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An Unpublished Manuscript of Ramanujan on Infinite Series Identities

12.1 Introduction

Published with Ramanujan's lost notebook [269, pp. 318–321] is a four-page, handwritten fragment on infinite series. Partial fraction expansions, the Riemann zeta function $\zeta(s)$, alternating sums over the odd integers, divisor sums $\sigma_k(n)$, Bernoulli numbers, and Euler numbers are featured in the formulas in this manuscript. The first result has the equation number (18) attached to it. Thus, the manuscript was likely intended to be the completion of either a published paper or another unpublished manuscript. We conjecture that this fragment was originally intended to be a part of Ramanujan's paper *Some formulae in the analytic theory of numbers*, [263], [267, pp. 133–135]. This paper contains several theorems featuring $\zeta(s)$ and $\sigma_k(n)$, and so the topics in the unpublished manuscript mesh well with those in the published paper. However, the last tagged equation in [263] is (22), whereas we would expect it to be (17) if our conjecture is correct. Often Ramanujan would think of additional results and add them to the paper as he was writing it, and so this could easily account for the discrepancy in equation numbers. We remark here that the manuscript does not provide any proofs, but Ramanujan usually gives an indication (in one line) how a particular formula may be deduced.

Why did Ramanujan not include this discarded piece in his paper [263], for the published paper is rather short, and the unpublished manuscript would add at most four pages to the length of the paper? We think that Ramanujan discovered that one of his claims, namely (21), was incorrect and that two of his deductions were not corollaries of his (incorrect) formula, as he had previously thought. Moreover, we suspect that he realized that some of his arguments were not rigorous. Since he had abandoned his intention to publish this portion, he did not bother to indicate that changes or corrections needed to be made in the fragment. He probably failed to discard it because he had wanted to return to it sometime in the future to attempt to correct his arguments.

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Ramanujan loved partial fraction expansions. Chapter 14 in his second notebook [38, 268], in particular, contains several such expansions, and others are scattered throughout all three earlier notebooks. See [40, Chap. 30] for some of these scattered partial fraction decompositions. However, Ramanujan's arguments were not always rigorous. Because of his apparent weakness in complex analysis, he evidently did not have a firm grasp of the Mittag–Leffler theorem, for claim (21) in his unpublished manuscript arises from an incorrect application of the Mittag–Leffler theorem, as we detail below. After claim (21), he then asserted several corollaries arising from this (incorrect) partial fraction decomposition. All of the corollaries are indeed correct, but two of them do not follow from this partial fraction expansion. Ramanujan undoubtedly had previously been familiar with all of these corollaries and almost certainly had derived them by other methods. Certain correct results were easily deduced from his expansion, and he must have been puzzled why two further known results could not be similarly deduced. It is interesting that the same incorrect partial fraction expansion occurs in Entry 19(i) of Chap. 14 of his second notebook [268], [38, p. 271], where it was derived by a different method, namely a general elementary theorem, Entry 18 of Chap. 14 [268], [38, pp. 267–268]. R. Sitaramachandrarao [289], [38, pp. 271–272] found an alternative version of Ramanujan's partial fraction expansion. After we provide Ramanujan's argument, we show that we can actually use Sitaramachandrarao's result to derive a corrected version of Ramanujan's partial fraction expansion. We shall see that Ramanujan's defective argument missed one expression; all other portions of Ramanujan's formula are correct. One of the two claims that did not follow from Ramanujan's expansion now is a corollary of the corrected version. However, this corrected version still does not allow us to rigorously deduce the other result.

The most celebrated result in this manuscript is probably claim (28), which is a famous formula for $\zeta(2n+1)$, where n is a positive integer. There is a large number of proofs of this result and many generalizations as well. References are given after we provide Ramanujan's proof of (28). Ramanujan's argument is rigorous and ironically is independent of whether his formula or the corrected version is used.

In (22), Ramanujan gives another partial fraction expansion, but this one is correct. All of its corollaries claimed by Ramanujan are correct, but not all the deductions can be rigorously established by Ramanujan's methods. These corollaries, like those arising from (19), are all well known, with some having been proved in the literature several times.

In the remainder of the chapter, we record all of Ramanujan's formulas, prove them rigorously in some cases, and "prove" them nonrigorously in other cases, i.e., we argue as Ramanujan most likely did. Most of the results appear in Ramanujan's notebooks, and for all theorems we provide references where proofs can be found. In providing references, we have adhered to the following rules. For each principal theorem, we locate it in Ramanujan's notebooks, indicate who gave the first proof, and lastly refer to the pages in the second author's books, primarily [38], where references to further proofs can be found. Since the publication of [38], additional proofs have been found in some instances, and so we provide references to those recent proofs of which we are aware.

The residue of a meromorphic function $f(z)$ at a pole z_0 will be denoted by $R(f, z_0) = R(z_0)$.

12.2 Three Formulas Containing Divisor Sums

Entry 12.2.1 (p. 318, formula (18)). *Let $\chi(n)$ denote the nonprincipal primitive character of modulus 4, i.e., $\chi(2n) = 0$ and $\chi(2n+1) = (-1)^n$, for each nonnegative integer n. Let $d(n)$ denote the number of positive divisors of the positive integer n. Then, if $x \neq in$, for each integer n,*

$$\sum_{n=1}^{\infty} \frac{\chi(n)d(n)n}{n^2+x^2} = \frac{\pi}{4}\sum_{n=1}^{\infty} \frac{\chi(n)}{n} \operatorname{sech}\left(\frac{\pi x}{2n}\right). \tag{12.2.1}$$

Proof. Recall the partial fraction expansion [126, p. 44, formula 1.422, no. 1]

$$\operatorname{sech}\left(\frac{\pi x}{2}\right) = \frac{4}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{2k-1}{(2k-1)^2+x^2}.$$

Thus,

$$\begin{aligned}\frac{\pi}{4}\sum_{n=1}^{\infty} \frac{\chi(n)}{n} \operatorname{sech}\left(\frac{\pi x}{2n}\right) &= \sum_{n=1}^{\infty}\frac{\chi(n)}{n}\sum_{k=1}^{\infty}\frac{\chi(k)k}{k^2+x^2/n^2}\\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\chi(nk)nk}{n^2k^2+x^2}\\ &= \sum_{r=1}^{\infty}\frac{\chi(r)d(r)r}{r^2+x^2}.\end{aligned}$$

This formally completes our argument. However, observe that in the penultimate line we rearranged the order of summation in the double sum, and this needs to be justified. The following argument was kindly supplied by Johann Thiel.

Proposition 12.2.1. *Let $\chi(n)$ denote the nonprincipal primitive character of modulus 4. Let $d(n)$ denote the number of divisors of the positive integer n. Then if $x \neq \imath n$, for each integer n,*

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\chi(nk)nk}{n^2k^2+x^2}=\sum_{r=1}^{\infty}\frac{\chi(r)d(r)r}{r^2+x^2}. \tag{12.2.2}$$

Proof. By the identity theorem, it suffices to show that (12.2.2) holds for $x \in [0, \frac{1}{4}]$.

