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A Partial Manuscript on Fourier and Laplace Transforms

13.1 Introduction

Pages 219–227 in the volume [269] containing Ramanujan's lost notebook are devoted to material "Copied from the Loose Papers." These "loose papers," in the handwriting of G.N. Watson, are housed in the Oxford University Library, while the original pages in Ramanujan's handwriting, from which the copy was made, are in the library at Trinity College, Cambridge. The three partial manuscripts on these nine pages are in rough form, with two perhaps being drafts of papers being prepared for publication. Most of these nine pages are connected with material in Ramanujan's published papers.

The first manuscript on pages 219–220 is the subject of this chapter. Most of the manuscript is discussed in the next section. Section 13.3 is reserved for the most interesting theorem in the manuscript, namely, a beautiful series transformation involving the logarithmic derivative of the gamma function, which in a second formula, is related to the Riemann zeta function. Our two proofs of this elegant transformation formula are taken from a paper by Berndt and A. Dixit [51]. These two formulas have an interesting history that we relate at the beginning of Sect. 13.3. Since all entries in this chapter can be found on either page 219 or 220 in [269], we refrain from giving page numbers beside entries in the sequel.

13.2 Fourier and Laplace Transforms

Following Ramanujan, we proceed formally without giving attention to such matters as inverting the order of integration in double integrals. It is clear that hypotheses are easily added to make any procedure rigorous.

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Entry 13.2.1. *If*

$$\int_0^\infty f(x)\sin(nx)dx =: \phi(n) \tag{13.2.1}$$

and

$$\int_0^\infty f(x)e^{-nx}dx =: \psi(n), \tag{13.2.2}$$

then

$$\int_0^\infty \phi(x)e^{-nx}dx = \int_0^\infty \psi(x)\cos(nx)dx \tag{13.2.3}$$

and

$$\int_0^\infty \phi\left(\frac{1}{x}\right)e^{-nx}dx = -\int_0^\infty \psi\left(\frac{1}{x}\right)\cos(nx)dx. \tag{13.2.4}$$

Proof. We employ the elementary integral evaluations [126, p. 512, Eqs. (3.893), no. 1, no. 2]

$$\int_0^\infty e^{-nx}\sin(xt)dx = \frac{t}{n^2+t^2}, \qquad n>0, \tag{13.2.5}$$

and

$$\int_0^\infty e^{-nx}\cos(xt)dx = \frac{n}{n^2+t^2}, \qquad n>0. \tag{13.2.6}$$

To prove (13.2.3), we use (13.2.1), (13.2.5), (13.2.6), and (13.2.2) to deduce that

$$\begin{aligned}\int_0^\infty \phi(x)e^{-nx}dx &= \int_0^\infty\int_0^\infty f(t)e^{-nx}\sin(xt)\,dt\,dx\\ &= \int_0^\infty f(t)\int_0^\infty e^{-nx}\sin(xt)\,dx\,dt\\ &= \int_0^\infty f(t)\int_0^\infty e^{-tx}\cos(nx)\,dx\,dt\\ &= \int_0^\infty \psi(x)\cos(nx)dx,\end{aligned}$$

which completes the proof of the first claim.

Using (13.2.1) and making the substitution $t=ux$, we find that

$$\begin{aligned}\int_0^\infty \phi\left(\frac{1}{x}\right)e^{-nx}dx &= \int_0^\infty\int_0^\infty f(t)e^{-nx}\sin(t/x)\,dt\,dx\\ &= \int_0^\infty\int_0^\infty xf(ux)e^{-nx}\sin u\,du\,dx\\ &= -\frac{d}{dn}\int_0^\infty\int_0^\infty f(ux)e^{-nx}\sin u\,dx\,du.\end{aligned} \tag{13.2.7}$$

Note that upon the replacement of n by n/t and x by tu in (13.2.2),

$$\psi\left(\frac{n}{t}\right) = t\int_0^\infty f(tu)e^{-nu}du. \tag{13.2.8}$$

Thus, from (13.2.7) and (13.2.8),

$$\begin{aligned}\int_0^\infty \phi\left(\frac{1}{x}\right)e^{-nx}dx &= -\frac{d}{dn}\int_0^\infty \psi\left(\frac{n}{u}\right)\frac{\sin u}{u}du\\ &= -\frac{d}{dn}\int_0^\infty \psi\left(\frac{1}{x}\right)\frac{\sin(nx)}{x}dx\\ &= -\int_0^\infty \psi\left(\frac{1}{x}\right)\cos(nx)dx,\end{aligned}$$

which completes the proof of (13.2.4). □

Entry 13.2.2. *If*

$$\int_0^\infty f(x)\cos(nx)dx =: \phi(n) \tag{13.2.9}$$

and

$$\int_0^\infty f(x)e^{-nx}dx =: \psi(n), \tag{13.2.10}$$

then

$$\int_0^\infty \phi(x)e^{-nx}dx = \int_0^\infty \psi(x)\sin(nx)dx \tag{13.2.11}$$

and

$$\int_0^\infty \phi\left(\frac{1}{x}\right)e^{-nx}dx = \int_0^\infty \psi\left(\frac{1}{x}\right)\sin(nx)dx. \tag{13.2.12}$$

Proof. The details of the proof of Entry 13.2.2 are completely analogous to those for the proof of Entry 13.2.1, and so there is no need to give them here. □

Suppose now that $f(x)$ is self-reciprocal in Entries 13.2.1 and 13.2.2, that is to say,

$$f(x) = \sqrt{\frac{2}{\pi}}\phi(x).$$

Hence, from (13.2.2),

$$\int_0^\infty \phi(x)e^{-nx}dx = \sqrt{\frac{\pi}{2}}\int_0^\infty f(x)e^{-nx}dx = \sqrt{\frac{\pi}{2}}\psi(n).$$

Then we see that (13.2.3) and (13.2.11) easily yield the next theorem.

Entry 13.2.3. *If*

$$\int_0^\infty \phi(x)\sin(nx)dx = \sqrt{\frac{\pi}{2}}\phi(n)$$

and

$$\int_0^\infty \phi(x)e^{-nx}dx =: \psi(n),$$

then

$$\int_0^\infty \psi(x)\cos(nx)dx = \sqrt{\frac{\pi}{2}}\psi(n). \tag{13.2.13}$$

If

$$\int_0^\infty \phi(x)\cos(nx)dx = \sqrt{\frac{\pi}{2}}\phi(n)$$

and

$$\int_0^\infty \phi(x)e^{-nx}dx =: \psi(n),$$

then

$$\int_0^\infty \psi(x)\sin(nx)dx = \sqrt{\frac{\pi}{2}}\psi(n). \tag{13.2.14}$$

Ramanujan then writes that (13.2.13) and (13.2.14) "enable us to find a number of reciprocal functions of the first and second kind out of one reciprocal function." He does not define what he means by "the first and second kind." Some examples of self-reciprocal functions are next recorded.

