10

Identities Related to the Riemann Zeta Function and Periodic Zeta Functions

10.1 Introduction

On page 196 in his lost notebook, Ramanujan lists several identities that are related to the Riemann zeta function, Dirichlet L-series, and periodic zeta functions. Some of the identities are connected to previous results of Ramanujan in [256] and [258], but none of the identities on page 196 can be found in these papers. Furthermore, all of the identities on page 196 are new. The purpose of this chapter is to examine all of these interesting identities. Two of the identities were examined and generalized in a paper that the second author wrote with H.H. Chan and Y. Tanigawa [47].

10.2 Identities for Series Related to $\zeta(2)$ and $L(1,\chi)$

At the top of page 196 in [269], Ramanujan records three identities related to $\zeta(2)$, and at the bottom of the page, he states a similar result related to $L(1,\chi)$, where χ is the nonprincipal primitive character modulo 4. In each of the first three identities, the coefficient 4 of the series on the right-hand side must be replaced by 2. We record the results in corrected form.

Entry 10.2.1 (p. 196). *Let* $\operatorname{Re} x \geq 0$. *Then*

$$\sum_{n=1}^{\infty} \frac{e^{-n^2\pi x}}{n^2} = \frac{\pi^2}{6} - \pi\sqrt{x} + \frac{1}{2}\pi x - 2\pi^2 x^{3/2} \sum_{n=1}^{\infty} \int_0^{\infty} t e^{-\pi(n+tx)^2/x} dt, \quad (10.2.1)$$

where the principal value of the square root is taken.

Proof. We assume throughout the proof that $x \geq 0$. The more general result for Re $x \geq 0$ will then hold by analytic continuation. We begin with the familiar theta transformation formula, which is found in Ramanujan's

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notebooks [268], [39, p. 43, Entry 27(i)]. It will be convenient, however, to use the formulation, for $\operatorname{Re} t > 0$,

$$\sum_{n=1}^{\infty} e^{-n^2\pi t} = -\frac{1}{2} + \frac{1}{2\sqrt{t}} + \frac{1}{2\sqrt{t}} \sum_{n=1}^{\infty} e^{-n^2\pi/t}, \tag{10.2.2}$$

which is found in Titchmarsh's treatise [306, p. 22, Eq. (2.6.3)], for example. Integrate both sides of (10.2.2) over $[0, x]$, invert the order of integration and summation by absolute convergence, and multiply both sides by $-\pi$ to reach the identity

$$\sum_{n=1}^{\infty} \frac{e^{-n^2\pi x}}{n^2} = \frac{\pi^2}{6} - \pi\sqrt{x} + \frac{1}{2}\pi x - \frac{\pi}{2} \sum_{n=1}^{\infty} \int_0^x \frac{e^{-n^2\pi t}}{\sqrt{t}} dt. \tag{10.2.3}$$

In comparing (10.2.3) with (10.2.1), we see that we must address the integrals on the right side of (10.2.3). First, set $t = x/u$ and then set $n^2 u = (n + tx)^2$. Hence,

$$\int_0^x \frac{e^{-n^2\pi t}}{\sqrt{t}} dt = \sqrt{x} \int_1^{\infty} \frac{e^{-n^2\pi u/x}}{u^{3/2}} du = 2x^{3/2} n \int_0^{\infty} \frac{e^{-\pi(n+tx)^2/x}}{(n+tx)^2} dt. \tag{10.2.4}$$

When examining (10.2.1) in relation to (10.2.3) and (10.2.4), we see that it remains to show that

$$2\pi \int_0^{\infty} t e^{-\pi(n+tx)^2/x} dt = n \int_0^{\infty} \frac{e^{-\pi(n+tx)^2/x}}{(n+tx)^2} dt. \tag{10.2.5}$$

