Chapter 4
Repeated Newsvendor Game with Transshipments

Xiao Huang and Greys Sošić

Abstract We study a repeated newsvendor game with transshipments. In every period n, retailers face a stochastic demand for an identical product and independently place their inventory orders before demand realization. After observing the actual demand, each retailer decides how much of her leftover inventory or unsatisfied demand she wants to share with the other retailers. Residual inventories are then transshipped in order to meet residual demands, and dual allocations are used to distribute residual profit. Unsold inventories are salvaged at the end of the period. While in a single-shot game retailers in an equilibrium withhold their residuals, we show that it is a subgame-perfect Nash equilibrium for the retailers to share all of the residuals when the discount factor is large enough and the game is repeated infinitely many times. We also study asymptotic behavior of the retailers' order quantities and discount factors when n is large. Finally, we provide conditions under which a system-optimal solution can be achieved in a game with n retailers, and develop a contract for achieving a system-optimal outcome when these conditions are not satisfied. This chapter is based on Huang and Sošić (European Journal of Operational Research 204(2):274–284, 2010).

Keywords Transshipment • Subgame perfect Nash equilibrium • Repeated game • Asymptotic behavior

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4.1 Introduction and Literature Review

As the intensity of the business competition grows, retailers and distributors want to achieve more flexibility and become more responsive to their customers. However, fulfilling the demand is a challenge given the high uncertainty of the market, the limited capacity, and the tight budget constraints. In many such situations, it is worthwhile for the distributors to form alliances that will share substitutable inventory or services. Such cooperation is even more beneficial when products have short life/sales cycles, become obsolete fast, face long suppliers' lead times, and customer's demands that are hard to predict. Examples of such products are apparel, pop music, and high-tech products, among others.

Many retail chains implement transshipments or inventory sharing (we will use both terms in this article) among their stores. For example, Takashimaya, a Japanese department store chain, adopts inventory sharing policies among its stores by allowing sales persons to search on their PDAs the inventories held by other branches when the product is not held in stock at their location. The requested product is received the following day. In this way, Takashimaya manages to optimize the inventory within specialized shops. Similar policies are implemented in Music Millenium, Guess, and others.

While inventory sharing within a company is, intuitively, feasible and profitable, it is worthwhile to mention that similar practices happen among independent parties as well. iSuppli.com markets itself as the "collaborative ground" and is trying to build up a network of unrelated parties that need the same electronic components.

When inventory sharing is introduced into the system, various questions need to be addressed:

1. *Inventory Decision*: One of the merits of inventory sharing is the reduction of the overstocking cost, because inventory-sharing parties usually hold less inventories.[1] An important question here is, to what extent are the inventory positions going to be reduced?
2. *Transshipment*: When multiple retailers participate in transshipments, how to allocate the inventory among them? The transshipping pattern can be either determined a priori by a contract (i.e., the retailer with surplus inventory may select where her inventory is going), or a posteriori according to some objective (i.e., maximize the total profit of all retailers).
3. *Profit Allocation*: How are the profits generated from transshipments allocated among the retailers? For example, there may be a flat-rate price for each unit transshipped, or the total profit can be divided evenly among all participating retailers.
4. *Sharing Decision*: How much of their leftover inventories or unsatisfied demands are the retailers willing to share with others? Are they going to put all their

[1]For some exceptions, see Yang and Schrage (2009), which show that the inventory levels can increase after centralization when demand follows right-skewed distributions, or when the newsvendor ratio is low.

leftovers (inventories or demands) on the table, or strategically withhold some of them? This decision may depend upon initial inventory position, transshipment policies, or profit allocation.

5. *Time Horizon*: Is inventory sharing a one-time event, or an activity in which the retailers will be engaged repeatedly? In the latter case, are the unsold inventories carried over to the next period, or salvaged at the end of the period?

Many of these questions have been addresses by the researchers in various combinations, as inventory sharing has been a subject of extensive research work. One stream of research focuses on inventory decisions. Parlar (1988) develops a game-theoretical model for substitutable products in which leftover inventory and unmet demand are matched through customer-driven search. This implicitly means that the party that holds the excess inventory receives the entire profit from inventory sharing. The paper proves that a first-best outcome (that is, a system-optimal solution) can be achieved in a two-retailer game. Wang and Parlar (1994) analyze a similar problem with three retailers. They find that the core of the game can be empty, and thus inventory sharing between sub-coalitions of players may occur. Lippman and McCardle (1997) consider an environment with aggregated stochastic industry demand, which has to be divided among different firms. They study the relationship between initial demand-sharing rules and equilibrium inventory decisions, and they determine conditions for a unique equilibrium.

Another stream of research analyzes transshipment of inventories. Among more recent papers, Dong and Rudi (2004) examine the impact of horizontal transshipments between the retailers on both the retailers and on the manufacturer, while Zhang (2005) generalizes their results. Rudi et al. (2001) and Hu et al. (2007) study decision making in decentralized systems and the significance of transshipment prices in local decisions. Wee and Dada (2005) consider a one-warehouse n-retailer system in which the retailers can receive inventory from the warehouse and from the other retailers. They analyze the impact of the number of retailers and demand correlation on transshipment decisions. Zhao et al. (2005) study a model in which the retailers determine both a base-stock policy (for inventory stocking) and a threshold policy (for inventory sharing) prior to demand realization. Shao et al. (2011) study a supply chain which is both vertically and horizontally decentralized. They show the importance of the transshipment price in determining whether firms benefit or lose from transshipment, and investigate how the control of the parameters of the transshipment decision affects firms' transshipment incentives.

If the retailers agree to share their residuals, a decision has to be made as to how to allocate the additional profit generated through transshipment of inventories. This decision can be made jointly by the retailers, or it can be, for instance, chosen by a manufacturer whose products they are selling, or a trade association, or a larger organization to which the retailers belong. Clearly, different allocation rules will have different impacts on the retailers' stocking quantities, on the amount of inventories shared among retailers, and on the profit levels realized in the system. Ideally, the retailers would want to choose an allocation rule that would maximize the additional profit from transshipments. In order to achieve this goal, it is sufficient that the allocation rule:

(a) Induces participation of *all* retailers.
(b) Motivates the retailers to share *all of their residuals* with others.

We will call condition (a) the *full participation* condition, and condition (b) the *complete sharing* condition. Anupindi et al. (ABZ 2001) and Granot and Sošić (G&S 2003) develop a multistage model for a problem in which n independent retailers face stochastic demands for identical products. In the first stage, before the demand is realized, retailers unilaterally determine their stocking quantities. After the demand is realized and the retailers fulfill their own demands with inventories on hand, some retailers are left with unsatisfied demand, while others have leftover supply. The retailers at this point cooperatively determine a transshipment pattern for distribution of residual inventories among themselves. The additional profit generated through transshipments (which we call *residual profit*) is divided according to an allocation rule specified at the beginning of the game. ABZ formulate a two-stage model for this problem and implicitly assume that the retailers share all of their residuals with the others. Thus, the complete sharing condition is automatically satisfied. They propose a core allocation rule based on the dual prices for the transshipment problem (referred hereafter as *dual allocations*; for detailed description, see Sect. 4.2), which satisfies the full participation condition. ABZ point out that dual allocations, in general, do not induce a first-best solution. When the retailers are allowed to withhold some of their residuals, G&S show that dual allocations may not be able to induce complete sharing of the residual supply/demand. This may, in turn, reduce the residual profit. On the other hand, monotonic allocation rules (such as the fractional rule and the Shapley value) satisfy the complete sharing condition, but these rules, in general, do not belong to the core, and thus they violate the full participation condition. Consequently, some retailers may form inventory sharing subcoalitions, which, in turn, may result in a reduced residual profit. Notice that all of the above conclusions hold in a myopic framework. If the retailers are farsighted and consider possible further reactions of their inventory-sharing partners to their actions, Sošić (2004) shows that complete inventory sharing among all retailers is a stable outcome when the residual profit is distributed according to the Shapely value allocations.

