14

Integral Analogues of Theta Functions and Gauss Sums

14.1 Introduction

In this chapter we discuss a second partial manuscript of two pages [269, pp. 221–222] as well as a related page from the original lost notebook [269, p. 198]. As previously indicated, this manuscript does not belong to the "official" lost notebook of Ramanujan, but instead is among the eight partial manuscripts in G.N. Watson's handwriting that were found in the Oxford University library and that were published along with the lost notebook; the original version for these two pages is in the library at Trinity College, Cambridge. Pages 221 and 222 provide a list of theorems, with no discourse, on integrals that are found in Ramanujan's two papers [256, 258], [267, pp. 59–67] and [194–199]; see also [247]. Indeed, most of the theorems can be found in these two papers, especially [258]. Since Ramanujan did not give many details in these two papers, we shall provide proofs for each claim, whether it is found in these two papers or not.

The objective in the two papers cited above and in the two page fragment is the study of the functions

$$\phi_w(t) := \int_0^\infty \frac{\cos(\pi t x)}{\cosh(\pi x)} e^{-\pi w x^2} dx, \tag{14.1.1}$$

$$\psi_w(t) := \int_0^\infty \frac{\sin(\pi t x)}{\sinh(\pi x)} e^{-\pi w x^2} dx. \tag{14.1.2}$$

It is clear from the definitions (14.1.1) and (14.1.2) that, respectively,

$$\phi_w(t) = \phi_w(-t) \quad \text{and} \quad \psi_w(t) = -\psi_w(-t). \tag{14.1.3}$$

Page 198 of [269] is an isolated page that is actually part of the original lost notebook, and its contents are related to pages 221–222. On this page,

G.E. Andrews and B.C. Berndt, *Ramanujan's Lost Notebook: Part IV*, DOI 10.1007/978-1-4614-4081-9_14,

Ramanujan records theorems, much in the spirit of those for $\phi_w(t)$ and $\psi_w(t)$, for the function

$$F_w(t) := \int_0^\infty \frac{\sin(\pi t x)}{\tanh(\pi x)} e^{-\pi w x^2} dx.$$

The theorems on page 198 are new and were first proved in a paper by Berndt and P. Xu [69].

The functions $\phi_w(t)$, $\psi_w(t)$, and $F_w(t)$ examined in this chapter and (for the former two functions) in [256, 258], [267, pp. 59–67, 202–207] can be regarded as continuous analogues of theta functions, because they each possess a transformation formula like that for the classical theta functions. For example, recall that the classical theta function

$$\theta_3(\tau) := \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}, \quad \operatorname{Im} \tau > 0,$$

satisfies the transformation formula [306, p. 22]

$$\theta_3(-1/\tau) = \sqrt{\tau/i}\, \theta_3(\tau). \tag{14.1.4}$$

On the other hand, because of the appearance of certain sums, which are reminiscent of Gauss sums, in the quasiperiodic relations, for example, in Entries 14.4.2 and 14.4.3, where the quasiperiods are $2i$ and $2w$, respectively, Ramanujan perhaps preferred the analogy with Gauss sums. Recall that the generalized Gauss sum $S(a,b,c)$, where a, b, and c are integers with $ac \neq 0$, is defined by

$$S(a,b,c) := \sum_{n=0}^{|c|-1} e^{\pi i (an^2+bn)/c}.$$

These sums satisfy a reciprocity theorem; namely, if $ac+b$ is even, then [54, p. 13]

$$S(a,b,c) = \sqrt{|c/a|} e^{\pi i \{\operatorname{sgn}(ac) - b^2/(ac)\}/4} S(-c,-b,a).$$

Note that on comparing the two sides of this identity, the roles of a and c are reversed. Moreover, $\sqrt{|c/a|}$ takes the place of $\sqrt{\tau}$ in (14.1.4) or $\sqrt{w}$ in the transformation formulas for $\phi_w(t)$, $\psi_w(t)$, and $F_w(t)$.

Because these functions possess quasiperiods $2i$ and $2w$, they can also be regarded as analogues of elliptic functions. For example, the Weierstrass σ-function is defined by

$$\sigma(z) := \sigma(z;\omega_1,\omega_2) := z \prod_{\omega \neq 0} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} - \frac{z^2}{2\omega^2}\right),$$

where $\omega = m\omega_1 + n\omega_2$, $-\infty < m,n < \infty$, and $\operatorname{Im} \omega_2/\omega_1 > 0$. Set $\omega_3 = \omega_1 + \omega_2$ and $\eta_j = \zeta(\omega_j/2)$, $j = 1,2,3$, where $\zeta(t)$ denotes the Weierstrass ζ-function. Then the Weierstrass σ-function obeys the quasiperiodic relations [88, p. 52]

$$\sigma(z+\omega_j) = -\sigma(z)e^{2\eta_j(z+\omega_j/2)}, \quad j=1,2,3.$$

Interesting analogues of the integrals studied by Ramanujan in [256] and [258] that involve Bessel functions have been derived by N.S. Koshliakov [192]. Those taking the qualifying examination in mathematics at Harvard University in fall 1998, day 2, were asked to evaluate a special case of $\phi_w(t)$.

14.2 Values of Useful Integrals

Throughout our proofs, we appeal to several integral evaluations, all of which can be found in the *Tables* of I.S. Gradshteyn and I.M. Ryzhik [126]. First [126, p. 515, formulas 3.898, nos. 1, 2], for $\operatorname{Re}\beta > 0$,

$$\int_0^\infty e^{-\beta x^2}\sin(ax)\sin(bx)dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left\{e^{-(a-b)^2/(4\beta)} - e^{-(a+b)^2/(4\beta)}\right\}, \tag{14.2.1}$$

$$\int_0^\infty e^{-\beta x^2}\cos(ax)\cos(bx)dx = \frac{1}{4}\sqrt{\frac{\pi}{\beta}}\left\{e^{-(a-b)^2/(4\beta)} + e^{-(a+b)^2/(4\beta)}\right\}. \tag{14.2.2}$$

Second [126, p. 400, formula 3.546, no. 2], for $\operatorname{Re}\beta > 0$,

$$\int_0^\infty e^{-\beta x^2}\cosh(ax)dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}e^{a^2/(4\beta)}. \tag{14.2.3}$$

Third [126, p. 536, formula 3.981, no. 1], for $\operatorname{Re}\beta > 0$ and $a > 0$,

$$\int_0^\infty \frac{\sin(ax)}{\sinh(\beta x)}dx = \frac{\pi}{2\beta}\tanh\left(\frac{a\pi}{2\beta}\right). \tag{14.2.4}$$

Fourth [126, p. 552, formula 4.133, no. 1], for $\operatorname{Re}\gamma > 0$,

$$\int_0^\infty e^{-x^2/(4\gamma)}\sin(ax)\sinh(\beta x)dx = \sqrt{\pi\gamma}e^{\gamma(\beta^2-a^2)}\sin(2a\beta\gamma). \tag{14.2.5}$$

14.3 The Claims in the Manuscript

We now examine in order the claims made by Ramanujan on pages 221 and 222.

