notation of Theorem 2 of [26], $q=0$, $r=m=1$, $\lambda_n=\mu_n=2\pi n/\sqrt{q}$, $a(n)=d_\chi(n)$, $b(n)=-i\tau(\chi)d_{\overline{\chi}}(n)/\sqrt{q}$, and $K_1(2\sqrt{\mu_n x};0;1)=J_1(2\sqrt{\mu_n x})$. We therefore record the following special case of [26, Theorem 2]. Let $x>0$. Then

$$\sum_{\lambda_n\leq x}{}' d_\chi(n) = \frac{-i\tau(\chi)}{\sqrt{q}}\sum_{n=1}^{\infty}d_{\overline{\chi}}(n)\left(\frac{x}{\mu_n}\right)^{1/2}J_1(2\sqrt{\mu_n x})+Q_0(x), \tag{2.3.5}$$

where

$$Q_0(x) = \frac{1}{2\pi i}\int_C \frac{(2\pi/\sqrt{q})^{-s}L(s,\chi)\zeta(s)x^s}{s}ds,$$

where C is a positively oriented closed contour with the singularities of the integrand in the interior.

We now replace x by $2\pi x/\sqrt{q}$ in (2.3.5) to obtain

$$\sum_{n\le x}{}' d_\chi(n) = \frac{-i\tau(\chi)}{\sqrt{q}}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\left(\frac{x}{n}\right)^{1/2} J_1(4\pi\sqrt{nx/q}) + Q_0(2\pi x/\sqrt{q}). \quad (2.3.6)$$

Now, since $\zeta(0) = -\frac{1}{2}$,

$$Q_0(2\pi x/\sqrt{q}) = \frac{1}{2\pi i}\int_C \frac{L(s,\chi)\zeta(s)x^s}{s}ds = -\frac{1}{2}L(0,\chi) + L(1,\chi)x. \quad (2.3.7)$$

From the functional equation (2.3.2),

$$\left(\frac{\pi}{q}\right)^{-1/2}\Gamma(1/2)L(0,\chi) = -i\frac{\tau(\chi)}{\sqrt{q}}\frac{q}{\pi}L(1,\overline{\chi}).$$

So,

$$L(0,\chi) = -\frac{i\tau(\chi)}{\pi}L(1,\overline{\chi}).$$

Thus, from (2.3.7),

$$Q_0(2\pi x/\sqrt{q}) = L(1,\chi)x + \frac{i\tau(\chi)}{2\pi}L(1,\overline{\chi}). \quad (2.3.8)$$

Lastly, putting (2.3.8) in (2.3.6) and using the identity $\tau(\chi)\tau(\overline{\chi}) = -q$, since χ is odd, we complete the proof of Theorem 2.3.2. □

After proving the following lemma, we show that Theorem 2.3.2 implies Theorem 2.3.1.

Lemma 2.3.1. *If* $0 < a < q$ *and* $(a,q) = 1$, *then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin\left(\frac{2\pi na}{q}\right) = -i\sum_{\substack{d|q\\ d>1}}\frac{1}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi \text{ odd}}}\chi(a)\tau(\overline{\chi})\sum_{1\le n\le dx/q}{}' d_\chi(n).$$

Proof. We have

$$\begin{aligned}\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin\left(\frac{2\pi na}{q}\right) &= \sum_{d|q}\sum_{(n,q)=q/d} F\left(\frac{x}{n}\right)\sin\left(\frac{2\pi na}{q}\right)\\ &= \sum_{d|q}\sum_{\substack{m=1\\(m,d)=1}}^{\infty} F\left(\frac{dx}{qm}\right)\sin\left(\frac{2\pi ma}{d}\right)\end{aligned}$$

$$= \sum_{m=1}^{\infty} F\left(\frac{x}{qm}\right) + \sum_{\substack{d|q \\ d>1}} \sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} F\left(\frac{dx}{qm}\right) \sin\left(\frac{2\pi ma}{d}\right)$$

$$= \sum_{m=1}^{\infty} F\left(\frac{x}{qm}\right) + \frac{1}{2} \sum_{\substack{d|q \\ d>1}} \sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} F\left(\frac{dx}{qm}\right) \left(e^{2\pi ima/d} - e^{-2\pi ima/d}\right). \tag{2.3.9}$$

We know that for any positive integers a_1, a_2, and q,

$$\sum_{\chi \bmod q} \chi(a_1)\overline{\chi}(a_2) = \begin{cases} \phi(q), & \text{if } a_1 \equiv a_2 \pmod q \text{ and } (a_1, q) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3.10}$$

Using (2.3.10) and the formula [101, p. 65]

$$\chi(n)\tau(\overline{\chi}) = \sum_{h=1}^{q} \overline{\chi}(h) e^{2\pi inh/q}, \tag{2.3.11}$$

for any character χ modulo q, we find that for m, d such that $(m, d) = 1$ and $d > 1$,

$$e^{2\pi ima/d} = \frac{1}{\phi(d)} \sum_{h=1}^{d} e^{2\pi imh/d} \sum_{\chi \bmod d} \chi(a)\overline{\chi}(h)$$

$$= \frac{1}{\phi(d)} \sum_{\chi \bmod d} \chi(a) \sum_{h=1}^{d} \overline{\chi}(h) e^{2\pi imh/d}$$

$$= \frac{1}{\phi(d)} \sum_{\chi \bmod d} \chi(a)\tau(\overline{\chi})\chi(m).$$

Thus,

$$\frac{1}{2} \sum_{\substack{d|q \\ d>1}} \sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} F\left(\frac{dx}{qm}\right) \left(e^{2\pi ima/d} - e^{-2\pi ima/d}\right)$$

$$= \sum_{\substack{d|q \\ d>1}} \frac{1}{2\phi(d)} \sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} F\left(\frac{dx}{qm}\right) \sum_{\chi \bmod d} \chi(a)\tau(\overline{\chi})(\chi(m) - \chi(-m))$$

$$= \sum_{\substack{d|q \\ d>1}} \frac{1}{\phi(d)} \sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} F\left(\frac{dx}{qm}\right) \sum_{\substack{\chi \bmod d \\ \chi \text{ even}}} \chi(a)\tau(\overline{\chi})\chi(m)$$

$$= \sum_{\substack{d|q \\ d>1}} \frac{1}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ even}}} \chi(a)\tau(\overline{\chi}) \sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} F\left(\frac{dx}{qm}\right) \chi(m)$$

$$= \sum_{\substack{d|q \\ d>1}} \frac{1}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ even}}} \chi(a)\tau(\overline{\chi}) \sum_{m=1}^{\infty} F\Big(\frac{dx}{qm}\Big)\chi(m),$$

since $\chi(m) = 0$ if $(m, d) > 1$. Hence, using the calculation above in (2.3.9), we obtain

$$\sum_{n=1}^{\infty} F\Big(\frac{x}{n}\Big) \sin\Big(\frac{2\pi na}{q}\Big)$$
$$= \sum_{m=1}^{\infty} F\Big(\frac{x}{qm}\Big) + \sum_{\substack{d|q \\ d>1}} \frac{1}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ odd}}} \chi(a)\tau(\overline{\chi}) \sum_{m=1}^{\infty} F\Big(\frac{dx}{qm}\Big)\chi(m)$$
$$= \sideset{}{'}\sum_{1\le n\le x/q} d(n) + \sum_{\substack{d|q \\ d>1}} \frac{1}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ odd}}} \chi(a)\tau(\overline{\chi}) \sideset{}{'}\sum_{1\le n\le dx/q} d_{\chi}(n),$$

where we used (2.1.12). Thus, our proof of Lemma 2.3.1 is complete. □

As promised, we now show that Theorem 2.3.2 implies Theorem 2.3.1.

Proof of Theorem 2.3.1. We easily see that $H(a, q, x) = -H(q-a, q, x)$ and $P(a, q, x) = -P(q-a, q, x)$, and so we can assume that $0 < a < q/2$. Consider

$$H(a, q, x)$$
$$= \frac{\sqrt{x}}{2} \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \left\{ \frac{J_1\big(4\pi\sqrt{m(n+a/q)x}\big)}{\sqrt{m(n+a/q)}} - \frac{J_1\big(4\pi\sqrt{m(n+1-a/q)x}\big)}{\sqrt{m(n+1-a/q)}} \right\}$$
$$= \frac{\sqrt{qx}}{2} \sum_{m=1}^{\infty} \left\{ \sum_{\substack{r=1 \\ r\equiv a \bmod q}}^{\infty} \frac{J_1\big(4\pi\sqrt{mrx/q}\big)}{\sqrt{mr}} - \sum_{\substack{r=1 \\ r\equiv -a \bmod q}}^{\infty} \frac{J_1\big(4\pi\sqrt{mrx/q}\big)}{\sqrt{mr}} \right\}$$
$$= \frac{\sqrt{qx}}{2\phi(q)} \sum_{m=1}^{\infty}\sum_{r=1}^{\infty} \frac{J_1\big(4\pi\sqrt{mrx/q}\big)}{\sqrt{mr}} \sum_{\chi \bmod q} \overline{\chi}(r)\big(\chi(a) - \chi(-a)\big)$$
$$= \frac{\sqrt{qx}}{\phi(q)} \sum_{m=1}^{\infty}\sum_{r=1}^{\infty} \frac{J_1\big(4\pi\sqrt{mrx/q}\big)}{\sqrt{mr}} \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \overline{\chi}(r)\chi(a)$$
$$= \frac{q}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \chi(a) \sum_{m=1}^{\infty}\sum_{r=1}^{\infty} \overline{\chi}(r)\sqrt{\frac{x}{qmr}}\, J_1\big(4\pi\sqrt{mrx/q}\big)$$
$$= \frac{q}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \chi(a) \sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\sqrt{\frac{x}{qn}}\, J_1\big(4\pi\sqrt{nx/q}\big). \tag{2.3.12}$$

On the other hand, by Lemma 2.3.1,

$$\begin{aligned} P(a,q,x) &= \sum_{n=1}^{\infty} F\Big(\frac{x}{n}\Big)\sin\Big(\frac{2\pi na}{q}\Big) - \pi x\Big(\frac{1}{2}-\frac{a}{q}\Big) + \frac{1}{4}\cot\Big(\frac{a\pi}{q}\Big) \\ &= \frac{-i}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \chi(a)\tau(\overline{\chi}) {\sum_{1\le n\le x}}' d_\chi(n) - \pi x\Big(\frac{1}{2}-\frac{a}{q}\Big) + \frac{1}{4}\cot\Big(\frac{a\pi}{q}\Big). \end{aligned}$$

Applying Theorem 2.3.2 and using (2.3.12), we only need to show that

$$\frac{i}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \chi(a)\tau(\overline{\chi})\Big(L(1,\chi)x + \frac{i\tau(\chi)}{2\pi}L(1,\overline{\chi})\Big) = -\pi x\Big(\frac{1}{2}-\frac{a}{q}\Big) + \frac{1}{4}\cot\Big(\frac{a\pi}{q}\Big). \tag{2.3.13}$$

We use the following formulas, which are (2.5) and (2.8) in [71]:

$$\tau(\overline{\chi})L(1,\chi) = 2\pi i \sum_{1\le h<q/2} \overline{\chi}(h)\Big(\frac{1}{2}-\frac{h}{q}\Big), \tag{2.3.14}$$

$$\tau(\chi)L(1,\overline{\chi}) = -\frac{\pi}{\tau(\overline{\chi})} \sum_{1\le h<q/2} \overline{\chi}(h)\cot\Big(\frac{\pi h}{q}\Big). \tag{2.3.15}$$

We also can easily deduce from (2.3.10) that

$$\sum_{\chi \text{ even}} \chi(a)\overline{\chi}(h) = \sum_{\chi \text{ odd}} \chi(a)\overline{\chi}(h) = \begin{cases} \phi(q)/2, & \text{if } h \equiv a \pmod{q}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.3.16}$$

since $(a,q)=1$.