We first examine the right-hand side of (12.2.2). If N is a positive integer, write

$$\sum_{r=1}^{\infty}\frac{\chi(r)d(r)r}{r^2+x^2}=\sum_{r=1}^{4N^2}\frac{\chi(r)d(r)r}{r^2+x^2}+\sum_{r=4N^2+1}^{\infty}\frac{\chi(r)d(r)r}{r^2+x^2}. \tag{12.2.3}$$

We want to show that as $N \to \infty$,

$$\sum_{r=4N^2+1}^{\infty}\frac{\chi(r)d(r)r}{r^2+x^2}=O\left(\frac{1}{N}\right). \tag{12.2.4}$$

To achieve this, we use the Dirichlet hyperbola method. Write

$$\begin{aligned}
\sum_{n\le y}\chi(n)d(n) &= \sum_{n\le y}\chi(n)\sum_{d|n}1=\sum_{d\le y}\sum_{\substack{n\le y\\ d|n}}\chi(n)\\
&=\sum_{d\le y}\sum_{m\le y/d}\chi(md)=\sum_{\substack{a,b\le y\\ ab\le y}}\chi(ab)\\
&=\sum_{a\le\sqrt{y}}\sum_{b\le y/a}\chi(a)\chi(b)+\sum_{b\le\sqrt{y}}\sum_{a\le y/b}\chi(a)\chi(b)-\sum_{a\le\sqrt{y}}\sum_{b\le\sqrt{y}}\chi(a)\chi(b)\\
&=2\sum_{a\le\sqrt{y}}\chi(a)\sum_{b\le y/a}\chi(b)-\sum_{a\le\sqrt{y}}\sum_{b\le\sqrt{y}}\chi(a)\chi(b)\\
&=O(\sqrt{y}),
\end{aligned} \tag{12.2.5}$$

as $y \to \infty$, where we used the fact that each of the inner sums in the penultimate line is $O(1)$. If we now apply partial summation in a straightforward fashion with the use of (12.2.5), we easily deduce (12.2.4). Using then (12.2.4) back in (12.2.3), we conclude that

$$\sum_{r=1}^{\infty}\frac{\chi(r)d(r)r}{r^2+x^2}=\sum_{r=1}^{4N^2}\frac{\chi(r)d(r)r}{r^2+x^2}+O\left(\frac{1}{N}\right). \tag{12.2.6}$$

Next, we examine the first sum on the right-hand side of (12.2.3), or the sum on the right-hand side in (12.2.6). Hence,

$$\sum_{r=1}^{4N^2} \frac{\chi(r)d(r)r}{r^2+x^2} = \sum_{nk\leq 4N^2} \frac{\chi(nk)nk}{n^2k^2+x^2}$$
$$= \sum_{n=1}^{2N}\sum_{k=1}^{2N} \frac{\chi(nk)nk}{n^2k^2+x^2} + 2\sum_{n=1}^{2N-1}\sum_{k=2N+1}^{\left\lfloor \frac{4N^2}{n}\right\rfloor} \frac{\chi(nk)nk}{n^2k^2+x^2}$$
$$= \sum_{n=1}^{2N}\sum_{k=1}^{2N} \frac{\chi(nk)nk}{n^2k^2+x^2} + 2\sum_{n=1}^{2N-1}\frac{\chi(n)}{n}\sum_{k=2N+1}^{\left\lfloor \frac{4N^2}{n}\right\rfloor} \frac{\chi(k)k}{k^2+(x/n)^2}. \tag{12.2.7}$$

Observe that the inner sum in the second series on the far right side of (12.2.7) is an alternating series and is consequently $O(1/N)$, as $N \to \infty$. Using this bound in (12.2.7) and then (12.2.7) in (12.2.6) gives

$$\sum_{r=1}^{\infty} \frac{\chi(r)d(r)r}{r^2+x^2} = \sum_{n=1}^{2N}\sum_{k=1}^{2N} \frac{\chi(nk)nk}{n^2k^2+x^2} + O\left(\frac{\log N}{N}\right). \tag{12.2.8}$$

We now examine the left-hand side of (12.2.2) and readily find that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{\chi(nk)nk}{n^2k^2+x^2} = \sum_{n=1}^{2N}\sum_{k=1}^{\infty} \frac{\chi(nk)nk}{n^2k^2+x^2} + \sum_{n=2N+1}^{\infty}\sum_{k=1}^{\infty} \frac{\chi(nk)nk}{n^2k^2+x^2}$$
$$= \sum_{n=1}^{2N}\sum_{k=1}^{\infty} \frac{\chi(nk)nk}{n^2k^2+x^2} + \sum_{n=2N+1}^{\infty}\frac{\chi(n)}{n}\sum_{k=1}^{\infty} \frac{\chi(k)k}{k^2+(x/n)^2}. \tag{12.2.9}$$

If we set

$$f(y) := \frac{1}{y}\sum_{k=1}^{\infty} \frac{\chi(k)k}{k^2+(x/y)^2},$$

for $y \in [1, \infty)$, by a straightforward calculation we see that $f'(y) < 0$ and consequently $\lim_{y\to\infty} f(y) = 0$. Therefore, we can apply the alternating series test to conclude that the inner sum of the second sum on the far right side of (12.2.9) is an alternating series that is $O(1/N)$, as $N \to \infty$. Therefore,

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{\chi(nk)nk}{n^2k^2+x^2} = \sum_{n=1}^{2N}\sum_{k=1}^{\infty} \frac{\chi(nk)nk}{n^2k^2+x^2} + O\left(\frac{1}{N}\right)$$
$$= \sum_{n=1}^{2N}\sum_{k=1}^{2N} \frac{\chi(nk)nk}{n^2k^2+x^2} + \sum_{n=1}^{2N}\sum_{k=2N+1}^{\infty} \frac{\chi(nk)nk}{n^2k^2+x^2} + O\left(\frac{1}{N}\right)$$
$$= \sum_{n=1}^{2N}\sum_{k=1}^{2N} \frac{\chi(nk)nk}{n^2k^2+x^2} + O\left(\frac{\log N}{N}\right), \tag{12.2.10}$$

where the last equality follows by an argument similar to the one used to deduce (12.2.8).

Taking the difference of (12.2.10) and (12.2.8), we complete the proof of (12.2.2). □

This then completes a rigorous proof of Entry 12.2.1. □

Entry 12.2.1 is a simple example of a large class of formulas involving the sech function and arithmetic functions. See papers by Berndt [34, Example 3] and P.V. Krishnaiah and R. Sita Rama Chandra Rao [201] for further examples.

Entry 12.2.2 (p. 318, formula (19)). *Let* $\sigma_k(n) = \sum_{d|n} d^k$. *Then, for* $\operatorname{Re} s > 1$ *and* $\operatorname{Re}(s-r) > 1$,

$$\zeta(s)\zeta(s-r) = \sum_{n=1}^{\infty} \frac{\sigma_r(n)}{n^s}. \tag{12.2.11}$$

The formula (12.2.11) is classical and simple to prove. Ramanujan [263], [267, pp. 133–135] found beautiful extensions of it. See also Titchmarsh's text [306, p. 8].