In this work, we study the extension of the above one-shot game from G&S to a repeated setting, in which each retailer faces her demand over several periods. In each period, the three-stage model corresponds to that described in the one-shot game. We want to point out that we are interested in studying the impact of the repeated interactions on the retailer's decisions in the second stage (how much of their residuals they want to share with others) and on selecting their partners for inventory-sharing (possible formation of subcoalitions). As a result, we continue to assume the newsvendor framework, in which unsold inventories are salvaged at the end of each period and no demand is backlogged. This setting is common, for example, for fashion goods or high-tech items. In addition, we assume that the retailers in each period sell a product with *identical* characteristics (demand distribution, cost, and price). This is a simplifying assumption, which nevertheless may approximate many real-life situations, in which items with *similar* characteristics are sold in different periods. For instance, every season apparel manufacturers introduce new collections. One can presume that items that fall into same categories (t-shirts or other casual clothing, business suits, or trendy items made by the same company) will have similar demand characteristics in different years. A similar conclusion can be made for Christmas toys (say, different versions of Barbie or Elmo dolls), music (new CDs released by Prince, Lady Gaga, or Carrie Underwood), etc. In the high-tech industry, new hard disk drives or new processors are introduced on a regular basis to replace the previous generation of corresponding products. As the technology advances and the models with better performance reach the market, one can assume that the new product will have demand and price similar to the original demand and price of the product that it is replacing. Note that our model also covers some instances in which the prices change in different periods—we discuss this in more detail in Sect. 4.3.

As mentioned earlier, when the retailers cooperatively generate additional profit, they have to decide how to distribute it among themselves. In our model, we assume that the retailers apportion this extra income according to the *dual allocations*. These allocations are based on the dual solution of the linear programming problem (4.1) used to determine the optimal shipping pattern for residuals, and are, therefore, easy to compute. For detailed description of the model and dual allocations, please see Sect. 4.2. As shown by ABZ, dual allocations are in the core of the corresponding game, which makes the coalition of all players stable, because no players (or subsets of players) benefit from a defection, and hence dual allocations satisfy the full participation condition. Thus, if each retailer shares all of her residuals, the profit from inventory sharing is maximized. However, if players are allowed to withhold some of their residuals, G&S show that players will not share all of their leftover inventory/unmet demand, which, in turn, reduces the profit obtained through inventory sharing. Note, however, that these results hold in a one-shot setting, where players do not consider future interactions. Now, in a repeated game, we want to address the following questions:

1. When the retailers interact repeatedly, what is the impact of the length of the time horizon on the retailers' decisions, and is it possible to induce the retailers to share all of their residuals with dual allocations?
2. Under what conditions can a first-best solution be achieved without additional enforcement mechanisms, and what type of contracts can induce system-optimal decisions when these conditions do not hold?

The answers to the first question are obtained through standard game-theoretical tools. We show that dual allocations induce the retailers to withhold residuals when the game is played a finite number of times. On the other hand, the retailers in the infinite-horizon model may be induced to share all of their residuals when they put enough weight on their future payoffs. As the number of retailers increases, calculation of the lower bound for the value of the discount factor that induces the complete sharing of residuals becomes intractable. However, we are able to obtain some asymptotic results for a large number of players. We also demonstrate that a complete sharing of residuals may be induced when the punishment (that is, nonsharing of inventories) is not enforced over an infinite horizon.

In answering the second question, we provide a condition for achieving a first-best outcome, and develop a contract that leads to a first-best outcome when some retailers' optimal stocking decisions differ from the system-optimal ones.

The structure of this article is as follows: we briefly introduce the one-shot inventory sharing game in Sect. 4.2, and in Sect. 4.3 we extend this model to a repeated setting. In Sect. 4.4, we develop some asymptotic results for the retailers' ordering quantities and lower bounds on discount factors that induce complete sharing of residuals for large number of players. In Sect. 4.5, we derive conditions for achieving a first-best outcome without additional enforcement mechanisms, while in Sect. 4.6 we develop a contract that induces a first-best solution in a more general setting. We conclude in Sect. 4.7. Longer proofs are given in a technical appendix.

4.2 One-Shot Inventory-Sharing Game

Each period in our repeated game corresponds to the three-stage inventory-sharing model from G&S and can be described as follows. We use $N = \{1, 2, \ldots, n\}$ to denote a set of retailers who are selling an identical product. We assume that the retailers face independent random demands, D_i, and that each retailer knows the distribution of her demand, F_i, and its density, f_i. After demands are realized and each retailer satisfies her own demand from inventory on hand, the retailers can share their residuals—leftover inventories or unsatisfied demands. The total profit from transshipments—residual profit—has to be divided among the retailers according to an allocation rule agreed upon by all of them before the game begins. We assume that there are no capacity constraints and that the game begins with zero inventory. The three stages are modeled as follows:

Stage 1: Before demand D_i is realized, each retailer independently makes her own ordering decision, X_i, contingent upon the demand distribution and the allocation rule that will be used to distribute the residual profit.

Stage 2: After demand is realized, each retailer decides how much of her residuals she would like to share with others. Let $\bar{H}_i = \max\{X_i - D_i, 0\}$ and $\bar{E}_i = \max\{D_i - X_i, 0\}$ denote the total leftover inventory and unsatisfied demand for retailer i, respectively. We will use bold letters to denote vectors, that is, $(\bar{\mathbf{H}}, \bar{\mathbf{E}}) = (\bar{H}_1, \ldots, \bar{H}_n, \bar{E}_1, \ldots, \bar{E}_n)$. We denote the retailers' sharing decisions (amounts of residual supply/demand that retailer i decides to share with the other retailers) by H_i and E_i, respectively. It is straightforward that H_i and E_i must satisfy $0 \le H_i \le \bar{H}_i$, $0 \le E_i \le \bar{E}_i$.

Stage 3: The shipping pattern for leftover inventory that maximizes the residual profit is determined. The resulting residual profit is then distributed among the retailers according to the allocation rule determined before the first stage takes place (in this article, we assume that the retailers use dual allocations). Any inventory left at the retailers is salvaged.

Let r_i, c_i, and v_i denote, respectively, the unit retail price, cost, and salvage value for retailer i, Y_{ij} and t_{ij} denote the amount of stock shipped and the unit cost of transshipment from retailer i to retailer j. We assume that $r_i > r_j - t_{ij}$, that is, each retailer satisfies her own demand first, and $v_i - t_{ji} < v_j$, that is, the retailers do not benefit from salvaging unsold items at other locations.