Integrating the latter integral by parts, in particular integrating $1/(n + tx)^2$ and differentiating the exponential, we readily find that for $\operatorname{Re} x > 0$,

$$n \int_0^{\infty} \frac{e^{-\pi(n+tx)^2/x}}{(n+tx)^2} dt = \frac{e^{-n^2\pi/x}}{x} - \frac{2\pi n}{x} \int_0^{\infty} e^{-\pi(n+tx)^2/x} dt.$$

On the other hand, after a little trickery and then a direct integration, we find that

$$\begin{aligned} 2\pi \int_0^{\infty} t e^{-\pi(n+tx)^2/x} dt &= \frac{2\pi}{x} \int_0^{\infty} (n+tx) e^{-\pi(n+tx)^2/x} dt \\ &\quad - \frac{2\pi n}{x} \int_0^{\infty} e^{-\pi(n+tx)^2/x} dt \\ &= \frac{e^{-n^2\pi/x}}{x} - \frac{2\pi n}{x} \int_0^{\infty} e^{-\pi(n+tx)^2/x} dt. \end{aligned}$$

From these two calculations, we see that (10.2.5) has been demonstrated, and so the proof is complete. □

In Chap. 15 of his second notebook [268], [38, p. 306, Theorem 3.1], Ramanujan stated a general asymptotic formula for

$$\sum_{n=1}^{\infty} e^{-xn^p} n^{m-1},$$

as $x \to 0^+$. If we set $p = 2$ and $m = -1$, and replace x by πx in this asymptotic formula, we find that

$$\sum_{n=1}^{\infty} \frac{e^{-n^2\pi x}}{n^2} = \frac{\pi^2}{6} - \pi\sqrt{x} + \frac{1}{2}\pi x + o(1), \tag{10.2.6}$$

as $x \to 0^+$, which should be compared with (10.2.1). In [38, pp. 306–308], a proof of Ramanujan's general asymptotic formula was obtained by contour integration. In the course of this proof, the error term, i.e., $o(1)$ in (10.2.6), is represented by a certain contour integral. It seems to be very difficult, however, to transform this contour integral into the expression involving the infinite series on the right-hand side of (10.2.1).

Entry 10.2.2 (p. 196). *Let* $\operatorname{Re} x \geq 0$. *Then*

$$\sum_{n=1}^{\infty} \frac{\cos(n^2\pi x)}{n^2} = \frac{\pi^2}{6} - \pi\sqrt{\frac{x}{2}} + 2\pi^2 x^{3/2} \times \sum_{n=1}^{\infty} \int_0^{\infty} te^{-2n\pi t} \cos\left(\frac{\pi}{4} - \frac{\pi n^2}{x} + \pi t^2 x\right) dt \tag{10.2.7}$$

and

$$\sum_{n=1}^{\infty} \frac{\sin(n^2\pi x)}{n^2} = \pi\sqrt{\frac{x}{2}} - \frac{1}{2}\pi x + 2\pi^2 x^{3/2} \times \sum_{n=1}^{\infty} \int_0^{\infty} te^{-2n\pi t} \sin\left(\frac{\pi}{4} - \frac{\pi n^2}{x} + \pi t^2 x\right) dt, \tag{10.2.8}$$

where the principal value of the square root is taken.

Proof. As in the previous proof, we shall assume that $x \geq 0$; an appeal to analytic continuation then establishes Entry 10.2.2 for $\operatorname{Re} x \geq 0$. We shall prove (10.2.7) and (10.2.8) with x replaced by y. In (10.2.1), replace x by $z = x + iy$, with $y \geq 0$. Let $\theta = \arg z$. Then (10.2.1) takes the form

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{e^{-n^2\pi z}}{n^2} &= \frac{\pi^2}{6} - \pi|z|^{1/2}(\cos\tfrac{1}{2}\theta + i\sin\tfrac{1}{2}\theta) + \frac{1}{2}\pi(x+iy) \\ &- 2\pi^2|z|^{3/2}(\cos\tfrac{3}{2}\theta + i\sin\tfrac{3}{2}\theta) \\ &\times \int_0^{\infty} t\exp\left\{-\frac{\pi}{|z|^2}\left((n+tx)^2 + 2it(n+tx)y - t^2y^2\right)(x-iy)\right\} dt.\end{aligned} \tag{10.2.9}$$