Entry 13.2.4. *For* $n > 0$*,*

$$\int_0^\infty \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x}\right)\sin(2\pi nx)dx = \frac{1}{2}\left(\frac{1}{e^{2\pi n}-1} - \frac{1}{2\pi n}\right). \tag{13.2.15}$$

Proof. This result is well known, and we shall be content with quoting from Titchmarsh's *Theory of Fourier Integrals* [305, p. 245]:

$$\begin{aligned}\frac{1}{e^{\sqrt{2\pi}\,x}-1} - \frac{1}{\sqrt{2\pi}\,x} &= \sqrt{\frac{2}{\pi}}\int_0^\infty \left(\frac{1}{e^{\sqrt{2\pi}\,y}-1} - \frac{1}{\sqrt{2\pi}\,y}\right)\sin(xy)dy \\ &= 2\int_0^\infty \left(\frac{1}{e^{2\pi u}-1} - \frac{1}{2\pi u}\right)\sin(\sqrt{2\pi}\,xu)du.\end{aligned}$$

Replacing x by $\sqrt{2\pi}\,n$, we immediately verify Ramanujan's claim. □

It will be convenient to use the familiar notation [126, p. 952, formulas 8.360, 8.362, no. 1]

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \sum_{k=0}^{\infty}\left(\frac{1}{k+x} - \frac{1}{k+1}\right), \tag{13.2.16}$$

where γ denotes Euler's constant. The notation $\psi(x)$ conflicts with the generic notation that we have utilized in Entries 13.2.1 and 13.2.2, but no confusion should arise in the sequel.

Entry 13.2.5. *For $n > 0$,*

$$\int_0^{\infty}\left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x}\right)e^{-2\pi n x}dx = \frac{1}{2\pi}\left(\log n - \psi(1+n)\right). \tag{13.2.17}$$

In the manuscript in [269], a factor of $-1/(2\pi)$ is missing on the right-hand side of (13.2.17).

Proof. We begin with the evaluation [126, p. 377, formula 3.427, no. 7]

$$\int_0^{\infty}\left(\frac{e^{-\nu x}}{1-e^{-x}} - \frac{e^{-\mu x}}{x}\right)dx = \log\mu - \psi(\nu),$$

where $\mu, \nu > 0$. Set $\nu = n+1$ and $\mu = n$ to deduce, after simplification, that

$$\begin{aligned}\log n - \psi(n+1) &= \int_0^{\infty}\left(\frac{1}{e^{x}-1} - \frac{1}{x}\right)e^{-nx}dx\\ &= 2\pi\int_0^{\infty}\left(\frac{1}{e^{2\pi u}-1} - \frac{1}{2\pi u}\right)e^{-2\pi n u}du.\end{aligned}$$

Thus, (13.2.17) is apparent. □

Entry 13.2.6. *If $n > 0$,*

$$\int_0^{\infty}\left(\psi(1+x) - \log x\right)\cos(2\pi n x)dx = \frac{1}{2}\left(\psi(1+n) - \log n\right). \tag{13.2.18}$$

Proof. Setting $u = 2\pi x$ in (13.2.15), we record that

$$\int_0^{\infty}\left(\frac{1}{e^{u}-1} - \frac{1}{u}\right)\sin(nu)du = \pi\left(\frac{1}{e^{2\pi n}-1} - \frac{1}{2\pi n}\right). \tag{13.2.19}$$

Thus, in the notation of Entry 13.2.1,

$$f(x) = \frac{1}{e^{x}-1} - \frac{1}{x} \qquad \text{and} \qquad \phi(n) = \pi\left(\frac{1}{e^{2\pi n}-1} - \frac{1}{2\pi n}\right).$$

By (13.2.17),

$$\begin{aligned}\int_0^{\infty} f(x)e^{-nx}dx &= \int_0^{\infty}\left(\frac{1}{e^{x}-1} - \frac{1}{x}\right)e^{-nx}dx\\ &= 2\pi\int_0^{\infty}\left(\frac{1}{e^{2\pi u}-1} - \frac{1}{2\pi u}\right)e^{-2\pi n u}du\\ &= \log n - \psi(1+n).\end{aligned} \tag{13.2.20}$$

Thus, in the notation of Entry 13.2.1,

$$\int_0^\infty \phi(x)e^{-nx}dx = \pi\int_0^\infty \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x}\right)e^{-nx}dx$$
$$= \int_0^\infty (\log x - \psi(1+x))\cos(nx)dx.$$

Replacing n by $2\pi n$ above, we find that

$$\pi\int_0^\infty \left(\frac{1}{e^{2\pi x}-1} - \frac{1}{2\pi x}\right)e^{-2\pi nx}dx = \int_0^\infty (\log x - \psi(1+x))\cos(2\pi nx)dx,$$

or, by (13.2.20),

$$\frac{1}{2}(\log n - \psi(1+n)) = \int_0^\infty (\log x - \psi(1+x))\cos(2\pi nx)dx,$$

as claimed. □

Ramanujan next quotes the following self-reciprocal Fourier cosine transform [126, p. 537, formula 3.981, no. 3].

Entry 13.2.7. *For real n,*

$$\int_0^\infty \frac{\cos(\frac{1}{2}\pi nx)}{\cosh(\frac{1}{2}\pi x)}dx = \frac{1}{\cosh(\frac{1}{2}\pi n)}. \tag{13.2.21}$$

Then he records the following entry.

Entry 13.2.8. *For $n > 0$,*

$$\int_0^\infty \frac{e^{-\frac{1}{2}\pi nx}}{\cosh(\frac{1}{2}\pi x)}dx = \frac{4}{\pi}\sum_{k=0}^\infty \frac{(-1)^k}{n+2k+1}. \tag{13.2.22}$$

This follows from the evaluation [126, p. 399, formula 3.541, no. 6]

$$\int_0^\infty \frac{e^{-\frac{1}{2}\pi nx}}{\cosh(\frac{1}{2}\pi x)}dx = \frac{1}{\pi}\left\{\psi\left(\frac{n+3}{4}\right) - \psi\left(\frac{n+1}{4}\right)\right\}$$
$$= \frac{4}{\pi}\sum_{k=0}^\infty \left(-\frac{1}{4k+n+3} + \frac{1}{4k+n+1}\right)$$
$$= \frac{4}{\pi}\sum_{k=0}^\infty \frac{(-1)^k}{2k+n+1},$$

where we utilized (13.2.16).