Entry 14.3.1 (p. 221). *For $w > 0$,*

$$\phi_w(t) = \frac{1}{\sqrt{w}}e^{-\pi t^2/(4w)}\phi_{1/w}(it/w). \tag{14.3.1}$$

Proof. Using (13.2.21), inverting the order of integration, employing (14.2.2), and simplifying, we find that

$$\begin{aligned}\phi_w(t) &= 2\int_0^\infty \int_0^\infty \frac{\cos(2\pi xz)}{\cosh(\pi z)} \cos(\pi tx) e^{-\pi w x^2} dz dx \\ &= 2\int_0^\infty \frac{dz}{\cosh(\pi z)} \int_0^\infty \cos(2\pi xz)\cos(\pi tx) e^{-\pi w x^2} dx \\ &= 2\int_0^\infty \frac{1}{\cosh(\pi z)} \frac{1}{4}\sqrt{\frac{1}{w}} \left\{ e^{-\frac{(2\pi z-\pi t)^2}{4\pi w}} + e^{-\frac{(2\pi z+\pi t)^2}{4\pi w}} \right\} \\ &= \frac{1}{\sqrt{w}} e^{-\pi t^2/(4w)} \int_0^\infty \frac{\cosh(\pi z t/w)}{\cosh(\pi z)} e^{-\pi z^2/w} dz, \end{aligned} \tag{14.3.2}$$

which is equivalent to (14.3.1). □

A different proof of Entry 14.3.1 has been given by Y. Lee [210].

Entry 14.3.2 (p. 221). *We have*

$$e^{\pi(t+w)^2/(4w)} \phi_w(t+w) = e^{\pi t^2/(4w)} \left(\frac{1}{2} + \psi_w(t) \right). \tag{14.3.3}$$

Proof. First observe from (14.2.3) that

$$\int_0^\infty \cosh(\pi t x/w) e^{-\pi x^2/w} dx = \frac{1}{2}\sqrt{w} e^{\pi t^2/(4w)} \tag{14.3.4}$$

and from (14.2.4) that

$$\int_0^\infty \frac{\sin(2\pi xz)}{\sinh(\pi z)} dz = \frac{1}{2}\tanh(\pi x). \tag{14.3.5}$$

Thus, using (14.3.2), (14.3.4), and (14.3.5), we find that

$$\begin{aligned}&\phi_w(t+w) \\ &= \frac{1}{\sqrt{w}} e^{-\pi(t+w)^2/(4w)} \qquad\qquad (14.3.6) \\ &\quad \times \int_0^\infty \frac{\cosh(\pi t x/w)\cosh(\pi x) + \sinh(\pi t x/w)\sinh(\pi x)}{\cosh(\pi x)} e^{-\pi x^2/w} dx \\ &= \frac{1}{\sqrt{w}} e^{-\pi(t+w)^2/(4w)} \left\{ \frac{1}{2}\sqrt{w} e^{\pi t^2/(4w)} \right. \\ &\qquad \left. + 2\int_0^\infty \int_0^\infty \frac{\sin(2\pi xz)}{\sinh(\pi z)} \sinh(\pi t x/w) e^{-\pi x^2/w} dz\, dx \right\}. \end{aligned}$$

Now, by (14.2.5),

$$\begin{aligned}
2\int_0^\infty\int_0^\infty &\frac{\sin(2\pi xz)}{\sinh(\pi z)}\sinh(\pi tx/w)e^{-\pi x^2/w}dz\,dx\\
&=2\int_0^\infty\frac{dz}{\sinh(\pi z)}\int_0^\infty \sin(2\pi xz)\sinh(\pi tx/w)e^{-\pi x^2/w}dx\\
&=2\int_0^\infty\frac{1}{\sinh(\pi z)}\frac{1}{2}\sqrt{w}e^{\pi t^2/(4w)}e^{-\pi z^2 w}\sin(\pi tz)dz\\
&=\sqrt{w}e^{\pi t^2/(4w)}\int_0^\infty\frac{\sin(\pi tz)}{\sinh(\pi z)}e^{-\pi z^2 w}dz\\
&=\sqrt{w}e^{\pi t^2/(4w)}\psi_w(t).
\end{aligned}$$

If we use this last calculation in (14.3.6) and manipulate slightly, we complete the proof of (14.3.3). □

Entry 14.3.3 (p. 221). *We have*

$$\frac{1}{2}+\psi_w(t+i)=\frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)}\left\{\frac{1}{2}-\psi_{1/w}\left(\frac{it}{w}+i\right)\right\}. \tag{14.3.7}$$

Proof. Rewrite (14.3.3) as

$$\frac{1}{2}+\psi_w(t)=e^{\pi t/2+\pi w/4}\phi_w(t+w). \tag{14.3.8}$$

Thus, using (14.3.8), (14.3.1), (14.1.3), and (14.3.3) with w replaced by $1/w$, we find that

$$\begin{aligned}
\frac{1}{2}&+\psi_w(t+i)=ie^{\pi t/2+\pi w/4}\phi_w(t+i+w)\\
&=ie^{\pi t/2+\pi w/4}\frac{1}{\sqrt{w}}e^{-\pi(t+i+w)^2/(4w)}\phi_{1/w}(i(t+i+w)/w)\\
&=\frac{i}{\sqrt{w}}e^{-\pi(t^2-1+2it+2iw)/(4w)}\phi_{1/w}\left(-\frac{it}{w}-i+\frac{1}{w}\right)\\
&=\frac{i}{\sqrt{w}}e^{-\pi(t^2-1+2it+2iw)/(4w)}e^{-\pi(-it/w-i)/2-\pi/(4w)}\left\{\frac{1}{2}+\psi_{1/w}\left(-\frac{it}{w}-i\right)\right\}\\
&=\frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)}\left\{\frac{1}{2}-\psi_{1/w}\left(\frac{it}{w}+i\right)\right\},
\end{aligned}$$

where in the last step we used (14.1.3). Hence, (14.3.7) has been established. □

Entry 14.3.4 (p. 221). *We have the evaluations*

$$\phi_w(i)=\frac{1}{2\sqrt{w}}, \tag{14.3.9}$$

$$\psi_w(i)=\frac{i}{2\sqrt{w}}, \tag{14.3.10}$$

$$\phi_w(w) = \frac{1}{2}e^{-\pi w/4}, \tag{14.3.11}$$

$$\frac{1}{2} - \psi_w(w) = e^{-\pi w/4}\phi_w(0). \tag{14.3.12}$$

Proof. From the definition (14.1.1),

$$\phi_w(i) = \int_0^\infty e^{-\pi w x^2}dx = \frac{1}{2\sqrt{w}},$$

and from the definition (14.1.2),

$$\psi_w(i) = i\int_0^\infty e^{-\pi w x^2}dx = \frac{i}{2\sqrt{w}}.$$

Next, by the functional equation (14.3.1),

$$\phi_w(w) = \frac{1}{\sqrt{w}}e^{-\pi w/4}\phi_{1/w}(i) = \frac{1}{2}e^{-\pi w/4},$$

upon the use of (14.3.9). Lastly, by (14.3.3) with $t = -w$ and by (14.1.3),

$$e^{\pi w/4}\left\{\frac{1}{2} + \psi_w(-w)\right\} = e^{\pi w/4}\left\{\frac{1}{2} - \psi_w(w)\right\} = \phi_w(0),$$

and so the final assertion (14.3.12) of our entry has been proved. □

Entry 14.3.5 (p. 221). *We have*

$$\phi_w(w \pm i) = \left(\frac{1}{2\sqrt{w}} \mp \frac{i}{2}\right)e^{-\pi w/4}, \tag{14.3.13}$$

$$\psi_w(w \pm i) = \frac{1}{2} \pm \frac{i}{2\sqrt{w}}e^{-\pi w/4}, \tag{14.3.14}$$

$$\phi_w\left(\frac{1}{2}w\right) + \psi_w\left(\frac{1}{2}w\right) = \frac{1}{2}. \tag{14.3.15}$$

Proof. Using (14.3.3), (14.1.3), and (14.3.10) and then simplifying, we find that

$$\phi_w(w\pm i) = e^{-\pi(\pm i+w)^2/(4w)-\pi/(4w)}\left\{\frac{1}{2} + \psi_w(\pm i)\right\} = e^{-\pi w/4}\left\{\mp\frac{i}{2} + \frac{1}{2\sqrt{w}}\right\},$$

which completes the proof of (14.3.13).