Then, using (2.3.14)–(2.3.16), we deduce that

$$\begin{aligned} &\frac{i}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \chi(a)\tau(\overline{\chi})\Big(L(1,\chi)x + \frac{i\tau(\chi)}{2\pi}L(1,\overline{\chi})\Big) \\ &= \left\{ -\frac{2\pi x}{\phi(q)} \sum_{1\le h<q/2} \Big(\frac{1}{2}-\frac{h}{q}\Big) + \frac{1}{2\phi(q)} \sum_{1\le h<q/2} \cot\Big(\frac{\pi h}{q}\Big) \right\} \sum_{\substack{\chi \bmod q \\ \chi \text{ odd}}} \chi(a)\overline{\chi}(h) \\ &= -\pi x\Big(\frac{1}{2}-\frac{a}{q}\Big) + \frac{1}{4}\cot\Big(\frac{a\pi}{q}\Big), \end{aligned}$$

which completes the proof of (2.3.13) and therefore also of Theorem 2.3.1. □

In fact, Theorem 2.3.1 is equivalent to the following theorem [57].

Theorem 2.3.3. *Let q be a positive integer, and let χ be an odd primitive character modulo q. Then, for any $x > 0$,*

$$\sum_{n\le x}{}' d_\chi(n) = L(1,\chi)x + \frac{i\tau(\chi)}{2\pi}L(1,\overline{\chi}) + \frac{i\sqrt{x}}{\tau(\overline{\chi})}\sum_{1\le h<q/2}\overline{\chi}(h)$$
$$\times \lim_{N\to\infty}\sum_{mn\le N}\left\{\frac{J_1\left(4\pi\sqrt{m\,(n+h/q)\,x}\right)}{\sqrt{m\,(n+h/q)}} - \frac{J_1\left(4\pi\sqrt{m\,(n+1-h/q)\,x}\right)}{\sqrt{m\,(n+1-h/q)}}\right\}. \tag{2.3.17}$$

2.4 Proof of Ramanujan's Second Bessel Function Identity (with the Order of Summation Reversed)

2.4.1 Preliminary Results

We now embark on a proof of Entry 2.1.2, where now we consider the double series on the right side of (2.1.6) to be an iterated double sum. As emphasized in the introduction, we will approach Entry 2.1.2 with the order of summation on the double series reversed. Our proof depends upon the following formulation of the Poisson summation formula due to A.P. Guinand [132, p. 595].

Theorem 2.4.1. *If $f(x)$ can be represented as a Fourier integral, $f(x)$ tends to 0 as $x \to \infty$, and $xf'(x) \in L^p(0,\infty)$ for some p, $1 < p \le 2$, then*

$$\lim_{N\to\infty}\left\{\sum_{n=1}^{N} f(n) - \int_0^N f(t)\,dt\right\} = \lim_{N\to\infty}\left\{\sum_{n=1}^{N} g(n) - \int_0^N g(t)\,dt\right\}, \tag{2.4.1}$$

where

$$g(x) := 2\int_0^\infty f(t)\cos(2\pi xt)\,dt.$$

We need the following two lemmas from [48, Lemmas 3.5, 3.4].

Lemma 2.4.1. *We have*

$$\int_0^\infty I_1(x)dx = 0.$$

Lemma 2.4.2. *With I_ν defined by (2.1.7) and $b, c > 0$,*

$$\int_0^\infty \cos(bx^2)I_1(cx)dx = \frac{1}{c}\sin\left(\frac{c^2}{4b}\right). \tag{2.4.2}$$

2.4.2 Reformulation of Entry 2.1.2

Theorem 2.4.2. *Let $F(x)$ be defined by* (2.1.4) *and let $I_1(x)$ be defined by* (2.1.7)*. Then, for $x > 0$ and $0 < \theta < 1$,*

$$\frac{1}{2}\sqrt{x}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{I_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}}+\frac{I_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}$$
$$=\frac{1}{2\pi}\left(\sum_{n=0}^{\infty}\frac{1}{n+\theta}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+\theta)x}{m}\right)-\int_0^M\sin\left(\frac{2\pi(n+\theta)x}{t}\right)dt\right\}\right.$$
$$+\sum_{n=0}^{\infty}\frac{1}{n+1-\theta}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+1-\theta)x}{m}\right)\right.$$
$$\left.\left.-\int_0^M\sin\left(\frac{2\pi(n+1-\theta)x}{t}\right)dt\right\}\right). \tag{2.4.3}$$

Proof. Let

$$f(t)=\frac{I_1(4\pi\sqrt{t(n+\theta)x})}{\sqrt{t(n+\theta)}}$$

in Theorem 2.4.1. First, setting $u = 4\pi\sqrt{t(n+\theta)x}$ and using Lemma 2.4.1, we find that

$$\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}}-\int_0^M\frac{I_1(4\pi\sqrt{t(n+\theta)x})}{\sqrt{t(n+\theta)}}dt\right\}$$
$$=\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}}\right\}-\frac{1}{2\pi(n+\theta)\sqrt{x}}\int_0^{\infty}I_1(u)du$$
$$=\sum_{m=1}^{\infty}\frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}}. \tag{2.4.4}$$

Second, putting $u = 4\pi\sqrt{t(n+\theta)x}$ and using Lemma 2.4.2, we find that

$$g(m)=2\int_0^{\infty}\frac{I_1(4\pi\sqrt{t(n+\theta)x})}{\sqrt{t(n+\theta)}}\cos(2\pi mt)dt$$
$$=\frac{1}{\pi(n+\theta)\sqrt{x}}\int_0^{\infty}I_1(u)\cos\left(\frac{mu^2}{8\pi(n+\theta)x}\right)du$$
$$=\frac{1}{\pi(n+\theta)\sqrt{x}}\sin\left(\frac{2\pi(n+\theta)x}{m}\right). \tag{2.4.5}$$

Hence,

$$\lim_{M\to\infty}\left\{\sum_{m=1}^{M} g(m)-\int_0^M g(t)\,dt\right\}$$
$$=\frac{1}{\pi(n+\theta)\sqrt{x}}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+\theta)x}{m}\right)-\int_0^M\sin\left(\frac{2\pi(n+\theta)x}{t}\right)dt\right\}. \tag{2.4.6}$$

We make a digression here to demonstrate conclusively that the limit in (2.4.6) actually does exist. Write, for $a>0$,

$$\begin{aligned}
&\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{a}{m}\right)-\int_0^M\sin\left(\frac{a}{t}\right)dt\right\}\\
&=\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\left(\sin\left(\frac{a}{m}\right)-\frac{a}{m}+\frac{a}{m}\right)-\int_1^M\left(\sin\left(\frac{a}{t}\right)-\frac{a}{t}+\frac{a}{t}\right)dt\right\}\\
&\quad-\int_0^1\sin\left(\frac{a}{t}\right)dt\\
&=L_1-L_2+\lim_{M\to\infty}\left\{a\sum_{m=1}^{M}\frac{1}{m}-a\int_1^M\frac{dt}{t}\right\}-\int_0^1\sin\left(\frac{a}{t}\right)dt\\
&=L_1-L_2-\int_0^1\sin\left(\frac{a}{t}\right)dt+a\left\{\log M+\gamma+o(1)-\log M\right\}\\
&=L_1-L_2-\int_0^1\sin\left(\frac{a}{t}\right)dt+a\gamma,
\end{aligned}$$

where γ denotes Euler's constant and where

$$L_1=\lim_{M\to\infty}\sum_{m=1}^{M}\left(\sin\left(\frac{a}{m}\right)-\frac{a}{m}\right),$$
$$L_2=\lim_{M\to\infty}\int_1^M\left(\sin\left(\frac{a}{t}\right)-\frac{a}{t}\right)dt.$$

Returning to our proof and putting together (2.4.4) and (2.4.6) in (2.4.1), we find that

$$\sum_{m=1}^{\infty}\frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}}$$
$$=\frac{1}{\pi(n+\theta)\sqrt{x}}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+\theta)x}{m}\right)-\int_0^M\sin\left(\frac{2\pi(n+\theta)x}{t}\right)dt\right\}. \tag{2.4.7}$$

Now in (2.4.7) replace θ by $1-\theta$ and add the result to (2.4.7). Sum both sides on n, $0 \le n < \infty$. Then multiply the resulting equality by $\frac{1}{2}\sqrt{x}$ to deduce (2.4.3) and thus complete the proof of Theorem 2.4.2. □

If we compare (2.1.6) with (2.4.3), we see that in order to prove Entry 2.1.2, *but with the order of summation reversed in the double series*, we need to prove that

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(2\pi n\theta) - \frac{1}{4} + x\log(2\sin(\pi\theta))$$
$$= \frac{1}{2\pi}\left(\sum_{n=0}^{\infty}\frac{1}{n+\theta}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+\theta)x}{m}\right) - \int_0^M \sin\left(\frac{2\pi(n+\theta)x}{t}\right)dt\right\}\right.$$
$$+\sum_{n=0}^{\infty}\frac{1}{n+1-\theta}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+1-\theta)x}{m}\right)\right.$$
$$\left.\left.-\int_0^M \sin\left(\frac{2\pi(n+1-\theta)x}{t}\right)dt\right\}\right).$$

2.4.3 The Convergence of (2.4.3)

Fix $x > 0$, and set $a = 2\pi x$. We are interested in the question of convergence (pointwise, or uniformly with respect to θ on compact subintervals of the interval $(0, 1)$) of the series

$$S(a,\theta) := \sum_{n=0}^{\infty}\frac{1}{n+\theta}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{a(n+\theta)}{m}\right) - \int_0^M \sin\left(\frac{a(n+\theta)}{t}\right)dt\right\}$$
$$+\sum_{n=0}^{\infty}\frac{1}{n+1-\theta}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{a(n+1-\theta)}{m}\right)\right.$$
$$\left.-\int_0^M \sin\left(\frac{a(n+1-\theta)}{t}\right)dt\right\}.$$

For $m > 2$,

$$\sin\left(\frac{a(n+\theta)}{m}\right) - \int_{m-1}^{m}\sin\left(\frac{a(n+\theta)}{t}\right)dt$$
$$= \int_{m-1}^{m}\left(\sin\left(\frac{a(n+\theta)}{m}\right) - \sin\left(\frac{a(n+\theta)}{t}\right)\right)dt$$
$$= \int_{m-1}^{m} 2\sin\frac{1}{2}\left(\frac{a(n+\theta)}{m} - \frac{a(n+\theta)}{t}\right)\cos\frac{1}{2}\left(\frac{a(n+\theta)}{m} - \frac{a(n+\theta)}{t}\right)dt.$$

Thus,

$$\begin{aligned}\left|\sin\left(\frac{a(n+\theta)}{m}\right)-\int_{m-1}^{m}\sin\left(\frac{a(n+\theta)}{t}\right)dt\right| \\ \leq 2\int_{m-1}^{m}\left|\sin\left(\frac{a(n+\theta)(t-m)}{2mt}\right)\right|dt \\ \leq \int_{m-1}^{m}\frac{a(n+\theta)(m-t)}{mt}dt<\frac{a(n+\theta)}{m(m-1)}.\end{aligned} \tag{2.4.8}$$

Fix $\delta_1>0$ and set $M_1=[n^{1+\delta_1}]$, where $[x]$ denotes the greatest integer $\leq x$. We write

$$\begin{aligned}&\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{a(n+\theta)}{m}\right)-\int_{0}^{M}\sin\left(\frac{a(n+\theta)}{t}\right)dt\right\}\\ &=\sum_{m=1}^{M_1}\sin\left(\frac{a(n+\theta)}{m}\right)-\int_{0}^{M_1}\sin\left(\frac{a(n+\theta)}{t}\right)dt\\ &\quad+\lim_{M\to\infty}\left\{\sum_{m=M_1+1}^{M}\sin\left(\frac{a(n+\theta)}{m}\right)-\int_{M_1}^{M}\sin\left(\frac{a(n+\theta)}{t}\right)dt\right\}.\end{aligned}$$

Here the last limit exists, and, by (2.4.8), is a real number bounded by

$$\left|\sum_{m=M_1+1}^{\infty}\frac{a(n+\theta)}{m(m-1)}\right|=\frac{a(n+\theta)}{M_1}\ll_a\frac{1}{n^{\delta_1}},$$

uniformly with respect to θ in $[0,1]$. Therefore the series

$$\sum_{n=0}^{\infty}\frac{1}{n+\theta}\lim_{M\to\infty}\left\{\sum_{m=M_1+1}^{M}\sin\left(\frac{a(n+\theta)}{m}\right)-\int_{M_1}^{M}\sin\left(\frac{a(n+\theta)}{t}\right)dt\right\}$$

converges uniformly with respect to θ, and the same holds for the other, similar series involving $n+1-\theta$. We deduce that the series

$$\begin{aligned}S_1(a,\theta,\delta_1):=&\sum_{n=0}^{\infty}\frac{1}{n+\theta}\left\{\sum_{m=1}^{M_1}\sin\left(\frac{a(n+\theta)}{m}\right)-\int_{0}^{M_1}\sin\left(\frac{a(n+\theta)}{t}\right)dt\right\}\\ &+\sum_{n=0}^{\infty}\frac{1}{n+1-\theta}\left\{\sum_{m=1}^{M_1}\sin\left(\frac{a(n+1-\theta)}{m}\right)-\int_{0}^{M_1}\sin\left(\frac{a(n+1-\theta)}{t}\right)dt\right\}\end{aligned}$$

converges pointwise if and only if the initial sum $S(a,\theta)$ converges pointwise, and $S_1(a,\theta,\delta_1)$ converges uniformly with respect to θ on compact subintervals of $(0,1)$ if and only if this holds for $S(a,\theta)$.

