19

Miscellaneous Results in Analysis

19.1 Introduction

Recall that when Ramanujan's lost notebook [269] was published in 1988, other fragments and partial manuscripts were also published with the lost notebook. In the first portion of this chapter, we examine two formulas found on page 336 of [269] that are clearly wrong. Undoubtedly, Ramanujan realized that these results are indeed incorrect as they stand. He possibly possessed correct identities and used some unknown formal procedure to replace certain expressions by divergent series in order to make the identities more attractive. Ramanujan frequently enjoyed stating identities in an unorthodox fashion in order to surprise or titillate his audience. We timorously conjecture that Ramanujan had established correct identities in each case, but we do not know what they are.

Following our discussion of these two intriguing but incorrect formulas, we consider various isolated results. Perhaps the most interesting are an integral-series identity on page 197 and a study of the integral $\int_0^x \frac{\sin u}{u} du$, for which Ramanujan determines the points where it achieves local maxima and minima.

19.2 Two False Claims

Entry 19.2.1 (p. 336). *Let* $\sigma_s(n) = \sum_{d|n} d^s$, *and let* $\zeta(s)$ *denote the Riemann zeta function. Then*

$$\Gamma(s+\tfrac{1}{2})\left\{\frac{\zeta(1-s)}{(s-\frac{1}{2})x^{s-\frac{1}{2}}}+\frac{\zeta(-s)\tan\frac{1}{2}\pi s}{2x^{s+\frac{1}{2}}}\right.$$
$$\left.+\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{2i}\left\{(x-in)^{-s-\frac{1}{2}}-(x+in)^{-s-\frac{1}{2}}\right\}\right\}$$

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$$= (2\pi)^s \left\{ \frac{\zeta(1-s)}{2\sqrt{\pi x}} - 2\pi\sqrt{\pi x}\zeta(-s)\tan\tfrac{1}{2}\pi s \right.$$
$$\left. +\sqrt{\pi}\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}+2\pi\sqrt{2nx}\right)\right\}. \qquad (19.2.1)$$

Entry 19.2.2 (p. 336). *Let $\sigma_s(n)$ and $\zeta(s)$ be as in the preceding entry. If α and β are positive numbers such that $\alpha\beta = 4\pi^2$, then*

$$\alpha^{(s+1)/2}\left\{\frac{1}{\alpha}\zeta(1-s)+\frac{1}{2}\zeta(-s)\tan\tfrac{1}{2}\pi s+\sum_{n=1}^{\infty}\sigma_s(n)\sin(n\alpha)\right\}$$
$$=\beta^{(s+1)/2}\left\{\frac{1}{\beta}\zeta(1-s)+\frac{1}{2}\zeta(-s)\tan\tfrac{1}{2}\pi s+\sum_{n=1}^{\infty}\sigma_s(n)\sin(n\beta)\right\}. \qquad (19.2.2)$$

Each of Ramanujan's claims is easily seen to be false in general, because each contains divergent series. In Sects. 19.3–19.6, we examine these two formulas. Formula (19.2.2) is especially intriguing because of its beautiful symmetry, because it appears to be a relation between Eisenstein series formally extended to the real line, and because it appears to be an analogue of the Poisson summation formula or a special instance of the Voronoï summation formula.

19.3 First Attempt: A Possible Connection with Eisenstein Series

A first examination of (19.2.2) reminds us of the transformation formulas for Eisenstein series when s is a positive odd integer. In [29], Berndt derived modular transformation formulas for a large class of analytic Eisenstein series. Specializing Theorem 2 of [29] for $r_1 = r_2 = 0$ and the modular transformation $Tz = -1/z$, for $z \in \mathcal{H} = \{z : \operatorname{Im} z > 0\}$ we find that for any complex number s,

$$z^{-s}(1+e^{\pi is})\sum_{n=1}^{\infty}\sigma_{s-1}(n)e^{-2\pi in/z} = (1+e^{\pi is})\sum_{n=1}^{\infty}\sigma_{s-1}(n)e^{2\pi inz}$$
$$- z^{-s}e^{\pi is}(2\pi i)^{-s}(1+e^{\pi is})\Gamma(s)\zeta(s) + (2\pi i)^{-s}(1+e^{\pi is})\Gamma(s)\zeta(s)$$
$$- (2\pi i)^{-s}\int_C u^{s-1}\frac{1}{e^{zu}-1}\frac{1}{e^{u}-1}du, \qquad (19.3.1)$$

where $\zeta(s)$ denotes the Riemann zeta function. Here C is a loop beginning at $+\infty$, proceeding to the left in $\mathcal{H}$, encircling the origin in the positive direction so that $u = 0$ is the only zero of $(e^{zu}-1)(e^{u}-1)$ lying "inside" the loop, and then returning to $+\infty$ in the lower half-plane. We choose the branch of u^s with $0 < \arg u < 2\pi$. Otherwise, outside the integrand, we choose the branch of $\log w$ such that $-\pi \leq \arg w < \pi$. Replacing s by $s+1$ in (19.3.1) and slightly simplifying, we find that

$$\begin{aligned} z^{-s-1}\sum_{n=1}^{\infty}\sigma_s(n)e^{-2\pi in/z} &= \sum_{n=1}^{\infty}\sigma_s(n)e^{2\pi inz} \\ &+ z^{-s-1}e^{\pi is}(2\pi i)^{-s-1}\Gamma(s+1)\zeta(s+1) + (2\pi i)^{-s-1}\Gamma(s+1)\zeta(s+1) \\ &- \frac{(2\pi i)^{-s-1}}{1-e^{\pi is}}\int_C u^s \frac{1}{e^{zu}-1}\frac{1}{e^u-1}du. \end{aligned} \tag{19.3.2}$$