Entry 13.2.9. *For $n > 0$,*

$$\int_0^\infty \sum_{k=0}^\infty \frac{(-1)^k}{x+2k+1} \sin(\tfrac{1}{2}\pi nx)dx = \sum_{k=0}^\infty \frac{(-1)^k}{n+2k+1}. \tag{13.2.23}$$

Proof. Rewrite (13.2.21) as

$$\int_0^\infty \frac{\cos(nu)}{\cosh u} du = \frac{\pi}{2\cosh(\frac{1}{2}\pi n)}.$$

We are thus going to apply Entry 13.2.2 with

$$f(x) = \frac{1}{\cosh x} \qquad \text{and} \qquad \phi(x) = \frac{\pi}{2\cosh(\frac{1}{2}\pi x)}.$$

From (13.2.22),

$$\begin{aligned}\int_0^\infty f(x)e^{-nx}dx = \int_0^\infty \frac{e^{-nx}}{\cosh x}dx &= \frac{\pi}{2}\int_0^\infty \frac{e^{-\frac{1}{2}\pi nu}}{\cosh(\frac{1}{2}\pi u)}du \\ &= 2\sum_{k=0}^\infty \frac{(-1)^k}{n+2k+1} := \psi(n).\end{aligned}$$

Hence, by Entry 13.2.2,

$$\frac{\pi}{2}\int_0^\infty \frac{e^{-nx}}{\cosh(\frac{1}{2}\pi x)}dx = 2\int_0^\infty \sum_{k=0}^\infty \frac{(-1)^k}{x+2k+1}\sin(nx)dx, \tag{13.2.24}$$

or, if we replace n by $\frac{1}{2}\pi n$,

$$\frac{\pi}{4}\int_0^\infty \frac{e^{-\frac{1}{2}\pi nx}}{\cosh(\frac{1}{2}\pi x)}dx = \int_0^\infty \sum_{k=0}^\infty \frac{(-1)^k}{x+2k+1}\sin(\tfrac{1}{2}\pi nx)dx.$$

Lastly, if we employ (13.2.22) in the foregoing equality, we conclude that

$$\sum_{k=0}^\infty \frac{(-1)^k}{n+2k+1} = \int_0^\infty \sum_{k=0}^\infty \frac{(-1)^k}{x+2k+1}\sin(\tfrac{1}{2}\pi nx)dx, \tag{13.2.25}$$

which is what we wanted to prove. □

Next Ramanujan restates Entries 13.2.1 and 13.2.2 under the assumption

$$f(x) = \sqrt{\frac{2}{\pi}}\phi(x),$$

that is to say, $\phi(x)$ is self-reciprocal. Since his claims are identical to those in Entry 13.2.3, we forego restating them here.

Ramanujan then provides some examples, which are essentially ones that he gave above. First,

$$\int_0^\infty \frac{\cos(nx)}{\cosh\left(x\sqrt{\frac{\pi}{2}}\right)}dx = \sqrt{\frac{\pi}{2}}\frac{1}{\cosh\left(n\sqrt{\frac{\pi}{2}}\right)},$$

which is an easy consequence of (13.2.21). Second,

$$\int_0^\infty \frac{e^{-nx}}{\cosh\left(x\sqrt{\frac{\pi}{2}}\right)}dx = 2\sum_{k=0}^\infty \frac{(-1)^k}{n+\sqrt{\frac{\pi}{2}}(2k+1)}.$$

To establish this identity, replace x by $\sqrt{2/\pi}\,x$ and n by $\sqrt{2/\pi}\,n$ in (13.2.22). Third,

$$\int_0^\infty \sum_{k=0}^\infty \frac{(-1)^k}{x+2k+1}\sin(\tfrac{1}{2}\pi nx)dx = \sum_{k=0}^\infty \frac{(-1)^k}{n+2k+1},$$

which is the same as (13.2.25). Fourth,

$$\int_0^\infty \left(\frac{1}{e^{\sqrt{2\pi}\,x}-1}-\frac{1}{\sqrt{2\pi}\,x}\right)\sin(nx)dx = \sqrt{\frac{\pi}{2}}\left(\frac{1}{e^{\sqrt{2\pi}\,n}-1}-\frac{1}{\sqrt{2\pi}\,n}\right).$$

This last identity follows easily from (13.2.15) upon replacing x by $x/\sqrt{2\pi}$ and n by $n/\sqrt{2\pi}$.

The next two examples contain errors. Ramanujan's fifth example asserts that

$$\int_0^\infty \left(\frac{1}{e^{\sqrt{2\pi}\,x}-1}-\frac{1}{\sqrt{2\pi}\,x}\right)e^{-nx}dx = \sqrt{2\pi}\left\{\gamma+\log\frac{n}{\sqrt{2\pi}}-\psi\left(1+\frac{n}{\sqrt{2\pi}}\right)\right\}, \tag{13.2.26}$$

where γ denotes Euler's constant and $\psi(x)$ is defined in (13.2.16). Return to (13.2.17) and replace x by $x/\sqrt{2\pi}$ and n by $n/\sqrt{2\pi}$. Because, as we previously noted, Ramanujan missed a factor of $-1/(2\pi)$ in (13.2.17), we see that the factor $\sqrt{2\pi}$ on the right-hand side above should be replaced by $-1/\sqrt{2\pi}$. However, there is another error in (13.2.26), because of the spurious appearance of γ on the right-hand side of (13.2.26). Lastly, Ramanujan asserts that

$$\int_0^\infty \{\gamma+\log x-\psi(1+x)\}\cos(2\pi nx)dx = \frac{1}{2}\{\gamma+\log n-\psi(1+n)\}. \tag{13.2.27}$$

To see that the claim (13.2.27) is false, we recall that [1, p. 259], as $x\to\infty$,

$$\psi(x+1) \sim \log x + \frac{1}{2x} + O\left(\frac{1}{x^2}\right). \tag{13.2.28}$$

Thus, we see that the integral in (13.2.27) diverges.

13.3 A Transformation Formula

The most interesting claim made by Ramanujan in the fragment on pages 219 and 220 of [269] is the next entry. To state this claim, we need to recall the following functions associated with Riemann's zeta function $\zeta(s)$. Let

$$\xi(s) := (s-1)\pi^{-\frac{1}{2}s}\Gamma(1+\tfrac{1}{2}s)\zeta(s).$$

Then Riemann's Ξ-function is defined by

$$\Xi(t) := \xi(\tfrac{1}{2}+it). \tag{13.3.1}$$

Entry 13.3.1. *Define*

$$\phi(x) := \psi(x) + \frac{1}{2x} - \log x. \tag{13.3.2}$$

If α and β are positive numbers such that $\alpha\beta = 1$, then

$$\sqrt{\alpha}\left\{\frac{\gamma-\log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty}\phi(n\alpha)\right\} = \sqrt{\beta}\left\{\frac{\gamma-\log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty}\phi(n\beta)\right\}$$
$$= -\frac{1}{\pi^{3/2}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}dt, \tag{13.3.3}$$

where γ denotes Euler's constant and $\Xi(x)$ denotes Riemann's Ξ-function.