Next, we need a bound for

$$\sum_{m=1}^{M_1} \sin\left(\frac{a(n+\theta)}{m}\right) - \int_0^{M_1} \sin\left(\frac{a(n+\theta)}{t}\right) dt.$$

We write this expression in the form

$$\sum_{m=1}^{[\sqrt{n}]} \sin\left(\frac{a(n+\theta)}{m}\right) - \int_0^{[\sqrt{n}]} \sin\left(\frac{a(n+\theta)}{t}\right) dt$$
$$+ \sum_{m=[\sqrt{n}]+1}^{M_1} \left(\sin\left(\frac{a(n+\theta)}{m}\right) - \int_{m-1}^{m} \sin\left(\frac{a(n+\theta)}{t}\right) dt\right).$$

Here the first sum is bounded in absolute value by $\sqrt{n}$. The same bound holds for the integral, i.e.,

$$\left|\int_0^{[\sqrt{n}]} \sin\left(\frac{a(n+\theta)}{t}\right) dt\right| < \sqrt{n}.$$

As for the last sum above, we use (2.4.8) to bound each term in order to conclude that

$$\sum_{m=[\sqrt{n}]+1}^{M_1} \left(\sin\left(\frac{a(n+\theta)}{m}\right) - \int_{m-1}^{m} \sin\left(\frac{a(n+\theta)}{t}\right) dt\right)$$
$$\ll \sum_{m=[\sqrt{n}]+1}^{M_1} \frac{a(n+\theta)}{m(m-1)} < \frac{a(n+\theta)}{[\sqrt{n}]}.$$

We thus have shown that

$$\left|\sum_{m=1}^{M_1} \sin\left(\frac{a(n+\theta)}{m}\right) - \int_0^{M_1} \sin\left(\frac{a(n+\theta)}{t}\right) dt\right| \ll_a \sqrt{n},$$

uniformly with respect to θ on compact subsets of $(0,1)$.

With this bound in hand, we now proceed to remove the dependence on θ from the coefficients $1/(n+\theta)$ and $1/(n+1-\theta)$ in $S_1(a,\theta,\delta_1)$. More specifically, we consider the sum

$$S_2(a,\theta,\delta_1) := \sum_{n=0}^{\infty} \frac{1}{n+\frac{1}{2}} \left\{\sum_{m=1}^{M_1} \sin\left(\frac{a(n+\theta)}{m}\right) - \int_0^{M_1} \sin\left(\frac{a(n+\theta)}{t}\right) dt\right.$$
$$\left. + \sum_{m=1}^{M_1} \sin\left(\frac{a(n+1-\theta)}{m}\right) - \int_0^{M_1} \sin\left(\frac{a(n+1-\theta)}{t}\right) dt\right\}.$$

Note that the sum

$$\sum_{n=0}^{\infty}\left\{\left(\frac{1}{n+\frac{1}{2}}-\frac{1}{n+\theta}\right)\left\{\sum_{m=1}^{M_1}\sin\left(\frac{a(n+\theta)}{m}\right)-\int_0^{M_1}\sin\left(\frac{a(n+\theta)}{t}\right)dt\right\}\right.$$
$$+\left(\frac{1}{n+\frac{1}{2}}-\frac{1}{n+1-\theta}\right)\left\{\sum_{m=1}^{M_1}\sin\left(\frac{a(n+1-\theta)}{m}\right)\right.$$
$$\left.\left.-\int_0^{M_1}\sin\left(\frac{a(n+1-\theta)}{t}\right)dt\right\}\right\} \tag{2.4.9}$$

is uniformly and absolutely convergent, since for each n,

$$\left|\frac{1}{n+\frac{1}{2}}-\frac{1}{n+\theta}\right|\left|\sum_{m=1}^{M_1}\sin\left(\frac{a(n+\theta)}{m}\right)-\int_0^{M_1}\sin\left(\frac{a(n+\theta)}{t}\right)dt\right|$$
$$\ll_a \frac{|\theta-\frac{1}{2}|}{(n+\frac{1}{2})(n+\theta)}\sqrt{n}\ll_a\frac{1}{n^{3/2}},$$

uniformly in θ. We obtain the same bound for the other sum in (2.4.9) by the same argument. It follows that the sum $S_2(a,\theta,\delta_1)$ is convergent for a given value of θ if and only if $S_1(a,\theta,\delta_1)$ is convergent for that value of θ. Also, $S_2(a,\theta,\delta_1)$ is uniformly convergent with respect to θ on closed subintervals of $(0,1)$ if and only if $S_1(a,\theta,\delta_1)$ has this property. Next, using the oscillatory behavior of the function $y\mapsto\sin y$, we perform another truncation of the inner sum in $S_2(a,\theta,\delta_1)$, by replacing M_1 by a smaller value M_2, to be determined later. Consider the sum

$$S_3(a,\theta):=\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{2}}\left\{\sum_{m=1}^{M_2}\left(\sin\left(\frac{a(n+\theta)}{m}\right)+\sin\left(\frac{a(n+1-\theta)}{m}\right)\right)\right.$$
$$\left.-\int_0^{M_2+\frac{1}{2}}\left(\sin\left(\frac{a(n+\theta)}{t}\right)+\sin\left(\frac{a(n+1-\theta)}{t}\right)\right)dt\right\}.$$

In order to relate the convergence of $S_3(a,\theta)$ to that of $S_2(a,\theta,\delta_1)$, we estimate, for each $m\in\{M_2+1,M_2+2,\dots,M_1\}$, the quantity

$$\sin\left(\frac{a(n+\theta)}{m}\right)+\sin\left(\frac{a(n+1-\theta)}{m}\right)$$
$$-\int_{m-\frac{1}{2}}^{m+\frac{1}{2}}\left(\sin\left(\frac{a(n+\theta)}{t}\right)+\sin\left(\frac{a(n+1-\theta)}{t}\right)\right)dt$$
$$=\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\sin\left(\frac{a(n+\theta)}{m}\right)-\sin\left(\frac{a(n+\theta)}{m+u}\right)\right.$$
$$\left.+\sin\left(\frac{a(n+1-\theta)}{m}\right)-\sin\left(\frac{a(n+1-\theta)}{m+u}\right)\right)du.$$

Here,

$$\begin{aligned}\frac{a(n+\theta)}{m+u} &= \frac{a(n+\theta)}{m(1+u/m)} = \frac{a(n+\theta)}{m}\left(1-\frac{u}{m}+O\left(\frac{1}{m^2}\right)\right)\\ &= \frac{a(n+\theta)}{m} - \frac{a(n+\theta)u}{m^2} + O_a\left(\frac{n}{m^3}\right),\end{aligned}$$

uniformly in θ. So,

$$\sin\left(\frac{a(n+\theta)}{m+u}\right) = \sin\left(\frac{a(n+\theta)}{m} - \frac{a(n+\theta)u}{m^2}\right) + O_a\left(\frac{n}{m^3}\right).$$

We will choose M_2 much larger than $\sqrt{n}$. Then the ratio $a(n+\theta)u/m^2$ will be small, a is fixed, $\theta \in [0,1]$, and $u \in [-\frac{1}{2}, \frac{1}{2}]$. Then, using the estimate

$$\sin(\alpha - \epsilon) = \sin\alpha - \epsilon\cos\alpha + O(\epsilon^2)$$

with $\alpha = a(n+\theta)/m$ and $\epsilon = a(n+\theta)u/m^2$, we see that

$$\begin{aligned}\sin\left(\frac{a(n+\theta)}{m+u}\right) &= \sin\left(\frac{a(n+\theta)}{m}\right) - \frac{a(n+\theta)u}{m^2}\cos\left(\frac{a(n+\theta)}{m}\right)\\ &\quad + O\left(\frac{n^2}{m^4}\right) + O\left(\frac{n}{m^3}\right).\end{aligned}$$

Since

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{a(n+\theta)u}{m^2}\cos\left(\frac{a(n+\theta)}{m}\right) du = 0,$$

it follows that

$$\sin\left(\frac{a(n+\theta)}{m}\right) - \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \sin\left(\frac{a(n+\theta)}{t}\right) dt = O\left(\frac{n^2}{m^4}\right) + O\left(\frac{n}{m^3}\right).$$

Similarly,

$$\sin\left(\frac{a(n+1-\theta)}{m}\right) - \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} \sin\left(\frac{a(n+1-\theta)}{t}\right) dt = O\left(\frac{n^2}{m^4}\right) + O\left(\frac{n}{m^3}\right).$$

We add up these relations for $m = M_2+1, \dots, M_1$ to find that

$$\begin{aligned}&\sum_{m=M_2+1}^{M_1}\left(\sin\left(\frac{a(n+\theta)}{m}\right) + \sin\left(\frac{a(n+1-\theta)}{m}\right)\right)\\ &\qquad - \int_{M_2+\frac{1}{2}}^{M_1+\frac{1}{2}}\left(\sin\left(\frac{a(n+\theta)}{t}\right) + \sin\left(\frac{a(n+1-\theta)}{t}\right)\right) dt\\ &\qquad = O\left(\frac{n^2}{M_2^3}\right) + O\left(\frac{n}{M_2^2}\right),\end{aligned}$$

uniformly for θ in compact subsets of $(0,1)$. Therefore, if we choose, for instance, $M_2 = [n^{2/3}\log n]$, then the series

$$\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{2}}\left(\sum_{m=M_2+1}^{M_1}\left(\sin\left(\frac{a(n+\theta)}{m}\right)+\sin\left(\frac{a(n+1-\theta)}{m}\right)\right)\right.$$
$$\left.-\int_{M_2+\frac{1}{2}}^{M_1+\frac{1}{2}}\left(\sin\left(\frac{a(n+\theta)}{t}\right)+\sin\left(\frac{a(n+1-\theta)}{t}\right)\right)dt\right)$$

is uniformly and absolutely convergent.

Let us also remark that for $t \in [M_1, M_1+\frac{1}{2}]$,

$$\frac{a(n+\theta)}{t} = O\left(\frac{1}{n^{\delta_1}}\right), \qquad \text{and so} \qquad \sin\left(\frac{a(n+\theta)}{t}\right) = O\left(\frac{1}{n^{\delta_1}}\right),$$

and also

$$\int_{M_1}^{M_1+\frac{1}{2}}\left(\sin\left(\frac{a(n+\theta)}{t}\right)+\sin\left(\frac{a(n+1-\theta)}{t}\right)\right)dt = O\left(\frac{1}{n^{\delta_1}}\right). \quad (2.4.10)$$

Hence, the series

$$\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{2}}\int_{M_1}^{M_1+\frac{1}{2}}\left(\sin\left(\frac{a(n+\theta)}{t}\right)+\sin\left(\frac{a(n+1-\theta)}{t}\right)\right)dt$$

is uniformly and absolutely convergent. Combining all of the above, we deduce that the initial series $S(a,\theta)$ is convergent for a given value of θ if and only if the series $S_3(a,\theta)$ is convergent for that value of θ. Moreover, $S(a,\theta)$ converges uniformly on compact subintervals of $(0,1)$ if and only if the same holds for $S_3(a,\theta)$.