Next, recall the functional equation of the Riemann zeta function (3.1.4) [306, p. 16, equation (2.1.8)],

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\tfrac{1}{2}\pi s\right)\Gamma(s)\zeta(s). \tag{19.3.3}$$

If we replace s by $s+1$ in (19.3.3), we easily see that

$$(2\pi i)^{-s-1}\Gamma(s+1)\zeta(s+1) = \frac{ie^{-\pi is/2}\zeta(-s)}{2\sin\left(\frac{1}{2}\pi s\right)}. \tag{19.3.4}$$

Using (19.3.4) in (19.3.2), we conclude that

$$\begin{aligned} z^{-s-1}\sum_{n=1}^{\infty}\sigma_s(n)e^{-2\pi in/z} &= \sum_{n=1}^{\infty}\sigma_s(n)e^{2\pi inz} + z^{-s-1}\frac{ie^{\pi is/2}\zeta(-s)}{2\sin\left(\frac{1}{2}\pi s\right)} \\ &+ \frac{ie^{-\pi is/2}\zeta(-s)}{2\sin\left(\frac{1}{2}\pi s\right)} - \frac{(2\pi i)^{-s-1}}{1-e^{\pi is}}\int_C u^s\frac{1}{e^{zu}-1}\frac{1}{e^u-1}du. \end{aligned} \tag{19.3.5}$$

Omitting n, note that the product of the arguments in the exponentials in the two infinite series in (19.3.5) is equal to $4\pi^2$, in accordance with the condition $\alpha\beta = 4\pi^2$ prescribed by Ramanujan. Equation (19.3.5) is as close to (19.2.2) as we can get using the chief theorem from [29].

19.4 Second Attempt: A Formula in Ramanujan's Paper [257]

We conjecture that Ramanujan's formula (19.2.2) arose from the research that produced his paper [257], [267, pp. 72–77]. On page 75 in [267], in formula (15), Ramanujan asserts that if $\operatorname{Re} s > -1$ and if α and β are positive numbers such that $\alpha\beta = 4\pi^2$, then

$$\begin{aligned} &\frac{\zeta(1-s)}{4\cos(\frac{1}{2}\pi s)}\alpha^{(s-1)/2} + \frac{\zeta(-s)}{8\sin(\frac{1}{2}\pi s)}\alpha^{(s+1)/2} \\ &\qquad + \alpha^{(s+1)/2}\int_0^{\infty}\int_0^{\infty}\frac{x^s\sin(\alpha xy)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}dx\,dy \end{aligned}$$

$$= \frac{\zeta(1-s)}{4\cos(\frac{1}{2}\pi s)}\beta^{(s-1)/2} + \frac{\zeta(-s)}{8\sin(\frac{1}{2}\pi s)}\beta^{(s+1)/2} + \beta^{(s+1)/2}\int_0^\infty\int_0^\infty \frac{x^s\sin(\beta xy)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}dx\,dy. \tag{19.4.1}$$

Suppose that we multiply both sides of (19.4.1) by $4\cos(\frac{1}{2}\pi s)$ to deduce that

$$\alpha^{(s+1)/2}\left\{\frac{1}{\alpha}\zeta(1-s) + \frac{1}{2}\zeta(-s)\cot(\tfrac{1}{2}\pi s) + 4\cos(\tfrac{1}{2}\pi s)\int_0^\infty\int_0^\infty \frac{x^s\sin(\alpha xy)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}dx\,dy\right\}$$
$$=\beta^{(s+1)/2}\left\{\frac{1}{\beta}\zeta(1-s) + \frac{1}{2}\zeta(-s)\cot(\tfrac{1}{2}\pi s) + 4\cos(\tfrac{1}{2}\pi s)\int_0^\infty\int_0^\infty \frac{x^s\sin(\beta xy)}{(e^{2\pi x}-1)(e^{2\pi y}-1)}dx\,dy\right\}. \tag{19.4.2}$$

We note that the first two expressions on each side of (19.4.2) are identical to the first two terms on each side of (19.2.2), except that $\tan(\frac{1}{2}\pi s)$ in (19.2.2) has been replaced by $\cot(\frac{1}{2}\pi s)$ in (19.4.2). However, we are unable to make any identification of the double integrals in (19.4.2) with the divergent sums in (19.2.2).

19.5 Third Attempt: The Voronoï Summation Formula

Our third attempt to prove Entries 19.2.2 and 19.2.1 depends on the Voronoï summation formula. We only briefly sketch the background and hypotheses needed for the statement of the Voronoï summation formula. For complete details, see the papers [26–28], and [89].