Appealing to (14.3.7) with $t = \pm w$ and using (14.1.3), we find that

$$\frac{1}{2} \pm \psi_w(w \pm i) = \frac{i}{\sqrt{w}}e^{-\pi w/4}\left\{\frac{1}{2} - \psi_{1/w}(\pm i + i)\right\}. \tag{14.3.16}$$

We need to distinguish two cases in (14.3.16). First,

$$\begin{aligned}\psi_{1/w}(2i) &= i\int_0^\infty \frac{\sinh(2\pi x)}{\sinh(\pi x)}e^{-\pi x^2/w}dx \\ &= 2i\int_0^\infty \cosh(\pi x)e^{-\pi x^2/w}dx = i\sqrt{w}e^{\pi w/4}, \qquad (14.3.17)\end{aligned}$$

by (14.2.3). Using the calculation from (14.3.17) in (14.3.16) and simplifying, we find that

$$\psi_w(w+i) = \frac{i}{2\sqrt{w}}e^{-\pi w/4} + \frac{1}{2},$$

as claimed in (14.3.14). Second, we observe that trivially $\psi_{1/w}(0) = 0$, and so in the second case, (14.3.16) reduces to

$$\frac{1}{2} - \psi_w(w-i) = \frac{i}{\sqrt{w}}e^{-\pi w/4}\frac{1}{2},$$

which immediately gives the other evaluation in (14.3.14). Third, return to (14.3.3) and set $t = -\frac{1}{2}w$ to deduce that

$$\frac{1}{2} - \psi_w\left(\frac{1}{2}w\right) = \phi_w\left(\frac{1}{2}w\right),$$

which is what we wanted to prove. □

Entry 14.3.6 (p. 221). *We have*

$$\phi_w(t+i) + \phi_w(t-i) = \frac{1}{\sqrt{w}}e^{-\pi t^2/(4w)}, \qquad (14.3.18)$$

$$\psi_w(t+i) - \psi_w(t-i) = \frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)}. \qquad (14.3.19)$$

Proof. Using the definition (14.1.1), elementary trigonometric identities, and (14.2.2), we find that

$$\phi_w(t+i) + \phi_w(t-i) = 2\int_0^\infty \cos(\pi t x)e^{-\pi w x^2}dx = \frac{1}{\sqrt{w}}e^{-\pi t^2/(4w)},$$

as claimed in (14.3.18).

Next, employing (14.1.2), further elementary trigonometric identities, and (14.2.2) once again, we see that

$$\psi_w(t+i) - \psi_w(t-i) = 2i\int_0^\infty \cos(\pi t x)e^{-\pi w x^2}dx = \frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)},$$

which is (14.3.19). □

Entry 14.3.7 (p. 221). *We have*

$$e^{\pi(t+w)^2/(4w)}\phi_w(t+w)+e^{\pi(t-w)^2/(4w)}\phi_w(t-w)=e^{\pi t^2/(4w)}. \tag{14.3.20}$$

Proof. Employing the identity (14.3.3) and the oddness of $\psi(t)$ noted in (14.1.3), we readily find that

$$\begin{aligned}&e^{\pi(t+w)^2/(4w)}\phi_w(t+w)+e^{\pi(t-w)^2/(4w)}\phi_w(t-w)\\&\qquad=e^{\pi t^2/(4w)}\left\{\frac{1}{2}+\psi_w(t)\right\}+e^{\pi t^2/(4w)}\left\{\frac{1}{2}+\psi_w(-t)\right\}=e^{\pi t^2/(4w)},\end{aligned}$$

which is identical to (14.3.20). ☐

Entry 14.3.8 (p. 221). *We have*

$$e^{\pi(t+w)^2/(4w)}\left\{\frac{1}{2}-\psi_w(t+w)\right\}=e^{\pi(t-w)^2/(4w)}\left\{\frac{1}{2}+\psi_w(t-w)\right\}. \tag{14.3.21}$$

Proof. Appealing to (14.3.3) and then using (14.1.3), we find that

$$\begin{aligned}e^{\pi t^2/(4w)}\left\{\frac{1}{2}+\psi_w(t)\right\}&=e^{\pi(t+w)^2/(4w)}\phi_w(t+w)\\&=e^{\pi(t+w)^2/(4w)}\phi_w(-t-w).\end{aligned} \tag{14.3.22}$$

Replacing t by $-t-w$ above and using (14.1.3), we arrive at

$$e^{\pi(t+w)^2/(4w)}\left\{\frac{1}{2}-\psi_w(t+w)\right\}=e^{\pi t^2/(4w)}\phi_w(t).$$

Replacing t by $t-w$ in (14.3.22) and using (14.1.3), we deduce that

$$e^{\pi(t-w)^2/(4w)}\left\{\frac{1}{2}+\psi_w(t-w)\right\}=e^{\pi t^2/(4w)}\phi_w(t).$$

The identity (14.3.21) is now an immediate consequence of the last two identities. ☐

Entry 14.3.9 (p. 221). *If n is any positive integer, then*

$$\phi_w(t)+(-1)^{n+1}\phi_w(t+2ni)=\frac{1}{\sqrt{w}}\sum_{k=0}^{n-1}(-1)^k e^{-\pi(t+(2k+1)i)^2/(4w)}. \tag{14.3.23}$$

Proof. We employ (14.3.18) with t successively replaced by $t+i, t+3i, \dots, t+(2n-1)i$ to deduce the array

$$\phi_w(t+2i)+\phi_w(t)=\frac{1}{\sqrt{w}}e^{-\pi(t+i)^2/(4w)},$$
$$\phi_w(t+4i)+\phi_w(t+2i)=\frac{1}{\sqrt{w}}e^{-\pi(t+3i)^2/(4w)},$$
$$\vdots$$
$$\phi_w(t+2ni)+\phi_w(t+(2n-2)i)=\frac{1}{\sqrt{w}}e^{-\pi(t+(2n-1)i)^2/(4w)}.$$

Alternately adding and subtracting the identities above, we immediately deduce (14.3.23). □

Entry 14.3.10 (p. 221). *If n is any positive integer,*

$$\psi_w(t)-\psi_w(t+2ni)=-\frac{i}{\sqrt{w}}\sum_{k=0}^{n-1}e^{-\pi(t+(2k+1)i)^2/(4w)}. \tag{14.3.24}$$