Entry 12.2.3 (p. 318, formula (20)). *Let* χ *be defined as in Entry* 12.2.1, *and let* $\sigma_k(n)$ *be as in Entry* 12.2.2. *Then, for* $\operatorname{Re} s > 1$ *and* $\operatorname{Re}(s-r) > 1$,

$$\sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{s-r}} = \sum_{n=1}^{\infty} \frac{\chi(n)\sigma_r(n)}{n^s}.$$

Proof. For $\operatorname{Re} s > 1$ and $\operatorname{Re}(s-r) > 1$,

$$\sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{s-r}} = \sum_{m,n=1}^{\infty} \frac{\chi(mn)n^r}{(mn)^s} = \sum_{k=1}^{\infty} \frac{\chi(k)\sigma_r(k)}{k^s},$$

which completes the proof for $\operatorname{Re} s > 1$ and $\operatorname{Re}(s-r) > 1$. We expect that the domain of validity can be extended to $\operatorname{Re} s > \sup\{0, \operatorname{Re} r\}$, but we are unable to prove this. □

There are many results in the literature generalizing or extending the last two results. The two most extensive papers in this direction are perhaps those by S. Chowla [91, 92], [95, pp. 92–115, 120–130].

12.3 Ramanujan's Incorrect Partial Fraction Expansion and Ramanujan's Celebrated Formula for $\zeta(2n+1)$

Prior to this next claim, Ramanujan writes, "By the theory of residues it can be shown that". Evidently, Ramanujan implied that he used the residue theorem to calculate the partial fraction decomposition that followed. His formal calculations should depend upon an application of the Mittag–Leffler theorem, which cannot be applied in this situation. We first state the incorrect expansion, indicate Ramanujan's probable approach, and then offer a correct version. Ramanujan used n to denote a complex variable; we replace it with the more natural notation $w = z^2$.

Entry 12.3.1 (p. 318, formula (21)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, then*

$$\frac{1}{2w} + \sum_{m=1}^{\infty} \left\{ \frac{m\alpha \coth(m\alpha)}{w + m^2\alpha} + \frac{m\beta \coth(m\beta)}{w - m^2\beta} \right\} = \frac{\pi}{2} \cot(\sqrt{w\alpha}) \coth(\sqrt{w\beta}). \tag{12.3.1}$$

Proof. (We emphasize that the following argument is not rigorous.) Consider

$$f(z) := \frac{\pi}{2} \cot(z\sqrt{\alpha}) \coth(z\sqrt{\beta}),$$

which has simple poles at $z = m\pi/\sqrt{\alpha}$, $-\infty < m < \infty$, $m \neq 0$, with residues

$$R(m\pi/\sqrt{\alpha}) = \frac{\pi}{2\sqrt{\alpha}} \coth(m\beta), \tag{12.3.2}$$

and simple poles at $z = m\pi i/\sqrt{\beta}, -\infty < m < \infty, m \neq 0$, with residues

$$R(m\pi i/\sqrt{\beta}) = -\frac{\pi i}{2\sqrt{\beta}} \coth(m\alpha), \tag{12.3.3}$$

where we used the fact $\alpha\beta = \pi^2$ in our calculations. Clearly $f(z)$ also has a double pole at $z = 0$. Using (12.3.2) and once again the relation $\alpha\beta = \pi^2$, we find that the contributions of the poles $z = m\pi/\sqrt{\alpha}$ and $z = -m\pi/\sqrt{\alpha}$, $1 \leq m < \infty$, to the partial fraction expansion of $f(z)$ are

$$\frac{\pi}{2\sqrt{\alpha}} \left(\frac{\coth(m\beta)}{z - m\pi/\sqrt{\alpha}} + \frac{\coth(-m\beta)}{z + m\pi/\sqrt{\alpha}} \right) = \frac{m\beta \coth(m\beta)}{z^2 - m^2\beta}. \tag{12.3.4}$$

Using (12.3.3) and once again the relation $\alpha\beta = \pi^2$, we find that the sum of the contributions of the poles $z = m\pi i/\sqrt{\beta}$ and $z = -m\pi i/\sqrt{\beta}$, $1 \leq m < \infty$, to the partial fraction decomposition of $f(z)$ equals

$$-\frac{\pi i}{2\sqrt{\beta}} \left(\frac{\coth(m\alpha)}{z - m\pi i/\sqrt{\beta}} - \frac{\coth(m\alpha)}{z + m\pi i/\sqrt{\beta}} \right) = \frac{m\alpha \coth(m\alpha)}{z^2 + m^2\alpha}. \tag{12.3.5}$$

That part of the partial fraction decomposition arising from the double pole at $z = 0$ clearly equals

$$\frac{\pi}{2\sqrt{\alpha\beta} z^2} = \frac{1}{2z^2}, \tag{12.3.6}$$

upon again using the relation $\alpha\beta = \pi^2$. Employing (12.3.4)–(12.3.6) and applying the Mittag–Leffler theorem, we find that there exists an entire function $g(z)$ such that

$$\frac{\pi}{2}\cot(z\sqrt{\alpha})\coth(z\sqrt{\beta}) = \frac{1}{2z^2} + \sum_{m=1}^{\infty}\left\{\frac{m\alpha\coth(m\alpha)}{z^2+m^2\alpha} + \frac{m\beta\coth(m\beta)}{z^2-m^2\beta}\right\} + g(z). \tag{12.3.7}$$

Here Ramanujan probably assumed that $g(z) \equiv 0$ and so completed his "proof" of (12.3.1). □

Normally, in applications of the Mittag–Leffler theorem, one lets $z \to \infty$ to conclude that $g(z) \equiv 0$. However, such an argument is invalid here, because $\cot(z\sqrt{\alpha})\coth(z\sqrt{\beta})$ oscillates and does not have a limit as $z \to \infty$. Moreover, one cannot justify taking the limit as $z \to \infty$ under the summation sign in (12.3.7).

In attempting to find a corrected version of (12.3.1), Sitaramachandrarao [289], [38, pp. 271–272] proved that

$$\begin{aligned}\pi^2 xy\cot(\pi x)\coth(\pi y) &= 1 + \frac{\pi^2}{3}(y^2-x^2) \\ &\quad - 2\pi xy\sum_{m=1}^{\infty}\left(\frac{y^2\coth(\pi mx/y)}{m(m^2+y^2)} + \frac{x^2\coth(\pi my/x)}{m(m^2-x^2)}\right).\end{aligned} \tag{12.3.8}$$

Using the elementary identities

$$\frac{y^2}{m(m^2+y^2)} = -\frac{m}{m^2+y^2} + \frac{1}{m}$$

and

$$\frac{x^2}{m(m^2-x^2)} = \frac{m}{m^2-x^2} - \frac{1}{m},$$

we find that (12.3.8) can be rewritten in the form

$$\begin{aligned}\pi^2 xy\cot(\pi x)\coth(\pi y) &= 1 + \frac{\pi^2}{3}(y^2-x^2) \\ &\quad + 2\pi xy\sum_{m=1}^{\infty}\left(\frac{m\coth(\pi mx/y)}{m^2+y^2} - \frac{m\coth(\pi my/x)}{m^2-x^2}\right) \\ &\quad - 2\pi xy\sum_{m=1}^{\infty}\frac{1}{m}\left(\coth(\pi mx/y) - \coth(\pi my/x)\right) \\ &= 1 + \frac{\pi^2}{3}(y^2-x^2)\end{aligned}$$

$$+ 2\pi xy \sum_{m=1}^{\infty} \left(\frac{m \coth(\pi m x/y)}{m^2 + y^2} - \frac{m \coth(\pi m y/x)}{m^2 - x^2} \right)$$
$$- 4\pi xy \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{e^{2\pi m x/y} - 1} - \frac{1}{e^{2\pi m y/x} - 1} \right), \tag{12.3.9}$$

where we used the elementary identity

$$\coth x = 1 + \frac{2}{e^{2x} - 1}. \tag{12.3.10}$$

We are now in a position to make simple changes of variables in (12.3.9) to derive a corrected version of (12.3.1).