Let $s = \sigma + it$, with σ and t real, and let

$$\phi(s) := \sum_{n=1}^\infty a(n)\lambda_n^{-s} \qquad \text{and} \qquad \psi(s) := \sum_{n=1}^\infty b(n)\mu_n^{-s}, \qquad 0 < \lambda_n, \mu_n \to \infty,$$

be two Dirichlet series with abscissas of absolute convergence σ_a and σ_a^*, respectively. Let $r > 0$, and suppose that $\phi(s)$ and $\psi(s)$ satisfy a functional equation of the type

$$\Gamma(s)\phi(s) = \Gamma(r-s)\psi(r-s). \tag{19.5.1}$$

Define also

$$Q(x) := \frac{1}{2\pi i}\int_C \frac{\phi(s)x^s}{s}ds, \tag{19.5.2}$$

where C is a simple closed curve(s) containing the integrand's poles in its interior.

The Voronoï summation formula in its original form with $a(n) = d(n)$, where $d(n)$ denotes the number of positive divisors of the positive integer n, was first proved by M.G. Voronoï in 1904 [310]. Since then, "Voronoï summation formulas" have been established for a variety of arithmetic functions under various hypotheses. In particular, Berndt [28] established various versions of the Voronoï summation formula, including the following theorem from [28, p. 142, Theorem 1], where several references to the literature on Voronoï summation formulas can be found.

Theorem 19.5.1. *Let $f \in C^{(1)}(0,\infty)$. Then, if $0 < a < \lambda_1 < x < \infty$,*

$$\sum_{\lambda_n \le x}{}' a(n) f(\lambda_n) = \int_a^x Q'(t) f(t)\,dt + \sum_{n=1}^{\infty} b(n) \int_a^x \left(\frac{t}{\mu_n}\right)^{(r-1)/2} J_{r-1}(2\sqrt{\mu_n t}) f(t)\,dt, \tag{19.5.3}$$

where the prime $\prime$ on the summation sign on the left-hand side indicates that if $x = \lambda_n$, for some integer n, then only $\frac{1}{2}a(n)f(\lambda_n)$ is counted, and where $J_\nu(x)$ denotes the ordinary Bessel function of order ν.

This is the simplest theorem of this sort. The two applications that we make of Theorem 19.5.1 are formal in the sense that there are no versions of the Voronoï summation formula that would ensure the validity of our applications; indeed, as we remarked above, both (19.2.1) and (19.2.2) contain divergent series. Possibly Ramanujan discovered some version of the Voronoï summation formula for $a(n) = \sigma_k(n)$, but if so, he apparently had established neither a precise version nor conditions for its validity. Under this assumption, we next see how Ramanujan *might* have been led to the two entries above.

In order to avoid possible confusion, we are going to replace s by k in our attempts to prove (19.2.1) and (19.2.2). It is well known and easy to prove that for any real number k,

$$\zeta(s)\zeta(s-k) = \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s}, \qquad \sigma > \sup\{1, k+1\}. \tag{19.5.4}$$

Then with the use of the functional equation (19.3.3) for $\zeta(s)$, it is not difficult to show that if k is an odd integer [89, p. 17],

$$(2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(k-s) = (-1)^{(k+1)/2}(2\pi)^{-(k+1-s)}\Gamma(k+1-s)\zeta(k+1-s)\zeta(1-s). \tag{19.5.5}$$

Thus, in the settings (19.5.1) and (19.5.5), we have

$$a(n) = \sigma_k(n), \qquad b(n) = (-1)^{(k+1)/2}\sigma_k(n), \qquad k \text{ odd}, \tag{19.5.6}$$

$$\lambda_n = \mu_n = 2\pi n, \qquad n \geq 1, \qquad r = k+1. \tag{19.5.7}$$

Furthermore, $Q(x)$ is the sum of the residues of

$$\frac{(2\pi)^{-s}\zeta(s)\zeta(s-k)x^s}{s}$$

taken over all its poles, which are at $s = 1$, $s = k+1$, and $s = 0$. Since $\zeta(s)$ has a simple pole at $s = 1$ with residue 1 and [306, p. 19]

$$\zeta(0) = -\frac{1}{2}, \tag{19.5.8}$$

we find that

$$Q(x) = -\frac{1}{2}\zeta(-k) + \frac{\zeta(1-k)x}{2\pi} + \frac{\zeta(k+1)x^{k+1}}{(2\pi)^{k+1}(k+1)}.$$

It follows that

$$Q'(x) = \frac{\zeta(1-k)}{2\pi} + \frac{\zeta(k+1)x^k}{(2\pi)^{k+1}}. \tag{19.5.9}$$

We first examine (19.2.2). In our formal application of (19.5.3), we clearly should set $a = 0$, $x = \infty$, and $f(t) = \sin(\alpha t/(2\pi))$. In order to apply (19.5.3), we need to employ the integral evaluation [126, p. 773, formula 6.728, no. 5]

$$\int_0^\infty x^{k+1} J_k(bx)\sin(ax^2)dx = \frac{b^k}{(2a)^{k+1}}\cos\left(\frac{b^2}{4a} - \frac{k\pi}{2}\right). \tag{19.5.10}$$