We next present some results from G&S (2003). The transshipment pattern in the third stage, given demand realizations and retailers' sharing decisions, can be solved through linear programming. Let $R(\mathbf{X},\mathbf{D},\mathbf{H},\mathbf{E})$ denote the residual profit from the transshipments; as the retailers sharing decisions, $(\mathbf{H},\mathbf{E})$, depend on the actual residual values, $(\bar{\mathbf{H}},\bar{\mathbf{E}})$, this profit is a function of the retailers' order sizes and demand realizations, $\mathbf{X}$ and $\mathbf{D}$. The optimal shipping pattern, $R^*(\mathbf{X},\mathbf{D},\mathbf{H},\mathbf{E})$, can be determined by solving the following linear programming problem.

$$R^*(\mathbf{X},\mathbf{D},\mathbf{H},\mathbf{E}) := \max_{\mathbf{Y}} \sum_{i,j=1}^{n} (r_j - v_i - t_{ij})Y_{ij}, \tag{4.1a}$$

$$\text{subject to: } \textstyle\sum_{j=1}^{n} Y_{ij} \le H_i \quad i = 1,2,\ldots,n, \tag{4.1b}$$

$$\textstyle\sum_{j=1}^{n} Y_{ji} \le E_i \quad i = 1,2,\ldots,n, \tag{4.1c}$$

$$Y_{ij} \ge 0 \qquad i,j = 1,2,\ldots,n. \tag{4.1d}$$

We denote the allocation of residual profit to retailer i by $\varphi_i^d(\mathbf{X},\mathbf{D},\mathbf{H},\mathbf{E})$. If λ_i and μ_i denote the dual prices corresponding to the constraints (4.1b) and (4.1c), respectively, then $\varphi_i^d(\mathbf{X},\mathbf{D},\mathbf{H},\mathbf{E}) = \lambda_i H_i + \mu_i E_i$, and the profit for a retailer, i, can be written as:

$$P_i^d(\mathbf{X},\mathbf{D},\mathbf{H},\mathbf{E}) = r_i \min\{X_i, D_i\} + v_i \bar{H}_i - c_i X_i + \varphi_i^d(\mathbf{X},\mathbf{D},\mathbf{H},\mathbf{E}).$$

Given the stocking quantity decisions and demand realizations, $\mathbf{X}$ and $\mathbf{D}$, the retailers in the second stage of the game make their sharing decisions according to the Nash equilibrium (NE), which we denote by $(\mathbf{H}^{\mathbf{X},\mathbf{D}},\mathbf{E}^{\mathbf{X},\mathbf{D}})$. Thus, they must satisfy the following inequalities:

$$P_i^d(\mathbf{X},\mathbf{D},\mathbf{H}^{\mathbf{X},\mathbf{D}},\mathbf{E}^{\mathbf{X},\mathbf{D}}) \ge P_i^d(\mathbf{X},\mathbf{D},H_i,H_{-i}^{\mathbf{X},\mathbf{D}},E_i,E_{-i}^{\mathbf{X},\mathbf{D}}),$$
$$\forall H_i \le \bar{H}_i,\ E_i \le \bar{E}_i,\ i = 1,2,\ldots,n,$$

where $x_{-i} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Finally, the first-stage NE ordering decisions, $\mathbf{X}^{\mathrm{d}}$, must satisfy

$$J_i^{\mathrm{d}}(\mathbf{X}^{\mathrm{d}}) \ge J_i^{\mathrm{d}}(X_i, \mathbf{X}_{-i}^d),$$

where $J_i^{\mathrm{d}}(\mathbf{X}) = \mathsf{E}[P_i^d(\mathbf{X},\mathbf{D},\mathbf{H}^{\mathbf{X},\mathbf{D}},\mathbf{E}^{\mathbf{X},\mathbf{D}})]$ is retailer i's expected profit when retailers' ordering decisions form vector $\mathbf{X}$. Huang and Sošić (2010a) provide conditions for existence of the NE in ordering quantities, $\mathbf{X}^{\mathrm{d}}$, for this game. As we are primarily interested in the effects of repeated interactions on players' decisions, in what follows we assume that these conditions are satisfied and that the NE exists.

We also mention, as benchmarks, two related models—the game without transshipments and the centralized model. If the retailers do not share their residuals, each retailer's profit can be described as

$$P_i^1(X_i, D_i) = r_i \min\{X_i, D_i\} + v_i \bar{H}_i - c_i X_i,$$

with the expectation $J_i^1(X_i) = \mathsf{E}[P_i^1(X_i, D_i)]$. Superscript 1 denotes the model in which each retailer acts individually. The optimal ordering decision, X_i^1, corresponds to the newsvendor solution. In the centralized model, in which a single decision maker optimizes the profit of the entire system, the total system profit can be written as

$$P^n(\mathbf{X}, \mathbf{D}) = \sum_{i=1}^{n} r_i \min\{X_i, D_i\} + v_i \bar{H}_i - c_i X_i + \bar{R}^*(\mathbf{X}, \mathbf{D}),$$

with the expectation $J^n(\mathbf{X}) = \mathsf{E}[P^n(\mathbf{X}, \mathbf{D})]$. Superscript n denotes that n retailers participate in inventory sharing. The optimal ordering amount for this model, $\mathbf{X}^n$, maximizes the total system profit and is referred to as a *first-best solution*.

4.3 Repeated Inventory-Sharing Game

In this section, we study the inventory-sharing game in a repeated setting. When the retailers do not expect future interactions with their inventory-sharing partners, dual allocations preclude them from formation of subcoalitions, but may also provide an incentive for some (or all) of them to withhold a portion of their residuals (which may increase their allocations). The main topic of our interest is to study the impact of repeated interactions on the retailers' sharing decisions in the second stage. Our repeated game is modeled identically in every period, following the steps described in the one-shot model. The goal of each retailer is to maximize her total discounted profit, and we consider both a finite and an infinite horizon. A solution concept commonly used in this setting is *subgame perfect Nash equilibrium* (SPNE)—a solution in which players' strategies constitute a NE in every subgame of the original game.

We assume that unsold inventories are salvaged at the end of each period and that inventory level at the beginning of each period is zero. If we allow the retailers to strategically increase their orders in one period and transfer a portion of inventory to the next period, the result would be a significantly more complicated model that is beyond the scope of this work. In addition, when making her decision, each retailer knows the entire history of previous decisions for all retailers. While this assumption may be rather strong, it is not uncommon in the repeated-game setting to assume that all players know the entire history (see, for instance, Bagwell and Staiger 1997;

Haltiwanger and Harrington 1991; Rotemberg and Saloner 1986). We feel that such an assumption may be appropriate, say, for settings in which the retailers belong to a larger organization, or within a trade association.

G&S (2003) show that the retailers who share inventory only once withhold some of their residuals. By using standard game-theoretical tools, it can be easily shown that the same is true when the game is repeated a finite number of times; hence, we state our next result without a proof.

Proposition 1. *Complete sharing is not achieved if the inventory-sharing game with n retailers is repeated a finite number of times.*

We next consider an infinitely repeated game and introduce the *Nash reversion* strategy (NRS), which can be described as follows: each retailer completely shares her residuals until one or more of them deviate by withholding some of their residuals. From that moment on, no residuals are shared in the subsequent periods by any of the retailers. We show that this strategy is an SPNE.

Let P_{it} and X_{it} denote the profit and the ordering quantity of retailer i in period t, respectively; we use similar notation for her shared and actual residuals in period t, H_{it}, E_{it} and $\bar{H}_{it}$, $\bar{E}_{it}$. The retailers' decisions are based on previous histories, $h_{t-1} = \{\mathbf{X}_l, \mathbf{H}_l, \mathbf{E}_l\}_{l=1}^{t-1}$, that include stocking quantities and shared residuals in all periods preceding t. Thus, we write $(X_{it}, E_{it}, H_{it})(h_{t-1})$ to denote that (X_{it}, E_{it}, H_{it}) depends on h_{t-1}. We let $(h_{t-1})_l = (\mathbf{X}_l, \mathbf{H}_l, \mathbf{E}_l)$ denote the retailers' decision in period l. Recall that X_i^{d} and X_i^{I} denote the optimal stocking quantities in one-shot games with dual allocations and without transshipments, respectively, and that δ denotes the discount factor. The following result can be shown through the application of the folk theorem.