Now,

$$\begin{aligned}E(x,y) &:= -2\pi^2|z|^{3/2}(\cos\tfrac{3}{2}\theta + i\sin\tfrac{3}{2}\theta)t\\ &\quad\times \exp\left\{-\frac{\pi}{|z|^2}\left((n+tx)^2 + 2it(n+tx)y - t^2y^2\right)(x-iy)\right\}\\ &= -2\pi^2|z|^{3/2}(\cos\tfrac{3}{2}\theta + i\sin\tfrac{3}{2}\theta)t\exp\Bigg(-\frac{\pi}{|z|^2}\left((n+tx)^2 - t^2y^2\right)x\\ &\quad + 2t(n+tx)y^2 + i\left(2tx(n+tx)y - y(n+tx)^2 - t^2y^3\right)\Bigg),\end{aligned}$$

from which we see that

$$\begin{aligned}\operatorname{Re} E(x,y) &= -2\pi^2|z|^{3/2}\cos\tfrac{3}{2}\theta\, t\exp\left\{-\frac{\pi}{|z|^2}\left((n+tx)^2 - t^2y^2\right)x\right\}\\ &\quad\times\cos\frac{\pi}{|z|^2}\left(2tx(n+tx)y - y(n+tx)^2 - t^2y^3\right)\\ &\quad + 2\pi^2|z|^{3/2}\sin\tfrac{3}{2}\theta\, t\exp\left\{-\frac{\pi}{|z|^2}\left((n+tx)^2 - t^2y^2\right)x\right\}\\ &\quad\times\sin\frac{\pi}{|z|^2}\left(2tx(n+tx)y - y(n+tx)^2 - t^2y^3\right).\end{aligned}$$

Setting $x = 0$ and $\theta = \frac{1}{2}\pi$, we find that

$$\begin{aligned}\operatorname{Re} E(0,y) &= 2\pi^2 y^{3/2}\frac{1}{\sqrt{2}}te^{-2n\pi t}\cos\left(-\frac{\pi n^2}{y} + \pi t^2 y\right)\\ &\quad - 2\pi^2 y^{3/2}\frac{1}{\sqrt{2}}te^{-2n\pi t}\sin\left(-\frac{\pi n^2}{y} + \pi t^2 y\right)\\ &= 2\pi^2 y^{3/2}te^{-2n\pi t}\cos\left(\frac{\pi}{4} - \frac{\pi n^2}{y} + \pi t^2 y\right).\end{aligned} \tag{10.2.10}$$

If we now use (10.2.10) in (10.2.9), we deduce (10.2.7) with x replaced by y.

A similar calculation of $\operatorname{Im} E(x,y)$ followed by setting $x = 0$ and $\theta = \frac{1}{2}\pi$ yields (10.2.8) with x replaced by y. □

Entry 10.2.3 (p. 196). *For $x \geq 0$,*

$$\sum_{n=0}^{\infty}(-1)^n\frac{e^{-(2n+1)^2\pi x/4}}{2n+1} = \frac{\pi}{4} - \pi\sqrt{x}\sum_{n=0}^{\infty}(-1)^n\int_0^{\infty} e^{-\pi(2n+1+2tx)^2/(4x)}dt. \tag{10.2.11}$$

Proof. We begin by specializing the well-known theta relation for an odd primitive character [101, p. 70, Eq. (9)]. In our case, this odd primitive character is the real nonprincipal character modulo 4. Accordingly, for $t > 0$,

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)e^{-(2n+1)^2\pi t/4} = t^{-3/2}\sum_{n=0}^{\infty}(-1)^n(2n+1)e^{-(2n+1)^2\pi/(4t)}. \tag{10.2.12}$$