Hence, using (19.5.10), we find that

$$\begin{aligned}\int_0^\infty t^{k/2} J_k(2\sqrt{2\pi n t})\sin\left(\frac{\alpha t}{2\pi}\right)dt &= 2\int_0^\infty u^{k+1} J_k(2\sqrt{\mu_n}\,u)\sin\left(\frac{\alpha u^2}{2\pi}\right)du \\ &= \frac{(2\pi)^{3k/2+1}n^{k/2}}{\alpha^{k+1}}\cos\left(\frac{4\pi^2 n}{\alpha} - \frac{k\pi}{2}\right) \\ &= (-1)^{(k-1)/2}\frac{(2\pi)^{3k/2+1}n^{k/2}}{\alpha^{k+1}}\sin\left(\frac{4\pi^2 n}{\alpha}\right) \\ &= (-1)^{(k-1)/2}\frac{(2\pi)^{3k/2+1}n^{k/2}}{\alpha^{k+1}}\sin(\beta n),\end{aligned} \tag{19.5.11}$$

since $\alpha\beta = 4\pi^2$.

With the preliminary details out of the way, we are now ready to apply the Voronoï summation formula (19.5.3). Using the calculations (19.5.9) and (19.5.11) and the parameters defined above, we formally find that

$$\sum_{n=1}^{\infty} \sigma_k(n)\sin(\alpha n) = \int_0^{\infty} \left(\frac{\zeta(1-k)}{2\pi} + \frac{\zeta(k+1)t^k}{(2\pi)^{k+1}} \right) \sin\left(\frac{\alpha t}{2\pi}\right) dt - \left(\frac{2\pi}{\alpha}\right)^{k+1} \sum_{n=1}^{\infty} \sigma_k(n)\sin(\beta n). \tag{19.5.12}$$

Thus, if we replace s by k in (19.2.2) and assume that k is an odd integer, then (19.5.12) is as close as we can get in our efforts to formally derive (19.2.2). Note that on the right side of (19.5.12) a minus sign appears, in contrast to the right side of (19.2.2), and that a divergent integral appears on the right-hand side of (19.5.12) in place of the expressions involving the Riemann zeta function appearing in (19.2.2).

We now turn to (19.2.1). Observe that the infinite series on the left-hand side are reminiscent of the *finite* Riesz sums $\sum_{n\leq x} \sigma_s(n)(x-n)^r$, for which identities have been derived by, for example, A. Oppenheim [239] and A. Laurinčikas [209]. Once more, we make an application of the Voronoï summation formula. Note that the series on the left-hand side of (19.2.1) does not converge for any real value of s, since $\sigma_s(n) \geq n^s$. Also note that for x sufficiently large and for $\sigma > \frac{1}{2}$, each expression in (19.2.1) tends to 0 as x tends to ∞, except for $-2\pi\sqrt{\pi x}\zeta(-s)\tan\frac{1}{2}\pi s$, which tends to ∞.

To effect our application of Theorem 19.5.1, we need the integral evaluation [126, p. 709, formula 6.565, no. 2]

$$\int_0^{\infty} x^{\nu+1} J_\nu(bx)(x^2+a^2)^{-\nu-1/2}dx = \frac{\sqrt{\pi}\, b^{\nu-1}}{2e^{ab}\Gamma(\nu+\frac{1}{2})}, \tag{19.5.13}$$

where $\operatorname{Re} a > 0$, $b > 0$, $\operatorname{Re}\nu > -\frac{1}{2}$, and $J_\nu(x)$ denotes the ordinary Bessel function of order ν. Apply the Voronoï summation formula (19.5.3) twice, with $a = 0$, $x = \infty$, and $f(t) = (x \mp it)^{-k-1/2}$, under the same conditions (19.5.6) and (19.5.7) as in our previous application. We do not provide further details but invite readers to consult the paper by Berndt, O.-Y. Chan, S.-G. Lim, and A. Zaharescu [48], where the remainder of the failed proof can be found. We eventually then arrive at the "identity"

$$\sum_{n=1}^{\infty}\sigma_k(n)\left\{(x-in)^{-k-1/2} - (x+in)^{-k-1/2}\right\} = \int_0^{\infty} \left(\frac{\zeta(1-k)}{2\pi} + \frac{\zeta(k+1)t^k}{(2\pi)^{k+1}} \right) \left((x-it)^{-k-1/2} - (x+it)^{-k-1/2}\right) dt - \frac{i\sqrt{2}}{\Gamma(k+\frac{1}{2})} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{\sqrt{n}} e^{-2\sqrt{\pi n x}} \sin\left(2\sqrt{\pi n x} + \frac{1}{4}\pi\right), \tag{19.5.14}$$

which should be compared with (19.2.1). Observe that the integral on the right-hand side of (19.5.14) diverges, although it can be subdivided into two improper integrals, one of which converges and is elementary, and the other of which diverges.