Theorem 1. *Suppose that an inventory sharing game with n retailers is repeated infinitely many times. Then, there exists* $\delta_n^* \in (0,1)$ *such that the NRS, in which*

$$(X_{it}, E_{it}, H_{it})(h_{t-1}) = \begin{cases} (X_i^{\mathrm{d}}, \bar{H}_{it}, \bar{E}_{it}) & \text{if } t = 1 \text{ or } (h_{t-1})_l = (\mathbf{X}^{\mathrm{d}}, \bar{\mathbf{H}}, \bar{\mathbf{E}}), \\ & \forall l = 1, \ldots, t-1 \\ (X_i^{\mathrm{I}}, 0, 0) & \text{otherwise,} \end{cases}$$

constitutes a SPNE of the infinitely repeated game whenever $\delta > \delta_n^*$.

The lower bound for the discount factor, δ_n^*, can in practice be difficult to evaluate, so in Sect. 4.4, we explore in more detail its asymptotic behavior. We illustrate our result with the following numerical example.

Example 1. Suppose that $n = 3$, all three retailers face two-point demand which can achieve 0 with probability 0.5 and 10 with probability 0.5, and c_i=3.7; r_i=10; $v_i = 1$, $i = 1,2,3; t_{ij} = 1,\ i,j = 1,2,3, i \neq j$. When the retailers share their inventory and distribute the residual profit according to dual allocations, their individual stocking quantities decrease from 10 to 7, and the corresponding expected profits increase from 18 to 22. The discount factors that induce complete residual sharing by all retailers satisfy $\delta > \delta_3^* = 0.93$.

4.3.1 Finite Punishment Period

The NRS represents the belief that "once the trust is lost, it is lost forever." However, one can object that infinite punishment may not be credible, because besides punishing the defecting retailer, it hurts the punishers as well. Hence, we consider a "milder" strategy in which punishment lasts only for a finite number of periods before the retailers recover from the "bad memories" and return to cooperation. In this framework, only the history of the past k periods, $h_{t-k}^{t-1} = \{\mathbf{X}_\tau, \mathbf{H}_\tau, \mathbf{E}_\tau\}_{\tau=t-k}^{t-1}$, has an impact on retailers' decisions.

Theorem 2. *Suppose that an inventory sharing game with n retailers is repeated infinitely many times. Then, there exists $k_n^* \in \mathbf{N}$ such that $\forall k > k_n^*$ there is a $\delta_n^*(k)$ such that the strategy in which*

$$(X_{it}, E_{it}, H_{it})(h_{t-k}^{t-1}) = \begin{cases} (X_i^{\mathrm{d}}, \bar{H}_{it}, \bar{E}_{it}) & \text{if } t=1 \text{ or } (h_{t-k}^{t-1})_\tau = (\mathbf{X^1}, \mathbf{0}, \mathbf{0})\ \forall \tau = 1, \ldots, k \\ & \text{or } (h_{t-k}^{t-1})_{t-1} = (\mathbf{X}^{\mathrm{d}}, \bar{\mathbf{H}}, \bar{\mathbf{E}}) \\ (X_i^1, 0, 0) & \text{otherwise,} \end{cases}$$

constitutes an SPNE of the infinitely repeated game whenever $\delta > \delta_n^*(k)$.

The proof is again obtained through the application of the folk theorem. If a player, j, considers a deviation from $(X_j^{\mathrm{d}}, \bar{H}_{jt}, \bar{E}_{jt})$, any momentary gain is canceled by future reduction in payoffs when the discount factor is large enough and the punishment is carried over an appropriate number of periods. During the punishment period, each retailer plays her optimal strategy for noncooperative setting, so a possible defection cannot increase her profits, while at the same time it prolongs the length of the punishment.

Theorem 2 implies that it is not necessary to impose infinite punishment to induce the retailers' cooperation. Intuitively, a longer punishment horizon requires lower discount factors—punishment that lasts only a few periods is effective only when the retailers' discount of the future is negligible. We illustrate this with the following example.

Example 2. Suppose that $n = 3$, all three retailers face two-point demand which can achieve 0 with probability 0.5 and 10 with probability 0.5, and c_i=3.7; r_i=10; v_i=1, $i = 1,2,3; t_{ij} = 1$, $i, j = 1,2,3, i \neq j$. We have shown in Example 1 that $\delta_3^* = 0.93$ when the punishment is enforced over an infinite horizon. The value of $\delta_3^*(k)$ as a function of k is depicted in Fig. 4.1. Note that, as k increases, $\delta_3^*(k)$ approaches δ_3^*.

4.3.2 Alternative Strategies for Achieving SPNEs

Note that strategies other than the NRS described in Theorem 1 can also lead to SPNEs. One such strategy can be defined as follows: let $\mathbf{X}^{\mathrm{d}(n-1)}$ be the optimal order quantity for decentralized system with $n-1$ retailers under dual allocations.

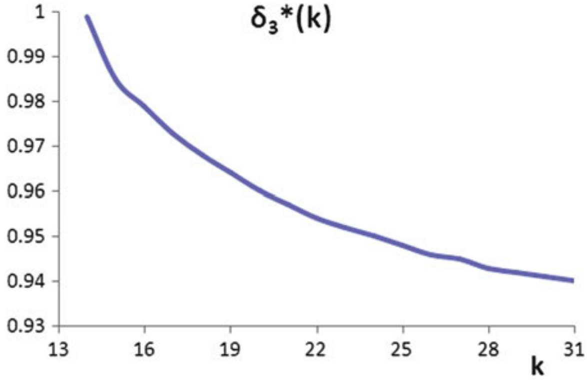

Fig. 4.1 $\delta_3^*(k)$ as a function of k

If a player, j, deviates from $(X_j^{\mathrm{d}}, \bar{H}_{jt}, \bar{E}_{jt})$ when $t = \bar{t}$, the remaining players follow strategy $(X_i^{\mathrm{d}(n-1)}, \bar{H}_{it}, \bar{E}_{it}), i \neq j, t > \bar{t}$, while retailer j adopts $(X_j^1, 0, 0), t > \bar{t}$. If a cooperating player, say l, deviates after a defection has already occurred, the punishment restarts and retailer l is excluded from future inventory sharing. Unlike the previous case (described in Theorem 1), the payoffs for the defecting player and for the cooperating players differ during the punishment period, and we need to consider them separately while checking if conditions for a SPNE are satisfied. When the discount factor is large enough, it can be shown that this strategy defines a SPNE, and that it leaves cooperating retailers with a larger payoff (during the punishment phase) than the strategy described in Theorem 1. However, observe that when the threat of punishment works, it is never actually carried out.

4.3.3 Decreasing/Increasing Costs and Prices

We would also like to mention that our model can be applied to some situations in which the costs and prices change in different periods. Let superscript t denote the values of costs/prices in period t, and suppose that $r_i^{t+1} = \rho r_i^t, v_i^{t+1} = \rho v_i^t, c_i^{t+1} = \rho c_i^t, t_{ij}^{t+1} = \rho t_{ij}^t$, for some $\rho > 0$. If $\rho < 1$, the parameters decrease with time, and our results hold if we replace δ with $\tilde{\delta} = \rho\delta$. If $\rho > 1$, the parameters increase over time, and our results will hold whenever $\tilde{\delta} < 1$, that is, when $1 < \rho < \delta^{-1}$.