Let us observe that the contribution of the integrals in (2.4.10) is small, while on the other hand, we do not have any cancellation inside the integrals

$$\int_{M_2}^{M_2+\frac{1}{2}}\left(\sin\left(\frac{a(n+\theta)}{t}\right)+\sin\left(\frac{a(n+1-\theta)}{t}\right)\right)dt.$$

Indeed, one can show that the integrand here is almost constant, in fact equal to

$$2\sin\left(\frac{an}{M_2}\right)+O\left(\frac{1}{n^{1/3}\log^2 n}\right) = \sin\left(\frac{an^{1/3}}{\log^2 n}\right)+O\left(\frac{1}{n^{1/3}\log^2 n}\right).$$

Moreover, one can show that the series

$$\sum_{n=2}^{\infty}\frac{1}{n+\frac{1}{2}}\sin\left(\frac{an^{1/3}}{\log^2 n}\right)$$

is *not* absolutely convergent. This forces us to keep at this stage $M_2 + \frac{1}{2}$ instead of M_2 as the upper limit of integration in the definition of $S_3(a,\theta)$. As a side remark, one can show that the series above, although not absolutely convergent, *is* convergent, via proving that the fractional parts

$$\left\{ \frac{an^{1/3}}{\pi \log^2 n} \right\}$$

are "very" uniformly distributed in the interval $[0,1]$, where "very" means that the discrepancy of the first N terms is $\ll N^{-c}$ for some absolute constant $c > 0$.

Next, we choose a new (integral) parameter M_3, whose precise value as a function of n will be given later, and consider the sum

$$\begin{aligned} S_4(a,\theta) := \sum_{n=0}^{\infty} \frac{1}{n+\frac{1}{2}} \Bigg\{ & \sum_{m=1}^{M_3} \left(\sin\left(\frac{a(n+\theta)}{m}\right) + \sin\left(\frac{a(n+1-\theta)}{m}\right) \right) \\ & - \int_0^{M_3+\frac{1}{2}} \left(\sin\left(\frac{a(n+\theta)}{t}\right) + \sin\left(\frac{a(n+1-\theta)}{t}\right) \right) dt \\ & + 2 \sum_{m=M_3+1}^{M_2} \sin\left(\frac{a(n+\frac{1}{2})}{m}\right) - 2 \int_{M_3+\frac{1}{2}}^{M_2+\frac{1}{2}} \sin\left(\frac{a(n+\frac{1}{2})}{t}\right) dt \Bigg\} . \end{aligned}$$

Note that the sum $S_4(a,\theta)$ differs from $S_3(a,\theta)$ by having θ replaced by $\frac{1}{2}$ in the range $M_3 + 1 \leq m \leq M_2$. In order to relate the convergence of these two sums, we write, for $m = M_3 + 1, \ldots, M_2$,

$$\begin{aligned} & \sin\left(\frac{a(n+\theta)}{m}\right) + \sin\left(\frac{a(n+1-\theta)}{m}\right) - 2\sin\left(\frac{a(n+\frac{1}{2})}{m}\right) \\ & = 2\sin\left(\frac{a(n+\frac{1}{2})}{m}\right) \cos\left(\frac{a(\theta-\frac{1}{2})}{m}\right) - 2\sin\left(\frac{a(n+\frac{1}{2})}{m}\right) \\ & = -4\sin\left(\frac{a(n+\frac{1}{2})}{m}\right) \sin^2\left(\frac{a(\theta-\frac{1}{2})}{2m}\right) . \end{aligned}$$

Therefore,

$$\begin{aligned} & \sum_{m=M_3+1}^{M_2} \left| \sin\left(\frac{a(n+\theta)}{m}\right) + \sin\left(\frac{a(n+1-\theta)}{m}\right) - 2\sin\left(\frac{a(n+\frac{1}{2})}{m}\right) \right| \\ & \leq 4 \sum_{m=M_3+1}^{M_2} \sin^2\left(\frac{a(\theta-\frac{1}{2})}{2m}\right) \ll_a \sum_{m=M_3+1}^{M_2} \frac{1}{m^2} \ll \frac{1}{M_3}, \end{aligned}$$

uniformly with respect to θ. Similarly,

$$\left|\int_{M_3+\frac{1}{2}}^{M_2+\frac{1}{2}}\left(\sin\left(\frac{a(n+\theta)}{t}\right)+\sin\left(\frac{a(n+1-\theta)}{t}\right)-2\sin\left(\frac{a(n+\frac{1}{2})}{t}\right)\right)dt\right|$$

$$=4\left|\int_{M_3+\frac{1}{2}}^{M_2+\frac{1}{2}}\sin\left(\frac{a(n+\frac{1}{2})}{t}\right)\sin^2\left(\frac{a(n+\frac{1}{2})}{2t}\right)dt\right| \ll_a \int_{M_3+\frac{1}{2}}^{M_2+\frac{1}{2}}\frac{dt}{t^2} \ll \frac{1}{M_3}.$$

If we now take $M_3=[\log^2 n]$, the sum

$$\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{2}}\left\{\sum_{m=M_3+1}^{M_2}\left|\sin\left(\frac{a(n+\theta)}{m}\right)+\sin\left(\frac{a(n+1-\theta)}{m}\right)\right.\right.$$
$$\left.-2\sin\left(\frac{a(n+\frac{1}{2})}{m}\right)\right|-\left|\int_{M_3+\frac{1}{2}}^{M_2+\frac{1}{2}}\left(\sin\left(\frac{a(n+\theta)}{t}\right)\right.\right.$$
$$\left.\left.+\ \sin\left(\frac{a(n+1-\theta)}{t}\right)-2\sin\left(\frac{a(n+\frac{1}{2})}{t}\right)\right)dt\right|\right\}$$

will be uniformly convergent with respect to θ. Consequently, the sum $S_3(a,\theta)$ will be convergent for a given θ if and only if the sum $S_4(a,\theta)$ converges for the same value of θ, and $S_3(a,\theta)$ converges uniformly on compact subintervals of $(0,1)$ if and only if $S_4(a,\theta)$ does.

In what follows, we define

$$S_5(a,\theta):=\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{2}}\left\{\sum_{m=1}^{M_3}\left(\sin\left(\frac{a(n+\theta)}{m}\right)+\sin\left(\frac{a(n+1-\theta)}{m}\right)\right)\right.$$
$$\left.-\int_0^{M_3+\frac{1}{2}}\left(\sin\left(\frac{a(n+\theta)}{t}\right)+\sin\left(\frac{a(n+1-\theta)}{t}\right)\right)dt\right\}$$

and

$$S_6(a):=\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{2}}\left\{\sum_{m=M_3+1}^{M_2}\sin\left(\frac{a(n+\frac{1}{2})}{m}\right)-\int_{M_3+\frac{1}{2}}^{M_2+\frac{1}{2}}\sin\left(\frac{a(n+\frac{1}{2})}{t}\right)dt\right\},$$

so that

$$S_4(a,\theta)=S_5(a,\theta)+2S_6(a).$$

Here the inner sum in $S_5(a,\theta)$ has a very short range, of the size of $\log^2 n$, while the inner sum in $S_6(a)$ has a larger range, but is independent of θ. We now turn our attention to $S_5(a,\theta)$ and see whether this sum is pointwise convergent, respectively uniformly convergent on compact subintervals of $(0,1)$. Set

$$A(a,\theta,N) := \sum_{n=0}^{N} \frac{1}{n+\frac{1}{2}} \sum_{m=1}^{M_3} \left(\sin\left(\frac{a(n+\theta)}{m}\right) + \sin\left(\frac{a(n+1-\theta)}{m}\right)\right)$$

and

$$B(a,\theta,N) := \sum_{n=0}^{N} \frac{1}{n+\frac{1}{2}} \int_0^{M_3+\frac{1}{2}} \left(\sin\left(\frac{a(n+\theta)}{t}\right) + \sin\left(\frac{a(n+1-\theta)}{t}\right)\right) dt.$$

Then $S_5(a,\theta)$ converges (respectively converges uniformly on compact subintervals of $(0,1)$), provided that for every $\epsilon > 0$, there exists an $N(\epsilon)$ such that for every $N_1, N_2 > N(\epsilon)$,

$$|A(a,\theta,N_1) + B(a,\theta,N_1) - A(a,\theta,N_2) - B(a,\theta,N_2)| < \epsilon$$

(respectively uniformly for all θ in a given compact subinterval of $(0,1)$).

Fix $\epsilon > 0$. For every positive integer N, we put $A(a,\theta,N)$ in the form

$$A(a,\theta,N) = 2\sum_{n=0}^{N} \frac{1}{n+\frac{1}{2}} \sum_{1\le m\le \log^2 n} \sin\left(\frac{a(n+\frac{1}{2})}{m}\right) \cos\left(\frac{a(2\theta-1)}{2m}\right).$$

Here the condition $m \le \log^2 n$ is equivalent to $e^{\sqrt{m}} \le n$. Thus, interchanging the order of summation above, we find that

$$\begin{aligned} A(a,\theta,N) &= 2 \sum_{1\le m\le \log^2 N} \cos\left(\frac{a(2\theta-1)}{2m}\right) \sum_{e^{\sqrt{m}}\le n\le N} \frac{1}{n+\frac{1}{2}} \sin\left(\frac{a(n+\frac{1}{2})}{m}\right) \\ &= 4 \sum_{1\le m\le \log^2 N} \cos\left(\frac{a(2\theta-1)}{2m}\right) \sum_{e^{\sqrt{m}}\le n\le N} \frac{1}{2n+1} \sin\left(\frac{a(2n+1)}{2m}\right). \end{aligned}$$

For two large positive integers $N_1 < N_2$, we put $A(a,\theta,N_2) - A(a,\theta,N_1)$ in the form

$$\begin{aligned} &A(a,\theta,N_2) - A(a,\theta,N_1) \\ &= 4 \sum_{1\le m\le \log^2 N_1} \cos\left(\frac{a(2\theta-1)}{2m}\right) \sum_{N_1+1\le n\le N_2} \frac{1}{2n+1} \sin\left(\frac{a(2n+1)}{2m}\right) \\ &\quad + 4 \sum_{\log^2 N_1 < m\le \log^2 N_2} \cos\left(\frac{a(2\theta-1)}{2m}\right) \sum_{e^{\sqrt{m}}\le n\le N_2} \frac{1}{2n+1} \sin\left(\frac{a(2n+1)}{2m}\right). \end{aligned}$$

For every positive real numbers $U < V$, consider the function

$$h_{U,V}(y) := \sum_{U\le n\le V} \frac{\sin\{(2n+1)y\}}{2n+1}.$$

With this notation, we may write

$$A(a,\theta,N_2)-A(a,\theta,N_1)=4\sum_{1\le m\le\log^2 N_1}\cos\left(\frac{a(2\theta-1)}{2m}\right)h_{N_1+1,N_2}\left(\frac{a}{2m}\right)$$
$$+4\sum_{\log^2 N_1<m\le\log^2 N_2}\cos\left(\frac{a(2\theta-1)}{2m}\right)h_{e^{\sqrt{m}},N_2}\left(\frac{a}{2m}\right). \quad (2.4.11)$$