19.6 Fourth Attempt: Mellin Transforms

Another effort to prove Entries 19.2.1 and 19.2.2 has utilized Mellin transforms. We refer readers to the aforementioned paper by Berndt, Chan, Lim, and Zaharescu [48] for the details of this failed attempt.

19.7 An Integral on Page 197

Entry 19.7.1 (p. 197). *Let* $n > 0$ *and let* $t > 0$*. Then*

$$\int_0^\infty \frac{\sin(\pi t x)}{x\cosh(\pi x)} e^{-i\pi n x^2} dx = \frac{\pi}{2} - 2\sum_{k=0}^\infty (-1)^k \frac{e^{-(2k+1)\pi t/2 + (2k+1)^2 i\pi n/4}}{2k+1}$$
$$- \frac{\pi}{\sqrt{n}} e^{-i\pi/4} \int_0^\infty \sum_{k=0}^\infty (-1)^k e^{(t+u+(2k+1)i)^2 i\pi/(4n)} du. \quad (19.7.1)$$

Ramanujan has a slight misprint in his formulation of (19.7.1) in [269]; he forgot the factor π in the exponents in the summands in the first series on the right-hand side.

Before proving Entry 19.7.1, we state the values of some integrals that we need in our proof. For $a, b > 0$ [126, p. 542, formulas 3.989, nos. 5, 6],

$$\int_0^\infty \frac{\sin(\pi a x^2)\cos(bx)}{\cosh(\pi x)} dx = -\sum_{k=0}^\infty (-1)^k e^{-(2k+1)b/2} \sin\left(\frac{(2k+1)^2 \pi a}{4}\right)$$
$$+ \frac{1}{\sqrt{a}} \sum_{k=0}^\infty (-1)^k e^{-(2k+1)b/(2a)} \sin\left(\frac{\pi}{4} - \frac{b^2}{4\pi a} + \frac{(2k+1)^2\pi}{4a}\right) \quad (19.7.2)$$

and

$$\int_0^\infty \frac{\cos(\pi a x^2)\cos(bx)}{\cosh(\pi x)} dx = \sum_{k=0}^\infty (-1)^k e^{-(2k+1)b/2} \cos\left(\frac{(2k+1)^2 \pi a}{4}\right)$$
$$+ \frac{1}{\sqrt{a}} \sum_{k=0}^\infty (-1)^k e^{-(2k+1)b/(2a)} \cos\left(\frac{\pi}{4} - \frac{b^2}{4\pi a} + \frac{(2k+1)^2\pi}{4a}\right). \quad (19.7.3)$$

In [126], the factor $(-1)^k$ has unfortunately been omitted from both sums in (19.7.2) and from the latter sum in (19.7.3). These formulas, including the mistakes, were copied from the tables of integral transforms [115, p. 36]. Next, for $a > 0$ and $\operatorname{Re} b > 0$ [126, p. 545, formula 4.111, no. 7],

$$\int_0^\infty \frac{\sin(ax)}{x\cosh(bx)} dx = 2\tan^{-1}\left(\exp\frac{\pi a}{2b}\right) - \frac{\pi}{2}. \quad (19.7.4)$$

Proof of Entry 19.7.1. Our uninspiring method of proof is undoubtedly not that used by Ramanujan, because our proof is a verification. We show that the derivatives of both sides of (19.7.1) as functions of t are equal. We then show that the limits of both sides of (19.7.1) as $t \to \infty$ are both equal to $\pi/2$ to conclude the proof. To that end, let $F(t)$ and $G(t)$ denote the left- and right-hand sides of (19.7.1). Then, using (19.7.2) and (19.7.3) with $a = n$ and $b = \pi t$, we find that

$$\begin{aligned} F'(t) &= \pi \int_0^\infty \frac{\sin(\pi t x)}{\cosh(\pi x)} e^{-i\pi n x^2} dx \\ &= \pi \left(\sum_{k=0}^\infty (-1)^k e^{-(2k+1)\pi t/2} e^{i(2k+1)^2 \pi n/4} \right. \qquad (19.7.5) \\ &\quad \left. + \frac{1}{\sqrt{n}} \sum_{k=0}^\infty (-1)^k e^{-(2k=1)\pi t/n} \exp\left(-i \left(\frac{\pi}{4} - \frac{\pi t^2}{4n} + \frac{(2k+1)^2 \pi}{4n} \right) \right) \right). \end{aligned}$$

On the other hand, by easily justified differentiations under the summation and integral signs and an inversion in order of integration and summation by absolute convergence,

$$\begin{aligned} G'(t) &= \pi \sum_{k=0}^\infty (-1)^k e^{-(2k+1)\pi t/2 + (2k+1)^2 \pi i n/4} \\ &\quad - 2 \frac{\pi i}{4n} \frac{\pi}{\sqrt{n}} \sum_{k=0}^\infty (-1)^k \int_0^\infty (t + u + (2k+1)i) e^{(t+u+(2k+1)i)^2 i\pi/(4n)} du \\ &= \pi \sum_{k=0}^\infty (-1)^k e^{-(2k+1)\pi t/2 + (2k+1)^2 \pi i n/4} \\ &\quad + \frac{\pi}{\sqrt{n}} \sum_{k=0}^\infty (-1)^k e^{(t+u+(2k+1)i)^2 i\pi/(4n)}. \qquad (19.7.6) \end{aligned}$$

A comparison of (19.7.5) and (19.7.6) shows that indeed $F'(t) = G'(t)$. So, it remains to show that $F(t)$ and $G(t)$ are equal for some value of t.