Although Ramanujan does not provide a proof of (13.3.3), he does indicate that (13.3.3) "can be deduced from" Entry 13.2.6, or (13.2.18). This remark might lead one to believe that his proof of (13.3.3) rests upon the Poisson summation formula. We provide below a proof of the first equality in (13.3.3) that naturally establishes the second equality as well. Then we give a proof of the first equality in (13.3.3) by means of the Poisson summation formula, but, as we indicated, no connection with $\zeta(s)$ and the integral in the second equality is obtained in this way. In both proofs, the self-reciprocal Fourier cosine transform in (13.2.18) is an essential ingredient.

The self-reciprocal property of $\psi(1+x) - \log x$ was rediscovered by A.P. Guinand [133] in 1947, and he later found a simpler proof of this result in [135]. In a footnote at the end of his paper [135], Guinand remarks that T.A. Brown had told him that he himself had proved the self-reciprocality of $\psi(1+x) - \log x$ some years ago, and that when he (Brown) communicated the result to G.H. Hardy, Hardy told him that the result was also given by Ramanujan in a progress report to the University of Madras, but was not published elsewhere. However, we cannot find this result in any of the three *Quarterly Reports* that Ramanujan submitted to the University of Madras [35–37]. In contrast to what Hardy recalled, it would appear that he saw (13.2.18) in the aforementioned manuscript that Watson had copied. We surmise that Hardy once possessed the original copies of both the *Quarterly Reports* and the present manuscript on pages 219–220 of [269], both of which were most likely mailed to him on August 30, 1923, by Francis Dewsbury, registrar at the University of Madras [64, p. 266]. It could be that the two documents were kept together, and so it is understandable that Hardy concluded that the manuscript was part of the *Quarterly Reports*. Unfortunately, the only copy of Ramanujan's *Quarterly Reports* that now exists is in Watson's handwriting.

The first equality in (13.3.3) was rediscovered by Guinand in [133] and appears in a footnote on the last page of his paper [133, p. 18]. It is interesting that Guinand remarks, "This formula also seems to have been overlooked." Here then is one more instance in which a mathematician thought that his or her theorem was new, but unbeknownst to the claimant, Ramanujan had beaten her/him to the punch! We now give Guinand's version of (13.3.3).

Theorem 13.3.1. *For any complex z such that $|\arg z| < \pi$, we have*

$$\sum_{n=1}^{\infty}\left(\psi(nz) - \log nz + \frac{1}{2nz}\right) + \frac{1}{2z}(\gamma - \log 2\pi z)$$
$$= \frac{1}{z}\sum_{n=1}^{\infty}\left(\psi\left(\frac{n}{z}\right) - \log\frac{n}{z} + \frac{z}{2n}\right) + \frac{1}{2}\left(\gamma - \log\frac{2\pi}{z}\right). \quad (13.3.4)$$

The first equality in (13.3.3) can be easily obtained from Guinand's version by multiplying both sides of (13.3.4) by $\sqrt{z}$ and then letting $z = \alpha$ and $1/z = \beta$. Although not offering a proof of (13.3.4) in [133], Guinand did remark that it can be obtained by using an appropriate form of Poisson's summation formula, namely the form given in Theorem 1 in [132]. Later Guinand gave another proof of Theorem 13.3.1 in [135], while also giving extensions of (13.3.4) involving derivatives of the ψ-function. He also established a finite version of (13.3.4) in [137]. However, Guinand apparently did not discover the connection of his work with Ramanujan's integral involving Riemann's Ξ-function.

We first provide a proof of both identities in Entry 13.3.1. Then we construct a second proof of the first equality in (13.3.3), or, more precisely, of (13.3.4), along the lines suggested by Guinand in [133]. We could have also provided another proof of (13.3.3) employing both (13.2.18) and (13.2.17), but this proof is similar but slightly more complicated than the first proof that we provide below. The two proofs of Entry 13.3.1 given here are from a paper by

A. Dixit and the second author [51]. In two further papers [107, 108], Dixit has found further proofs of Entry 13.3.1.

Although the Riemann zeta function appears at various instances throughout Ramanujan's notebooks [268] and lost notebook [269], he wrote only one paper in which the zeta function plays the leading role [257], [267, pp. 72–77]. In fact, a result proved by Ramanujan in [257], namely Eq. (13.3.18) below, is a key to proving (13.3.3). About the integral involving Riemann's Ξ-function in this result, Hardy [143] comments that "the properties of this integral resemble those of one which Mr. Littlewood and I have used, in a paper to be published shortly in the *Acta Mathematica*, to prove that

$$\int_{-T}^{T} \left|\zeta\left(\frac{1}{2}+ti\right)\right|^2 dt \sim 2T\log T.\text{''} \tag{13.3.5}$$

(We have corrected a misprint in Hardy's version of (13.3.5).)

In a paper immediately following Ramanujan's paper [257], Hardy [143] remarks that the integral on the right-hand side in Ramanujan's formula [257, p. 75, Eq. (13)] can be used to prove that there are infinitely many zeros of $\zeta(s)$ on the critical line $\operatorname{Re} s = \frac{1}{2}$, and then he concludes his note by stating (13.3.6) below, which he says is not unlike the aforementioned formula of Ramanujan. However, Hardy does not give a proof of his formula. Proofs were independently supplied by N.S. Koshliakov [190],[193, Eq. (20)], [194, Chap. 9, Sect. 36], [196, Eq. (34.10)] and Dixit [107]. In Hardy's formulation, the sign of $\frac{1}{2}\gamma$ should be $+$ and not $-$. The sign error was corrected in the papers by Koshliakov and Dixit, but there is an erroneous added factor of $\log 2$ in Koshliakov's formulation in [196]. Koshliakov [190, 195] and Dixit [111] also have given generalizations of Hardy's result.