4.4 Asymptotic Behavior for Large *n*

In this section, we consider the optimal retailers' ordering quantity and discount factors for large values of n. All proofs are given in the Appendix.

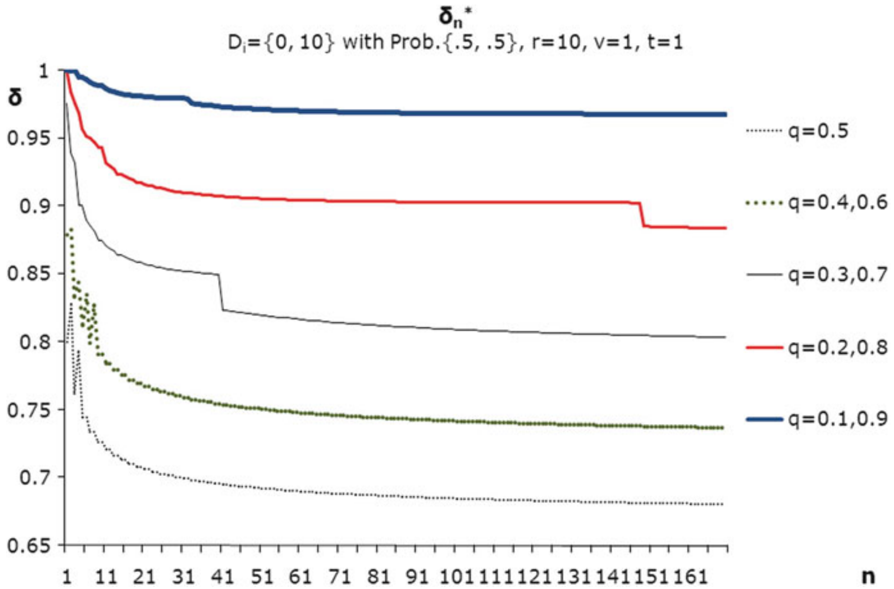

Fig. 4.2 δ_n^* for different values of the critical fractile $q = \frac{r-c}{r-v}$

We say that the retailers are *symmetric* if they face the same demand distribution F_i, cost c_i, retailer price r_i, and salvage value v_i, along with equal transportation costs in both directions, $t_{ij} = t_{ji}$. In this part of the analysis we focus on symmetric retailers, so we omit indices from notation.

Our next result provides a characterization of the lower bound for the discount factors that induces complete sharing in the NRS described in Theorem 1, δ_n^*.

Theorem 3. *In an inventory-sharing game with n symmetric retailers facing strictly increasing and independent distribution functions, there is an* $M > 0$ *such that* δ_n^* *is decreasing in n for* $n \geq \hat{n}$, *where* $\hat{n} = \min\{n \in Z : nX^{\mathrm{d}} \geq M\}$.[2]

Thus, with enough retailers participating in inventory sharing, δ_n^* is decreasing in n. Note that in many real-life situations this number can be as low as two or three. As the number of retailers increases, it is more likely for an individual retailer to benefit from inventory sharing and she is willing to participate in transshipments when she discounts her future payoffs more. We illustrate in Fig. 4.2 the behavior of δ_n^* for discrete demand that can achieve two values, 0 or 10, with equal probabilities. The two-point format of this distribution is the reason why we observe some "jumps" in the value of δ for small n. We fix r, v, and t, and change the value of c to obtain different values of the critical fractile, $q = (r-c)/(r-v)$. The values of δ_n^* are equal for "symmetric" critical fractiles

[2]If D has a finite support with upper bound $\bar{D}$, then $M = \bar{D}$.

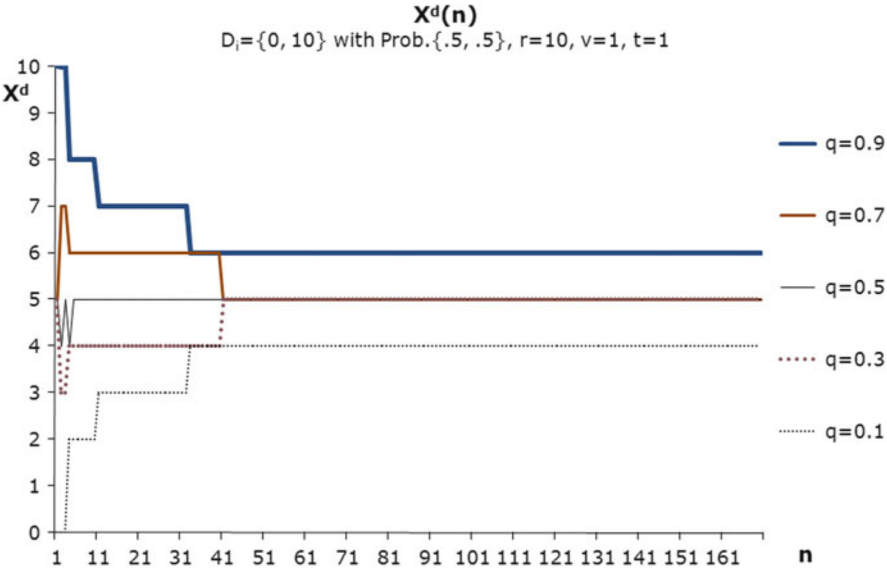

Fig. 4.3 X^{d} for different values of the critical fractile $q = \frac{r-c}{r-v}$

(q and $1-q$). As the number of retailers increases, δ_n^* shows a decreasing trend, and converges to a positive value. In addition, the discount factor that induces complete sharing increases as the critical fractile moves further from 0.5. When the critical fractile is close to 0.5, the ordering quantities at each retailer are close to the mean demand, and each retailer is more likely to benefit from transshipments. As the critical fractile moves further below (resp., above) from 0.5, each retailer orders less (resp., more), which leads to constant undersupply (resp., oversupply) and makes cooperation less useful. Therefore, cooperation is less beneficial and a larger δ is needed to incentivize the retailers.

We next characterize the retailers' ordering quantity, X^{d}.

Proposition 2. *In an inventory-sharing game with n symmetric retailers and strictly increasing distribution function $F(\cdot)$, the asymptotic behavior of the equilibrium ordering quantity can be described by*

$$\lim_{n\to\infty} X^{\mathrm{d}}(n) = \begin{cases} \mu, & \text{if } t\text{=0 or } \frac{r-c-p}{t} \le F(\mu) \le \frac{r-c}{t}, \\ \sup\{x : F(x) < \frac{r-c}{t}\} & \text{if } F(\mu) > \frac{r-c}{t}, \\ \inf\{x : F(x) > \frac{r-c-p}{t}\} & \text{if } F(\mu) < \frac{r-c-p}{t}. \end{cases}$$

Thus, when the cost of transshipment is not too high and the margin $r-c$ is not too low, the retailers will order the mean demand value. Once again, we conduct numerical analysis with a two-point demand distribution to explore the behavior of the optimal ordering quantities and illustrate it in Fig. 4.3. One can note that in this case the optimal order quantity converges to the mean demand value, and the values corresponding to different critical fractiles are symmetric with respect to the line

$X^{\mathrm{d}} = \mu$. We note that the convergence is faster for the value of the critical fractile closer to 0.5. Due to the special nature of our demand (two-point), we may see that the optimal order quantity can exhibit some jumps initially, but eventually starts monotonic convergence toward its limit.