We are interested in the behavior of the function $h_{U,V}(y)$. This function is odd and periodic modulo 2π, and so it is sufficient to study the function on the interval $[0,\pi]$. Also, we note that $h_{U,V}(y)=h_{U,V}(\pi-y)$, and so furthermore, it is sufficient to consider this function on the interval $[0,\frac{1}{2}\pi]$. Observe that $h_{U,V}(0)=0$. Next, since the series is alternating with decreasing terms,

$$\left|h_{U,V}(\tfrac{1}{2}\pi)\right|=\left|\sum_{U\le n\le V}\frac{(-1)^n}{2n+1}\right|\le\frac{1}{2U+1}.$$

For $0<y<\frac{1}{2}\pi$, we write $h_{U,V}(y)$ in the form

$$h_{U,V}(y)=h_{U,V}(\tfrac{1}{2}\pi)+h_{U,V}(y)-h_{U,V}(\tfrac{1}{2}\pi)=h_{U,V}(\tfrac{1}{2}\pi)-\int_y^{\frac{1}{2}\pi}h'_{U,V}(t)dt. \quad (2.4.12)$$

Here we write [126, p. 36, formula 1.342, no. 4]

$$h'_{U,V}(t)=\sum_{U\le n\le V}\cos\{(2n+1)t\}=\frac{1}{2\sin t}\left(\sin\{2(\lfloor V\rfloor+1)t\}-\sin(2\lceil U\rceil t)\right), \quad (2.4.13)$$

where $\lfloor V\rfloor$ is the floor of V, that is, the largest integer $\le V$, and $\lceil U\rceil$ is the ceiling of U, that is, the smallest integer $\ge U$. From (2.4.12) and (2.4.13) and an integration by parts,

$$\begin{aligned}h_{U,V}(y)&=h_{U,V}(\tfrac{1}{2}\pi)-\int_y^{\frac{1}{2}\pi}\frac{1}{2\sin t}\left(\sin\{2(\lfloor V\rfloor+1)t\}-\sin(2\lceil U\rceil t)\right)dt\\&=h_{U,V}(\tfrac{1}{2}\pi)+\frac{1}{2\sin t}\left(\frac{\cos\{2(\lfloor V\rfloor+1)t\}}{2(\lfloor V\rfloor+1)}-\frac{\cos(2\lceil U\rceil t)}{2\lceil U\rceil}\right)\Bigg|_y^{\frac{1}{2}\pi}\\&\quad+\int_y^{\frac{1}{2}\pi}\frac{\cos t}{2\sin^2 t}\left(\frac{\cos\{2(\lfloor V\rfloor+1)t\}}{2(\lfloor V\rfloor+1)}-\frac{\cos(2\lceil U\rceil t)}{2\lceil U\rceil}\right)dt\\&=O\left(\frac{1}{U}\right)+O\left(\frac{1}{Uy}\right)+O\left(\frac{1}{Uy^2}\right)\\&=O\left(\frac{1}{U}\left(1+\frac{1}{y^2}\right)\right),\end{aligned}$$

uniformly for $0 < y \le \frac{1}{2}\pi$. If we need a bound that holds for all $y > 0$, we may write

$$|h_{U,V}(y)| = O\left(\frac{1}{U} \cdot \frac{1}{\|y/\pi\|^2}\right),$$

where $\|y/\pi\|$ denotes the distance from y/π to the nearest integer, which is proportional (via a factor of π) to the distance from y to the set $\pi\mathbb{Z} = \{\dots, -\pi, 0, \pi, 2\pi, \dots\}$. Recall that at these points $\pi\mathbb{Z}$, the function $h_{U,V}(y)$ vanishes.

We are now ready to apply these considerations to our expression for $A(a,\theta,N_2) - A(a,\theta,N_1)$ from (2.4.11). For $\log^2 N_1 < m \le \log^2 N_2$ and a fixed, $a/(2m)$ is a small positive number, which belongs to $(0, \frac{1}{2}\pi)$. Hence,

$$\left|h_{e^{\sqrt{m}},N_2}\left(\frac{a}{2m}\right)\right| = O\left(\frac{1}{e^{\sqrt{m}}}\left(1 + \frac{4m^2}{a^2}\right)\right) = O\left(\frac{m^2}{e^{\sqrt{m}}}\right).$$

It follows that

$$\begin{aligned}
&4\left|\sum_{\log^2 N_1 < m \le \log^2 N_2} \cos\left(\frac{a(2\theta-1)}{2m}\right) h_{e^{\sqrt{m}},N_2}\left(\frac{a}{2m}\right)\right| \\
&\quad \le 4 \sum_{\log^2 N_1 < m \le \log^2 N_2} \left|h_{e^{\sqrt{m}},N_2}\left(\frac{a}{2m}\right)\right| \\
&\quad = O\left(\sum_{\log^2 N_1 < m \le \log^2 N_2} \frac{m^2}{e^{\sqrt{m}}}\right) \\
&\quad = O\left(\int_{\log^2 N_1}^{\infty} \frac{x^2}{e^{\sqrt{x}}} dx\right) = O\left(\int_{\log N_1}^{\infty} \frac{2t^5}{e^t} dt\right) = O\left(\frac{\log^5 N_1}{N_1}\right).
\end{aligned}$$

Next, we similarly examine the sum

$$4 \sum_{1 \le m \le \log^2 N_1} \cos\left(\frac{a(2\theta-1)}{2m}\right) h_{N_1+1,N_2}\left(\frac{a}{2m}\right),$$

at least as far as the terms with large m, so that $a/(2m) \in (0, \frac{1}{2}\pi]$, are concerned. These are terms for which $m \ge a/\pi$. To that end,

$$\begin{aligned}
&4\left|\sum_{a/\pi \le m \le \log^2 N_1} \cos\left(\frac{a(2\theta-1)}{2m}\right) h_{N_1+1,N_2}\left(\frac{a}{2m}\right)\right| \\
&\quad \le 4 \sum_{a/\pi \le m \le \log^2 N_1} \left|h_{N_1+1,N_2}\left(\frac{a}{2m}\right)\right| \\
&\quad = O\left(\sum_{a/\pi \le m \le \log^2 N_1} \frac{1}{N_1}\left(1 + \frac{4m^2}{a^2}\right)\right)
\end{aligned}$$

$$= O\left(\frac{1}{N_1}\sum_{a/\pi\le m\le \log^2 N_1} m^2\right) = O\left(\frac{\log^6 N_1}{N_1}\right).$$

Lastly, the sum

$$4\sum_{1\le m<a/\pi}\cos\left(\frac{a(2\theta-1)}{2m}\right)h_{N_1+1,N_2}\left(\frac{a}{2m}\right) \tag{2.4.14}$$

has a bounded number of terms. For each m, with $1\le m<a/\pi$, we distinguish two cases. Either $a/(2m)$ is an integral multiple of π, or it is not. In the former case, we know that

$$h_{N_1+1,N_2}\left(\frac{a}{2m}\right)=0,$$

and hence these terms do not have any contribution to the sum (2.4.14). For all the other values of m, with $1\le m<a/\pi$, we examine the distances between the numbers $a/(2m\pi)$ and the set $\mathbb{Z}$. These distances, no matter how small, are some fixed strictly positive numbers, which are independent of N_1 and N_2. If we let $\delta>0$ denote the smallest such distance, in other words,

$$\delta=\min\left\{\left\|\frac{a}{2\pi m}\right\| : 1\le m<\frac{a}{\pi},\quad \frac{a}{2m}\notin\mathbb{Z}\right\},$$

then

$$\begin{aligned}4\left|\sum_{1\le m<a/\pi}\cos\left(\frac{a(2\theta-1)}{2m}\right)h_{N_1+1,N_2}\left(\frac{a}{2m}\right)\right| &\le 4\sum_{1\le m<a/\pi}\left|h_{N_1+1,N_2}\left(\frac{a}{2m}\right)\right|\\ &= O\left(\sum_{\substack{1\le m<a/\pi\\ a/(2m)\notin\mathbb{Z}}}\frac{1}{N_1\delta^2}\right)\\ &= O\left(\frac{1}{N_1\delta^2}\right).\end{aligned}$$

Thus this sum too tends to 0 as $N_1<N_2$ tend to infinity, since $\delta>0$ is fixed.

In conclusion, for every $\epsilon>0$, there exists $N(\epsilon)$ such that for all $N_1,N_2>N(\epsilon)$,

$$|A(a,\theta,N_1)-A(a,\theta,N_2)|<\epsilon,$$

uniformly for all θ in any given compact subinterval of $(0,1)$, as desired. Similarly, working with integrals instead of sums, we find that

$$|B(a,\theta,N_1)-B(a,\theta,N_2)|<\epsilon,$$

for N_1,N_2 sufficiently large. This implies that $S_5(a,\theta)$ is uniformly convergent on compact subsets of $(0,1)$. The conclusion is that the initial sum $S(a,\theta)$ is uniformly convergent on compact subintervals of $(0,1)$ if and only if $S_6(a)$ is. *But $S_6(a)$ does not depend on θ. So the convergence at one single value of θ implies uniform convergence in compact subintervals of* $(0,1)$.

2.4.4 Reformulation and Proof of Entry 2.1.2

In view of Entry 2.1.2, Theorem 2.4.2, and the proof of convergence in Sect. 2.4.3, we now reformulate and prove the following theorem.

Theorem 2.4.3. *Fix $x > 0$ and set $\theta = u + \frac{1}{2}$, where $-\frac{1}{2} < u < \frac{1}{2}$. Recall that $F(x)$ is defined in (2.1.4). If the identity below is valid for at least one value of θ, then it is valid for all values of θ, and*

$$\sum_{1\le n\le x} (-1)^n F\left(\frac{x}{n}\right)\cos(2\pi n u) - \frac{1}{4} + x\log(2\cos(\pi u))$$
$$= \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{2}+u}\lim_{M\to\infty}\left\{\sum_{m=1}^{\infty}\sin\left(\frac{2\pi(n+\frac{1}{2}+u)x}{m}\right) - \int_0^M \sin\left(\frac{2\pi(n+\frac{1}{2}+u)x}{t}\right)dt\right\}$$
$$+ \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{2}-u}\lim_{M\to\infty}\left\{\sum_{m=1}^{\infty}\sin\left(\frac{2\pi(n+\frac{1}{2}-u)x}{m}\right) - \int_0^M \sin\left(\frac{2\pi(n+\frac{1}{2}-u)x}{t}\right)dt\right\}. \quad (2.4.15)$$

Moreover, the series on the right-hand side of (2.4.15) converges uniformly on compact subintervals of $(-\frac{1}{2}, \frac{1}{2})$.