Theorem 13.3.2 (Correct version). *For real n,*

$$\int_0^\infty \frac{\Xi(\frac{1}{2}t)}{1+t^2}\frac{\cos nt}{\cosh\frac{1}{2}\pi t}\,dt = \frac{1}{4}e^{-n}\left(2n+\frac{1}{2}\gamma+\frac{1}{2}\log\pi+\log 2\right) + \frac{1}{2}e^{n}\int_0^\infty \psi(x+1)e^{-\pi x^2 e^{4n}}\,dx. \tag{13.3.6}$$

Inexplicably, this short note [143] is not reproduced in any of the seven volumes of the *Collected Papers of G.H. Hardy*!

First Proof of Entry 13.3.1. We first collect several well-known theorems that we use in our proof. First, from [99, p. 191], for $t \neq 0$,

$$\sum_{n=1}^{\infty}\frac{1}{t^2+4n^2\pi^2} = \frac{1}{2t}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right). \tag{13.3.7}$$

Second, from [315, p. 251], we find that for $\operatorname{Re} z > 0$,

$$\phi(z) = -2\int_0^\infty \frac{t\,dt}{(t^2+z^2)(e^{2\pi t}-1)}. \tag{13.3.8}$$

Third, we require Binet's integral for $\log\Gamma(z)$, i.e., for $\operatorname{Re} z > 0$ [315, p. 249], [126, p. 377, formula 3.427, no. 4],

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-zt}}{t}dt. \tag{13.3.9}$$

Fourth, from [126, p. 377, formula 3.427, no. 2], we find that

$$\int_0^\infty \left(\frac{1}{1-e^{-x}}-\frac{1}{x}\right)e^{-x}dx = \gamma, \tag{13.3.10}$$

where γ denotes Euler's constant. Fifth, by Frullani's integral [126, p. 378, formula 3.434, no. 2],

$$\int_0^\infty \frac{e^{-\mu x}-e^{-\nu x}}{x}dx = \log\frac{\nu}{\mu}. \tag{13.3.11}$$

Our first goal is to establish an integral representation for the far left side of (13.3.3). Replacing z by $n\alpha$ in (13.3.8) and summing on n, $1 \le n < \infty$, we find that

$$\begin{aligned}\sum_{n=1}^\infty \phi(n\alpha) &= -2\sum_{n=1}^\infty \int_0^\infty \frac{t\,dt}{(t^2+n^2\alpha^2)(e^{2\pi t}-1)}\\ &= -\frac{2}{\alpha^2}\int_0^\infty \frac{t}{(e^{2\pi t}-1)}\sum_{n=1}^\infty \frac{1}{(t/\alpha)^2+n^2}.\end{aligned} \tag{13.3.12}$$

Invoking (13.3.7) in (13.3.12), we see that

$$\sum_{n=1}^\infty \phi(n\alpha) = -\frac{2\pi}{\alpha}\int_0^\infty \frac{1}{(e^{2\pi t}-1)}\left(\frac{1}{e^{2\pi t/\alpha}-1}-\frac{\alpha}{2\pi t}+\frac{1}{2}\right)dt. \tag{13.3.13}$$

Next, setting $x = 2\pi t$ in (13.3.10), we readily find that

$$\gamma = \int_0^\infty \left(\frac{2\pi}{e^{2\pi t}-1}-\frac{e^{-2\pi t}}{t}\right)dt. \tag{13.3.14}$$

By Frullani's integral (13.3.11),

$$\int_0^\infty \frac{e^{-t/\alpha}-e^{-2\pi t}}{t}dt = \log\left(\frac{2\pi}{1/\alpha}\right) = \log(2\pi\alpha). \tag{13.3.15}$$

Combining (13.3.14) and (13.3.15), we arrive at

$$\gamma - \log(2\pi\alpha) = \int_0^\infty \left(\frac{2\pi}{e^{2\pi t}-1} - \frac{e^{-t/\alpha}}{t} \right) dt. \tag{13.3.16}$$

Hence, from (13.3.13) and (13.3.16), we deduce that

$$\begin{aligned} &\sqrt{\alpha}\left(\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^\infty \phi(n\alpha) \right) \qquad (13.3.17)\\ &\quad = \frac{1}{2\sqrt{\alpha}} \int_0^\infty \left(\frac{2\pi}{e^{2\pi t}-1} - \frac{e^{-t/\alpha}}{t} \right) dt \\ &\qquad - \frac{2\pi}{\sqrt{\alpha}} \int_0^\infty \frac{1}{(e^{2\pi t}-1)} \left(\frac{1}{e^{2\pi t/\alpha}-1} - \frac{\alpha}{2\pi t} + \frac{1}{2} \right) dt \\ &\quad = \int_0^\infty \left(\frac{\sqrt{\alpha}}{t(e^{2\pi t}-1)} - \frac{2\pi}{\sqrt{\alpha}(e^{2\pi t/\alpha}-1)(e^{2\pi t}-1)} - \frac{e^{-t/\alpha}}{2t\sqrt{\alpha}} \right) dt. \end{aligned}$$

Now from [257, p. 260, Eq. (22)] or [267, p. 77], for n real,

$$\begin{aligned} &\int_0^\infty \Gamma\left(\frac{-1+it}{4} \right) \Gamma\left(\frac{-1-it}{4} \right) \left(\Xi\left(\frac{1}{2}t \right) \right)^2 \frac{\cos nt}{1+t^2}\, dt \\ &\quad = \int_0^\infty \left| \Xi\left(\frac{1}{2}t \right) \Gamma\left(\frac{-1+it}{4} \right) \right|^2 \frac{\cos nt}{1+t^2}\, dt \\ &\quad = \pi^{3/2} \int_0^\infty \left(\frac{1}{e^{xe^n}-1} - \frac{1}{xe^n} \right) \left(\frac{1}{e^{xe^{-n}}-1} - \frac{1}{xe^{-n}} \right) dx. \qquad (13.3.18) \end{aligned}$$

Letting $n = \frac{1}{2}\log\alpha$ and $x = 2\pi t/\sqrt{\alpha}$ in (13.3.18), we deduce that

$$\begin{aligned} &-\frac{1}{\pi^{3/2}} \int_0^\infty \left| \Xi\left(\frac{1}{2}t \right) \Gamma\left(\frac{-1+it}{4} \right) \right|^2 \frac{\cos(\frac{1}{2}t\log\alpha)}{1+t^2} dt \qquad (13.3.19)\\ &= -\frac{2\pi}{\sqrt{\alpha}} \int_0^\infty \left(\frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t} \right) \left(\frac{1}{e^{2\pi t/\alpha}-1} - \frac{\alpha}{2\pi t} \right) dt \\ &= \int_0^\infty \left(\frac{-2\pi/\sqrt{\alpha}}{(e^{2\pi t/\alpha}-1)(e^{2\pi t}-1)} + \frac{\sqrt{\alpha}}{t(e^{2\pi t}-1)} + \frac{1}{t\sqrt{\alpha}(e^{2\pi t/\alpha}-1)} - \frac{\sqrt{\alpha}}{2\pi t^2} \right) dt. \end{aligned}$$