While in the previous case we assumed that $t = 1$ and have changed the values of c to manipulate critical fractile, we now fix the value of c and look at the impact of changes in the transshipment cost. Figure 4.4 depicts two sample cases: the graph on the left looks at the low product cost ($c = 3.7$), while the graph on the right looks at the high product cost ($c = 7.3$). In both cases, the changes in the transshipment cost determine the limiting quantity. With both low and high product cost, the limiting order quantity corresponds to the mean demand when the transshipment cost is low. However, as the transshipment cost increases, inventory sharing is less likely to occur, and the limiting order quantity moves away from the mean value—with low product cost, it moves up, and with high product cost, it moves down, which is consistent with the results from Proposition 2. When the high transshipment cost makes inventory sharing prohibitive ($t > 5$), each retailer facing high product cost (low critical fractile) orders zero, while each retailer facing low cost orders 10, which coincides with their ordering quantities without transshipments.

An immediate corollary of Proposition 2 characterizes the relationship between the retailers' optimal ordering quantities in models with and without transshipments: while the asymptotic ordering quantity may go below (resp., above) the mean demand value when the cost c becomes large (resp., small), it will never go below (resp., above) the ordering level without transshipment.

Corollary 1. *In an inventory-sharing game with n symmetric retailers and strictly increasing distribution function $F(\cdot)$, the following relationships hold when n is large:*

1. *When $t > 0$: if $F(\mu) > \frac{r-c}{t}$, then $X^1 \le X^{\mathrm{d}}(n) < \mu$; if $F(\mu) < \frac{r-c-p}{t}$, then $\mu < X^{\mathrm{d}}(n) \le X^1$.*
2. *When $t = 0$: if $F(\mu) > \frac{r-c}{r-v}$, then $X^1 \le X^{\mathrm{d}}(n) = \mu$; if $F(\mu) < \frac{r-c}{r-v}$, then $X^1 \ge X^{\mathrm{d}}(n) = \mu$.*

The results obtained so far help us in determining asymptotic behavior of δ_n^* when n is large.

Theorem 4. *In an inventory-sharing game with n symmetric retailers and strictly increasing distribution function $F(\cdot)$, $\delta_n^* \to \delta_\infty^* > 0$. More specifically, let M be as defined in Theorem 3, and let $\xi(x) = \int_0^x yf(y)\mathrm{d}y$ and $\rho(x) = p\max\{x, M-x\}$. Then,*

$$\delta_\infty^* = \begin{cases} \frac{\rho(\mu)}{\rho(\mu)+(r-c-tF(\mu))\mu+t\xi(\mu)-(r-v)\xi(X^1)}, & \text{if } \frac{r-c-p}{t} \le F(\mu) \le \frac{r-c}{t} \text{ or } t=0; \\ \frac{\rho(X^{\mathrm{d}})}{\rho(X^{\mathrm{d}})+t\xi(X^{\mathrm{d}})-(r-v)\xi(X^1)}, & \text{if } F(\mu) > \frac{r-c}{t} \text{ and } X^{\mathrm{d}} = \sup\{x : F(x) < \frac{r-c}{t}\}; \\ \frac{\rho(X^{\mathrm{d}})}{\rho(X^{\mathrm{d}})+t(\xi(X^{\mathrm{d}})-\mu)-(r-v)(\xi(X^1)-\mu)}, & \text{if } F(\mu) < \frac{r-c-p}{t} \text{ and } X^{\mathrm{d}} = \sup\{x : F(x) > \frac{r-c-p}{t}\}. \end{cases}$$

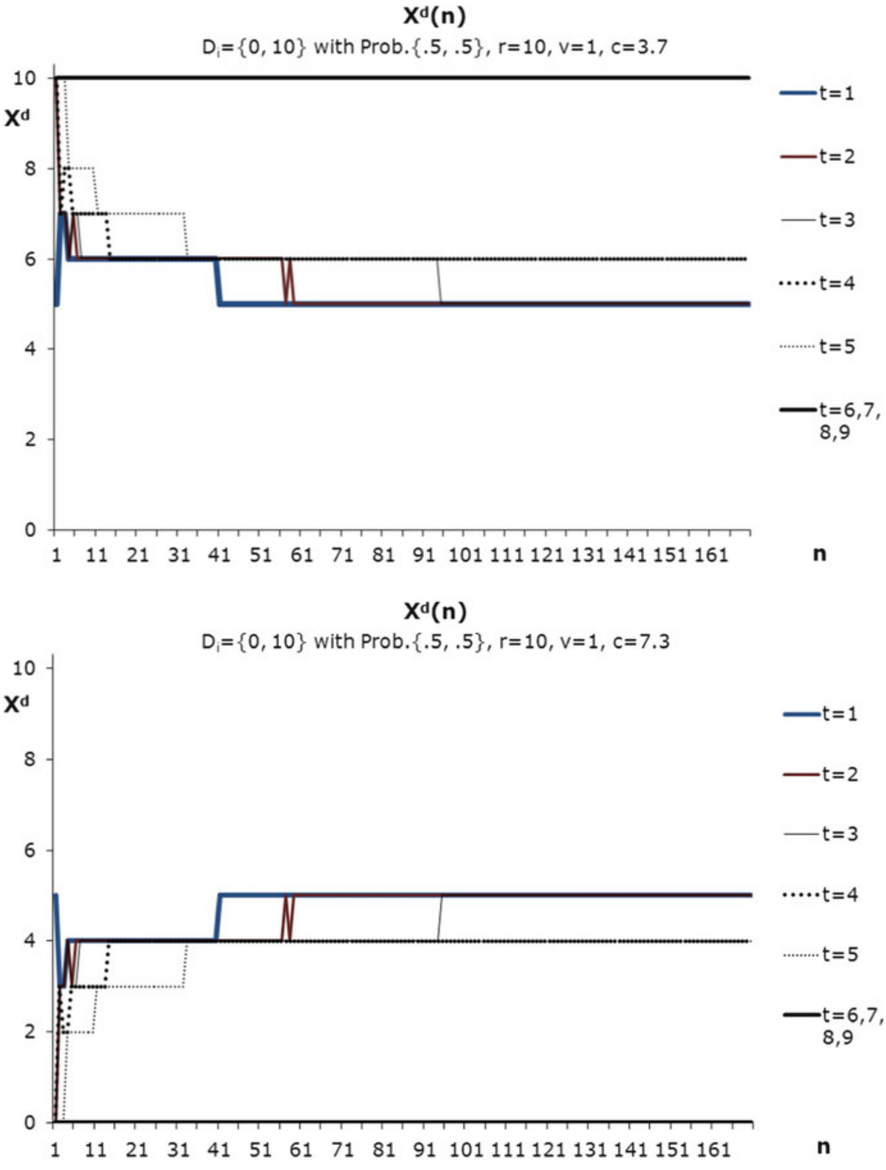

Fig. 4.4 X^{d} for different values of the transshipment cost with low and high product cost

Theorem 4 can be used to evaluate the limiting values of discount factors that induce complete sharing of residuals. An illustrative analysis is given in the following example.

Example 3. Suppose that $n \to \infty$, all retailers face demand uniformly distributed on [0,10], and $r = 10; v = 1; t = 1$. We consider different values of c, which lead to different values of the critical fractile $q = (r-c)/(r-v)$, and obtain the following results:

q	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
δ^*_∞	0.935	0.871	0.830	0.807	0.800	0.807	0.830	0.871	0.935

4.5 Achieving a First-Best Solution

Unfortunately, even if the retailers share all of their residuals, it is not easy to coordinate the system (except in some special cases that we discuss below) without some additional incentives, because some retailers may see a reduction in their individual profits as a result of ordering system-optimal quantities. We first discuss the cases under which a first-best outcome can be achieved without additional coordinating mechanisms, and then discuss what happens when this is not the case.