Integrate both sides of (10.2.12) over $[0, x]$, and then multiply both sides by $-\pi/4$ to deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n \frac{e^{-(2n+1)^2\pi x/4}}{2n+1} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} - \frac{\pi}{4}\sum_{n=0}^{\infty}(-1)^n(2n+1)\int_0^x t^{-3/2}e^{-(2n+1)^2\pi/(4t)}dt \\ &= \frac{\pi}{4} - \frac{\pi\sqrt{x}}{4}\sum_{n=0}^{\infty}(-1)^n(2n+1)\int_1^{\infty}\frac{e^{-(2n+1)^2\pi u/(4x)}}{\sqrt{u}}du, \qquad (10.2.13)\end{aligned}$$

where we utilized Leibniz's series for $\pi/4$ and made the substitution $t = x/u$ in the integrals on the right side. Next, set $(2n+1)^2u = (2n+1+2tx)^2$. Then

$$\int_1^{\infty}\frac{e^{-(2n+1)^2\pi u/(4x)}}{\sqrt{u}}du = \frac{4x}{2n+1}\int_0^{\infty} e^{-(2n+1+2tx)^2\pi/(4x)}dt. \qquad (10.2.14)$$

If we substitute (10.2.14) into (10.2.13), we obtain (10.2.11) to complete the proof. □

10.3 Analogues of Gauss Sums

We now offer three claims from the middle of page 196 of [269]. These were first proved in a more general setting by Berndt, Chan, and Tanigawa [47]. More precisely, they evaluate the sum

$$\sum_{n=1}^{\infty}\frac{e^{2\pi i n^2/k}}{n^{2m}},$$

where m and k are positive integers, in several ways, obtaining evaluations in terms of trigonometric functions, Stirling numbers of the second kind, and ballot numbers. On page 196, Ramanujan considers only the case $m = 1$.

Entry 10.3.1 (p. 196). *Let a be an even positive integer. Then*

$$\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi n^2}{a}\right)}{n^2} = \frac{\pi^2}{6} - \frac{\pi^2}{\sqrt{a}}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\sin\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right), \qquad (10.3.1)$$

$$\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi n^2}{a}\right)}{n^2} = -\frac{\pi^2}{\sqrt{a}}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\cos\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right), \qquad (10.3.2)$$

$$\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi}{4}+\frac{\pi n^2}{a}\right)}{n^2} = \frac{\pi^2}{6\sqrt{2}} - \frac{\pi^2}{\sqrt{a}}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\cos\left(\frac{\pi r^2}{a}\right). \qquad (10.3.3)$$

We first prove (10.3.3) assuming the truth of (10.3.1) and (10.3.2).

Proof of (10.3.3) *of Entry* 10.3.1. Using the addition formulas for sin and cos, we easily find that

$$\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi}{4}+\frac{\pi n^2}{a}\right)}{n^2}=\frac{1}{\sqrt{2}}\left(\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi n^2}{a}\right)}{n^2}+\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi n^2}{a}\right)}{n^2}\right)$$
$$=\frac{\pi^2}{6\sqrt{2}}-\frac{\pi^2}{\sqrt{2}a}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\left\{\cos\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right)+\sin\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right)\right\}$$
$$=\frac{\pi^2}{6\sqrt{2}}-\frac{\pi^2}{\sqrt{a}}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\cos\left(\frac{\pi r^2}{a}\right).$$

□

We evaluate the more general series

$$S_a(r):=\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi n^2}{a}\right)}{n^r}\qquad\text{and}\qquad T_a(r):=\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi n^2}{a}\right)}{n^r},\tag{10.3.4}$$

where r is an even positive integer. In order to effect these evaluations, we need to introduce periodic Bernoulli numbers.