Hence, combining (13.3.17) and (13.3.19), in order to prove that the far left side of (13.3.3) equals the far right side of (13.3.3), we see that it suffices to show that

$$\begin{aligned} &\int_0^\infty \left(\frac{1}{t\sqrt{\alpha}(e^{2\pi t/\alpha}-1)} - \frac{\sqrt{\alpha}}{2\pi t^2} + \frac{e^{-t/\alpha}}{2t\sqrt{\alpha}} \right) dt \\ &\qquad = \frac{1}{\sqrt{\alpha}} \int_0^\infty \left(\frac{1}{u(e^u-1)} - \frac{1}{u^2} + \frac{e^{-u/(2\pi)}}{2u} \right) du = 0, \qquad (13.3.20) \end{aligned}$$

where we made the change of variable $u = 2\pi t/\alpha$. In fact, more generally, we show that

$$\int_0^\infty \left(\frac{1}{u(e^u-1)} - \frac{1}{u^2} + \frac{e^{-ua}}{2u}\right) du = -\frac{1}{2}\log(2\pi a), \tag{13.3.21}$$

so that if we set $a = 1/(2\pi)$ in (13.3.21), we deduce (13.3.20).

Consider the integral, for $t > 0$,

$$\begin{aligned} F(a,t) :&= \int_0^\infty \left\{\left(\frac{1}{e^u-1} - \frac{1}{u} + \frac{1}{2}\right)\frac{e^{-tu}}{u} + \frac{e^{-ua}-e^{-tu}}{2u}\right\} du \\ &= \log \Gamma(t) - \left(t - \frac{1}{2}\right)\log t + t - \frac{1}{2}\log(2\pi) + \frac{1}{2}\log\frac{t}{a}, \end{aligned} \tag{13.3.22}$$

where we applied (13.3.9) and (13.3.11). Upon the integration of (13.2.16), it is easily gleaned that as $t \to 0^+$,

$$\log \Gamma(t) \sim -\log t - \gamma t,$$

where γ denotes Euler's constant. Using this in (13.3.22), we find, upon simplification, that as $t \to 0^+$,

$$F(a,t) \sim -\gamma t - t\log t + t - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log a.$$

Hence,

$$\lim_{t\to 0^+} F(a,t) = -\frac{1}{2}\log(2\pi a). \tag{13.3.23}$$

Letting t approach 0^+ in (13.3.22), taking the limit under the integral sign on the right-hand side using Lebesgue's dominated convergence theorem, and employing (13.3.23), we immediately deduce (13.3.21). As previously discussed, this is sufficient to prove the equality of the first and third expressions in (13.3.3), namely,

$$\begin{aligned} &\sqrt{\alpha}\left\{\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^\infty \phi(n\alpha)\right\} \\ &\quad = -\frac{1}{\pi^{3/2}}\int_0^\infty \left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}\, dt. \end{aligned} \tag{13.3.24}$$

Lastly, using (13.3.24) with α replaced by β and employing the relation $\alpha\beta = 1$, we conclude that

$$\begin{aligned} &\sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^\infty \phi(n\beta)\right\} \\ &\quad = -\frac{1}{\pi^{3/2}}\int_0^\infty \left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\beta\right)}{1+t^2} dt \end{aligned}$$

$$= -\frac{1}{\pi^{3/2}} \int_0^\infty \left| \Xi\left(\frac{1}{2}t\right) \Gamma\left(\frac{-1+it}{4}\right) \right|^2 \frac{\cos\left(\frac{1}{2}t\log(1/\alpha)\right)}{1+t^2} dt$$
$$= -\frac{1}{\pi^{3/2}} \int_0^\infty \left| \Xi\left(\frac{1}{2}t\right) \Gamma\left(\frac{-1+it}{4}\right) \right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2} dt.$$

Hence, the equality of the second and third expressions in (13.3.3) has been demonstrated, and so the proof is complete. □

We next give our second proof of the first identity in (13.3.3) using Guinand's generalization of Poisson's summation formula in [132]. We emphasize that this route does not take us to the integral involving Riemann's Ξ-function in the second identity of (13.3.3). First, we reproduce the needed version of the Poisson summation formula from Theorem 1 in [132].

Theorem 13.3.3. *If $f(x)$ has a Fourier integral representation, $f(x)$ tends to zero as $x \to \infty$, and $xf'(x)$ belongs to $L^p(0,\infty)$, for some p, $1 < p \leq 2$, then*

$$\lim_{N\to\infty}\left(\sum_{n=1}^N f(n) - \int_0^N f(t)\,dt\right) = \lim_{N\to\infty}\left(\sum_{n=1}^N g(n) - \int_0^N g(t)\,dt\right),$$

where

$$g(x) = 2\int_0^\infty f(t)\cos(2\pi x t)\,dt. \tag{13.3.25}$$

We first state a lemma[1] that will subsequently be used in our proof of (13.3.3).

Lemma 13.3.1. *If $\psi(x)$ is defined by (13.2.16), then*

$$\int_0^\infty \left(\psi(t+1) - \frac{1}{2(t+1)} - \log t\right) dt = \frac{1}{2}\log 2\pi. \tag{13.3.26}$$

Proof. Let I denote the integral on the left-hand side of (13.3.26). Then,

$$\begin{aligned}
I &= \int_0^\infty \frac{d}{dt}\left(\log\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}}\right) dt \\
&= \lim_{t\to\infty}\log\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} - \lim_{t\to 0}\log\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} \\
&= \log\lim_{t\to\infty}\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}} - \log\left(\lim_{t\to 0} e^t\Gamma(t+1)\right) - \lim_{t\to 0} t\log t - \lim_{t\to 0}\frac{1}{2}\log(t+1) \\
&= \log\lim_{t\to\infty}\frac{e^t\Gamma(t+1)}{t^t\sqrt{t+1}}.
\end{aligned} \tag{13.3.27}$$

[1] The authors are indebted to M.L. Glasser for the proof of this lemma. The authors' original proof of this lemma was substantially longer than Glasser's given here.