Proof. For each nonnegative integer n, set

$$f_n(u) := \frac{1}{n+\frac{1}{2}+u}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+\frac{1}{2}+u)x}{m}\right) - \int_0^M \sin\left(\frac{2\pi(n+\frac{1}{2}+u)x}{t}\right)dt\right\}$$
$$+ \frac{1}{n+\frac{1}{2}-u}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+\frac{1}{2}-u)x}{m}\right) - \int_0^M \sin\left(\frac{2\pi(n+\frac{1}{2}-u)x}{t}\right)dt\right\}. \quad (2.4.16)$$

From our work in Sect. 2.4.3, we know that the series $\sum_{n=0}^{\infty} f_n(u)$ either diverges for each value of u or converges for each value of u with the convergence being uniform in every compact subinterval of $(-\frac{1}{2}, \frac{1}{2})$. Assuming that the latter holds, we define

$$f(u) := \sum_{n=0}^{\infty} f_n(u),$$

and we endeavor to prove that the two sides of (2.4.15) have the same Fourier coefficients. If $\tilde{f}(u)$ denotes the left-hand side of (2.4.15), then we want to show that

$$\frac{1}{2\pi}\sum_{n=0}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} f_n(u)e^{2\pi iku}du = \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{f}(u)e^{2\pi iku}du, \tag{2.4.17}$$

for each integer k. Since $\tilde{f}(u)$ as well as each of the functions $f_n(u)$, $n \geq 0$, is an even function of u, it is sufficient to show that for every integer $k \geq 0$,

$$\sum_{n=0}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} f_n(u)\cos(2\pi ku)du = 2\pi\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{f}(u)\cos(2\pi ku)du. \tag{2.4.18}$$

In what follows, k is fixed, and we proceed under the aforementioned assumption of uniform convergence of the series $\sum_{n=0}^{\infty} f_n(u)$, so that the convergence at the left side of (2.4.18) is assured. Let us denote, for each positive integer N,

$$I_N := \sum_{n=0}^{N-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} f_n(u)\cos(2\pi ku)du,$$

so that (2.4.18) is equivalent to

$$\lim_{N\to\infty} I_N = 2\pi\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{f}(u)\cos(2\pi ku)du. \tag{2.4.19}$$

Next, for N large, write I_N in the form

$$\begin{aligned}
I_N = \sum_{n=0}^{N-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\cos(2\pi ku)}{n+\frac{1}{2}+u}\Bigg(&\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+\frac{1}{2}+u)x}{m}\right)\right. \\
&\left.-\int_0^M \sin\left(\frac{2\pi(n+\frac{1}{2}+u)x}{t}\right)dt\right\} \\
&+\frac{\cos(2\pi ku)}{n+\frac{1}{2}-u}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+\frac{1}{2}-u)x}{m}\right)\right. \\
&\left.-\int_0^M \sin\left(\frac{2\pi(n+\frac{1}{2}-u)x}{t}\right)dt\right\}\Bigg)\,du.
\end{aligned} \tag{2.4.20}$$

From Sect. 2.4.3, we know that for each fixed n, we have uniform convergence with respect to u on compact subintervals of $(-\frac{1}{2},\frac{1}{2})$ as $M\to\infty$. Thus, in (2.4.20), we may interchange the order of summation, integration, and taking the limit as $M\to\infty$ to deduce that

$$\begin{aligned} I_N = \lim_{M\to\infty}\sum_{m=1}^{M}\sum_{n=0}^{N-1}\Bigg\{ &\int_{-\frac12}^{\frac12}\frac{\cos(2\pi ku)}{n+\frac12+u}\sin\left(\frac{2\pi(n+\frac12+u)x}{m}\right)du \\ &+\int_{-\frac12}^{\frac12}\frac{\cos(2\pi ku)}{n+\frac12-u}\sin\left(\frac{2\pi(n+\frac12-u)x}{m}\right)du \\ &-\int_0^M\int_{-\frac12}^{\frac12}\frac{\cos(2\pi ku)}{n+\frac12+u}\sin\left(\frac{2\pi(n+\frac12+u)x}{t}\right)du\,dt \\ &-\int_0^M\int_{-\frac12}^{\frac12}\frac{\cos(2\pi ku)}{n+\frac12-u}\sin\left(\frac{2\pi(n+\frac12-u)x}{t}\right)du\,dt\Bigg\}. \end{aligned} \tag{2.4.21}$$

For each n, $0\le n\le N-1$, we rewrite the integrals with respect to u on the right side of (2.4.21) in the forms

$$\begin{aligned} &\int_{-\frac12}^{\frac12}\frac{\cos(2\pi ku)}{n+\frac12+u}\sin\left(\frac{2\pi(n+\frac12+u)x}{m}\right)du \\ &\quad=\int_n^{n+1}\frac{\cos(2\pi k(w-n-\frac12))}{w}\sin\left(\frac{2\pi wx}{m}\right)dw \\ &\quad=(-1)^k\int_n^{n+1}\frac{\cos(2\pi kw)}{w}\sin\left(\frac{2\pi wx}{m}\right)dw \end{aligned}$$

and

$$\begin{aligned} &\int_{-\frac12}^{\frac12}\frac{\cos(2\pi ku)}{n+\frac12-u}\sin\left(\frac{2\pi(n+\frac12-u)x}{m}\right)du \\ &\quad=\int_n^{n+1}\frac{\cos(2\pi k(n+\frac12-w))}{w}\sin\left(\frac{2\pi wx}{m}\right)dw \\ &\quad=(-1)^k\int_{-n-1}^{-n}\frac{\cos(2\pi kw)}{w}\sin\left(\frac{2\pi wx}{m}\right)dw. \end{aligned}$$

Similar calculations hold for the remaining two integrals in (2.4.21) with m replaced by t. Hence, (2.4.21) can be rewritten in the form

$$\begin{aligned} I_N=(-1)^k\lim_{M\to\infty}\Bigg\{&\sum_{m=1}^{M}\int_{-N}^{N}\frac{\cos(2\pi kw)}{w}\sin\left(\frac{2\pi wx}{m}\right)dw \\ &-\int_0^M\int_{-N}^{N}\frac{\cos(2\pi kw)}{w}\sin\left(\frac{2\pi wx}{t}\right)dw\,dt\Bigg\}. \end{aligned} \tag{2.4.22}$$

The first integral on the right side of (2.4.22) can be rewritten as

$$\begin{aligned}&\int_{-N}^{N}\frac{\cos(2\pi kw)}{w}\sin\left(\frac{2\pi wx}{m}\right)dw\\&=\frac{1}{2}\int_{-N}^{N}\frac{\sin\left((2\pi k+2\pi x/m)w\right)}{w}dw-\frac{1}{2}\int_{-N}^{N}\frac{\sin\left((2\pi k-2\pi x/m)w\right)}{w}dw\\&=\frac{1}{2}\int_{-(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/m)N}\frac{\sin y}{y}dy-\frac{1}{2}\int_{-(2\pi k-2\pi x/m)N}^{(2\pi k-2\pi x/m)N}\frac{\sin y}{y}dy.\end{aligned}$$

A similar representation holds for the last integral on the right-hand side of (2.4.22) with m replaced by t. Therefore, (2.4.22) can be recast in the form

$$\begin{aligned}I_N=\frac{(-1)^k}{2}\lim_{M\to\infty}&\left\{\sum_{m=1}^{M}\int_{-(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/m)N}\frac{\sin y}{y}dy-\int_{-(2\pi k-2\pi x/m)N}^{(2\pi k-2\pi x/m)N}\frac{\sin y}{y}dy\right.\\&\left.-\int_0^M\int_{-(2\pi k+2\pi x/t)N}^{(2\pi k+2\pi x/t)N}\frac{\sin y}{y}dy\,dt+\int_0^M\int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N}\frac{\sin y}{y}dy\,dt\right\}.\end{aligned}\tag{2.4.23}$$

In the following we now need to assume that $k>0$. For large m,

$$\begin{aligned}J_N(m):&=\int_{-(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/m)N}\frac{\sin y}{y}dy-\int_{m-1}^{m}\int_{-(2\pi k+2\pi x/t)N}^{(2\pi k+2\pi x/t)N}\frac{\sin y}{y}dy\,dt\\&=\int_{m-1}^{m}\left(\int_{-(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/m)N}\frac{\sin y}{y}dy-\int_{-(2\pi k+2\pi x/t)N}^{(2\pi k+2\pi x/t)N}\frac{\sin y}{y}dy\right)dt\\&=-\int_{m-1}^{m}\int_{(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/t)N}\frac{\sin y}{y}dy\,dt-\int_{m-1}^{m}\int_{-(2\pi k+2\pi x/t)N}^{-(2\pi k+2\pi x/m)N}\frac{\sin y}{y}dy\,dt.\end{aligned}\tag{2.4.24}$$

Note that

$$(2\pi k+2\pi x/t)N\geq(2\pi k+2\pi x/m)N\geq 2\pi kN,$$

and so the integrand in each of the double integrals on the far right side of (2.4.24) is $O(1/N)$. Also, the two double integrals are over domains of area bounded by

$$\frac{2\pi xN}{t}-\frac{2\pi xN}{m}=O\left(\frac{N}{mt}\right)=O\left(\frac{N}{m^2}\right).$$

Hence, we see that the first double integral on the extreme right side of (2.4.24) is

$$O\left(\frac{1}{m^2}\right).$$

We now consider the second double integral on the far right side of (2.4.24). Note that

$$(2\pi k - 2\pi x/m)N \geq (2\pi k - 2\pi x/t)N \gg N.$$

Thus, it is easy to see that we will obtain the same estimates for the second double integral on the right-hand side of (2.4.24). We now sum both sides of (2.4.24), $[\log N] + 1 \leq m \leq M$, to find that

$$\sum_{m=[\log N]+1}^{M} J_N(m) = O\left(\frac{1}{\log N}\right).$$

We now use the bound above in (2.4.23), so that (2.4.23) now reduces to

$$\begin{aligned} I_N = \frac{(-1)^k}{2} \sum_{m=1}^{[\log N]} &\left(\int_{-(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/m)N} \frac{\sin y}{y} dy - \int_{-(2\pi k-2\pi x/m)N}^{(2\pi k-2\pi x/m)N} \frac{\sin y}{y} dy \right) \\ - \frac{(-1)^k}{2} \int_0^{[\log N]} &\left(\int_{-(2\pi k+2\pi x/t)N}^{(2\pi k+2\pi x/t)N} \frac{\sin y}{y} dy - \int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N} \frac{\sin y}{y} dy \right) dt \\ + O\left(\frac{1}{\log N}\right). \end{aligned} \tag{2.4.25}$$

Next, we divide the sum on m into two parts, $m \leq \lceil 2x \rceil$ and $\lceil 2x \rceil < m \leq [\log N]$, and we similarly divide the interval of integration with respect to t. Note that for each $m \geq \lceil 2x \rceil + 1$ and every $t \in [m-1, m]$,

$$2\pi k - \frac{2\pi x}{m} \geq 2\pi k - \frac{2\pi x}{t} \geq 2\pi k - \frac{2\pi x}{\lceil 2x \rceil} \geq 2\pi k - \pi \geq \pi,$$

for all $k \geq 1$. Therefore, for such m, all the integrals in (2.4.25) are of the type, for $B \geq \pi N$,

$$\int_{-B}^{B} \frac{\sin y}{y} dy = \pi + O\left(\frac{1}{N}\right).$$

This estimate is uniform in m, for $m \geq \lceil 2x \rceil + 1$, and uniform in t, for $t \in [m-1, m]$. It follows that

$$\begin{aligned} &\int_{-(2\pi k\pm 2\pi x/m)N}^{(2\pi k\pm 2\pi x/m)N} \frac{\sin y}{y} dy - \int_{m-1}^{m} \int_{-(2\pi k\pm 2\pi x/t)N}^{(2\pi k\pm 2\pi x/t)N} \frac{\sin y}{y} dy\, dt \\ &= \left(\pi + O\left(\frac{1}{N}\right)\right) - \left(\pi + O\left(\frac{1}{N}\right)\right) = O\left(\frac{1}{N}\right), \end{aligned}$$

uniformly for $m \geq \lceil 2x \rceil + 1$, where the $\pm$ signs above are the same in all four places, i.e., either all of the signs are plus, or all of the signs are minus. It follows that the ranges of summation and integration in (2.4.25) can be further reduced to a bounded range. Thus,

$$I_N = \frac{(-1)^k}{2} \sum_{m=1}^{\lceil 2x \rceil} \left(\int_{-(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/m)N} \frac{\sin y}{y} dy - \int_{-(2\pi k-2\pi x/m)N}^{(2\pi k-2\pi x/m)N} \frac{\sin y}{y} dy \right)$$
$$- \frac{(-1)^k}{2} \int_0^{\lceil 2x \rceil} \left(\int_{-(2\pi k+2\pi x/t)N}^{(2\pi k+2\pi x/t)N} \frac{\sin y}{y} dy - \int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N} \frac{\sin y}{y} dy \right) dt$$
$$+ O\left(\frac{1}{\log N}\right). \tag{2.4.26}$$

Inside the sum on m, each integral has a limit as $N \to \infty$, and these limits are

$$\lim_{N\to\infty} \int_{-(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/m)N} \frac{\sin y}{y} dy = \pi, \qquad 1 \le m \le \lceil 2x \rceil,$$
$$\lim_{N\to\infty} \int_{-(2\pi k-2\pi x/m)N}^{(2\pi k-2\pi x/m)N} \frac{\sin y}{y} dy = \begin{cases} \pi, & \text{if } 2\pi k > 2\pi x/m, \\ 0, & \text{if } 2\pi k = 2\pi x/m, \\ -\pi, & \text{if } 2\pi k < 2\pi x/m. \end{cases}$$