Proof. We employ (14.3.19) with t successively replaced by $t+i, t+3i, \dots, t+(2n-1)i$, and so record the identities

$$\psi_w(t+2i)-\psi_w(t)=\frac{i}{\sqrt{w}}e^{-\pi(t+i)^2/(4w)},$$
$$\psi_w(t+4i)-\psi_w(t+2i)=\frac{i}{\sqrt{w}}e^{-\pi(t+3i)^2/(4w)},$$
$$\vdots$$
$$\psi_w(t+2ni)-\psi_w(t+(2n-2)i)=\frac{i}{\sqrt{w}}e^{-\pi(t+(2n-1)i)^2/(4w)}.$$

Adding the identities above, we deduce (14.3.24) forthwith. □

Entry 14.3.11 (p. 221). *For any positive integer n,*

$$\begin{aligned}e^{\pi t^2/(4w)}\phi_w(t)&+(-1)^{n+1}e^{\pi(t+2nw)^2/(4w)}\phi_w(t+2nw)\\&=\sum_{k=0}^{n-1}(-1)^k e^{\pi(t+(2k+1)w)^2/(4w)}.\end{aligned} \tag{14.3.25}$$

Proof. We return to (14.3.20) and successively replace t by $t+w, t+3w, \dots, t+(2n-1)w$ to deduce the n equations

$$e^{\pi(t+2w)^2/(4w)}\phi_w(t+2w)+e^{\pi t^2/(4w)}\phi_w(t)=e^{\pi(t+w)^2/(4w)},$$
$$e^{\pi(t+4w)^2/(4w)}\phi_w(t+4w)+e^{\pi(t+2w)^2/(4w)}\phi_w(t+2w)=e^{\pi(t+3w)^2/(4w)},$$
$$\vdots$$
$$\begin{aligned}e^{\pi(t+2nw)^2/(4w)}\phi_w(t+2nw)&+e^{\pi(t+(2n-2)w)^2/(4w)}\phi_w(t+(2n-2)w)\\&=e^{\pi(t+(2n-1)w)^2/(4w)}.\end{aligned}$$

If we now alternately add and subtract the identities above, we readily deduce (14.3.25). □

Entry 14.3.12 (p. 221). *For any positive integer* n,

$$e^{\pi t^2/(4w)}\left\{\frac{1}{2}+\psi_w(t)\right\}+(-1)^{n+1}e^{\pi(t+2nw)^2/(4w)}\left\{\frac{1}{2}+\psi_w(t+2nw)\right\}$$
$$=\sum_{k=1}^{n}(-1)^{k-1}e^{\pi(t+2kw)^2/(4w)}. \tag{14.3.26}$$

Proof. We apply (14.3.21) with t successively replaced by $t+w, t+3w, \dots, t+(2n-1)w$ in order to derive the set of equations

$$e^{\pi(t+2w)^2/(4w)}\left\{\frac{1}{2}-\psi_w(t+2w)\right\}=e^{\pi t^2/(4w)}\left\{\frac{1}{2}+\psi_w(t)\right\},$$
$$e^{\pi(t+4w)^2/(4w)}\left\{\frac{1}{2}-\psi_w(t+4w)\right\}=e^{\pi(t+2w)^2/(4w)}\left\{\frac{1}{2}+\psi_w(t+2w)\right\},$$
$$\vdots$$
$$e^{\pi(t+2nw)^2/(4w)}\left\{\frac{1}{2}-\psi_w(t+2nw)\right\}$$
$$=e^{\pi(t+(2n-2)w)^2/(4w)}\left\{\frac{1}{2}+\psi_w(t+(2n-2)w)\right\}.$$

We now alternately add and subtract these identities to achieve (14.3.26). □

Entry 14.3.13 (p. 222). *Let* m *and* n *denote any positive integers and set* $s=t+2mw\pm 2ni$. *Then*

$$\phi_w(s)+(-1)^{(m+1)(n+1)}e^{-\frac{1}{2}\pi m(s+t)}\phi_w(t)$$
$$=e^{-\pi s^2/(4w)}\sum_{k=0}^{m-1}(-1)^k e^{\pi(s-(2k+1)w)^2/(4w)}$$
$$+\frac{(-1)^{(m+1)(n+1)}}{\sqrt{w}}e^{-\frac{1}{2}\pi m(s+t)}\sum_{k=0}^{n-1}(-1)^k e^{-\pi(t\pm(2k+1)i)^2/(4w)}. \tag{14.3.27}$$

Proof. We first observe that an analogue to Entry 14.3.9 can be obtained by beginning the proof with the relation

$$\phi_w(t)+\phi_w(t-2i)=\frac{1}{\sqrt{w}}e^{-\pi(t-i)^2/(4w)}.$$

Proceeding as before, we can then deduce that

$$\phi_w(t)+(-1)^{n+1}\phi_w(t-2ni)=\frac{1}{\sqrt{w}}\sum_{k=0}^{n-1}(-1)^k e^{-\pi(t-(2k+1)i)^2/(4w)}. \tag{14.3.28}$$

We apply Entry 14.3.11 with n replaced by m, where m is a positive integer, and then with t replaced by $t \pm 2ni$, where n is a positive integer. After rearranging and using the definition of s, we find that

$$\begin{aligned}
&\phi_w(s) + (-1)^{m+1} e^{-\pi m s + \pi m^2 w} \phi_w(t \pm 2ni) \\
&\qquad = \phi_w(s) + (-1)^{m+1} e^{-\pi s^2/(4w) + \pi (t \pm 2ni)^2/(4w)} \phi_w(t \pm 2ni) \\
&\qquad = (-1)^{m+1} e^{-\pi s^2/(4w)} \sum_{j=1}^{m} (-1)^{j-1} e^{\pi (t \pm 2ni + (2j-1)w)^2/(4w)} \\
&\qquad = e^{-\pi s^2/(4w)} \sum_{r=0}^{m-1} (-1)^r e^{\pi (s-(2r+1)w)^2/(4w)}, \qquad (14.3.29)
\end{aligned}$$

where we changed the index of summation by setting $j = m - r$.

Next, we apply Entry 14.3.9 and its analogue (14.3.28) to see that

$$(-1)^{n+1} \phi_w(t \pm 2ni) + \phi_w(t) = \frac{1}{\sqrt{w}} \sum_{k=0}^{n-1} (-1)^k e^{-\pi (t \pm (2k+1)i)^2/(4w)}.$$

Upon multiplying both sides by

$$(-1)^{(m+1)(n+1)} e^{-\frac{1}{2}\pi m (s+t)},$$

we find that

$$\begin{aligned}
&(-1)^{n+1+(m+1)(n+1)} e^{-\frac{1}{2}\pi m (s+t)} \phi_w(t \pm 2ni) + (-1)^{(m+1)(n+1)} e^{-\frac{1}{2}\pi m (s+t)} \phi_w(t) \\
&\qquad = \frac{(-1)^{(m+1)(n+1)} e^{-\frac{1}{2}\pi m (s+t)}}{\sqrt{w}} \sum_{k=0}^{n-1} (-1)^k e^{-\pi (t \pm (2k+1)i)^2/(4w)}. \qquad (14.3.30)
\end{aligned}$$

We now add (14.3.29) and (14.3.30) and observe, with the aid of the definition of s, that the coefficient of $\phi_w(t \pm 2ni)$ is equal to

$$(-1)^{m+1} e^{-\pi m s + \pi m^2 w} + (-1)^{m(n+1)} e^{-\frac{1}{2}\pi m (s+t)} = 0. \qquad (14.3.31)$$