We let t tend to ∞ to deduce the desired equality. Because of absolute and uniform convergence with respect to t in a neighborhood of ∞, we can let $t \to \infty$ under the integral and summation signs on the right side of (19.7.1) and readily deduce that

$$\lim_{t\to\infty} G(t) = \frac{\pi}{2}. \qquad (19.7.7)$$

On the other hand, write, with the use of (19.7.4),

$$F(t) = \int_0^\infty \left(\frac{\sin(\pi t x)}{x \cosh(\pi x)} e^{-i\pi n x^2} - \frac{\sin(\pi t x)}{x \cosh(\pi x)} \right) dx + \int_0^\infty \frac{\sin(\pi t x)}{x \cosh(\pi x)} dx$$

$$= \int_0^\infty \sin(\pi t x) \left(\frac{1}{x \cosh(\pi x)} e^{-i\pi n x^2} - \frac{1}{x \cosh(\pi x)} \right) dx \\ + 2 \tan^{-1} \left(e^{\pi t/2} \right) - \frac{\pi}{2}. \qquad (19.7.8)$$

Clearly, the function

$$\frac{1}{x \cosh(\pi x)} e^{-i\pi n x^2} - \frac{1}{x \cosh(\pi x)}$$

is in $L(-\infty, \infty)$. Hence, by (19.7.8) and a standard theorem from the theory of Fourier integrals [305, p. 11],

$$\lim_{t \to \infty} F(t) = 0 + 2\frac{\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2}. \qquad (19.7.9)$$

Thus, we see from (19.7.7) and (19.7.9) that $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} G(t)$, and so the proof is complete. □

19.8 On the Integral $\int_0^x \frac{\sin u}{u} du$

On page 256 in [269], Ramanujan obtains explicit representations for the values of the local maxima and minima of the integral

$$\mathcal{S}(x) := \int_0^x \frac{\sin u}{u} du, \qquad (19.8.1)$$

when $x > 0$. The integral $\mathcal{S}(x)$ is intimately connected with the sine and cosine integrals defined for $x > 0$ by [126, p. 936, formulas 8.230, nos. 1,2]

$$\mathrm{si}(x) := -\int_x^\infty \frac{\sin t}{t} dt \qquad \text{and} \qquad \mathrm{ci}(x) := \int_x^\infty \frac{\cos t}{t} dt. \qquad (19.8.2)$$

Ramanujan first defines r, $r > 0$, and θ, $0 < \theta < \frac{1}{2}\pi$, by

$$r \cos\theta := \int_0^\infty \frac{e^{-xt}}{1+t^2} dt \qquad \text{and} \qquad r \sin\theta := \int_0^\infty \frac{t e^{-xt}}{1+t^2} dt, \qquad (19.8.3)$$

where $x > 0$. His first claim is the following identity.

Entry 19.8.1 (p. 256). *If r is defined by (19.8.3), then*

$$r^2 = \int_0^\infty \frac{e^{-xt}}{t} \log(1+t^2) dt.$$

Proof. From [126, p. 359, formula 3.354, nos. 1,2],

$$\int_0^\infty \frac{e^{-xt}}{1+t^2} dt = \text{ci}(x) \sin x - \text{si}(x) \cos x \tag{19.8.4}$$

and

$$\int_0^\infty \frac{te^{-xt}}{1+t^2} dt = -\text{ci}(x) \cos x - \text{si}(x) \sin x, \tag{19.8.5}$$

where $\text{ci}(x)$ and $\text{si}(x)$ are defined by (19.8.2). Using the definitions (19.8.3) in conjunction with the foregoing identities, we easily see that

$$\begin{aligned} r^2 &= r^2 \cos^2 \theta + r^2 \sin^2 \theta \\ &= \{\text{ci}(x) \sin x - \text{si}(x) \cos x\}^2 + \{-\text{ci}(x) \cos x - \text{si}(x) \sin x\}^2 \\ &= \text{ci}^2(x) + \text{si}^2(x) \\ &= \int_0^\infty \frac{e^{-xt}}{t} \log(1+t^2) dt, \end{aligned}$$

where we have used another integral evaluation from the *Tables* [126, p. 609, formula 4.366, no. 1]. This completes the proof. □

Entry 19.8.2 (p. 256). *If* r *and* θ *are defined by* (19.8.3) *and* $x > 0$, *then*

$$\int_0^x \frac{\sin u}{u} du = \frac{\pi}{2} - r \cos(x - \theta) \tag{19.8.6}$$

and

$$\int_0^x \frac{1 - \cos u}{u} du = \gamma + \log x - r \sin(x - \theta), \tag{19.8.7}$$

where γ *denotes Euler's constant.*

Proof. Again using (19.8.3)–(19.8.5), we easily find that

$$\begin{aligned} r \cos(x - \theta) &= r \cos x \cos \theta + r \sin x \sin \theta \\ &= \cos x \{\text{ci}(x) \sin x - \text{si}(x) \cos x\} \\ &\quad + \sin x \{-\text{ci}(x) \cos x - \text{si}(x) \sin x\} \\ &= - \text{si}(x). \end{aligned} \tag{19.8.8}$$

The result (19.8.6) now follows from the definition (19.8.2) of $\text{si}(x)$.

