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Introduction

In contrast to our first three volumes [12–14] devoted to *Ramanujan's Lost Notebook and Other Unpublished Papers* [269], this volume does not focus on q-series. Number theory and classical analysis are in the spotlight in the present book, which is the fourth of five projected volumes, wherein the authors plan to discuss all the claims made by Ramanujan in [269]. As in our previous volumes, in the sequel, we liberally interpret *lost notebook* not only to include the original lost notebook found by the first author in the library at Trinity College, Cambridge, in March 1976, but also to include all of the material published in [269]. This includes letters that Ramanujan wrote to G.H. Hardy from nursing homes, several partial manuscripts, and miscellaneous papers. Some of these manuscripts are located at Oxford University, are in the handwriting of G.N. Watson, and are "copied from loose papers." However, it should be emphasized that the original manuscripts in Ramanujan's handwriting can be found at Trinity College Library, Cambridge.

We now relate some of the highlights in this volume, while at the same time offering our thanks to several mathematicians who helped prove some of these results.

Chapter 2 is devoted to two intriguing identities involving double series of Bessel functions found on page 335 of [269]. One is connected with the classical *circle problem*, while the other is conjoined to the equally famous *Dirichlet divisor problem.* The double series converge very slowly, and the identities were extremely difficult to prove. Initially, the second author and his collaborators, Sun Kim and Alexandru Zaharescu, were not able to prove the identities with the order of summation as prescribed by Ramanujan, i.e., the identities were proved with the order of summation reversed [57, 71]. It is possible that Ramanujan intended that the summation indices should tend to infinity "together." The three authors therefore also proved the two identities with the product of the summation indices tending to ∞ [57]. Finally, these authors proved Ramanujan's first identity with the order of summation as prescribed by Ramanujan [60]. It might be remarked here that the proofs under the three interpretations of the summation indices are entirely different; the authors

G.E. Andrews and B.C. Berndt, *Ramanujan's Lost Notebook: Part IV*, DOI 10.1007/978-1-4614-4081-9_1,

did not use any idea from one proof in the proofs of the same identity under different interpretations. In Chap. 2, we provide proofs of the two identities with the order of summation indicated by Ramanujan in the first identity and with the order of summation reversed in the second identity. We also establish the identities when the product of the two indices of summation tends to infinity. In addition to thanking Sun Kim and Alexandru Zaharescu for their collaborations, the present authors also thank O-Yeat Chan, who performed several calculations to discern the convergence of these and related series.

It came as a huge surprise to us while examining pages in [269] when we espied famous formulas of N.S. Koshliakov and A.P. Guinand, although Ramanujan wrote them in slightly disguised forms. Moreover, we discovered that Ramanujan had found some consequences of these formulas that had not theretofore been found by any other authors. We are grateful to Yoonbok Lee and Jaebum Sohn for their collaboration on these formulas, which are the focus of Chap. 3.

Chapter 4, on the classical gamma function, features two sets of claims. We begin the chapter with some integrals involving the gamma function in the integrands. Secondly, we examine a claim that reverts to a problem [260] that Ramanujan submitted to the *Journal of the Indian Mathematical Society*, which was never completely solved. On page 339 in [269], Ramanujan offers a refinement of this problem, which was proved by the combined efforts of Ekaterina Karatsuba [177] and Horst Alzer [4].

Hypergeometric functions are featured in Chap. 5. This chapter contains two particularly interesting results. The first is an explicit representation for a quotient of two particular bilateral hypergeometric series, which was proved in a paper [50] by the second author and Wenchang Chu, whom we thank for his expert collaboration. We also appreciate correspondence with Tom Koornwinder about one particular formula on bilateral series that was crucial in our proof. Ramanujan's formula is so unexpected that no one but Ramanujan could have discovered it! The second is a beautiful continued fraction, for which Soon-Yi Kang, Sung-Geun Lim, and Sohn [175] found two entirely different proofs, each providing a different understanding of the entry. A further beautiful continued fraction of Ramanujan was only briefly examined in [175], but Kang supplied us with a very nice proof, which appears here for the first time.

Chapter 6 contains accounts of two incomplete manuscripts on Euler's constant γ, one of which was coauthored by the second author with Doug Bowman [46] and the other of which was coauthored by the second author with Tim Huber [55].

Sun Kim kindly collaborated with the second author on Chap. 7, on an unusual problem examined in a rough manuscript by Ramanujan on Diophantine approximation [56]. She also worked with the second author and Zaharescu on another partial manuscript providing the best possible Diophantine approximation to $e^{2/a}$, where a is any nonzero integer [61].

This manuscript was another huge surprise to us, for it had never been noticed by anyone, to the best of our knowledge, that Ramanujan had derived the best possible Diophantine approximation to $e^{2/a}$, which was first proved in print approximately 60 years after Ramanujan had found his proof. A third manuscript on Diophantine approximation in [269] turned out to be without substance, unless we have grossly misinterpreted Ramanujan's claims on page 343 of [269].