Entry 12.3.2 (Corrected Version of (21)). *Under the hypotheses of Entry 12.3.1,*

$$\frac{\pi}{2} \cot(\sqrt{w\alpha}) \coth(\sqrt{w\beta}) = \frac{1}{2w} + \frac{1}{2} \log \frac{\beta}{\alpha} + \sum_{m=1}^{\infty} \left\{ \frac{m\alpha \coth(m\alpha)}{w + m^2\alpha} + \frac{m\beta \coth(m\beta)}{w - m^2\beta} \right\}. \tag{12.3.11}$$

Proof. Let $\pi x = \sqrt{w\alpha}$ and $\pi y = \sqrt{w\beta}$ in (12.3.9) to deduce that

$$\begin{aligned}
&\frac{\pi}{2} \cot(\sqrt{w\alpha}) \coth(\sqrt{w\beta}) \\
&\quad = \frac{1}{2w} + \frac{1}{6}(\beta - \alpha) + \sum_{m=1}^{\infty} \left(\frac{m\alpha \coth(m\alpha)}{m^2\alpha + w} - \frac{m\beta \coth(m\beta)}{\beta m^2 - w} \right) \\
&\quad\quad - 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{e^{2m\alpha} - 1} - \frac{1}{e^{2m\beta} - 1} \right) \\
&\quad = \frac{1}{2w} + \frac{1}{6}(\beta - \alpha) + \sum_{m=1}^{\infty} \left(\frac{m\alpha \coth(m\alpha)}{m^2\alpha + w} - \frac{m\beta \coth(m\beta)}{\beta m^2 - w} \right) \\
&\quad\quad - 2 \left(\frac{1}{4} \log \alpha - \frac{\alpha}{12} - \frac{1}{4} \log \beta + \frac{\beta}{12} \right) \\
&\quad = \frac{1}{2w} + \frac{1}{2} \log \frac{\beta}{\alpha} + \sum_{m=1}^{\infty} \left\{ \frac{m\alpha \coth(m\alpha)}{w + m^2\alpha} + \frac{m\beta \coth(m\beta)}{w - m^2\beta} \right\},
\end{aligned}$$

where we have used an equivalent formulation for the transformation of the Dedekind eta function, namely [68],

$$\sum_{m=1}^{\infty} \frac{1}{m(e^{2m\alpha}-1)} - \frac{1}{4}\log\alpha + \frac{\alpha}{12} = \sum_{m=1}^{\infty} \frac{1}{m(e^{2m\beta}-1)} - \frac{1}{4}\log\beta + \frac{\beta}{12}, \quad (12.3.12)$$

under the condition $\alpha\beta = \pi^2$. This completes the proof of (12.3.11). □

Thus, Ramanujan's claim (21) was correct except for the missing term $\frac{1}{2}\log\frac{\beta}{\alpha}$.

We now proceed to examine the four deductions Ramanujan made from (12.3.1). We first examine the claim that cannot be formally deduced from either (12.3.1) or the corrected version (12.3.11), and provide Ramanujan's argument. Ramanujan asserts that "Equating the coefficients of $1/n$ ($1/w$ in our notation) in both sides in (21) we have ...."

Entry 12.3.3 (p. 318, formula (23)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, then*

$$\alpha \sum_{m=1}^{\infty} \frac{m}{e^{2m\alpha}-1} + \beta \sum_{m=1}^{\infty} \frac{m}{e^{2m\beta}-1} = \frac{\alpha+\beta}{24} - \frac{1}{4}. \quad (12.3.13)$$

Proof. (incorrect) Following Ramanujan, we equate coefficients of $1/w$ on both sides of (12.3.11). Observe from the Laurent expansion of $\cot(\sqrt{w\alpha})\coth(\sqrt{w\beta})$ about $w=0$ that the coefficient of $1/w$ equals $\frac{1}{2}$ on the left side of (12.3.11). Note also the term $1/(2w)$ on the right side of (12.3.11). Hence, the only contribution of $1/w$ that remains must come from

$$\sum_{m=1}^{\infty} \left\{ \frac{m\alpha}{w+m^2\alpha}\left(1+\frac{2}{e^{2m\alpha}-1}\right) + \frac{m\beta}{w-m^2\beta}\left(1+\frac{2}{e^{2m\beta}-1}\right) \right\}, \quad (12.3.14)$$

upon the use of (12.3.10), and this contribution must equal 0.

Proceeding formally, we have

$$\frac{m\alpha}{w+m^2\alpha} = \frac{m\alpha}{w}\sum_{r=0}^{\infty}\left(-\frac{m^2\alpha}{w}\right)^r \qquad \text{and} \qquad \frac{m\beta}{w-m^2\beta} = \frac{m\beta}{w}\sum_{r=0}^{\infty}\left(\frac{m^2\beta}{w}\right)^r.$$

Thus, from (12.3.14) we find that a contribution to the coefficient of $1/w$ equals

$$2\alpha \sum_{m=1}^{\infty} \frac{m}{e^{2m\alpha}-1} + 2\beta \sum_{m=1}^{\infty} \frac{m}{e^{2m\beta}-1}. \quad (12.3.15)$$

The remaining contribution to the coefficient of $1/w$ in (12.3.14) is given by

$$(\alpha+\beta)\sum_{m=1}^{\infty} m = (\alpha+\beta)\zeta(-1) = -\frac{\alpha+\beta}{12}. \quad (12.3.16)$$

Of course, this agrument is not rigorous. The value $\zeta(-1) = -\frac{1}{12}$ can be found in Titchmarsh's book [306, p. 19, Eq. (2.4.3)], for example. Alternatively, the "constant" for the series $\sum_{m=1}^{\infty} m$ in Ramanujan's terminology is equal to $-\frac{1}{12}$ [37, p. 135, Example 2]. Recalling that the contributions of the coefficients of $1/w$ in (12.3.14) must equal 0, we find from (12.3.15) and (12.3.16) that

$$\alpha \sum_{m=1}^{\infty} \frac{m}{e^{2m\alpha}-1} + \beta \sum_{m=1}^{\infty} \frac{m}{e^{2m\beta}-1} = \frac{\alpha+\beta}{24}. \tag{12.3.17}$$

In comparing (12.3.17) with (12.3.13), we find that the term $-\frac{1}{4}$ in (12.3.13) does not appear in (12.3.17). This concludes what we think must have been Ramanujan's argument. □

Entry 12.3.4 (pp. 318–319, formula (24)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if $\sigma(m) = \sum_{d|m} d$, then*

$$\alpha \sum_{m=1}^{\infty} \sigma(m) e^{-2m\alpha} + \beta \sum_{m=1}^{\infty} \sigma(m) e^{-2m\beta} = \frac{\alpha+\beta}{24} - \frac{1}{4}. \tag{12.3.18}$$

Proof. Entry 12.3.4 is simply another version of Entry 12.3.3. To that end, expand the summands of (12.3.13) into geometric series and collect the coefficients of $e^{-2m\alpha}$ and $e^{-2m\beta}$ to complete the proof. □

Ramanujan offered Entry 12.3.3 as Corollary (i) in Sect. 8 of Chap. 14 in his second notebook [268], [38, p. 255]. To the best of our knowledge, Entry 12.3.3 was first proved by O. Schlömilch [279, 280] in 1877. There now exist many proofs; see [38, p. 256] for references to several proofs. One of the most common proofs of the special case $\alpha = \beta = \pi$ of both Entries 12.3.3 and 12.3.6 was recently rediscovered by O. Ogievetsky and V. Schechtman [236]. Entry 12.3.3 is equivalent to the transformation formula for Ramanujan's Eisenstein series $P(q)$.