In summary,

$$\lim_{N\to\infty} \frac{(-1)^k}{2} \sum_{m=1}^{\lceil 2x \rceil} \left(\int_{-(2\pi k+2\pi x/m)N}^{(2\pi k+2\pi x/m)N} \frac{\sin y}{y} dy - \int_{-(2\pi k-2\pi x/m)N}^{(2\pi k-2\pi x/m)N} \frac{\sin y}{y} dy \right)$$
$$= \frac{(-1)^k}{2} \left(\lceil 2x \rceil \pi - \# \{1 \le m \le \lceil 2x \rceil : m > x/k\} \pi \right.$$
$$\left. + \# \{1 \le m \le \lceil 2x \rceil : m < x/k\} \pi \right)$$
$$= \frac{(-1)^k \pi}{2} \left(\lceil 2x \rceil - \lceil 2x \rceil - \# \{1 \le m \le \lceil 2x \rceil : m = x/k\} \right.$$
$$\left. + 2\# \{1 \le m \le \lceil 2x \rceil : m \le x/k\} \right)$$
$$= (-1)^k \pi \left[\frac{x}{k}\right] - \frac{(-1)^k \pi}{2} \delta, \tag{2.4.27}$$

where

$$\delta = \begin{cases} 1, & \text{if } x/k \text{ is an integer}, \\ 0, & \text{otherwise}. \end{cases}$$

Hence, by (2.4.26) and (2.4.27),

$$\lim_{N\to\infty} I_N = (-1)^k \pi \left[\frac{x}{k}\right] - \frac{(-1)^k \pi}{2} \delta$$
$$- \lim_{N\to\infty} \frac{(-1)^k}{2} \int_0^{\lceil 2x \rceil} \left(\int_{-(2\pi k+2\pi x/t)N}^{(2\pi k+2\pi x/t)N} \frac{\sin y}{y} dy - \int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N} \frac{\sin y}{y} dy \right) dt, \tag{2.4.28}$$

provided that the limit on the right-hand side of (2.4.28) indeed does exist. As we have seen above, the first integral on the right-hand side of (2.4.28) equals $\pi + O(1/N)$, uniformly in t, $t \in (0, \lceil 2x \rceil)$. Therefore,

$$\lim_{N\to\infty}\frac{(-1)^k}{2}\int_0^{\lceil 2x\rceil}\int_{-(2\pi k+2\pi x/t)N}^{(2\pi k+2\pi x/t)N}\frac{\sin y}{y}dy=\lim_{N\to\infty}\frac{(-1)^k}{2}\left(\lceil 2x\rceil\pi+O\left(\frac{1}{N}\right)\right)$$
$$=\frac{(-1)^k}{2}\lceil 2x\rceil\pi. \tag{2.4.29}$$

For the remaining double integral in (2.4.28), we subdivide the outer range of integration $[0,\lceil 2x\rceil]$ into the three ranges

$$\left[0,\frac{x}{k}-\frac{1}{\log N}\right],\qquad \left[\frac{x}{k}-\frac{1}{\log N},\frac{x}{k}+\frac{1}{\log N}\right],\qquad \left[\frac{x}{k}+\frac{1}{\log N},\lceil 2x\rceil\right].$$

Using the fact that

$$\sup_{B\in\mathbf{R}}\left|\int_{-B}^{B}\frac{\sin y}{y}dy\right|<\infty,$$

we find that

$$\int_{\frac{x}{k}-\frac{1}{\log N}}^{\frac{x}{k}+\frac{1}{\log N}}\int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N}\frac{\sin y}{y}dydt=O\left(\frac{1}{\log N}\right). \tag{2.4.30}$$

Next, uniformly for $t\in\left[\frac{x}{k}+\frac{1}{\log N},\lceil 2x\rceil\right]$, we see that

$$\int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N}\frac{\sin y}{y}dy=\pi+O\left(\left(\left|2\pi k-\frac{2\pi x}{\frac{x}{k}+\frac{1}{\log N}}\right|N\right)^{-1}\right)$$
$$=\pi+O\left(\frac{\log N}{N}\right),$$

and hence

$$\int_{\frac{x}{k}+\frac{1}{\log N}}^{\lceil 2x\rceil}\int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N}\frac{\sin y}{y}dydt$$
$$=\left(\lceil 2x\rceil-\left(\frac{x}{k}+\frac{1}{\log N}\right)\right)\left(\pi+O\left(\frac{\log N}{N}\right)\right)$$
$$=\lceil 2x\rceil\pi-\frac{\pi x}{k}+O\left(\frac{1}{\log N}\right). \tag{2.4.31}$$

Lastly, uniformly for $t\in\left(0,\frac{x}{k}-\frac{1}{\log N}\right)$,

$$\int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N}\frac{\sin y}{y}dy=-\pi+O\left(\left(\left|2\pi k-\frac{2\pi x}{\frac{x}{k}-\frac{1}{\log N}}\right|N\right)^{-1}\right)$$
$$=-\pi+O\left(\frac{\log N}{N}\right),$$

and hence

$$\int_0^{\frac{x}{k}-\frac{1}{\log N}} \int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N} \frac{\sin y}{y} dydt = \left(\frac{x}{k} - \frac{1}{\log N}\right)\left(-\pi + O\left(\frac{\log N}{N}\right)\right)$$
$$= -\frac{\pi x}{k} + O\left(\frac{1}{\log N}\right). \tag{2.4.32}$$

Combining (2.4.29)–(2.4.32), we conclude that

$$\lim_{N\to\infty} \frac{(-1)^k}{2} \int_0^{\lceil 2x \rceil} \left(\int_{-(2\pi k+2\pi x/t)N}^{(2\pi k+2\pi x/t)N} \frac{\sin y}{y} dy - \int_{-(2\pi k-2\pi x/t)N}^{(2\pi k-2\pi x/t)N} \frac{\sin y}{y} dy \right) dt$$
$$= \frac{(-1)^k}{2} \left(\lceil 2x \rceil \pi - \lceil 2x \rceil \pi + \frac{\pi x}{k} + \frac{\pi x}{k} \right)$$
$$= \frac{(-1)^k \pi x}{k}. \tag{2.4.33}$$

Combining (2.4.33) and (2.4.28), we finally deduce that

$$\lim_{N\to\infty} I_N = (-1)^k \pi \left[\frac{x}{k}\right] - \frac{(-1)^k \pi}{2}\delta + \frac{(-1)^k \pi x}{k}. \tag{2.4.34}$$

So, assuming that the right-hand side of (2.4.15) converges for at least one value of θ, we see that either (2.4.15) or (2.4.19) is equivalent to the proposition that

$$(-1)^k \left[\frac{x}{k}\right] - \frac{(-1)^k}{2}\delta - \frac{(-1)^k x}{k} = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{f}(u)\cos(2\pi k u) du, \tag{2.4.35}$$

for each $k \geq 1$, where

$$\delta = \begin{cases} 1, & \text{if } x/k \text{ is an integer,} \\ 0, & \text{otherwise.} \end{cases}$$

There remains the calculation of the integral on the right-hand side of (2.4.35). First, for each $k \geq 1$,

$$2\int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{1\leq n\leq x} (-1)^n F\left(\frac{x}{n}\right) \cos(2\pi n u)\cos(2\pi k u) du = (-1)^k F\left(\frac{x}{k}\right)$$
$$= (-1)^k \left(\left[\frac{x}{k}\right] - \frac{1}{2}\delta \right). \tag{2.4.36}$$

Trivially, for each $k \geq 1$,

$$2\int_{-\frac{1}{2}}^{\frac{1}{2}} -\frac{1}{4}\cos(2\pi ku)du = 0. \tag{2.4.37}$$

Next, recall the Fourier series [126, p. 46, formula 1.441, no. 2]

$$\log(2\cos(\pi u)) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(2\pi nu)}{n}, \qquad -\frac{1}{2} < u < \frac{1}{2}.$$

Because the series on the right-hand side above is boundedly convergent on $[-\frac{1}{2}, \frac{1}{2}]$, we may invert the order of summation and integration to deduce that

$$\begin{aligned} 2x&\int_{-\frac{1}{2}}^{\frac{1}{2}} \log(2\cos(\pi u))\cos(2\pi ku)du \\ &= 2x\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_{-\frac{1}{2}}^{\frac{1}{2}}\cos(2\pi nu)\cos(2\pi ku)du \\ &= x\frac{(-1)^{k-1}}{k}. \end{aligned} \tag{2.4.38}$$

Bringing together (2.4.36)–(2.4.38), we find that

$$2\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{f}(u)\cos(2\pi ku)du = (-1)^k\left(\left[\frac{x}{k}\right] - \frac{1}{2}\delta\right) + x\frac{(-1)^{k-1}}{k}. \tag{2.4.39}$$

Comparing (2.4.39) with (2.4.35), we see that indeed (2.4.35) has been proven for $k \geq 1$.

Let us summarize what we have accomplished. We have assumed that (2.4.15) holds for one particular value of θ. We have shown that the right side of (2.4.15) converges uniformly on compact subsets of $(-\frac{1}{2}, \frac{1}{2})$. Thus, the right side is a well-defined, continuous function of θ on $(-\frac{1}{2}, \frac{1}{2})$, and we need to check that it is equal to the function on the left side of (2.4.15). Consider the difference of these two functions, which is a continuous function of θ on $(-\frac{1}{2}, \frac{1}{2})$. We have proved that all its Fourier coefficients for $k \neq 0$ vanish. Then, as a function of θ, this function will be constant. Moreover, since the two sides of (2.4.15) are equal for one particular value of θ, the aforementioned constant must be zero. And so (2.4.15) holds for all θ. This then completes the proof of Theorem 2.4.3. □

2.5 Proof of Ramanujan's Second Bessel Function Identity (Symmetric Form)

In this section, we prove Ramanujan's second assertion on page 335 of [269], i.e., Entry 2.1.2, under the assumption that the product of the indices of the double series tends to infinity. As in our proof of the first identity in symmetric form, it will be sufficient to prove Entry 2.1.2 for rational $\theta = a/q$, where q is prime and $0 < a < q$.

We define

$$G(a,q,x) := \frac{\sqrt{x}}{2}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{I_1\left(4\pi\sqrt{m(n+a/q)x}\right)}{\sqrt{m(n+a/q)}}+\frac{I_1\left(4\pi\sqrt{m(n+1-a/q)x}\right)}{\sqrt{m(n+1-a/q)}}\right\}$$
$$= \frac{\sqrt{qx}}{2}\sum_{m=1}^{\infty}\sum_{\substack{r=0\\ r\equiv\pm a \bmod q}}^{\infty}\frac{I_1\left(4\pi\sqrt{mrx/q}\right)}{\sqrt{mr}}. \tag{2.5.1}$$

Thus, Entry 2.1.2 is equivalent to the following theorem.

Theorem 2.5.1. *If q is prime and $0 < a < q$, then*

$$G(a,q,x) = \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos\left(\frac{2\pi na}{q}\right) - \frac{1}{4} + x\log(2\sin(\pi a/q)) =: K(a,q,x). \tag{2.5.2}$$

Our first task in reaching our goal of proving Entry 2.1.2 or Theorem 2.5.1 is to establish the following theorem.