Let $A=\{a_n\}$, $-\infty<n<\infty$, denote a sequence of numbers with period k. Then the periodic Bernoulli numbers $B_n(A)$, $n\geq 0$, can be defined [66, p. 55, Proposition 9.1], for $|z|<2\pi/k$, by

$$\frac{z\sum_{n=0}^{k-1}a_ne^{nz}}{e^{kz}-1}=\sum_{n=0}^{\infty}\frac{B_n(A)}{n!}z^n.$$

Furthermore [66, p. 56, Eq. (9.5)], for each positive integer n,

$$B_n(A)=k^{n-1}\sum_{j=0}^{k-1}a_{-j}B_n\left(\frac{j}{k}\right),\tag{10.3.5}$$

where $B_n(x)$, $n\geq 0$, denotes the nth Bernoulli polynomial. We say that A is even if $a_n=a_{-n}$ for every integer n.

The complementary sequence $B=\{b_n\}$, $-\infty<n<\infty$, is defined by [66, p. 32]

$$b_n=\frac{1}{k}\sum_{j=0}^{k-1}a_je^{-2\pi ijn/k}.\tag{10.3.6}$$

It is easily checked that if A is even, then B is even, and that (10.3.6) holds if and only if

$$a_n = \sum_{j=0}^{k-1} b_j e^{2\pi ijn/k}, \quad -\infty < n < \infty. \tag{10.3.7}$$

Now set

$$\zeta(s;A) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \operatorname{Re} s > 1.$$

If A and r are even and if $r \geq 2$, then [66, p. 49, Eq. (6.25)]

$$\zeta(r;B) = \frac{(-1)^{r+1} B_r(A)}{2\,r!} \left(\frac{2\pi i}{k}\right)^r.$$

From (10.3.6) and (10.3.7), we see that the sequences A and B are not symmetric. Thus, we note from above that since A is even,

$$\zeta(r;A) = \frac{(-1)^{r+1} B_r(B)k}{2\,r!} \left(\frac{2\pi i}{k}\right)^r. \tag{10.3.8}$$

We are now ready to state general evaluations in closed form for $S_a(r)$ and $T_a(r)$.

Theorem 10.3.1. *If $S_a(r)$ and $T_a(r)$ are defined by (10.3.4) and if r and a are even positive integers, then*

$$S_a(r) = \frac{(-1)^{1+r/2} 2^{r-1} \pi^r}{r!\sqrt{a}} \sum_{m=0}^{a-1} B_r\left(\frac{m}{a}\right) \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) \tag{10.3.9}$$

and

$$T_a(r) = \frac{(-1)^{1+r/2} 2^{r-1} \pi^r}{r!\sqrt{a}} \sum_{m=0}^{a-1} B_r\left(\frac{m}{a}\right) \cos\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right). \tag{10.3.10}$$

In our work below, we need the value of the Gauss sum [54, p. 43, Exercise 5]

$$\sum_{n=0}^{c-1} e^{\pi i n^2/c} = e^{\pi i/4}\sqrt{c}, \tag{10.3.11}$$

where c is an even positive integer.

Before proceeding further, we show that (10.3.1) and (10.3.2) are special cases of (10.3.9) and (10.3.10), respectively. Let $r = 2$ in Theorem 10.3.1. Recall that $B_2(x) = x^2 - x + \frac{1}{6}$. Then

$$\begin{aligned}
S_a(2) &= \frac{\pi^2}{\sqrt{a}} \sum_{m=0}^{a-1} \left\{ \left(\frac{m}{a}\right)^2 - \frac{m}{a} + \frac{1}{6} \right\} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) \\
&= \frac{\pi^2}{6\sqrt{a}} \sum_{m=0}^{a-1} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) + \frac{\pi^2}{\sqrt{a}} \sum_{m=0}^{a-1} \left\{ \left(\frac{m}{a}\right)^2 - \frac{m}{a} \right\} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) \\
&= \frac{\pi^2}{6} + \frac{\pi^2}{\sqrt{a}} \sum_{m=0}^{a-1} \left\{ \left(\frac{m}{a}\right)^2 - \frac{m}{a} \right\} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right),
\end{aligned}$$

upon the use of (10.3.11) twice.

