Chapter 197
Boundedness Estimates to a Steady State Nonlinear Fourth Order Elliptic Equation

Bo Liang, Xiaoming Dai, and Meishan Wang

Abstract In the paper, we study a fourth order partial differential equation which appears in the description of the motion of a very thin layer of viscous incompressible fluids and in the phase transformation theory. In order to prove the existence, a truncation system is studied. By applying the test function method and an iteration technique, some a-prior estimates of solutions to the steady state problem are obtained. Finally, the boundedness estimates are gained for the truncation problem. The results will have important in the existence of steady state thin film equations.

Keywords Fourth order • Parabolic • Higher order

197.1 Introduction

Many people have begun to study the nonlinear fourth order partial differential equations including the thin film equation and Cahn-Hilliard equation. The thin film equation reads

$$u_t + (u^n u_{xxx})_x = 0$$

Which appears in the description of the motion of a very thin layer of viscous incompressible fluids along an inclined plane and is derived from a lubrication approximation. Here the index $n > 0$ is a constant and the unknown function u represents the thickness of some flow films. The readers may refer to [1–8].

If letting $n = 0$, the thin film equation is the classic Cahn-Hilliard-type equation (see [3]):

B. Liang (✉) • X. Dai • M. Wang
School of Science, Dalian Jiaotong University, Dalian 116028, People's Republic of China
e-mail: cnliangbo@163.com; dxmbird@163.com; wangmeishanTina@126.com

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$$u_t + (A(u)u_{xxx} - B(u)u_x)_x = 0$$

Bernis and Friedman [1] firstly proved the existence of weak solutions in the distributional sense for the thin film equation. Bertozzi and Pugh [2] studied the problem with a second-order diffusion term. They obtained the existence and long time decay.

In the paper [4] and [5], the authors have studied the existence and blow-up behavior for Cahn-Hilliard equation. For the Cahn-Hilliard model with a grads mobility, Xu, Zhou in [9, 10] applied the semi-discrete method to get the existence and stability results.

We introduce some notations: $C^{m,y}(\overline{\Omega})$ denotes the Hölder space. $C^m(\overline{\Omega})$ denotes the uniformly continuous space. $W^{m,p}(\Omega)$ denotes the Sobolev space. (see [11]) C is denoted as a positive constant and may change from line to line.

197.2 Main Results

In the paper, we study the following steady state thin film model with a second-order diffusion term in the multi-dimensional space:

$$\nabla(u^n \nabla \Delta u) - \delta u^\sigma \Delta u = f(x, u) \quad in \quad \Omega, \tag{197.1}$$

$$u = u_D, \Delta u = 0 \quad on \quad \partial\Omega, \tag{197.2}$$

where $\Omega \subset R^N$ is a bound domain and $\partial\Omega$ is smooth enough. n, δ, σ are all nonnegative real constants.

We will employ the truncation function method, which has been successfully applied to the steady state quantum hydrodynamic model (see [7]), to get the existence of the stationary thin film equation. By letting

$$V = -\Delta u,$$

the (197.1) can be transformed into a second-order elliptic system:

$$-\nabla(u^n \nabla V) + \delta u^\sigma V = f(x, u) \text{ in } \Omega, \tag{197.3}$$

$$-\Delta u = V, \quad \text{in} \quad \Omega \tag{197.4}$$

$$u = u_D, V = 0 \quad \text{on} \quad \partial\Omega, \tag{197.5}$$

If we solve the problem (197.3), (197.4), (197.5) completely, the original problem (197.1)–(197.2) could be solved in the same time.

The following assumption is needed for the two variable function $f(t, s)$ defined in $\Omega \times (0, +\infty)$:

$$0 \le f(x, s) \le \eta_1 + \eta_2 s^\alpha \text{ for some } \alpha \in (0, 1).$$

For this purpose of existence, we study the following truncation problem in the paper:

$$\begin{aligned} -\nabla(g_{m,K}(u)^n \nabla V) + \delta g_{m,K}(u)^\sigma V = f(x, g_{m,K}(u)) \quad &\text{in} \quad \Omega, \\ -\Delta u = V \quad &\text{in} \quad \Omega, \\ u = u_D, V = 0 \quad &\text{on} \quad \partial\Omega \end{aligned} \tag{197.6}$$

where m and K are both undetermined positive constants and the truncation function $g_{m,K}(\cdot)$ is defined by

$$g_{m,K}(s) = \begin{cases} K, s \ge K; \\ s,\ m < s < K; \\ m, s \le m. \end{cases} \tag{197.7}$$

Now we list main results for the steady state problem as below:

Theorem 1. *(Boundedness of weak solutions) Suppose* $\inf_{x \in \partial\Omega} u_D(x) > 0$ and $u_D \in W^{2,p}(\Omega)$ *for* $p > \frac{N}{2}$ Let $(u, V) \in H^1(\Omega) \times H_0^1(\Omega)$ *be a weak solution of*

$$\begin{aligned} -\nabla(g_{m,K}(u)^n \nabla V) + \delta g_{m,K}(u)^\sigma V = f(x, g_{m,K}(u)) \quad &in \quad \Omega, \\ -\Delta u = V \ \ &in \quad \Omega, \\ u = u_D, V = 0 \quad &on \quad \partial\Omega \end{aligned} \tag{197.8}$$

Then

$$\begin{aligned} m \le u \le K, 0 \le V \le C(1 + K), \\ \| u\|_{W^{2,p}} + \| V\|_{C^{1,y}} \le C. \end{aligned} \tag{197.9}$$

197.3 Proof

The proof is arranged as follows. We will prove the a-prior estimates of solutions to the steady state problem.