Entry 12.3.5 (p. 320, formula (29)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if $\sigma_k(m) = \sum_{d|m} d^k$, then*

$$\begin{aligned} &\sum_{m=1}^{\infty} \frac{1}{m(e^{2m\alpha}-1)} - \sum_{m=1}^{\infty} \frac{1}{m(e^{2m\beta}-1)} \\ &\quad = \sum_{m=1}^{\infty} \sigma_{-1}(m) e^{-2m\alpha} - \sum_{m=1}^{\infty} \sigma_{-1}(m) e^{-2m\beta} = \frac{1}{4}\log\frac{\alpha}{\beta} - \frac{\alpha-\beta}{12}. \end{aligned} \tag{12.3.19}$$

Proof. Following but altering Ramanujan's directions, we equate the terms independent of w in (12.3.11) (not (12.3.1)) and use (12.3.10) to deduce that

$$\begin{aligned} \frac{\pi}{2}\left(-\frac{\sqrt{\alpha}}{3\sqrt{\beta}} + \frac{\sqrt{\beta}}{3\sqrt{\alpha}}\right) &= \frac{1}{2}\log\frac{\beta}{\alpha} \\ &\quad + \sum_{m=1}^{\infty}\left\{\frac{1}{m}\left(1+\frac{2}{e^{2m\alpha}-1}\right) - \frac{1}{m}\left(1+\frac{2}{e^{2m\beta}-1}\right)\right\}. \end{aligned}$$

The desired result (12.3.19) now follows upon simplification, with the use of the identity $\alpha\beta = \pi^2$. □

Entry 12.3.5 is stated by Ramanujan as Corollary (ii) in Sect. 8 of Chap. 14 in his second notebook [268], [38, p. 256] and as Entry 27(iii) in Chap. 16 of his second notebook [268], [39, p. 43]. It is equivalent to the transformation formula for the Dedekind eta function. Note that we already used (12.3.19) in the equivalent form (12.3.12) in order to obtain a corrected version of Entry 12.3.1.

The Bernoulli numbers B_m, $m \geq 0$, are defined by

$$\frac{z}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} z^m, \qquad |z| < 2\pi.$$

This convention for Bernoulli numbers is not the same as that used by Ramanujan in his unpublished manuscript.

Entry 12.3.6 (p. 319, formula (25)). *Let α and β be positive numbers such that $\alpha\beta = \pi^2$, and let B_m, $m \geq 0$, denote the mth Bernoulli number. Then, if r is a positive integer with $r \geq 2$,*

$$\alpha^r \left(\sum_{m=1}^{\infty} \frac{m^{2r-1}}{e^{2m\alpha} - 1} - \frac{B_{2r}}{4r} \right) = (-\beta)^r \left(\sum_{m=1}^{\infty} \frac{m^{2r-1}}{e^{2m\beta} - 1} - \frac{B_{2r}}{4r} \right). \tag{12.3.20}$$

Proof. (nonrigorous) Return to (12.3.11), use (12.3.10), and formally expand the summands into geometric series to arrive at

$$\begin{aligned} \frac{\pi}{2} \cot(\sqrt{w\alpha}) \coth(\sqrt{w\beta}) &= \frac{1}{2w} + \frac{1}{2} \log \frac{\beta}{\alpha} \\ &+ \sum_{m=1}^{\infty} \left\{ \frac{m\alpha}{w} \sum_{k=0}^{\infty} \left(-\frac{m^2\alpha}{w} \right)^k \left(1 + \frac{2}{e^{2m\alpha} - 1} \right) \right. \\ &\qquad \left. + \frac{m\beta}{w} \sum_{k=0}^{\infty} \left(\frac{m^2\beta}{w} \right)^k \left(1 + \frac{2}{e^{2m\beta} - 1} \right) \right\}. \end{aligned} \tag{12.3.21}$$

Following Ramanujan's directions, we equate coefficients of $1/w^r$, $r \geq 2$, on both sides of (12.3.21) to formally deduce that

$$\begin{aligned} 0 = (-1)^{r-1} \alpha^r \zeta(1 - 2r) &+ 2(-1)^{r-1} \alpha^r \sum_{m=1}^{\infty} \frac{m^{2r-1}}{e^{2m\alpha} - 1} \\ &+ \beta^r \zeta(1 - 2r) + 2\beta^r \sum_{m=1}^{\infty} \frac{m^{2r-1}}{e^{2m\beta} - 1}. \end{aligned}$$

Using the relation [306, p. 19, Eq. (2.4.3)]

$$\zeta(1 - 2r) = -\frac{B_{2r}}{2r}, \quad r \geq 1,$$

dividing both sides by $2(-1)^r$, and simplifying, we deduce (12.3.20). □

Entry 12.3.6 is identical to Entry 13 in Chap. 14 of Ramanujan's second notebook [268], [38, p. 261]. To the best of our knowledge, the first published proof of Entry 12.3.6 was given by M.B. Rao and M.V. Ayyar [271] in 1923. There exist many proofs of Entry 12.3.6, and even more proofs for the special case $\alpha = \beta = \pi$; see [38, pp. 261–262] for references. N.S. Koshliakov [189, 192] has derived interesting analogues of Entry 12.3.6 and other entries in this section.

Expanding the summands in geometric series, we deduce, as in previous entries, the following corollary, which is, in essence, the transformation formula for classical Eisenstein series.