The proof of (10.3.2) follows along the same lines, but note that in this case, by (10.3.11),

$$\sum_{m=0}^{a-1} \cos\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) = 0.$$

Proof of Theorem 10.3.1. Let

$$a_n = \cos\left(\frac{\pi n^2}{a}\right), \qquad -\infty < n < \infty,$$

which is an even periodic sequence with period a, since a is even. Then, from (10.3.6) and (10.3.11),

$$\begin{aligned}
b_{-m} &= \frac{1}{a} \sum_{j=0}^{a-1} \cos\left(\frac{\pi j^2}{a}\right) e^{2\pi i j m/a} \\
&= \frac{1}{2a} e^{-\pi i m^2/a} \sum_{j=0}^{a-1} e^{\pi i (j+m)^2/a} + \frac{1}{2a} e^{\pi i m^2/a} \sum_{j=0}^{a-1} e^{-\pi i (j+m)^2/a} \\
&= \frac{1}{2a} e^{-\pi i m^2/a} \sum_{j=0}^{a-1} e^{\pi i j^2/a} + \frac{1}{2a} e^{\pi i m^2/a} \sum_{j=0}^{a-1} e^{-\pi i j^2/a} \\
&= \frac{1}{2a} e^{-\pi i m^2/a + \pi i/4} \sqrt{a} + \frac{1}{2a} e^{\pi i m^2/a - \pi i/4} \sqrt{a} \\
&= \frac{1}{\sqrt{a}} \cos\left(\frac{\pi m^2}{a} - \frac{\pi}{4}\right) \\
&= \frac{1}{\sqrt{a}} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right).
\end{aligned}$$

Therefore, by (10.3.5), with B in place of A,

$$B_n(B) = a^{n-3/2} \sum_{m=0}^{a-1} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) B_n\left(\frac{m}{a}\right). \tag{10.3.12}$$

If we substitute (10.3.12) into (10.3.8) and simplify, we deduce (10.3.9).

The proof of (10.3.10) is analogous to that for (10.3.9). Now we set

$$a_n = \sin\left(\frac{\pi n^2}{a}\right), \qquad -\infty < n < \infty,$$

which of course is even, and repeat the same kind of argument that we gave above. □

We now provide another evaluation of the series on the left-hand sides of (10.3.1) and (10.3.2) in closed form. However, we obtain evaluations in entirely different forms from those claimed by Ramanujan in Entry 10.3.1.

Theorem 10.3.2. *Let a be an even positive integer, $a \geq 2$. Then*

$$S_a(2) = \frac{\pi^2}{6a^2} + \frac{\pi^2\cos(a\pi/4)}{2a^2} + \frac{\pi^2}{a^2}\sum_{j=1}^{a/2-1}\cos\left(\frac{j^2\pi}{a}\right)\csc^2\left(\frac{j\pi}{a}\right) \tag{10.3.13}$$

and

$$T_a(2) = \frac{\pi^2\sin(a\pi/4)}{2a^2} + \frac{\pi^2}{a^2}\sum_{j=1}^{a/2-1}\sin\left(\frac{j^2\pi}{a}\right)\csc^2\left(\frac{j\pi}{a}\right). \tag{10.3.14}$$

Proof. Setting $n = ka + j$, $0 \leq k < \infty$, $1 \leq j \leq a$, we find that

$$\begin{aligned} S_a(2) &= \sum_{j=1}^{a}\cos\left(\frac{j^2\pi}{a}\right)\sum_{k=0}^{\infty}\frac{1}{(ka+j)^2} \\ &= \frac{\pi^2}{6a^2} + \frac{1}{a^2}\sum_{j=1}^{a-1}\cos\left(\frac{j^2\pi}{a}\right)\sum_{k=0}^{\infty}\frac{1}{(k+j/a)^2}. \end{aligned} \tag{10.3.15}$$