Next, Stirling's formula [126, p. 945, formula 8.327] tells us that

$$\Gamma(z) \sim \sqrt{2\pi}\, z^{z-1/2} e^{-z}, \tag{13.3.28}$$

as $|z| \to \infty$ for $|\arg z| \le \pi - \delta$, where $0 < \delta < \pi$. Hence, employing (13.3.28), we find that

$$\frac{e^t \Gamma(t+1)}{t^t \sqrt{t+1}} \sim \frac{\sqrt{2\pi}}{e} \left(1 + \frac{1}{t}\right)^t, \tag{13.3.29}$$

so that

$$\lim_{t\to\infty} \frac{e^t \Gamma(t+1)}{t^t \sqrt{t+1}} = \sqrt{2\pi}. \tag{13.3.30}$$

Thus, from (13.3.27) and (13.3.30), we conclude that

$$I = \frac{1}{2} \log 2\pi. \tag{13.3.31}$$

□

Second Proof of the first equality of (13.3.3), *or of* (13.3.4). We first prove (13.3.3) for $\operatorname{Re} z > 0$. Let

$$f(x) = \psi(xz+1) - \log xz. \tag{13.3.32}$$

We show that $f(x)$ satisfies the hypotheses of Theorem 13.3.3. From (13.2.18), we see that $f(x)$ has the required integral representation. Next, we need two formulas for $\psi(x)$. First, from [1, p. 259, formula 6.3.18], for $|\arg z| < \pi$, as $z \to \infty$,

$$\psi(z) \sim \log z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots. \tag{13.3.33}$$

Second, from [315, p. 250],

$$\psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}. \tag{13.3.34}$$

Using the easily verified equality

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{13.3.35}$$

(13.3.32), and (13.3.33), we see that

$$f(x) \sim \frac{1}{2xz} - \frac{1}{12x^2z^2} + \frac{1}{120x^4z^4} - \frac{1}{252x^6z^6} + \cdots, \tag{13.3.36}$$

so that

$$\lim_{x\to\infty} f(x) = 0. \tag{13.3.37}$$

Next, we show that $xf^{'}(x)$ belongs to $L^p(0,\infty)$ for some p such that $1 < p \leq 2$. Using (13.3.36), we find that as $x \to \infty$,

$$xf^{'}(x) \sim -\frac{1}{2xz}, \tag{13.3.38}$$

so that $|xf^{'}(x)|^p \sim (2x|z|)^{-p}$. Thus, for $p > 1$, we see that $xf^{'}(x)$ is locally integrable near ∞. Also, using (13.3.35) and (13.3.34), we have

$$\begin{aligned}\lim_{x\to 0} xf^{'}(x) &= \lim_{x\to 0}\left(xz\sum_{n=0}^{\infty}\frac{1}{(xz+n)^2} - \frac{1}{xz} - 1\right)\\ &= \lim_{x\to 0}\left(xz\sum_{n=1}^{\infty}\frac{1}{(xz+n)^2} - 1\right)\\ &= -1. \end{aligned} \tag{13.3.39}$$

This proves that $xf^{'}(x)$ is locally integrable near 0. Hence, we have shown that $xf^{'}(x)$ belongs to $L^p(0,\infty)$ for some p such that $1 < p \leq 2$.

Now from (13.3.25) and (13.3.32), we find that

$$g(x) = 2\int_0^{\infty}(\psi(tz+1) - \log tz)\cos(2\pi xt)\,dt.$$

Employing the change of variable $y = tz$ and using (13.2.18), we find that

$$\begin{aligned} g(x) &= \frac{2}{z}\int_0^{\infty}(\psi(y+1) - \log y)\cos(2\pi xy/z)\,dy\\ &= \frac{1}{z}\left(\psi\left(\frac{x}{z}+1\right) - \log\left(\frac{x}{z}\right)\right). \end{aligned} \tag{13.3.40}$$

Substituting the expressions for $f(x)$ and $g(x)$ from (13.3.32) and (13.3.40), respectively, in Theorem 13.3.3, we find that

$$\begin{aligned} &\lim_{N\to\infty}\left(\sum_{n=1}^{N}(\psi(nz+1) - \log nz) - \int_0^N(\psi(tz+1) - \log tz)\,dt\right) \qquad (13.3.41)\\ &= \frac{1}{z}\left[\lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(\psi\left(\frac{n}{z}+1\right) - \log\frac{n}{z}\right) - \int_0^N\left(\psi\left(\frac{t}{z}+1\right) - \log\frac{t}{z}\right)dt\right)\right]. \end{aligned}$$

Thus, with the use of (13.3.35),

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(\frac{\Gamma'}{\Gamma}(nz)+\frac{1}{2nz}-\log nz\right)\right.$$
$$\left.+\sum_{n=1}^{N}\frac{1}{2nz}-\int_0^N(\psi(tz+1)-\log tz)\,dt\right)$$
$$=\frac{1}{z}\left[\lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(\frac{\Gamma'}{\Gamma}\left(\frac{n}{z}\right)+\frac{z}{2n}-\log\frac{n}{z}\right)\right.\right.$$
$$\left.\left.+\sum_{n=1}^{N}\frac{z}{2n}-\int_0^N\left(\psi\left(\frac{t}{z}+1\right)-\log\frac{t}{z}\right)dt\right)\right]. \tag{13.3.42}$$

Now if we can show that

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{2nz}-\int_0^N(\psi(tz+1)-\log tz)\,dt\right)=\frac{\gamma-\log 2\pi z}{2z}, \tag{13.3.43}$$

then replacing z by $1/z$ in (13.3.43) will give us

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{z}{2n}-\int_0^N\left(\psi\left(\frac{t}{z}+1\right)-\log\frac{t}{z}\right)dt\right)=\frac{z(\gamma-\log(2\pi/z))}{2}. \tag{13.3.44}$$

Then substituting (13.3.43) and (13.3.44) in (13.3.42) will complete the proof of Theorem 13.3.1. To that end,

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{2nz}-\int_0^N(\psi(tz+1)-\log tz)\,dt\right)$$
$$=\lim_{N\to\infty}\left(\frac{1}{2z}\left(\sum_{n=1}^{N}\frac{1}{n}-\log N\right)+\frac{\log N}{2z}-\int_0^N(\psi(tz+1)-\log tz)\,dt\right)$$
$$=\frac{\gamma}{2z}+\lim_{N\to\infty}\left(-\frac{\log z}{2z}+\frac{\log Nz}{2z}-\int_0^N(\psi(tz+1)-\log tz)\,dt\right)$$
$$=\frac{\gamma}{2z}-\frac{\log z}{2z}+\lim_{N\to\infty}\left(\frac{\log(Nz+1)}{2z}-\frac{1}{z}\int_0^{Nz}(\psi(t+1)-\log t)\,dt\right.$$
$$\left.-\frac{1}{2z}\log\left(1+\frac{1}{Nz}\right)\right)$$
$$=\frac{\gamma}{2z}-\frac{\log z}{2z}+\frac{1}{z}\lim_{N\to\infty}\left(\frac{\log(Nz+1)}{2}-\int_0^{Nz}(\psi(t+1)-\log t)\,dt\right)$$
$$=\frac{\gamma}{2z}-\frac{\log z}{2z}+\frac{1}{z}\lim_{N\to\infty}\left(\frac{1}{2}\int_0^{Nz}\frac{1}{t+1}\,dt-\int_0^{Nz}(\psi(t+1)-\log t)\,dt\right)$$