Entry 12.3.7 (p. 319, formula (26)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if r is a positive integer with $r \geq 2$, then*

$$\alpha^r \left(\sum_{m=1}^{\infty} \sigma_{2r-1}(m) e^{-2m\alpha} - \frac{B_{2r}}{4r} \right) = (-\beta)^r \left(\sum_{m=1}^{\infty} \sigma_{2r-1}(m) e^{-2m\beta} - \frac{B_{2r}}{4r} \right).$$

Entry 12.3.8 (p. 319, formula (27)). *We have*

$$\sum_{m=1}^{\infty} \sigma_5(m) e^{-2\pi m} = \frac{1}{504}.$$

Proof. Entry 12.3.8 follows immediately from Entry 12.3.7 by setting $r = 3$ and $\alpha = \beta = \pi$, and then using the fact that $B_6 = \frac{1}{42}$. □

Entry 12.3.9 (pp. 319–320, formula (28)). *If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if r is a positive integer, then*

$$\begin{aligned}
&(4\alpha)^{-r} \left(\frac{1}{2}\zeta(2r+1) + \sum_{m=1}^{\infty} \frac{1}{m^{2r+1}(e^{2m\alpha}-1)} \right) \\
&\quad - (-4\beta)^{-r} \left(\frac{1}{2}\zeta(2r+1) + \sum_{m=1}^{\infty} \frac{1}{m^{2r+1}(e^{2m\beta}-1)} \right) \\
&= (4\alpha)^{-r} \left(\frac{1}{2}\zeta(2r+1) + \sum_{m=1}^{\infty} \sigma_{-1-2r}(m) e^{-2m\alpha} \right) \\
&\quad - (-4\beta)^{-r} \left(\frac{1}{2}\zeta(2r+1) + \sum_{m=1}^{\infty} \sigma_{-1-2r}(m) e^{-2m\beta} \right) \\
&= -\sum_{k=0}^{r+1} \frac{(-1)^k B_{2k} B_{2r+2-2k} \alpha^{r+1-k} \beta^k}{(2k)!(2r+2-2k)!}.
\end{aligned} \tag{12.3.22}$$

Proof. Return to (12.3.11), use (12.3.10), and expand the summands into geometric series to arrive at

$$\frac{\pi}{2}\cot(\sqrt{w\alpha})\coth(\sqrt{w\beta}) = \frac{1}{2w} + \frac{1}{2}\log\frac{\beta}{\alpha}$$
$$+\sum_{m=1}^{\infty}\left\{\frac{1}{m}\sum_{k=0}^{\infty}\left(-\frac{w}{m^2\alpha}\right)^k\left(1+\frac{2}{e^{2m\alpha}-1}\right)\right.$$
$$\left.-\frac{1}{m}\sum_{k=0}^{\infty}\left(\frac{w}{m^2\beta}\right)^k\left(1+\frac{2}{e^{2m\beta}-1}\right)\right\}. \tag{12.3.23}$$

Following Ramanujan's advice, we equate coefficients of w^r, $r \geq 1$, on both sides of (12.3.23). On the right side, the coefficient of w^r equals

$$(-\alpha)^{-r}\zeta(2r+1) + 2(-\alpha)^{-r}\sum_{m=1}^{\infty}\frac{1}{m^{2r+1}(e^{2m\alpha}-1)}$$
$$-\beta^{-r}\zeta(2r+1) + 2\beta^{-r}\sum_{m=1}^{\infty}\frac{1}{m^{2r+1}(e^{2m\beta}-1)}. \tag{12.3.24}$$

Using the Laurent expansions for $\cot z$ and $\coth z$ about $z=0$, we find that on the left side of (12.3.23),

$$\frac{\pi}{2}\cot(\sqrt{w\alpha})\coth(\sqrt{w\beta}) = \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k}B_{2k}}{(2k)!}(w\alpha)^{k-1/2}$$
$$\times\sum_{j=0}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!}(w\beta)^{j-1/2}. \tag{12.3.25}$$

The coefficient of w^r in (12.3.25) is easily seen to be equal to

$$2^{2r+1}\sum_{k=0}^{r+1}\frac{(-1)^k B_{2k}B_{2r+2-2k}}{(2k)!(2r+2-2k)!}\alpha^k\beta^{r+1-k}, \tag{12.3.26}$$

where we used the equality $\alpha\beta = \pi^2$. Now equate the expressions in (12.3.24) and (12.3.26), then multiply both sides by $(-1)^r 2^{-2r-1}$, and lastly replace k by $r+1-k$ in the finite sum. We then have shown the equality of the first and third expressions in (12.3.22). The first equality of (12.3.22) follows as before by expanding the summands on the left side into geometric series. □

Entry 12.3.9 is the same as Entry 21(i) in Chap. 14 of Ramanujan's second notebook [268], [38, pp. 275–276]. An extensive generalization of Entry 12.3.9 can be found in Entry 20 of Chap. 16 in Ramanujan's first notebook [268], [40, pp. 429–432]. The special case $\alpha = \beta = \pi$ of Entry 12.3.9 was first established by M. Lerch [215] in 1901, but the general theorem was not proved in print until S.L. Malurkar [220] did so in 1925. Inspired by two papers by E. Grosswald [130, 131], the second author established a proof of Entry 12.3.9, the first claim in Ramanujan's notebooks that the second author had ever examined;

his first paper on Ramanujan's work was the survey paper [30] on Ramanujan's formula for $\zeta(2n+1)$. However, at about the same time, the second author had established another proof of Ramanujan's formula for $\zeta(2n+1)$ as well as a far-ranging generalization [33, Theorem 5.2]. The former paper and the second author's book [38, p. 276] contain a multitude of references for the many proofs and generalizations of Entry 12.3.9. Sitaramachandrarao [289] gave a proof of Entry 12.3.9 based on his partial fraction decomposition (12.3.8), and so his proof is similar to that of Ramanujan. Further proofs and generalizations have been given by D. Bradley [74], L. Vepštas [308], and S. Kanemitsu, Y. Tanigawa, and M. Yoshimoto [171, 172]. A very engaging proof, in fact of a significant generalization, via Barnes's multiple zeta functions, was devised by Y. Komori, K. Matsumoto, and H. Tsumura [186]. An especially interesting proof, arising out of a very general asymptotic formula, has been devised by M. Katsurada [179]; see also interesting remarks in his paper [180]. A discussion of Ramanujan's formula in conjunction with numerical calculations has been made by B. Ghusayni [122].

The two infinite series on the far left side of (12.3.22) converge very rapidly. If we "ignore" these two series and let r be odd, then we see that $\zeta(2r+1)$ is "almost" a rational multiple of π^{2r+1}. Continuing this line of thought, suppose that we set $\alpha = \pi z$ and $\beta = \pi z$, and now require that z be a root of

$$\sum_{m=1}^{\infty} \frac{1}{m^{2r+1}(e^{2m\pi z}-1)} + \sum_{m=1}^{\infty} \frac{1}{m^{2r+1}(e^{2m\pi/z}-1)} = 0.$$

Next, multiply both sides of (12.3.22) by $(-1)^r 2^{2r+1}\pi^r z^{r+1}$ and replace k by $r+1-k$ in the finite sum on the far right-hand side. Hence, for such values of z, we deduce that

$$P_k(z) := \frac{(2\pi)^{2k-1}}{(2k)!} \sum_{k=0}^{r+1} (-1)^k \frac{B_{2k}B_{2r+2-2k}}{(2k)!(2r+2-2k)!} z^{2k} = 0. \tag{12.3.27}$$

Accordingly, S. Gun, M.R. Murty, and P. Rath [138] defined the related polynomials

$$R_{2k+1}(z) := \sum_{j=0}^{k+1} \frac{B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!} z^{2j}$$

and showed that all of their nonreal roots lie on the unit circle. Murty, C.J. Smyth, and R.J. Wang [230] discovered further properties of these polynomials. In particular, they discovered bounds for their real zeros, and they proved that the largest real zero approaches 2 from above, as $k \to \infty$. M. Lalín and M.D. Rogers [205] studied polynomials that are similar to $R_{2k+1}(z)$ and that are also related to further identities of Ramanujan, and showed that their zeros lie on the unit circle. The study of the polynomials $P_k(z)$ turns out to be more difficult, and in [205], only partial results were obtained. In particular, for $2 \leq k \leq 1{,}000$, the aforementioned authors showed that all of the roots of $P_k(z)$ lie on the unit circle. Finally, Lalín and Smyth [206] proved that all zeros of $P_k(z)$ are indeed located on $|z| = 1$.