Singling out the term for $j = a/2$ and noting that the terms in the outer sum with indices j and $a - j$ are identical, we find from (10.3.15) that

$$\begin{aligned} S_a(2) &= \frac{\pi^2}{6a^2} + \frac{\pi^2\cos(a\pi/4)}{2a^2} + \frac{1}{a^2}\sum_{j=1}^{a/2-1}\cos\left(\frac{j^2\pi}{a}\right) \\ &\quad\times\left(\sum_{k=0}^{\infty}\frac{1}{(k+j/a)^2} + \sum_{k=0}^{\infty}\frac{1}{(k+(a-j)/a)^2}\right) \\ &=: \frac{\pi^2}{6a^2} + \frac{\pi^2\cos(a\pi/4)}{2a^2} + \frac{1}{a^2}\sum_{j=1}^{a/2-1}\cos\left(\frac{j^2\pi}{a}\right)U(j,a), \end{aligned} \tag{10.3.16}$$

say. There remains the evaluation of $U(j, a)$.

First observe that if for $-\infty < k \leq -1$, we set $k = -r - 1$, then

$$\sum_{k=-\infty}^{\infty}\left(\frac{1}{(k+j/a)^2}+\frac{1}{(k+(a-j)/a)^2}\right)$$
$$=\sum_{k=0}^{\infty}\left(\frac{1}{(k+j/a)^2}+\frac{1}{(k+(a-j)/a)^2}\right)$$
$$+\sum_{r=0}^{\infty}\left(\frac{1}{(-r-1+j/a)^2}+\frac{1}{(-r-j/a)^2}\right)=2U(j,a). \tag{10.3.17}$$

It therefore suffices to evaluate the bilateral sum in (10.3.17).

To evaluate $U(j,a)$, recall the partial fraction decomposition

$$\pi\,\cot(\pi z)=\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}+\frac{1}{z-n}\right).$$

Differentiating once above, we find that

$$\pi^2\csc^2(\pi z)=\sum_{n=-\infty}^{\infty}\frac{1}{(z+n)^2}. \tag{10.3.18}$$

Putting $z=r/k$ in (10.3.18), we deduce that

$$U(j,a)=\pi^2\csc^2(\pi r/k). \tag{10.3.19}$$

Putting (10.3.19) in (10.3.16), we complete the proof of (10.3.13).

The proof of (10.3.14) follows along exactly the same lines. In analogy with (10.3.17), we now easily deduce that

$$T_a(2)=\frac{1}{a^2}\sum_{j=1}^{a}\sin\left(\frac{j^2\pi}{a}\right)\sum_{k=0}^{\infty}\frac{1}{(k+j/a)^2}.$$

By the same identical argument that we used above, we conclude that

$$T_a(2)=\frac{\pi^2\sin(a\pi/4)}{2a^2}+\frac{\pi^2}{a^2}\sum_{j=1}^{a/2-1}\sin\left(\frac{j^2\pi}{a}\right)\csc^2\left(\frac{j\pi}{a}\right).$$

□

We record a few examples to illustrate Theorem 10.3.2, namely,

$$S_2(2)=\frac{\pi^2}{24},\qquad S_4(2)=-\frac{\pi^2}{48}+\frac{\pi^2\sqrt{2}}{16},\qquad S_6(2)=-\frac{\pi^2}{72}+\frac{\pi^2\sqrt{3}}{18},$$
$$T_2(2)=\frac{\pi^2}{8},\qquad T_4(2)=\frac{\pi^2\sqrt{2}}{16},\qquad T_6(2)=\frac{\pi^2}{24}+\frac{\pi^2\sqrt{3}}{54}.$$