We thus immediately obtain (14.3.27) to complete the proof. □

Entry 14.3.14 (p. 222). *Let m and n denote positive integers. Then, if $s = 2mw \pm 2ni$,*

$$\begin{aligned}
&\frac{1}{2} - \psi_w(s) + (-1)^{mn+m+1} e^{-\frac{1}{2}\pi m (s+t)} \left\{ \frac{1}{2} - \psi_w(t) \right\} \\
&\qquad = e^{-\pi s^2/(4w)} \sum_{j=1}^{m} (-1)^{j-1} e^{\pi (s - 2jw)^2/(4w)} \\
&\qquad\quad \pm \frac{(-1)^{mn+m+1} i}{\sqrt{w}} e^{-\frac{1}{2}\pi m (s+t)} \sum_{j=0}^{n-1} e^{-\pi (t \pm (2j+1)i)^2/(4w)}. \qquad (14.3.32)
\end{aligned}$$

Proof. If we examine the proof of Entry 14.3.10, we see that we can obtain an analogue of (14.3.24), just as we previously obtained (14.3.28), except that now the right-hand side is multiplied by -1. Hence,

$$\psi_w(t) - \psi_w(t \pm 2ni) = \mp \frac{i}{\sqrt{w}} \sum_{j=0}^{n-1} e^{-\pi(t+(2j+1)i)^2/(4w)}. \tag{14.3.33}$$

We apply Entry 14.3.12 with n replaced by m, and then with t replaced by $t \pm 2ni$. Next multiply both sides by $(-1)^{m+1}e^{-\pi s^2/(4w)}$. Setting also $j = m+1-r$ below, we find that

$$\begin{aligned}\frac{1}{2} + \psi_w(s) &+ (-1)^{m+1} e^{\pi(t\pm 2ni)^2/(4w) - \pi s^2/(4w)} \left\{ \frac{1}{2} + \psi_w(t \pm 2ni) \right\} \\ &= (-1)^{m+1} e^{-\pi s^2/(4w)} \sum_{j=1}^{m} (-1)^{j-1} e^{\pi(t \pm 2ni + 2jw)^2/(4w)} \\ &= -e^{-\pi s^2/(4w)} \sum_{r=1}^{m} (-1)^r e^{\pi(s+2(1-r)w)^2/(4w)} \\ &= 1 - e^{-\pi s^2/(4w)} \sum_{r=2}^{m} (-1)^r e^{\pi(s+2(1-r)w)^2/(4w)}.\end{aligned}$$

Rearranging, we deduce that

$$\begin{aligned}\frac{1}{2} - \psi_w(s) &- (-1)^{m+1} e^{\pi(t\pm 2ni)^2/(4w) - \pi s^2/(4w)} \left\{ \frac{1}{2} + \psi_w(t \pm 2ni) \right\} \\ &= e^{-\pi s^2/(4w)} \sum_{r=2}^{m} (-1)^r e^{\pi(s+2(1-r)w)^2/(4w)}.\end{aligned} \tag{14.3.34}$$

Observe that with the definition of s,

$$(-1)^{m+1} e^{\pi(t\pm 2ni)^2/(4w) - \pi s^2/(4w)} = (-1)^{m+1} e^{-\pi m s + \pi m^2 w}. \tag{14.3.35}$$

We also observe that if $r = m+1$, the corresponding expression (including $e^{-\pi s^2/(4w)}$) on the right-hand side of (14.3.34) is also equal to the right-hand side of (14.3.35). We add this expression to both sides of (14.3.34) and replace r by $r+1$, so that we can rewrite (14.3.34) in the form

$$\begin{aligned}\frac{1}{2} - \psi_w(s) &+ (-1)^{m+1} e^{-\pi m s + \pi m^2 w} \left\{ \frac{1}{2} - \psi_w(t \pm 2ni) \right\} \\ &= e^{-\pi s^2/(4w)} \sum_{r=1}^{m} (-1)^{r-1} e^{\pi(s-2rw)^2/(4w)}.\end{aligned} \tag{14.3.36}$$

Multiply both sides of (14.3.33) by

$$-(-1)^{mn+m+1}e^{-\frac{1}{2}\pi m(s+t)}$$

to deduce that

$$(-1)^{mn+m+1}e^{-\frac{1}{2}\pi m(s+t)}\left(\left\{\frac{1}{2}-\psi_w(t)\right\}-\left\{\frac{1}{2}-\psi_w(t\pm 2ni)\right\}\right)$$
$$=\pm\frac{(-1)^{mn+m+1}i}{\sqrt{w}}e^{-\frac{1}{2}\pi m(s+t)}\sum_{j=0}^{n-1}e^{-\pi(t+(2j+1)i)^2/(4w)}. \qquad (14.3.37)$$

We now add (14.3.36) and (14.3.37). Observe that, with the definition of s, the coefficient of $\frac{1}{2}-\psi_w(t\pm 2ni)$ equals

$$(-1)^{m+1}e^{-\pi ms+\pi m^2 w}-(-1)^{mn+m+1}e^{-\frac{1}{2}\pi m(s+t)}=0,$$

by the same calculation as in (14.3.31). We thus immediately deduce (14.3.32) to complete the proof. □

Entry 14.3.15 (p. 222). *Let $t = mw \pm ni$, where m and n are positive integers. If m is odd and n is odd, or if m is even and n is odd, or if m is odd and n is even, then*

$$\phi_w(t)=\frac{1}{2}e^{-\pi t^2/(4w)}\sum_{j=0}^{m-1}(-1)^j e^{\pi(t-(2j+1)w)^2/(4w)}$$
$$+\frac{1}{2\sqrt{w}}\sum_{j=0}^{n-1}(-1)^j e^{\pi(t\mp(2j+1)i)^2/(4w)}. \qquad (14.3.38)$$

Proof. In Entry 14.3.13, replace t by $-t$ and then set $s=t$. Thus, t has the form stated in the present entry. In all three cases, (14.3.27) readily reduces to (14.3.38). □

Entry 14.3.16 (p. 222). *Let $t = mw \pm ni$, where m and n are positive integers. If m is odd and n is odd, or if m is even and n is odd, or if m is even and n is even, then*

$$\psi_w(t)=-\frac{1}{2}e^{-\pi t^2/(4w)}\sum_{j=1}^{m}(-1)^{j-1}e^{\pi(t-2jw)^2/(4w)}$$
$$\pm\frac{i}{2\sqrt{w}}\sum_{j=0}^{n-1}e^{\pi(t\mp(2j+1)i)^2/(4w)}. \qquad (14.3.39)$$

Proof. The proof is similar to the previous proof. In Entry 14.3.14, replace t by $-t$ and then set $s=t$. In all three cases, (14.3.32) simplifies to (14.3.39). □

We quote Ramanujan for the last claim on page 222 of [269]. If $t = mw \pm ni$, then

$$\phi_w(t) = e^{-\frac{1}{4}\pi m^2 w} \left\{ \frac{1}{2} \left(\frac{1}{\sqrt{w}} + e^{\mp\frac{1}{2}\pi imn} \right) \sin \tfrac{1}{2}\pi m ?. \right. \tag{14.3.40}$$

Evidently, the presence of the question mark indicates that Ramanujan was unsure of his claim and that further terms (possibly unknown to Ramanujan) were needed to complete the identity. As (14.3.40) is presently stated, it is not true in general. For example, if $m = 2$ and $n = 1$, (14.3.40) is false.