12.4 A Correct Partial Fraction Decomposition and Hyperbolic Secant Sums

As in the previous section, we alter Ramanujan's notation by setting $n = w = z^2$.

Entry 12.4.1 (p. 318, formula (22)). *If α and β are positive numbers such that $\alpha\beta = \pi^2/4$, and if $w \neq -(2m+1)^2\alpha, (2m+1)^2\beta$, $0 \leq m < \infty$, then*

$$\frac{\pi}{4}\sec(\sqrt{w\alpha})\operatorname{sech}(\sqrt{w\beta}) = \sum_{m=0}^{\infty}(-1)^m \left\{ \frac{(2m+1)\alpha \operatorname{sech}(2m+1)\alpha}{w+(2m+1)^2\alpha} - \frac{(2m+1)\beta \operatorname{sech}(2m+1)\beta}{w-(2m+1)^2\beta} \right\}. \tag{12.4.1}$$

Proof. We apply the Mittag–Leffler theorem to

$$f(z) := \frac{\pi}{4}\sec(z\sqrt{\alpha})\operatorname{sech}(z\sqrt{\beta}),$$

which has simple poles at $z = (2m+1)\pi/(2\sqrt{\alpha})$ and $z = (2m+1)\pi i/(2\sqrt{\beta})$, for each integer m. The residues are easily calculated to be

$$R((2m+1)\pi/(2\sqrt{\alpha})) = -\frac{(-1)^m\pi}{4\sqrt{\alpha}}\operatorname{sech}(2m+1)\beta \tag{12.4.2}$$

and

$$R((2m+1)\pi i/(2\sqrt{\beta})) = \frac{(-1)^m\pi}{4i\sqrt{\beta}}\operatorname{sech}(2m+1)\alpha, \tag{12.4.3}$$

where we used the relation $\alpha\beta = \pi^2/4$. By (12.4.2), the contributions from the poles $z = (2m+1)\pi/(2\sqrt{\alpha})$ and $z = -(2m+1)\pi/(2\sqrt{\alpha})$, $m \geq 0$, to the partial fraction decomposition of $f(z)$ are

$$\begin{aligned} \frac{(-1)^m\pi}{4\sqrt{\alpha}} &\left(-\frac{\operatorname{sech}(2m+1)\beta}{z-(2m+1)\pi/(2\sqrt{\alpha})} + \frac{\operatorname{sech}(2m+1)\beta}{z+(2m+1)\pi/(2\sqrt{\alpha})} \right) \\ &= -\frac{(-1)^m(2m+1)\beta \operatorname{sech}(2m+1)\beta}{z^2-(2m+1)^2\beta}, \end{aligned} \tag{12.4.4}$$

where we used the equality $\alpha\beta = \pi^2/4$. Next, by (12.4.3), the contributions of the poles $z = (2m+1)\pi i/(2\sqrt{\beta})$ and $z = -(2m+1)\pi i/(2\sqrt{\beta})$, $m \geq 0$, to the partial fraction decomposition of $f(z)$ are

$$\frac{(-1)^m\pi}{4i\sqrt{\beta}}\left(\frac{\operatorname{sech}(2m+1)\alpha}{z-(2m+1)\pi i/(2\sqrt{\beta})}-\frac{\operatorname{sech}(2m+1)\alpha}{z+(2m+1)\pi i/(2\sqrt{\beta})}\right)$$
$$=\frac{(-1)^m(2m+1)\alpha\operatorname{sech}(2m+1)\alpha}{z^2+(2m+1)^2\alpha}, \tag{12.4.5}$$

upon using the equality $\alpha\beta=\pi^2/4$. Thus, applying the Mittag–Leffler theorem and using (12.4.4) and (12.4.5), we find that there exists an entire function $g(z)$ such that

$$\frac{\pi}{4}\sec(z\sqrt{\alpha})\operatorname{sech}(z\sqrt{\beta})=\sum_{m=0}^{\infty}(-1)^m\left\{\frac{(2m+1)\alpha\operatorname{sech}(2m+1)\alpha}{z^2+(2m+1)^2\alpha}\right.$$
$$\left.-\frac{(2m+1)\beta\operatorname{sech}(2m+1)\beta}{z^2-(2m+1)^2\beta}\right\}+g(z). \tag{12.4.6}$$

Letting $z\to\infty$, we find that $\lim_{z\to\infty}g(z)=0$. Hence, $g(z)\equiv 0$, and thus (12.4.1) follows to complete the proof. □

An equivalent formulation of Entry 12.4.1 is found as Entry 19(iv) in Chap. 14 of Ramanujan's second notebook [268], [38, p. 273], where a different kind of proof was indicated by Ramanujan.

Entry 12.4.2 (p. 320, formula (30)). *If $\alpha\beta=\pi^2/4$, where α and β are positive numbers, and if r is any positive integer, then*

$$\alpha^r\sum_{m=0}^{\infty}\frac{(-1)^m(2m+1)^{2r-1}}{\cosh(2m+1)\alpha}+(-\beta)^r\sum_{m=0}^{\infty}\frac{(-1)^m(2m+1)^{2r-1}}{\cosh(2m+1)\beta}=0. \tag{12.4.7}$$

Proof. (nonrigorous) Return to (12.4.1) and formally expand the summands on the right side into geometric series to deduce that

$$\frac{\pi}{4}\sec(\sqrt{w\alpha})\operatorname{sech}(\sqrt{w\beta})$$
$$=\sum_{m=0}^{\infty}(-1)^m\left\{\frac{(2m+1)\alpha}{w}\operatorname{sech}(2m+1)\alpha\sum_{k=0}^{\infty}\left(-\frac{(2m+1)^2\alpha}{w}\right)^k\right.$$
$$\left.-\frac{(2m+1)\beta}{w}\operatorname{sech}(2m+1)\beta\sum_{k=0}^{\infty}\left(\frac{(2m+1)^2\beta}{w}\right)^k\right\}. \tag{12.4.8}$$

Equating coefficients of $1/w^r$, $r\geq 1$, on both sides of (12.4.8), we find that

$$0=\sum_{m=0}^{\infty}(-1)^{m+r-1}(2m+1)^{2r-1}\alpha^r\operatorname{sech}(2m+1)\alpha$$
$$-\sum_{m=0}^{\infty}(-1)^m(2m+1)^{2r-1}\beta^r\operatorname{sech}(2m+1)\beta,$$

which is easily seen to be equivalent to (12.4.7). □

Entry 12.4.2 is Entry 14 of Chap. 14 in Ramanujan's second notebook [268], [38, p. 262], and the first proof known to us was given by Malurkar [220]. See [38, p. 262] for further references and comments.