Note that even when the retailers share their entire residuals, the maximum system profit is not achieved unless the retailers order the amount optimal for the centralized model, $\mathbf{X}^n$, in each period. Thus, although the full participation and complete sharing conditions are satisfied, dual allocations may, in general, result in inefficiencies. We, therefore, start by analyzing the conditions under which decentralized stocking quantities, $\mathbf{X}^{\mathrm{d}}$, may coincide with the centralized ones, $\mathbf{X}^n$.

We first assume that the retailers are symmetric. Then, there is an equilibrium in which all retailers order the same quantity, $X_i^{\mathrm{d}} = X^{\mathrm{d}}, \forall i$, and $J_i^{\mathrm{d}}(\mathbf{X}^{\mathrm{d}}) = J^{\mathrm{d}}(\mathbf{X}^{\mathrm{d}}), \forall i$. If we consider the centralized system, there is an equilibrium in which all retailers order the same quantity, $X_i^n = X^n, \forall i$. Because the centralized model maximizes the expected profit, $J^n(\mathbf{X}^n) \geq nJ^{\mathrm{d}}(\mathbf{X}^{\mathrm{d}})$, and it is optimal for symmetric retailers to order at the first-best level, $\mathbf{X}^{\mathrm{d}} = \mathbf{X}^n$. We formalize this analysis in the following result.

Proposition 3. *If n retailers in the repeated inventory-sharing game are symmetric and $\delta > \delta_n^*$, a first-best solution can be achieved through dual allocation.*

Proposition 3 says that it is sufficient to have *symmetric* retailers to achieve a first-best outcome. This condition may be satisfied if, for instance, all retailers belong to the same organization; hence, they face the same costs/prices, and cover similar territories. However, in many realistic cases, this condition may not hold. Thus, we want to find more general conditions under which a first-best outcome can be achieved. Recall that the expected profit for retailer i is

$$J_i^{\mathrm{d}}(\mathbf{X}) = r_i \mathsf{E}[\min\{X_i, D_i\}] - c_i X_i + v_i \mathsf{E}[\bar{H}_i] + \mathsf{E}[\varphi_i^d(\mathbf{X})].$$

The total expected profit for the system of retailers is then $J^n(\mathbf{X}) = \sum_i J_i^{\mathrm{d}}(\mathbf{X})$. The optimal ordering strategy for the centralized model, $\mathbf{X}^n$, satisfies the following first-order conditions:

$$\frac{\partial J^n(\mathbf{X})}{\partial X_i} = r_i - c_i - (r_i - v_i)F_i(X_i) + \frac{\partial \mathsf{E}[\varphi_i(\mathbf{X})]}{\partial X_i} + \frac{\partial \mathsf{E}[\varphi_{-i}(\mathbf{X})]}{\partial X_i} = 0 \quad \forall i, \tag{4.2}$$

while the optimal order of an individual retailer in the decentralized system, X_i^{d}, satisfies

$$\frac{\partial J_i^{\mathrm{d}}(\mathbf{X})}{\partial X_i} = r_i - c_i - (r_i - v_i)F_i(X_i) + \frac{\partial \mathsf{E}[\varphi_i(\mathbf{X})]}{\partial X_i} = 0 \qquad \forall i. \tag{4.3}$$

Equations (4.2) and (4.3) give us a sufficient and necessary condition for a retailer in the decentralized system with an arbitrary number of retailers to order a system-optimal quantity.

Proposition 4. *If the expected total profit for the system of retailers, $J^n(\mathbf{X})$, is unimodal in $\mathbf{X}$, the sufficient and necessary condition for achieving a first-best solution is*

$$\frac{\partial \mathsf{E}[\varphi_{-i}(X_i, X_{-i}^n)]}{\partial X_i} = 0 \qquad \forall i. \tag{4.4}$$

For example, when $n = 3$, one can evaluate that the retailers with $D_i \sim U[0, 100]$; $i = 1,2,3$; $p_{12} = p_{23} = p_{31} = 6$; and $p_{21} = p_{32} = p_{13} = 8$ satisfy the above condition, and a first-best outcome can be achieved. However, through various numerical experiments we were able to observe that even small differences among parameters of different retailers may prevent us from coordinating the system. One of our analytical results is given in the following proposition.

Proposition 5. *If n retailers face i.i.d. demand distributions and differ only in their material costs (that is, $r_i = r_j = r, v_i = v_j = v, t_{ij} = t_{ji} = t$ for $i, j \in \{1,\dots,n\}$), a first-best outcome cannot be achieved.*

We conducted a numerical analysis to study what is the impact of retailers' diversity on efficiency losses; as in Proposition 5, we assume that the retailers differ only in their cost, and study the impact of the mean and standard deviation of material cost, of the number of retailers, of the retail price, and of the salvage value. Although the system cannot be coordinated, we observe that the efficiency losses are rather small, even with a very few retailers. Some of our results are depicted in Fig. 4.5.

Our analysis indicates that, as expected, the efficiency improves as the standard deviation of cost decreases, and as the number of retailers increases. Additional simulations, in which we fix either the mean value of the cost, c, or the salvage value, v, while we vary the other parameter, indicate that the efficiency also improves with the increase of the critical fractile, which can be partially observed in Fig. 4.5. On one hand, as the decrease of the mean product cost, c, translates into larger profit

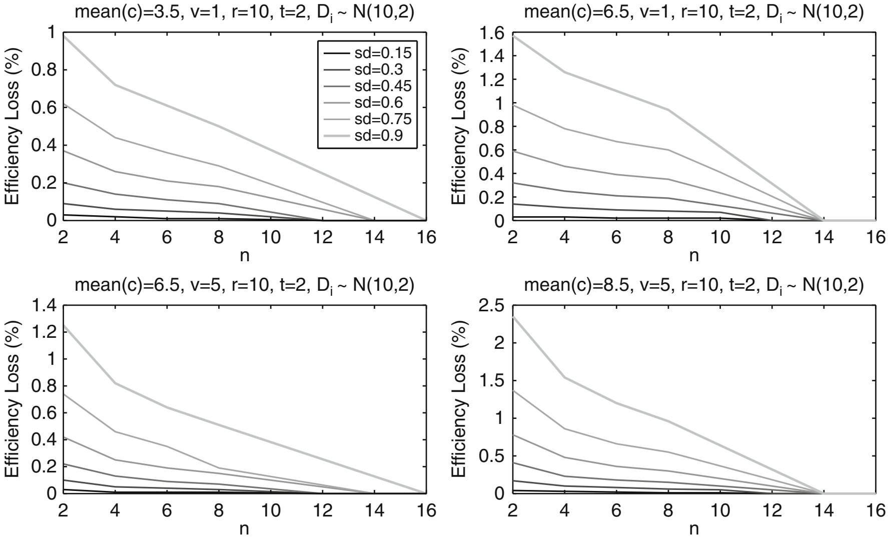

Fig. 4.5 Efficiency losses for different levels of differentiation among retailers

margin, benefits from transshipments are increased; on the other hand, the increase in the salvage value, v, hedges off the risk in demand uncertainty. In either case, the retailers' decisions become closer to those of the centralized system.