Next, from [126, p. 936, formula 8.230, no. 2],

$$\int_0^x \frac{1 - \cos u}{u} du = \gamma + \log x - \text{ci}(x). \tag{19.8.9}$$

A comparison of (19.8.9) with (19.8.7) indicates that in order to prove (19.8.7), all we need to do is to show that

$$r \sin(x-\theta) = \mathrm{ci}(x). \tag{19.8.10}$$

The demonstration of (19.8.10) follows along the same lines as the calculation in (19.8.8), and so this completes the proof. □

Entry 19.8.3 (p. 256). *The function* $\mathcal{S}(x)$ *defined in* (19.8.1) *has local maxima at* $x = (2n+1)\pi$, $n \geq 0$, *with the maximum values being*

$$\mathcal{S}(2n+1) = \frac{\pi}{2} + \int_0^\infty \frac{e^{-(2n+1)\pi t}}{1+t^2} dt, \tag{19.8.11}$$

while the local minima are at $x = 2n\pi$, $n \geq 1$, *with the minimum values being*

$$\mathcal{S}(2n) = \frac{\pi}{2} - \int_0^\infty \frac{e^{-2n\pi t}}{1+t^2} dt. \tag{19.8.12}$$

Proof. From elementary calculus, it is trivial to see that the critical points of $\mathcal{S}(x)$ are at $x = n\pi$, $n > 0$, when x is positive. Furthermore, it is easy to see that when n is odd, a local maximum is reached, and when n is even, a local minimum is obtained. Furthermore, from (19.8.6) and (19.8.3),

$$\begin{aligned}\mathcal{S}(2n+1) = \frac{\pi}{2} - r\cos((2n+1)\pi - \theta) &= \frac{\pi}{2} + r\cos\theta \\ &= \frac{\pi}{2} + \int_0^\infty \frac{e^{-(2n+1)\pi t}}{1+t^2} dt,\end{aligned}$$

and so (19.8.11) is established. Similarly, (19.8.6) and (19.8.3) immediately yield (19.8.12). □

19.9 Two Infinite Products

Entry 19.9.1 (p. 370). *If* $|\operatorname{Re}\beta| < 1$, $|\operatorname{Im}\alpha| < 1$, *and*

$$\cosh\left(\tfrac{1}{2}\pi\beta\right) = \sec\left(\tfrac{1}{2}\pi\alpha\right), \tag{19.9.1}$$

then

$$\prod_{n=0}^\infty \left(\frac{(2n+1)^2+\alpha^2}{(2n+1)^2-\beta^2}\right)^{(-1)^n(2n+1)} = e^{\frac{1}{2}\pi\alpha\beta}. \tag{19.9.2}$$

With the roles of α and β reversed, Entry 19.9.1 is identical to equation (17) in Ramanujan's paper [250], [267, p. 41]. See also (17.2.13) of the present volume. In fact, in place of the condition (19.9.1), Ramanujan wrote the hypothesis

$$\frac{\pi\alpha}{2} = gd\left(\frac{\pi\beta}{2}\right).$$

(Possibly, gd denotes the Gudermannian function.) Since Ramanujan only sketched a proof of (19.9.2) in [250], the editors of [267, pp. 336–337] supplied a more detailed proof. An equivalent form of Entry 19.9.1 can be found on page 286 in Ramanujan's second notebook [268], and a proof of Entry 19.9.1 in this form can be found in Berndt's book [41, p. 461, Entry 30]. Lastly, Ramanujan also submitted Entry 19.9.1 as a problem to the *Journal of the Indian Mathematical Society* [248].

Entry 19.9.2 (p. 370 (incorrect)). *For* $|x| < 1$*,*

$$\prod_{n=1}^{\infty}\left\{\left(1+\frac{x}{n^2}\right)^{n^2} e^{-x}\right\} = e^{\frac{1}{2}x}, \tag{19.9.3}$$

provided that

$$x = \left\{\frac{1}{\pi}\log\left(\frac{1+\sqrt{5}}{2}\right)\right\}^2. \tag{19.9.4}$$

If we take the logarithm of both sides of (19.9.3), employ the Maclaurin series for $\log(1-z)$, and interchange the order of summation, we deduce that

$$\sum_{j=2}^{\infty}\frac{(-1)^{j-1}}{j}\zeta(2j-2)x^j = \frac{x}{2}, \tag{19.9.5}$$

where $\zeta(s)$ denotes the Riemann zeta function. Since $\zeta(0) = -\frac{1}{2}$ [306, p. 19], we can rewrite (19.9.5) in the form

$$\sum_{j=0}^{\infty}\frac{(-1)^j}{j+1}\zeta(2j)x^j = 0. \tag{19.9.6}$$

Hence, combining (19.9.6) with (19.9.4), we see that Ramanujan claimed that a root of (19.9.6) is (19.9.4), which, if true, would be a remarkable result.