As with previous theorems, Ramanujan provides an alternative version of Entry 12.4.2 in terms of divisor sums. The details are similar to those above, and so we do not give them, but we remark that careful attention to the signs of the summands should be taken.

Entry 12.4.3 (p. 321, formula (31)). *If α and β are positive numbers such that $\alpha\beta = \pi^2/4$, and if r is any positive integer, then*

$$\alpha^r \sum_{m=0}^{\infty}(-1)^m \sigma_{2r-1}(m) e^{-(2m+1)\alpha} + (-\beta)^r \sum_{m=0}^{\infty}(-1)^m \sigma_{2r-1}(m) e^{-(2m+1)\beta} = 0.$$

Recall that the Euler numbers E_{2k}, $k \geq 0$, are defined by [126, p. 42, formula 1.411, no. 10]

$$\operatorname{sech} z = \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} z^{2k}, \qquad |z| < \pi/2. \tag{12.4.9}$$

Entry 12.4.4 (p. 321, formula (32)). *If α and β are positive numbers such that $\alpha\beta = \pi^2/4$, if r is any positive integer, and if χ denotes the nonprincipal primitive character of modulus 4, as in Sect. 12.2, then*

$$\begin{aligned}
&2\alpha^{1-r} \sum_{m=1}^{\infty} \frac{\chi(m) m^{1-2r}}{\cosh(m\alpha)} + 2(-\beta)^{1-r} \sum_{m=1}^{\infty} \frac{\chi(m) m^{1-2r}}{\cosh(m\beta)} \\
&\quad = 4\alpha^{1-r} \sum_{m=1}^{\infty} \chi(m) \sigma_{1-2r}(m) e^{-m\alpha} + 2(-\beta)^{1-r} \sum_{m=1}^{\infty} \chi(m) \sigma_{1-2r}(m) e^{-m\beta} \\
&\quad = \frac{\pi}{2} \sum_{k=0}^{r-1} (-1)^k \frac{E_{2k} E_{2r-2-2k}}{(2k)!(2r-2-2k)!} \alpha^{r-1-k} \beta^k.
\end{aligned} \tag{12.4.10}$$

Proof. Return to (12.4.1) and expand both sides in Taylor series about 0. Using (12.4.9), we find that

$$\begin{aligned}
&\frac{\pi}{4} \sum_{j=0}^{\infty} (-1)^j \frac{E_{2j}}{(2j)!} (w\alpha)^j \cdot \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} (w\beta)^k \\
&\quad = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \operatorname{sech}(2m+1)\alpha \sum_{r=0}^{\infty} (-1)^r \left(\frac{w}{(2m+1)^2 \alpha} \right)^r \\
&\qquad + \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \operatorname{sech}(2m+1)\beta \sum_{r=0}^{\infty} (-1)^r \left(\frac{w}{(2m+1)^2 \beta} \right)^r.
\end{aligned} \tag{12.4.11}$$

In (12.4.11) we equate coefficients of w^{r-1}, $r \geq 1$, on both sides to deduce that

$$\frac{\pi}{4}\sum_{j=0}^{r-1}(-1)^j \frac{E_{2j}E_{2r-2j-2}}{(2j)!(2r-2j-2)!}\alpha^j\beta^{r-j-1}$$
$$= \alpha^{1-r}\sum_{m=0}^{\infty}\frac{(-1)^{m+1-r}\operatorname{sech}(2m+1)\alpha}{(2m+1)^{2r-1}} + \beta^{1-r}\sum_{m=0}^{\infty}\frac{(-1)^m\operatorname{sech}(2m+1)\beta}{(2m+1)^{2r-1}}. \tag{12.4.12}$$

Now set $j = r-1-k$ in the sum on the left side of (12.4.12) and multiply both sides of (12.4.12) by $2(-1)^{r-1}$. We then readily deduce the equality of the first and third expressions in (12.4.10). The first equality of (12.4.10) follows as usual from expanding the summands on the left side into geometric series. □

Entry 12.4.4 appears in two formulations, Entries 21(ii), (iii), in Chap. 14 of Ramanujan's second notebook [268], [38, pp. 276–277]. The first proofs of Entry 12.4.4 were found by Malurkar [220] and Chowla [93], [95, pp. 143–170], and further references can be found in [38, p. 277].

Entry 12.4.5 (p. 321, formula (33)). *We have*

$$4\sum_{m=0}^{\infty}(-1)^m\sigma_{-1}(m)e^{-(2m+1)\alpha} + 4\sum_{m=0}^{\infty}(-1)^m\sigma_{-1}(m)e^{-(2m+1)\beta} = \frac{\pi}{2}.$$

Proof. Set $r = 1$ in Entry 12.4.4. □

S.-G. Lim [216] has generalized many of Ramanujan's theorems on infinite series identities from Ramanujan's notebooks [268], in particular from Chap. 14 in his second notebook, [38, Chap. 14]. For Example, Lim [216, Corollaries 3.33, 3.35] has proved the following two results that generalize Entries 12.3.3 and 12.3.5, respectively. Let α and β be positive numbers such that $\alpha\beta = \pi^2$. Suppose that c is any positive integer. Then

$$\alpha\sum_{m=1}^{\infty}\frac{m}{e^{2m(\alpha-i\pi)/c}-1} + \beta\sum_{m=1}^{\infty}\frac{m}{e^{2m(\beta+i\pi)/c}-1} = \frac{\alpha+\beta}{24} - \frac{c}{4}$$

and

$$\sum_{m=1}^{\infty}\frac{1}{m(e^{2m(\alpha-i\pi)/c}-1)} - \sum_{m=1}^{\infty}\frac{1}{m(e^{2m(\beta+i\pi)/c}-1)}$$
$$= \frac{1}{4}\log\frac{\alpha}{\beta} - \frac{\alpha-\beta}{12c} + \frac{(c-1)(c-2)\pi i}{12c}.$$

When $c = 1$ in the identities above, we deduce Entries 12.3.3 and 12.3.5, respectively.

In another paper [217], Lim has found generalizations of the results in Sect. 12.4. We state one of his general theorems.

Theorem 12.4.1. *Let α and β be positive numbers such that $\alpha\beta = \pi^2$. Let r be any real number such that $0 < r < 1$. Then, for any integer n,*

$$\begin{aligned}\alpha^{-n}\sum_{k=1}^{\infty}\frac{(-1)^k\sinh((1-2r)\alpha k)\sin((1-2r)\pi k)}{k^{2n+1}\sinh(\alpha k)}\\ = -(-\beta)^{-n}\sum_{k=1}^{\infty}\frac{(-1)^k\sinh((1-2r)\beta k)\sin((1-2r)\pi k)}{k^{2n+1}\sinh(\beta k)}\\ - 2^{2n+1}\pi\sum_{k=0}^{n}\frac{B_{2k+1}(r)B_{2n+1-2k}(r)}{(2k+1)!(2n+1-2k)!}\alpha^{n-k}(-\beta)^k,\end{aligned} \tag{12.4.13}$$

where $B_j(r)$, $j \geq 0$, denotes the jth Bernoulli polynomial.

Although we avoid providing details, setting $r = \frac{1}{4}$ in (12.4.13) yields Entries 12.4.2 and 12.4.4 [217, Corollary 3.23, Proposition 3.21].