The following a-prior estimates will play an important role to treat the existence.

Lemma1. *Let* $(u, V) \in H^1(\Omega) \times H_0^1(\Omega)$ *be a weak solution. Then*

$$(u,V)\in H^1(\Omega)\times H_0^1(\Omega)$$

Proof. (Positivity) Taking $V_-=\min\{V,0\}$ as a test function for the problem, we have

$$\begin{aligned} m^n\int_\Omega|\nabla V_-|^2dx+\delta m^\sigma\int_\Omega|V_-|^2dx \\ \le \int_\Omega f(x,g_{m,K}(u))V_-dx \\ \le 0. \end{aligned} \tag{197.10}$$

It means $V_-=0$ a.e. in Ω and so we conclude $V\ge 0$ a.e. in Ω.

Let $m=\inf\limits_{x\in\partial\Omega}u_D(x)>0$ and choose $(u-m)_-$ as a test function to get

$$\int_\Omega|\nabla(u-m)_-|^2dx=\int_\Omega V(u-m)_-dx\le 0. \tag{197.11}$$

Poincaré inequality yields $u\ge m\ge 0$ a.e. in Ω.

(L^∞-estimate) Next we will use the De Giorgi iteration technique to gain the L^∞-bound of the solution. Taking as a test function, we have

$$\begin{aligned} m^n\int_\Omega|\nabla(V-K)_+|^2dx+\delta m^\sigma\int V(V-K)_+dx \\ =\int_\Omega f(x,g_{m,K}(u))(V-K)_+dx \end{aligned} \tag{197.12}$$

and then

$$\begin{aligned} m^n\int_\Omega|\nabla(V-K)_+|^2dx \\ \le\int_\Omega f(x,g_{m,K}(u))(V-K)_+dx. \end{aligned} \tag{197.13}$$

If f satisfies the condition of the assumption (a), we apply Hölder inequality to get

$$\begin{aligned} &\int_\Omega f(x,g_{m,K}(u))(V-K)_+dx \\ &\le (\eta_1+\eta_2K^\alpha)\int_{\{V>K\}}(V-K) \\ &\le (\eta_1+\eta_2K^\alpha)\left(\int_{\{V>K\}}|V-K|^{2^*}dx\right)^{2^*}|\{V>K\}|^{1-\frac{1}{2^*}} \end{aligned} \tag{197.14}$$

where the positive constant 2^* is defined by

$$2^* = \begin{cases} \frac{2N}{N-2}, N > 2; \\ q(> 2), N \leq 2. \end{cases} \tag{197.15}$$

The expression $\{V > K\}$ denotes the set $\{x \in \Omega | V(x) > K\}$ and the representation $|\cdot|$ denotes the set measure.

In view of the Sobolev embedding $H^1(\Omega) \to L^{2^*}(\Omega)$, we have

$$\left(\int_{\{V>K\}} |(V-K)|^{2^*} dx\right)^{\frac{1}{2^*}} \tag{197.16}$$

$$\leq C(1+K^\alpha)|\{V > K\}|^{1-\frac{1}{2^*}} \tag{197.17}$$

where the constant $2^* > 2$ ensures $1 - \frac{1}{2^*} > 0$ Introducing a new constant H and for H > K > 0, we get

$$\left(\int_{\{V>K\}} |(V-K)|^{2^*} dx\right)^{\frac{1}{2^*}} \geq |\{V > H\}|^{\frac{1}{2^*}}(H-K). \tag{197.18}$$

Combining (197.5) and (197.6) yields

$$|\{V > K\}| \leq \frac{(C(1+K^\alpha))^{2^*}}{(H-K)^{2^*}} |\{V > K\}|^{2^*-1}. \tag{197.19}$$

Apply Stampacchia's Lemma (see [6]) to get

$$0 \leq V \leq c_1(1+K^\alpha) \tag{197.20}$$

where the constant is independent of K.

Similarly, using (197.2) and (197.3), we can receive the following estimate

$$m \leq u \sup_{x\in\partial\Omega} u_D(x) + c_2(1+K^\alpha) \tag{197.21}$$

where the constant c_2 is independent of K. Taking advantage of the assumption (a), we conclude that there exists a large enough constant K such that the inequality

$$\sup_{x\in\partial\Omega} u_D(x) + c_2(1+K^\alpha) \leq K \tag{197.22}$$

holds. Hence, we have obtained the estimate $m \leq u \leq K$.

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