Equating the evaluations of $S_a(2)$ and $T_a(2)$ in (10.3.13) and (10.3.14), respectively, with those in (10.3.1) and (10.3.2), we obtain identities that would be surprising if we had not known of their origins, namely,

$$\frac{\pi^2}{6a^2}+\frac{\pi^2\cos(\pi a/4)}{2a^2}+\frac{\pi^2}{a^2}\sum_{j=1}^{\frac{1}{2}a-1}\cos\left(\frac{\pi j^2}{a}\right)\csc^2\left(\frac{\pi j}{a}\right)$$
$$=\frac{\pi^2}{6}-\frac{\pi^2}{\sqrt{a}}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\sin\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right)$$

and

$$\frac{\pi^2\sin(\pi a/4)}{2a^2}+\frac{\pi^2}{a^2}\sum_{j=1}^{\frac{1}{2}a-1}\sin\left(\frac{\pi j^2}{a}\right)\csc^2\left(\frac{\pi j}{a}\right)$$
$$=-\frac{\pi^2}{\sqrt{a}}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\cos\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right).$$

Note that on the left-hand sides above, the sums contain only trigonometric functions, while on the right-hand sides the sums contain both polynomials and trigonometric functions. Trigonometric identities involving polynomials in the summands appear to be rare. The sums on both sides of the identities may be regarded as new analogues of Gauss sums.

In fact, Ramanujan states a second equality for the sum on the left side of (10.3.3). We slightly reformulate this result in the next entry.

Entry 10.3.2 (p. 196). *If a is an even positive integer, then*

$$\frac{4\pi^2}{a^{3/2}}\left\{\frac{1}{8\pi}+\sum_{n=1}^{\infty}\frac{n\cos(\pi n^2/a)}{e^{2n\pi}-1}\right\}-2^{3/2}\pi^2\left\{\frac{1}{8\pi a}+\sum_{n=1}^{\infty}\frac{n}{e^{2n\pi a}-1}\right\}$$
$$=-\frac{\pi^2}{a^{5/2}}\sum_{r=1}^{a}r(a-r)\cos\left(\frac{\pi r^2}{a}\right). \quad (10.3.20)$$

Proof. Our proof depends on two results from Ramanujan's papers [256, 262]. First, if a is an even positive integer [262, Eq. (17)], [267, p. 132], then

$$\frac{1}{8\pi}+\sum_{n=1}^{\infty}\frac{n\cos(\pi n^2/a)}{e^{2n\pi}-1}=\int_0^{\infty}\frac{x\cos(\pi x^2/a)}{e^{2\pi x}-1}dx+a\sqrt{\tfrac{1}{2}a}\sum_{n=1}^{\infty}\frac{n}{e^{2n\pi a}-1}. \quad (10.3.21)$$

Now, from [256, Eq. (50)], [267, p. 67],

$$\int_0^{\infty}\frac{x\cos(\pi x^2/a)}{e^{2\pi x}-1}dx=\frac{1}{2}\int_0^{\infty}\frac{\cos(\pi u/a)}{e^{2\pi\sqrt{u}}-1}du$$
$$=\frac{1}{2}\left(\frac{\sqrt{a/2}}{4\pi}-\frac{1}{2a}\sum_{r=1}^{a}r(a-r)\cos\left(\frac{\pi r^2}{a}\right)\right)$$
$$=\frac{\sqrt{a}}{8\pi\sqrt{2}}-\frac{1}{4a}\sum_{r=1}^{a}r(a-r)\cos\left(\frac{\pi r^2}{a}\right). \quad (10.3.22)$$

If we substitute (10.3.22) in (10.3.21) and then multiply both sides of the resulting equality by $4\pi^2/a^{3/2}$, we deduce that

$$\begin{aligned}\frac{\pi}{2a^{3/2}} + \frac{4\pi^2}{a^{3/2}} \sum_{n=1}^{\infty} \frac{n\cos(\pi n^2/a)}{e^{2n\pi}-1} \\ = \frac{\pi}{2\sqrt{2}a} - \frac{\pi^2}{a^{5/2}} \sum_{r=1}^{a} r(a-r)\cos\left(\frac{\pi r^2}{a}\right) + \frac{4\pi^2}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{n}{e^{2n\pi a}-1},\end{aligned}$$

which is easily seen to be equivalent to (10.3.20). □