14.4 Page 198

Page 198 in the lost notebook is devoted to properties of the function

$$F_w(t) := \int_0^\infty \frac{\sin(\pi tx)}{\tanh(\pi x)} e^{-\pi w x^2} dx. \tag{14.4.1}$$

The formulas claimed by Ramanujan on page 198 are difficult to read, partly because the original page was perhaps a thin, colored piece of paper, for example, a piece of parchment paper, that was difficult to photocopy.

It is clear from the definition (14.4.1) that

$$F_w(t) = -F_w(-t). \tag{14.4.2}$$

Entry 14.4.1 (p. 198). *We have*

$$F_w(t) = -\frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} F_{1/w}(it/w). \tag{14.4.3}$$

Proof. Write

$$\begin{aligned} F_w(t) &= \int_0^\infty \frac{\sin(\pi tx)\cosh(\pi x)}{\sinh(\pi x)} e^{-\pi w x^2} dx \\ &= \int_0^\infty \frac{\sin(\pi tx)\cos(i\pi x)}{\sinh(\pi x)} e^{-\pi w x^2} dx \\ &= \frac{1}{2} \int_0^\infty \frac{\sin(t+i)\pi x + \sin(t-i)\pi x}{\sinh(\pi x)} e^{-\pi w x^2} dx \\ &= \frac{1}{2} \left\{ \psi_w(t+i) + \psi_w(t-i) \right\}, \end{aligned} \tag{14.4.4}$$

by (14.1.2). Recall from (14.3.7) that

$$\frac{1}{2} + \psi_w(t+i) = \frac{i}{\sqrt{w}} e^{-\pi t^2/(4w)} \left\{ \frac{1}{2} - \psi_{1/w}\left(\frac{it}{w} + i \right) \right\}. \tag{14.4.5}$$

Since $\psi(t)$ is odd, we find from (14.4.5) that

$$-\frac{1}{2}+\psi_w(t-i) = -\frac{1}{2}-\psi_w(-t+i) = -\frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)}\left\{\frac{1}{2}-\psi_{1/w}\left(-\frac{it}{w}+i\right)\right\}. \tag{14.4.6}$$

Hence, from (14.4.4)–(14.4.6),

$$\begin{aligned}
F_w(t) &= \frac{1}{2}\left\{\frac{1}{2}+\psi_w(t+i)-\frac{1}{2}+\psi_w(t-i)\right\}\\
&= \frac{1}{2}\left(\frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)}\left\{\frac{1}{2}-\psi_{1/w}\left(\frac{it}{w}+i\right)\right\}\right.\\
&\qquad\left.-\frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)}\left\{\frac{1}{2}-\psi_{1/w}\left(-\frac{it}{w}+i\right)\right\}\right)\\
&= \frac{i}{2\sqrt{w}}e^{-\pi t^2/(4w)}\left(-\psi_{1/w}\left(\frac{it}{w}+i\right)+\psi_{1/w}\left(-\frac{it}{w}+i\right)\right)\\
&= -\frac{i}{2\sqrt{w}}e^{-\pi t^2/(4w)}\left(\psi_{1/w}\left(\frac{it}{w}+i\right)+\psi_{1/w}\left(\frac{it}{w}-i\right)\right)\\
&= -\frac{i}{\sqrt{w}}e^{-\pi t^2/(4w)}F_{1/w}(it/w),
\end{aligned}$$

by (14.4.4), and this completes the proof. □

Entry 14.4.2 (p. 198). *If n is any positive integer, then*

$$F_w(t)-F_w(t+2ni) = -\frac{i}{\sqrt{w}}\sum_{j=0}^{n}{}' e^{-\pi(t+2ji)^2/(4w)}, \tag{14.4.7}$$

where the prime $'$ on the summation sign indicates that the terms with $j=0,n$ are to be multiplied by $\frac{1}{2}$.

Proof. Recall from (14.4.4) that

$$F_w(t) = \frac{1}{2}\{\psi_w(t+i)+\psi_w(t-i)\}, \tag{14.4.8}$$

and so

$$\begin{aligned}
&F_w(t)-F_w(t+2ni)\\
&\quad= \frac{1}{2}\{\psi_w(t+i)-\psi_w(t+(2n+1)i)\}+\frac{1}{2}\{\psi_w(t-i)-\psi_w(t+(2n-1)i)\}.
\end{aligned}$$

Applying Entry 14.3.10 on the right side above, we see that

$$\begin{aligned}
&F_w(t) - F_w(t+2ni)\\
&\quad= \frac{1}{2}\left\{-\frac{i}{\sqrt{w}}\sum_{k=0}^{n-1} e^{-\pi(t+(2k+2)i)^2/(4w)} - \frac{i}{\sqrt{w}}\sum_{k=0}^{n-1} e^{-\pi(t+2ki)^2/(4w)}\right\}\\
&\quad= -\frac{i}{\sqrt{w}}\sum_{j=0}^{n}{}' e^{-\pi(t+2ji)^2/(4w)}.
\end{aligned}$$

This concludes the proof. □

Entry 14.4.3 (p. 198). *If n is a positive integer, then*

$$F_w(t) - e^{\pi n(t+nw)}F_w(t+2nw) = -e^{-\pi t^2/(4w)}\sum_{j=0}^{n}{}' e^{\pi(t+2jw)^2/(4w)}, \quad (14.4.9)$$

where the prime on the summation sign has the same meaning as in Entry 14.4.2.

Proof. Replacing t by $t+i$ and $t-i$ in Entry 14.3.8, we deduce, respectively, that

$$\begin{aligned}
&e^{\pi(t+i+w)^2/(4w)}\psi_w(t+i+w) + e^{\pi(t+i-w)^2/(4w)}\psi_w(t+i-w)\\
&\qquad= \frac{1}{2}\left(e^{\pi(t+i+w)^2/(4w)} - e^{\pi(t+i-w)^2/(4w)}\right) \quad (14.4.10)
\end{aligned}$$

and

$$\begin{aligned}
&e^{\pi(t-i+w)^2/(4w)}\psi_w(t-i+w) + e^{\pi(t-i-w)^2/(4w)}\psi_w(t-i-w)\\
&\qquad= \frac{1}{2}\left(e^{\pi(t-i+w)^2/(4w)} - e^{\pi(t-i-w)^2/(4w)}\right). \quad (14.4.11)
\end{aligned}$$

Now observe that $e^{4\pi i(t+w)/(4w)} = e^{4\pi i(t-w)/(4w)}$. We multiply $e^{\pi(t-i+w)^2/(4w)}$ in its two appearances in (14.4.11) by $e^{4\pi i(t+w)/(4w)}$, and we multiply $e^{\pi(t-i-w)^2/(4w)}$ in its two appearances in (14.4.11) by $e^{4\pi i(t-w)/(4w)}$. Thus, (14.4.11) can be recast in the form

$$\begin{aligned}
&e^{\pi(t+i+w)^2/(4w)}\psi_w(t-i+w) + e^{\pi(t+i-w)^2/(4w)}\psi_w(t-i-w)\\
&\qquad= \frac{1}{2}\left(e^{\pi(t+i+w)^2/(4w)} - e^{\pi(t+i-w)^2/(4w)}\right). \quad (14.4.12)
\end{aligned}$$