Unfortunately, Entry 19.9.2 is incorrect. This entry also appears in Ramanujan's third notebook [268, p. 365], and in [41, pp. 488–490] Berndt showed that Ramanujan's claim in Entry 19.9.2 is false. In particular, Ramanujan also claimed on the same page in [268] that for $|x| < 1$,

$$\sum_{j=0}^{\infty}\frac{(-1)^j}{j+1}\zeta(2j)x^{2j+2} = -\frac{1}{\pi^2}\int_0^{\pi x} t^2\coth t\,dt. \tag{19.9.7}$$

If we set

$$x = \frac{1}{\pi}\log\left(\frac{1+\sqrt{5}}{2}\right)$$

above, Ramanujan's claim in Entry 19.9.2 would be equivalent to asserting that the integral on the right side of (19.9.7) equals 0, which is obviously untrue.

19.10 Two Formulas from the Theory of Elliptic Functions

We recall some needed notation from the theory of elliptic functions [39, Chaps. 17, 18, in particular, pp. 101–102]. The incomplete elliptic integral of the first kind is defined, for $0 < \varphi \leq \frac{1}{2}\pi$, by

$$\int_0^{\varphi} \frac{dt}{\sqrt{1-k^2\sin^2 t}}, \tag{19.10.1}$$

where k, $0 < k < 1$, is the modulus. The complementary modulus k' is defined by $k' = \sqrt{1-k^2}$. For brevity, we set $x = k^2$. The complete elliptic integral of the first kind is given by (19.10.1) when $\varphi = \frac{1}{2}\pi$ and is denoted by $K = K(k)$. Define $K' := K'(k) := K(k')$. Then in the theory of elliptic functions, we set

$$q := \exp\left(-\pi\frac{K'}{K}\right) =: e^{-y}. \tag{19.10.2}$$

Define, for $0 < \theta \leq \frac{1}{2}\pi$,

$$\theta = \frac{1}{z}\int_0^{\varphi} \frac{dt}{\sqrt{1-k^2\sin^2 t}}.$$

Incomplete elliptic integrals satisfy Jacobi's imaginary transformation. If $0 < \varphi < \frac{1}{2}\pi$, then

$$\int_0^{i\log(\tan(\pi/4+\varphi/2))} \frac{dt}{\sqrt{1-x\sin^2 t}} = i\int_0^{\varphi} \frac{dt}{\sqrt{1-(1-x)\sin^2 t}}. \tag{19.10.3}$$

Entry 19.10.1 (p. 346). *Set, in the notation above,*

$$\frac{2K\theta}{\pi} = \int_0^{\varphi} \frac{dt}{\sqrt{1-k^2\sin^2 t}}. \tag{19.10.4}$$

Then

$$\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + 4\sum_{n=0}^{\infty}(-1)^n \frac{q^{2n+1}\sin\{(2n+1)\theta\}}{(2n+1)(1-q^{2n+1})} = \log\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right). \tag{19.10.5}$$

Entry 19.10.1 coincides with Entry 16(v) of Chap. 18 of Ramanujan's second notebook [268], and a proof can be found in [39, p. 175].

Entry 19.10.2 (p. 346). *In addition to the notation set above, also put*

$$\frac{2K\theta'}{\pi} = \int_0^{\varphi} \frac{dt}{\sqrt{1 - k'^2 \sin^2 t}}. \tag{19.10.6}$$

Then,

$$\theta' + 2\sum_{n=1}^{\infty} \frac{q^n \sinh(2n\theta')}{n(1+q^{2n})} = \log\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right). \tag{19.10.7}$$

Proof. In the notation above, in particular (19.10.2), in Entry 15(iv) in Chap. 18 of his second notebook [268], Ramanujan claims that

$$\theta + \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n\cosh(ny)} = \varphi; \tag{19.10.8}$$

see [39, pp. 172–173] for a proof. Now in the notation of (19.10.4) and (19.10.6), we will restate (19.10.3) in greater detail, inserting the arguments of the functions θ and θ'. To that end,

$$\theta\left(i\log\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\right) = i\,\theta'(\varphi). \tag{19.10.9}$$

Next, in (19.10.8), we substitute (19.10.9) in the form $\theta = i\theta'$. Keeping in mind that φ is defined by (19.10.6), we see from (19.10.9) that we must also replace φ with $i\log\tan(\pi/4 + \varphi/2)$. Hence,

$$i\,\theta' + \sum_{n=1}^{\infty} \frac{\sin(2ni\theta')}{n\cosh(ny)} = i\log\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right). \tag{19.10.10}$$

Dividing both sides of (19.10.10) by i and recalling from (19.10.2) that $q = e^{-y}$, we deduce (19.10.7). □

Slightly more complicated proofs of the two preceding entries were given by the authors in Part II [13, pp. 238–240].