$$= \frac{\gamma}{2z} - \frac{\log z}{2z} - \frac{1}{z} \lim_{N\to\infty} \int_0^{NZ} \left(\psi(t+1) - \frac{1}{2(t+1)} - \log t \right) dt$$
$$= \frac{\gamma}{2z} - \frac{\log z}{2z} - \frac{1}{z} \int_0^{\infty} \left(\psi(t+1) - \frac{1}{2(t+1)} - \log t \right) dt$$
$$= \frac{\gamma}{2z} - \frac{\log z}{2z} - \frac{\log 2\pi}{2z}$$
$$= \frac{\gamma - \log 2\pi z}{2z}, \tag{13.3.45}$$

where in the antepenultimate line we have made use of Lemma 13.3.1. This completes the proof of (13.3.43) and hence the proof of Theorem 13.3.1 for $\operatorname{Re} z > 0$. But both sides of (13.3.4) are analytic for $|\arg z| < \pi$. Hence, by analytic continuation, the theorem is true for all complex z such that $|\arg z| < \pi$. ☐

Y. Lee [210] has also devised a proof of Entry 13.3.1.

13.4 Page 195

On page 195 in [269], Ramanujan defines

$$\phi(x) := \psi(x) + \frac{1}{2x} - \gamma - \log x \tag{13.4.1}$$

and then concludes that

$$\sqrt{\alpha}\left\{\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty} \phi(n\alpha)\right\} = \sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty} \phi(n\beta)\right\} \tag{13.4.2}$$
$$= -\sqrt{\alpha}\int_0^{\infty} \left(\frac{1}{e^x - 1} - \frac{1}{x}\right)\left(\frac{1}{e^{x\alpha} - 1} - \frac{1}{x\alpha}\right) dx \tag{13.4.3}$$
$$= -\frac{1}{\pi^{3/2}} \int_0^{\infty} \left| \Xi\left(\frac{1}{2}t\right) \Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t \log\alpha\right)}{1+t^2} dt. \tag{13.4.4}$$

First, in view of the asymptotic expansion (13.2.28) and the definition (13.4.1), the series in (13.4.2) do not converge. Second, the equality of the expressions in (13.4.3) and (13.4.4) does not hold. For equality to exist, the expression in (13.4.3) must be replaced by (see equation (22) of [257])

$$-\int_0^{\infty} \left(\frac{1}{e^{x\sqrt{\beta}} - 1} - \frac{1}{x\sqrt{\beta}}\right)\left(\frac{1}{e^{x\sqrt{\alpha}} - 1} - \frac{1}{x\sqrt{\alpha}}\right) dx.$$

13.5 Analogues of Entry 13.3.1

A. Dixit [108, 109] has established two beautiful analogues of Entry 13.3.1. Previously, a finite analogue of Theorem 13.5.1 was established by L. Carlitz [86].

Theorem 13.5.1. *Let $\zeta(z,a)$ denote the Hurwitz zeta function defined for $a>0$ and $\operatorname{Re} z>1$ by*

$$\zeta(z,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^z}.$$

If α and β are positive numbers such that $\alpha\beta=1$, then for $\operatorname{Re} z>2$ and $1<c<\operatorname{Re} z-1$,

$$\begin{aligned}\alpha^{-z/2}\sum_{k=1}^{\infty}\zeta\left(z,1+\frac{k}{\alpha}\right)&=\beta^{-z/2}\sum_{k=1}^{\infty}\zeta\left(z,1+\frac{k}{\beta}\right)\\&=\frac{\alpha^{z/2}}{2\pi i\Gamma(z)}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\zeta(s)\Gamma(z-s)\zeta(z-s)\alpha^{-s}\,ds\\&=\frac{8(4\pi)^{(z-4)/2}}{\Gamma(z)}\int_0^{\infty}\Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\\&\quad\times\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}dt,\end{aligned}$$

where $\Xi(t)$ is defined in (13.3.1).

Theorem 13.5.2. *Let $0<\operatorname{Re} z<2$. Define $\varphi(z,x)$ by*

$$\varphi(z,x)=\zeta(z,x)-\frac{1}{2}x^{-z}+\frac{x^{1-z}}{1-z},$$

where $\zeta(z,x)$ denotes the Hurwitz zeta function. Then if α and β are any positive numbers such that $\alpha\beta=1$,

$$\begin{aligned}&\alpha^{z/2}\left(\sum_{n=1}^{\infty}\varphi(z,n\alpha)-\frac{\zeta(z)}{2\alpha^z}-\frac{\zeta(z-1)}{\alpha(z-1)}\right)\\&=\beta^{z/2}\left(\sum_{n=1}^{\infty}\varphi(z,n\beta)-\frac{\zeta(z)}{2\beta^z}-\frac{\zeta(z-1)}{\beta(z-1)}\right)\\&=\frac{8(4\pi)^{(z-4)/2}}{\Gamma(z)}\int_0^{\infty}\Gamma\left(\frac{z-2+it}{4}\right)\Gamma\left(\frac{z-2-it}{4}\right)\\&\quad\times\Xi\left(\frac{t+i(z-1)}{2}\right)\Xi\left(\frac{t-i(z-1)}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{z^2+t^2}\,dt,\end{aligned}\tag{13.5.1}$$

where $\Xi(t)$ is defined in (13.3.1).

If we let $z \to 1$ in (13.5.1), then we obtain Ramanujan's transformation (13.3.3). Thus Theorem 13.5.2 is a generalization of Entry 13.3.1. In [109], Dixit also obtained an analogue of Theorem 13.5.2 for $-3 < \operatorname{Re} z < -1$. Another generalization of the first identity in Entry 13.3.1 has been found by O. Oloa [237]. Another proof, employing a theorem on the double cotangent function, has been given by H. Tanaka [299].

13.6 Added Note: Pages 193, 194, 250

On pages 193 and 194 in [269], Ramanujan offers several Fourier and Laplace transforms, most of which are found in Entries 13.2.1 and 13.2.2. Since all of the results are standard in the theory of Fourier transforms, there is no need to repeat them here. On page 250 there appears some scratch work on Laplace transforms; no identities are recorded. The third integral on the page appears to be related to [255, Eq. (16)], [267, p. 56].