Using (14.4.8), (14.4.10), and (14.4.12), we find that

$$
\begin{aligned}
&e^{\pi(t+i+w)^2/(4w)}F_w(t+w)+e^{\pi(t+i-w)^2/(4w)}F_w(t-w)\\
&\quad=\frac{1}{2}\Big\{e^{\pi(t+i+w)^2/(4w)}\psi_w(t+i+w)+e^{\pi(t+i+w)^2/(4w)}\psi_w(t-i+w)\\
&\qquad+e^{\pi(t+i-w)^2/(4w)}\psi_w(t+i-w)+e^{\pi(t+i-w)^2/(4w)}\psi_w(t-i-w)\Big\}\\
&\quad=\frac{1}{2}\left(e^{\pi(t+i+w)^2/(4w)}-e^{\pi(t+i-w)^2/(4w)}\right).
\end{aligned}
\tag{14.4.13}
$$

We now apply (14.4.13) with t successively replaced by $t+w$, $t+3w$, $\dots$, $t+(2n-1)w$ to deduce the n equations

$$
\begin{aligned}
&e^{\pi(t+i+2w)^2/(4w)}F_w(t+2w)+e^{\pi(t+i)^2/(4w)}F_w(t)\\
&\qquad\qquad=\frac{1}{2}\left(e^{\pi(t+i+2w)^2/(4w)}-e^{\pi(t+i)^2/(4w)}\right),\\
&e^{\pi(t+i+4w)^2/(4w)}F_w(t+4w)+e^{\pi(t+i+2w)^2/(4w)}F_w(t+2w)\\
&\qquad\qquad=\frac{1}{2}\left(e^{\pi(t+i+4w)^2/(4w)}-e^{\pi(t+i+2w)^2/(4w)}\right),\\
&\qquad\qquad\vdots\\
&e^{\pi(t+i+2nw)^2/(4w)}F_w(t+2nw)+e^{\pi(t+i+(2n-2)w)^2/(4w)}F_w(t+(2n-2)w)\\
&\qquad\qquad=\frac{1}{2}\left(e^{\pi(t+i+2nw)^2/(4w)}-e^{\pi(t+i+(2n-2)w)^2/(4w)}\right).
\end{aligned}
$$

Alternately adding and subtracting the identities above, we conclude that

$$
\begin{aligned}
&e^{\pi(t+i)^2/(4w)}F_w(t)+(-1)^{n+1}e^{\pi(t+i+2nw)^2/(4w)}F_w(t+2nw)\\
&\qquad=\sum_{j=0}^{n}{}'(-1)^{j+1}e^{\pi(t+i+2jw)^2/(4w)},
\end{aligned}
$$

that is to say,

$$
F_w(t)-e^{\pi n(t+nw)}F_w(t+2nw)=-e^{-\pi t^2/(4w)}\sum_{j=0}^{n}{}'e^{\pi(t+2jw)^2/(4w)},
$$

which completes our proof. □

Entry 14.4.4 (p. 198). *Let $s=t+2\eta_1 mw+2\eta_2 ni$, where $\eta_1^2=\eta_2^2=1$, and where m and n are positive integers. Then*

$$
\begin{aligned}
F_w(s)+(-1)^{mn-1}e^{-\frac{1}{2}\pi\eta_1 m(s+t)}F_w(t)&=\eta_1 e^{-\pi s^2/(4w)}\sum_{j=0}^{m}{}'e^{\pi(s-2j\eta_1 w)^2/(4w)}\\
&\quad+\eta_2(-1)^{mn}\frac{i}{\sqrt{w}}e^{-\frac{1}{2}\pi\eta_1 m(s+t)}\sum_{j=0}^{n}{}'e^{-\pi(t+2\eta_2 ji)^2/(4w)},
\end{aligned}
\tag{14.4.14}
$$

where the primes on the summation signs have the same meaning as in the two previous entries.

Proof. If we examine the proof of Entry 14.4.3, we see that we can similarly obtain an expression for $F_w(t) - e^{-\pi n(t-nw)} F_w(t-2nw)$, but with the right-hand side multiplied by -1 and the exponents j in the summands being replaced by $-j$. Thus, we shall apply Entry 14.4.3 and its just described analogue with n replaced by m and t replaced by $t + 2\eta_2 ni$. Note that the right-hand side will be multiplied by η_1, and so we obtain

$$F_w(t+2\eta_2 ni) - e^{\pi\eta_1 m(t+2\eta_2 ni+\eta_1 mw)} F_w(t+2\eta_2 ni+2\eta_1 mw)$$
$$= -\eta_1 e^{-\pi(t+2\eta_2 ni)^2/(4w)} {\sum_{j=0}^{m}}' e^{\pi(t+2\eta_2 ni+2\eta_1 jw)^2/(4w)}.$$

Using the definition of s, we can reformulate the foregoing equality as

$$\begin{aligned} F_w(t+2\eta_2 ni) &- e^{\pi\eta_1 m(s-\eta_1 mw)} F_w(s) \\ &= -\eta_1 e^{-\pi(s-2\eta_1 mw)^2/(4w)} {\sum_{j=0}^{m}}' e^{\pi(s-2\eta_1 mw+2\eta_1 jw)^2/(4w)} \\ &= -\eta_1 e^{-\pi(s-2\eta_1 mw)^2/(4w)} {\sum_{j=0}^{m}}' e^{\pi(s-2\eta_1 jw)^2/(4w)}. \end{aligned} \tag{14.4.15}$$

If we examine the proof of Entry 14.4.2, we see that we can obtain an analogue for $F_w(t) - F_w(t-2ni)$ with the right-hand side now being multiplied by -1 and with the summand exponents j replaced by $-j$. Then if we apply Entry 14.4.2 and its analogue that we just described above to $F_w(t+2\eta_2 ni)$, we must multiply the right-hand side by η_2. Hence, using (14.4.7), its analogue, and (14.4.15), we find that

$$F_w(t) - e^{\pi\eta_1 m(s-\eta_1 mw)} F_w(s)$$
$$= -\eta_1 e^{-\pi(s-2\eta_1 mw)^2/(4w)} {\sum_{j=0}^{m}}' e^{\pi(s-2\eta_1 jw)^2/(4w)} - \eta_2 \frac{i}{\sqrt{w}} {\sum_{j=0}^{n}}' e^{-\pi(t+2\eta_2 ji)^2/(4w)}.$$

Upon multiplying both sides above by $e^{-\pi\eta_1 m(s-\eta_1 mw)}$ and simplifying, we find that

$$F_w(s) + (-1)^{mn-1} e^{-\frac{1}{2}\pi\eta_1 m(s+t)} F_w(t) = -\eta_1 e^{-\pi s^2/(4w)} {\sum_{j=0}^{m}}' e^{\pi(s-2j\eta_1 w)^2/(4w)}$$
$$+ \eta_2 (-1)^{mn} \frac{i}{\sqrt{w}} e^{-\frac{1}{2}\pi\eta_1 m(s+t)} {\sum_{j=0}^{n}}' e^{\pi(t+2\eta_2 ji)^2/(4w)},$$

where we used the fact that

$$(-1)^{mn}e^{-\frac{1}{2}\pi\eta_1 m(s+t)} = e^{-\pi\eta_1 m(s-\eta_1 mw)}.$$

This completes our proof. □

14.5 Examples

If we set $s = t$ in Entry 14.4.4, it follows that $w = -(\eta_2 ni)/(\eta_1 m)$. If we further suppose that both m and n are odd, then (14.4.14) reduces to the identity

$$(1+e^{-\pi\eta_1 mt})F_w(t) = \eta_1 e^{-\pi t^2/(4w)}\sum_{j=0}^{m}{}' e^{\pi(t-2j\eta_1 w)^2/(4w)} - \eta_2\frac{i}{\sqrt{w}}e^{-\pi\eta_1 mt}\sum_{j=0}^{n}{}' e^{-\pi(t+2\eta_2 ji)^2/(4w)}.$$