Theorem 2.5.2. *If χ is a nonprincipal even primitive character modulo q, then*

$$\sum_{n\le x}{}' d_\chi(n) = \frac{\sqrt{q}}{\tau(\overline{\chi})}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\sqrt{\frac{x}{n}}I_1\left(4\pi\sqrt{nx/q}\right)$$
$$- \frac{x}{\tau(\overline{\chi})}\sum_{h=1}^{q-1}\overline{\chi}(h)\log\left(2\sin(\pi h/q)\right). \tag{2.5.3}$$

Proof. Recall the functional equation of $\zeta(2s)$ [101, p. 59],

$$\pi^{-s}\Gamma(s)\zeta(2s) = \pi^{-(\frac{1}{2}-s)}\Gamma(\tfrac{1}{2}-s)\zeta(1-2s).$$

Recall also that if χ is an even nonprincipal primitive character of modulus q, then the Dirichlet L-function $L(x,\chi)$ satisfies the functional equation [101, p. 69]

$$(\pi/q)^{-s}\Gamma(s)L(2s,\chi) = \frac{\tau(\chi)}{\sqrt{q}}(\pi/q)^{-(\frac{1}{2}-s)}\Gamma(\tfrac{1}{2}-s)L(1-2s,\overline{\chi}).$$

Then, if

$$F(s,\chi) := \zeta(2s)L(2s,\chi) = \sum_{n=1}^{\infty} d_\chi(n)n^{-2s}$$

and

$$\xi(s,\chi) := (\pi/\sqrt{q})^{-2s}\Gamma^2(s)F(s,\chi),$$

the functional equations of $\zeta(s)$ and $L(s,\chi)$ yield the functional equation

$$\xi(s,\chi) = \frac{\tau(\chi)}{\sqrt{q}}\xi\left(\tfrac{1}{2}-s,\overline{\chi}\right).$$

We next state a special case of [26, p. 351, Theorem 2; p. 356, Theorem 4]. In the notation of those theorems from [26], $q=0$, $r=\frac{1}{2}$, $m=2$, $\lambda_n = \mu_n = \pi^2 n^2/q$, $a(n) = d_\chi(n)$, and $b(n) = \tau(\chi)d_{\overline{\chi}}(n)/\sqrt{q}$. Also, as above, $J_\nu(x)$ denotes the ordinary Bessel function of order ν. Let $x>0$. Then

$$\sum_{\lambda_n \le x}{}' d_\chi(n) = \frac{\tau(\chi)}{\sqrt{q}}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\left(\frac{x}{\mu_n}\right)^{1/4} K_{1/2}(4\sqrt{\mu_n x}; -\tfrac{1}{2}; 2) + Q_0(x), \quad (2.5.4)$$

where [26, p. 348, Definition 4]

$$K_\nu(x;\mu;2) = \int_0^\infty u^{\nu-\mu-1}J_\mu(u)J_\nu(x/u)du$$

and

$$Q_0(x) = \frac{1}{2\pi i}\int_C \frac{(\pi/\sqrt{q})^{-2s}F(s,\chi)x^s}{s}ds,$$

where C is a positively oriented closed curve encircling the poles of the integrand. Moreover, the series on the right-hand side of (2.5.4) is uniformly convergent on compact intervals not containing values of λ_n.

We calculate $Q_0(x)$. Since $L(s,\chi)$ is an entire function, and since $L(0,\chi) = 0$, when the character χ is even, the only pole of the integrand is at $s=\frac{1}{2}$, arising from the simple pole of $\zeta(2s)$. Thus,

$$Q_0(x) = \frac{\sqrt{qx}}{\pi}L(1,\chi) = -\frac{\tau(\chi)}{\pi}\sqrt{\frac{x}{q}}\sum_{n=1}^{q-1}\overline{\chi}(n)\log|1-\zeta_q^n|, \quad (2.5.5)$$

where $\zeta_q = e^{2\pi i/q}$, and where we have used an evaluation for $L(1,\chi)$ found in [104].

Next, recall that [314, p. 54]

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos z \qquad \text{and} \qquad J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z.$$

Thus, anticipating a later change of variable and using a result that can readily be derived from [314, p. 184, formula (3)], we find that

$$K_{1/2}(4\pi^2 nx/q; -\tfrac{1}{2}; 2) = \frac{1}{\pi^2}\sqrt{\frac{q}{nx}}\int_0^\infty \cos u \, \sin\left(\frac{4\pi^2 nx}{qu}\right) du$$
$$= -\frac{1}{\pi^2}\sqrt{\frac{q}{nx}}2\pi\sqrt{\frac{nx}{q}}\left(\frac{\pi}{2}Y_1(4\pi\sqrt{nx/q}) + K_1(4\pi\sqrt{nx/q})\right)$$
$$= I_1(4\pi\sqrt{nx/q}). \tag{2.5.6}$$

We now replace x by π^2x^2/q and substitute the values $\lambda_n = \mu_n = \pi^2n^2/q$ in (2.5.4). Using (2.5.5) and (2.5.6) in (2.5.4), we conclude that

$$\sum_{n\le x}{}' d_\chi(n) = \frac{\tau(\chi)}{\sqrt{q}}\sum_{n=1}^\infty d_{\overline{\chi}}(n)\sqrt{\frac{x}{n}}I_1(4\pi\sqrt{nx/q}) - \frac{\tau(\chi)x}{q}\sum_{n=1}^{q-1}\overline{\chi}(n)\log|1-\zeta_q^n|. \tag{2.5.7}$$

Using the fact that $\tau(\chi)\tau(\overline{\chi}) = q$ and the simple identity

$$\log|1-\zeta_q^n| = \log|\zeta_q^{-n/2} - \zeta_q^{n/2}| = \log(2\sin(\pi n/q)),$$

we obtain

$$\sum_{n\le x}{}' d_\chi(n) = \frac{\sqrt{q}}{\tau(\overline{\chi})}\sum_{n=1}^\infty d_{\overline{\chi}}(n)\sqrt{\frac{x}{n}}I_1\left(4\pi\sqrt{nx/q}\right)$$
$$- \frac{x}{\tau(\overline{\chi})}\sum_{n=1}^{q-1}\overline{\chi}(n)\log\left(2\sin(\pi n/q)\right),$$

which completes the proof. □

We need one further result before commencing our proof of Theorem 2.5.1.

Lemma 2.5.1. *If* $0 < a < q$ *and* $(a, q) = 1$, *then*

$$\sum_{n=1}^\infty F\left(\frac{x}{n}\right)\cos\left(\frac{2\pi na}{q}\right)$$
$$= \sum_{1\le n\le x/q}{}' d(n) + \sum_{\substack{d|q\\d>1}}\frac{1}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi \text{ even}}}\chi(a)\tau(\overline{\chi})\sum_{1\le n\le dx/q}{}' d_\chi(n).$$

The proof of Lemma 2.5.1 is very similar to that of Lemma 2.3.1, and so we omit the proof.

Proof of Theorem 2.5.1. First, using (2.3.10) and the fact that χ is even, we see that

$$G(a, q, x) = \frac{q}{2}\sum_{m=1}^\infty \sum_{\substack{r=1\\ r\equiv \pm a \bmod q}}^\infty \sqrt{\frac{x}{qmr}}I_1\left(4\pi\sqrt{mrx/q}\right)$$

$$= \frac{q}{2\phi(q)} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \sqrt{\frac{x}{qmr}} I_1\left(4\pi\sqrt{mrx/q}\right) \sum_{\chi \bmod q} \overline{\chi}(r)\left(\chi(a)+\chi(-a)\right)$$

$$= \frac{q}{\phi(q)} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \sqrt{\frac{x}{qmr}} I_1\left(4\pi\sqrt{mrx/q}\right) \sum_{\substack{\chi \bmod q \\ \chi \text{ even}}} \chi(a)\overline{\chi}(r)$$

$$= \frac{q}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ even}}} \chi(a) \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \overline{\chi}(r) \sqrt{\frac{x}{qmr}} I_1\left(4\pi\sqrt{mrx/q}\right)$$

$$= \frac{q}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ even}}} \chi(a) \sum_{n=1}^{\infty} d_{\overline{\chi}}(n) \sqrt{\frac{x}{qn}} I_1\left(4\pi\sqrt{nx/q}\right).$$

So, if q is prime and χ_0 denotes the principal character modulo q, then

$$\begin{aligned} G(a,q,x) &= \frac{q}{\phi(q)} \sum_{m=1}^{\infty} \sum_{\substack{r=1 \\ q\nmid r}}^{\infty} \sqrt{\frac{x}{qmr}} I_1\left(4\pi\sqrt{mrx/q}\right) \\ &\quad + \frac{q}{\phi(q)} \sum_{\substack{\chi \neq \chi_0 \\ \chi \text{ even}}} \chi(a) \sum_{n=1}^{\infty} d_{\overline{\chi}}(n) \sqrt{\frac{x}{qn}} I_1\left(4\pi\sqrt{nx/q}\right) \\ &= \frac{q}{\phi(q)} \Delta(x/q) - \frac{1}{\phi(q)} \Delta(x) \\ &\quad + \frac{q}{\phi(q)} \sum_{\substack{\chi \neq \chi_0 \\ \chi \text{ even}}} \chi(a) \sum_{n=1}^{\infty} d_{\overline{\chi}}(n) \sqrt{\frac{x}{qn}} I_1\left(4\pi\sqrt{nx/q}\right) \\ &= -\frac{1}{\phi(q)} {\sum_{n\le x}}' d(n) + \frac{q}{\phi(q)} {\sum_{n \le x/q}}' d(n) - \frac{1}{4} + \frac{x}{\phi(q)} \log q \\ &\quad + \frac{q}{\phi(q)} \sum_{\substack{\chi \neq \chi_0 \\ \chi \text{ even}}} \chi(a) \sum_{n=1}^{\infty} d_{\overline{\chi}}(n) \sqrt{\frac{x}{qn}} I_1\left(4\pi\sqrt{nx/q}\right). \end{aligned} \tag{2.5.8}$$

On the other hand, by Lemma 2.5.1 with q prime,

$$\begin{aligned} K(a,q,x) &= -\frac{1}{\phi(q)} {\sum_{n\le x}}' d(n) + \frac{1+\phi(q)}{\phi(q)} {\sum_{n\le x/q}}' d(n) - \frac{1}{4} \\ &\quad + \frac{1}{\phi(q)} \sum_{\substack{\chi \neq \chi_0 \\ \chi \text{ even}}} \chi(a)\tau(\overline{\chi}) {\sum_{1\le n\le x}}' d_{\chi}(n) + x\log(2\sin \pi a/q). \end{aligned} \tag{2.5.9}$$

Thus, in view of (2.5.8), (2.5.9), and (2.5.2), it suffices to show that

$$\sum_{\substack{\chi\neq\chi_0\\ \chi \text{ even}}} \chi(a)\tau(\overline{\chi}) \sideset{}{'}\sum_{1\leq n\leq x} d_\chi(n) + (q-1)x\log(2\sin\pi a/q)$$
$$= q \sum_{\substack{\chi\neq\chi_0\\ \chi \text{ even}}} \chi(a) \sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\sqrt{\frac{x}{qn}} I_1\left(4\pi\sqrt{nx/q}\right) + x\log q.$$

By Theorem 2.5.2, we now only have to show that

$$\sum_{\substack{\chi\neq\chi_0\\ \chi \text{ even}}} \chi(a) \sum_{h=1}^{q-1} \overline{\chi}(h)\log\left(2\sin(\pi h/q)\right) = (q-1)\log(2\sin\pi a/q) - \log q. \quad (2.5.10)$$

Now

$$\sum_{\substack{\chi\neq\chi_0\\ \chi \text{ even}}} \chi(a) \sum_{h=1}^{q-1} \overline{\chi}(h)\log\left(2\sin(\pi h/q)\right) = \sum_{h=1}^{q-1} \log\left(2\sin(\pi h/q)\right) \sum_{\substack{\chi\neq\chi_0\\ \chi \text{ even}}} \chi(a)\overline{\chi}(h)$$
$$= \sum_{h=1}^{q-1} \log\left(2\sin(\pi h/q)\right) \sum_{\chi \text{ even}} \chi(a)\overline{\chi}(h) - \sum_{h=1}^{q-1} \log\left(2\sin(\pi h/q)\right)$$
$$= (q-1)\log(2\sin\pi a/q) - \log\left(2^{q-1}\prod_{h=1}^{q-1}\sin(\pi h/q)\right)$$
$$= (q-1)\log(2\sin\pi a/q) - \log q,$$

where we have used the familiar formula [126, p. 41, formula 1.392, no. 1]

$$\prod_{h=1}^{q-1}\sin(\pi h/q) = \frac{q}{2^{q-1}}.$$

Thus, (2.5.10) has been established, and we have completed the proof. □