We next collect some results from number theory, not all of which are correct. At the beginning of Chap. 8, in Sect. 8.1, we relate that Ramanujan had anticipated the famous work of L.G. Sathe [275–278] and A. Selberg [281] on the distribution of primes, although Ramanujan did not state any specific theorems. In prime number theory, Dickman's function is a famous and useful function, but in Sect. 8.2, we see that Ramanujan had discovered Dickman's function at least 10 years before Dickman did in 1930 [106]. A.J. Hildebrand, a colleague of the second author, supplied a clever proof of Ramanujan's formula for, in standard notation, $\Psi(x, x^{\epsilon})$ and then provided us with a heuristic argument that might have been the approach used by Ramanujan. We then turn to a formula for $\zeta(\frac{1}{2})$, first given in Sect. 8 of Chap. 15 in Ramanujan's second notebook. In [269], Ramanujan offers an elegant reinterpretation of this formula, which renders an already intriguing result even more fascinating. Next, we examine a fragment on sums of powers that was very difficult to interpret; our account of this fragment is taken from a paper by D. Schultz and the second author [67]. One of the most interesting results in the chapter yields an unusual algorithm for generating solutions to Euler's diophantine equation $a^3 + b^3 = c^3 + d^3$. This result was established in different ways by Mike Hirschhorn in a series of papers [141, 158–160].

Chapter 9 is devoted to discarded fragments of manuscripts and partial manuscripts concerning the divisor functions $\sigma_k(n)$ and $d(n)$, respectively, the sum of the kth powers of the divisors of n, and the number of divisors of n. Some of this work is related to Ramanujan's paper [265]. An account of one of these fragments appeared in a paper that the second author coauthored with Prapanpong Pongsriiam [63].

In the next chapter, Chap. 10, we prove all of the results on page 196 of [269]. Two of the results evaluating certain Dirichlet series are especially interesting. A more detailed examination of these results can be found in a paper that the second author coauthored with Heng Huat Chan and Yoshio Tanigawa [47].

Chapter 11 contains some unusual old and new results on primes arranged in two rough, partial manuscripts. Ramanujan's manuscripts contain several errors, and we conjecture that this work predates his departure for England in 1914. Harold Diamond helped us enormously in both interpreting and correcting the claims made by Ramanujan in the two partial manuscripts examined in Chap. 11.

In Chap. 12, we discuss a manuscript that was either intended to be a paper by itself or, more probably, was slated to be the concluding portion of

Ramanujan's paper [263]. The results in this paper hark back to Ramanujan's early preoccupation with infinite series identities and the material in Chap. 14 of his second notebook [38, 268]. The second author had previously published an account of this manuscript [42]. Our account here includes a closer examination of two of Ramanujan's series by Johann Thiel, to whom we are very grateful for his contributions.

Perhaps the most fascinating formula found in the three manuscripts on Fourier analysis in the handwriting of Watson is a transformation formula involving the Riemann Ξ-function and the logarithmic derivative of the gamma function in Chap. 13. We are pleased to thank Atul Dixit, who collaborated with the second author on several proofs of this formula. One of the hallmarks of Ramanujan's mathematics is that it frequently generates further interesting mathematics, and this formula is no exception. In a series of papers [108–111], Dixit found analogues of this formula and found new bonds with the Ξ-function, in particular, with the beautiful formulas of Guinand and Koshliakov.

The second of the aforementioned manuscripts features integrals that possess transformation formulas like those satisfied by theta functions. Two of the integrals were examined by Ramanujan in two papers [256, 258], [267, pp. 59–67, 202–207], where he considered the integrals to be analogues of Gauss sums, a view that we corroborate in Chap. 14. One of the integrals, to which page 198 of [269] is devoted, was not examined earlier by Ramanujan. Ping Xu and the second author established Ramanujan's claims for this integral in [69]; the account given in Chap. 14 is slightly improved in places over that in [69]. (The authors are grateful to Noam Elkies for a historical note at the end of Sect. 14.1.)

In the third manuscript, on Fourier analysis, which we discuss in Chap. 15, Ramanujan considers some problems on Mellin transforms.

The next three chapters pertain to some of Ramanujan's earlier published papers. We then consider miscellaneous collections of results in classical analysis and elementary mathematics in the next two chapters.

Chapter 21 is devoted to some strange, partially incorrect claims of Ramanujan that likely originate from an early part of his career.

In summary, the second author is exceedingly obliged to his coauthors Doug Bowman, O.-Yeat Chan, Wenchang Chu, Atul Dixit, Tim Huber, Sun Kim, Yoonbok Lee, Sung-Geun Lim, Prapanpong Pongsriiam, Dan Schultz, Jaebum Sohn, Ping Xu, and Alexandru Zaharescu for their contributions.

As with earlier volumes, Jaebum Sohn carefully read several chapters and offered many corrections and helpful comments, for which we are especially grateful. Michael Somos offered several proofs for Chap. 20 and numerous corrections in several other chapters. Mike Hirschhorn also contributed many useful remarks.

We offer our gratitude to Harold Diamond, Andrew Granville, A.J. Hildebrand, Pieter Moree, Kannan Soundararajan, Gérald Tenenbaum, and Robert Vaughan for their comments that greatly enhanced our discussion of the

Sathe–Selberg results and Dickman's function in Sects. 8.1 and 8.2, respectively. Atul Dixit uncovered several papers by Guinand and Koshliakov of which the authors had not previously been aware. Most of Sect. 8.7 was kindly supplied to us by Jean-Louis Nicolas, with M.Tip Phaovibul also providing valuable insights. Useful correspondence with Ron Evans about Mersenne primes is greatly appreciated.

We offer our sincere thanks to Springer's TEX experts, Suresh Kumar and Rajiv Monsurate, for much technical advice, and to Springer copy editor David Kramer for several corrections and helpful remarks.

We thank the Master and Fellows of Trinity College, Cambridge, for providing the authors with the manuscripts to which this book is devoted.

The first author thanks both the National Science Foundation and the National Security Agency for financial support. The second author also thanks the NSA for support, and is particularly grateful to the Sloan Foundation for one year of financial support that enabled the authors to complete this book.