In the identity above, first let $\eta_1 = 1, \eta_2 = -1$ and multiply both sides by e^{mt}. Second, let $\eta_1 = -1, \eta_2 = 1$ and multiply both sides by e^{-mt}. Replace t by $2t/\pi$ in each identity. We then respectively obtain the two identities

$$\begin{aligned}2\cosh(mt)&\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)}e^{-\frac{\pi nx^2}{m}i}dx\\ &= \frac{1}{2}e^{mt} + e^{(m-2)t+\frac{\pi n}{m}i} + e^{(m-4)t+\frac{4\pi n}{m}i} + \cdots + \frac{1}{2}e^{-mt+\pi mni}\\ &\quad + \sqrt{\frac{m}{n}}\left\{\frac{1}{2}e^{-mt+\left(\frac{mt^2}{\pi n}+\frac{\pi}{4}\right)i} + e^{\left(\frac{2}{n}-1\right)mt+\left[\left(\frac{t^2}{\pi^2}-1\right)\frac{\pi m}{n}+\frac{\pi}{4}\right]i}\right.\\ &\quad\left. + \cdots + \frac{1}{2}e^{mt+\left[\left(\frac{t^2}{\pi^2}-n^2\right)\frac{\pi m}{n}+\frac{\pi}{4}\right]i}\right\}\end{aligned} \tag{14.5.1}$$

and

$$\begin{aligned}2\cosh(mt)&\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)}e^{-\frac{\pi nx^2}{m}i}dx\\ &= -\frac{1}{2}e^{-mt} - e^{(2-m)t+\frac{\pi n}{m}i} - e^{(4-m)t+\frac{4\pi n}{m}i} + \cdots - \frac{1}{2}e^{mt+\pi mni}\\ &\quad - \sqrt{\frac{m}{n}}\left\{\frac{1}{2}e^{mt+\left(\frac{mt^2}{\pi n}+\frac{\pi}{4}\right)i} + e^{\left(1-\frac{2}{n}\right)mt+\left[\left(\frac{t^2}{\pi^2}-1\right)\frac{\pi m}{n}+\frac{\pi}{4}\right]i}\right.\\ &\quad\left. + \cdots + \frac{1}{2}e^{-mt+\left[\left(\frac{t^2}{\pi^2}-n^2\right)\frac{\pi m}{n}+\frac{\pi}{4}\right]i}\right\}.\end{aligned} \tag{14.5.2}$$

Next add (14.5.1) and (14.5.2), divide both sides by 2, and equate the real and imaginary parts on both sides to obtain the two identities

$$\begin{aligned}
&2\cosh(mt)\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)}\cos\frac{\pi n x^2}{m}dx \\
&\quad= \frac{1}{2}\sinh\{mt\} + \sinh\{(m-2)t\}\cos\frac{\pi n}{m} + \sinh\{(m-4)t\}\cos\frac{4\pi n}{m} \\
&\quad\quad + \cdots + \frac{1}{2}\sinh\{-mt\}\cos(\pi m n) \\
&\quad\quad + \sqrt{\frac{m}{n}}\left\{\frac{1}{2}\sinh\{-mt\}\cos\left(\frac{mt^2}{\pi n}+\frac{\pi}{4}\right) + \sinh\left\{\left(\frac{2}{n}-1\right)mt\right\}\right. \\
&\quad\quad\quad\quad \times\cos\left(\left(\frac{t^2}{\pi^2}-1\right)\frac{\pi m}{n}+\frac{\pi}{4}\right) \\
&\quad\quad\quad\quad \left. + \cdots + \frac{1}{2}\sinh\{mt\}\cos\left(\left(\frac{t^2}{\pi^2}-n^2\right)\frac{\pi m}{n}+\frac{\pi}{4}\right)\right\} \qquad (14.5.3)
\end{aligned}$$

and

$$\begin{aligned}
&-2\cosh(mt)\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)}\sin\frac{\pi n x^2}{m}dx \\
&\quad= \sinh\{(m-2)t\}\sin\frac{\pi n}{m} + \sinh\{(m-4)t\}\sin\frac{4\pi n}{m} \\
&\quad\quad + \cdots + \frac{1}{2}\sinh\{-mt\}\sin(\pi m n) \\
&\quad\quad + \sqrt{\frac{m}{n}}\left\{\frac{1}{2}\sinh\{-mt\}\sin\left(\frac{mt^2}{\pi n}+\frac{\pi}{4}\right) + \sinh\left\{\left(\frac{2}{n}-1\right)mt\right\}\right. \\
&\quad\quad\quad\quad \times\sin\left(\left(\frac{t^2}{\pi^2}-1\right)\frac{\pi m}{n}+\frac{\pi}{4}\right) \\
&\quad\quad\quad\quad \left. + \cdots + \frac{1}{2}\sinh\{mt\}\sin\left(\left(\frac{t^2}{\pi^2}-n^2\right)\frac{\pi m}{n}+\frac{\pi}{4}\right)\right\}. \qquad (14.5.4)
\end{aligned}$$

Using (14.5.3) and (14.5.4), we can evaluate several definite integrals. For example, if we set $m = n = 1$ in (14.5.3) and (14.5.4), we find that, respectively,

$$\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)}\cos(\pi x^2)dx = \frac{\sinh t}{2\cosh t}\left(1-\cos\left(\frac{t^2}{\pi}+\frac{\pi}{4}\right)\right)$$

and

$$\int_0^\infty \frac{\sin(2tx)}{\tanh(\pi x)}\sin(\pi x^2)dx = \frac{\sinh t}{2\cosh t}\sin\left(\frac{t^2}{\pi}+\frac{\pi}{4}\right).$$

These evaluations can be found in [126, p. 542, formulas 3.991, nos. 1, 2], respectively. No further cases of (14.5.3) and (14.5.4) can be found in [126].

14.6 One Further Integral

There is one further integral, namely,

$$G_w(t) := \int_0^\infty \frac{\sin(\pi t x)}{\coth(\pi x)} e^{-\pi w x^2} dx,$$

that can be placed in the theory of $\phi_w(t)$, $\psi_w(t)$, and $F_w(t)$. Note that

$$\begin{aligned} G_w(t) &= \int_0^\infty \frac{\sin(\pi t x)\sinh(\pi x)}{\cosh(\pi x)} e^{-\pi w x^2} dx \\ &= -i \int_0^\infty \frac{\sin(\pi t x)\sin(i\pi x)}{\cosh(\pi x)} e^{-\pi w x^2} dx \\ &= \frac{i}{2} \int_0^\infty \frac{\cos\{\pi x(t+i)\} - \cos\{\pi x(t-i)\}}{\cosh(\pi x)} e^{-\pi w x^2} dx \\ &= \frac{i}{2}\{\phi_w(t+i) - \phi_w(t-i)\}, \end{aligned} \tag{14.6.1}$$

by (14.1.1). The formula (14.6.1) should be compared with (14.3.18).