4.6 A Contractual Mechanism that Induces a First-Best Solution

In Sect. 4.5, we have shown that a first-best solution can be achieved if condition (4.4) is satisfied. However, when this condition is not satisfied, the retailers' individually optimal decisions may lead to significant efficiency losses. Achieving a first-best solution in a decentralized system may not be possible in many realistic situations without the use of some additional enforcing mechanisms.[3]

In what follows, we assume that the discount factors satisfy $\delta_i > \delta_n^*$ (hence, complete sharing is achieved), and develop a contract that leads to system-optimal order quantities without any additional constraints. Although the total system profit increases if the retailers order a first-best solution, the profit of some retailers may decrease so that they need to be induced to cooperate by some type of side payments.

[3] Note that in our repeated-game setting we were able to achieve $\mathbf{X}^{\mathrm{d}}$ as a SPNE, by utilizing the fact that $J_i^{\mathrm{d}}(X_i^D) \geq J_i^1(X_i^1)$. Unfortunately, $J_i^{\mathrm{d}}(X_i^C)$ can be greater or smaller than $J_i^1(X_i^1)$, hence a first-best ordering quantity cannot, in general, be obtained as a SPNE.

In addition, in order to prevent those retailers from defection in the future, deviations from the contract should be penalized. Thus, our contract consists of the following parts:

1. Retailer i's ordering strategy, X_{it}, and her residual-sharing amount, E_{it}, H_{it}, in every period. When a retailer orders inventory and shares residuals as prescribed by the contract, she is included in cooperation (inventory sharing) in the next period. If she breaks the contract and orders a different quantity or shares a different amount in period t, she is excluded from cooperation in all subsequent periods, $t+1, t+2, \ldots$
2. Discretionary transfer payments at the end of each period. A retailer, i, who breaks the contract in period t makes a positive payment, d_{it}. This value is distributed among the retailers who have followed the contract, $\sum_i d_{it} = 0$.
3. Contract activation bonus, B_i, upon signing the contract. This one-time bonus can be positive or negative, with $\sum_i B_i = 0$. The retailers who benefit from cooperation are those that may be required to have a negative activation bonus in order to induce participation of retailers who would individually prefer not to order a first-best quantity.

We will refer to this contract as the *eviction contract* because the most severe punishment for a defecting retailer is her eviction from the inventory-sharing system. Changes in the cooperative behavior of the system can be described through coalition structures, in which cooperating retailers belong to a coalition. Each time a retailer is evicted, the remaining retailers form a new inventory-sharing system and completely share residuals in this reduced system. Thus, if none of the retailers has ever defected, the system operates as the grand coalition. We assume that the retailers who are evicted do not form new inventory-sharing groups. This implies that each evicted retailer constitutes a one-member coalition. In other words, suppose that the current system is described by coalition structure $Z = \{S_1, S_2, \ldots, S_{n-k+1}\}$. Then, $|S_j| = 1$ for $n-k$ coalitions, and $|S_i| = k$ for some coalition S_i. We will use Z^k to denote a coalition structure in which exactly one coalition has k members, while the remaining $n-k$ coalitions consist of a single retailer. Thus, Z^n denotes the grand coalition, while Z^1 denotes the coalition structure with no inventory sharing. Clearly, the system-optimal stocking quantity in state Z^n is $\mathbf{X}^n$, while $\mathbf{X}^1$ maximizes the system profit under state Z^1. We denote by $\mathbf{X}^k$ the system-optimal stocking quantities for coalition structure Z^k. For an arbitrary coalition structure, Z, we denote the system-optimal order quantity by $\mathbf{X}^Z$.

In order to induce a system-optimal solution, the eviction contract requires the retailers to order system-optimal quantities and share all of their residuals. Thus, given a coalition structure, the orders placed and residuals shared by the retailers in period t, we can determine the coalition structure in period $t+1$ as follows:

$$Z_{t+1}(Z_t, \mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t | Z_t = Z^k) = \begin{cases} Z^k, & \text{if } \mathbf{X}_t = \mathbf{X}^k, \mathbf{H}_t = \bar{\mathbf{H}}_t, \mathbf{E}_t = \bar{\mathbf{E}}_t; \\ Z^{k-l}, & \text{if } (X_{it} = X_i^k, H_{it} = \bar{H}_{it}, E_{it} = \bar{E}_{it}) \text{ does} \\ & \text{not hold for } l \text{ coalition members in } Z_t. \end{cases} \tag{4.5}$$

Now, the eviction contract can be described by

$$(\mathbf{X}_t(\hat{\mathbf{h}}_{t-1}), \mathbf{H}_t(\hat{\mathbf{h}}_{t-1}), \mathbf{E}_t(\hat{\mathbf{h}}_{t-1}), \mathbf{d}_t(\hat{\mathbf{h}}_t), \mathbf{B}),$$

where $\hat{\mathbf{h}}_t$ denotes the history up to period t, $\hat{\mathbf{h}}_t = \{Z_\tau, \mathbf{X}_\tau, \mathbf{H}_\tau, \mathbf{E}_\tau\}_{\tau=1}^t$.

Recall that we use J_i^{d} to denote the expected profit for retailer i under dual allocations when all retailers participate in inventory sharing. We now introduce some additional notation. We denote by $J_i^Z(\mathbf{X}, \mathbf{H}, \mathbf{E})$ the expected profit for i under coalition structure Z, and by $J^Z(\mathbf{X}, \mathbf{H}, \mathbf{E})$ the expected total system profit under coalition structure Z. The following theorem describes how a first-best solution can be achieved through an eviction contract. Its proof is given in the Appendix.

Theorem 5. *Suppose that all retailers participate in inventory sharing and $J^n(\mathbf{X})$ is unimodal. Then, the eviction contract $(\mathbf{X}_t(\hat{\mathbf{h}}_{t-1}), \mathbf{H}_t(\hat{\mathbf{h}}_{t-1}), \mathbf{E}_t(\hat{\mathbf{h}}_{t-1}), \mathbf{d}_t(\hat{\mathbf{h}}_t), \mathbf{B})$ is a contract that induces a first-best solution if the retailers' ordering strategies, $\mathbf{X}_t$, are given by*

$$\mathbf{X}_t(\hat{\mathbf{h}}_{t-1}|Z_t = Z^k) = \mathbf{X}^k,$$

all coalition members share their entire residuals, the evicted members share nothing, the discretionary transfer payments are

$$d_{it}(\hat{\mathbf{h}}_t) = \begin{cases} \frac{\Delta_{it}(\hat{\mathbf{h}}_t)}{\sum_{I_t^+} \Delta_{jt}(\hat{\mathbf{h}}_t)} \times \sum_{I_t^-} (-\Delta_{jt}(\hat{\mathbf{h}}_t)) & i \in I_t^+ \\ \Delta_{it}(\hat{\mathbf{h}}_t) & i \in I_t^-, \end{cases}$$

where

$$\Delta_{it}(\hat{\mathbf{h}}_t) = \frac{1}{1-\delta_i}\left[J_i^{Z_t}(\mathbf{X}^{Z_t}) - \delta J_i^1(X_i^1)\right] - J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t),$$

$$I_t^+ = \{i : \Delta_{it}(\hat{\mathbf{h}}_t) > 0\} \;\; \textit{and} \;\; I_t^- = \{i : \Delta_{it}(\hat{\mathbf{h}}_t) \le 0\},$$

and the one-time contract activation bonus is given as

$$B_i = \begin{cases} \frac{\Lambda_i}{\sum_{K^-} \Lambda_i} \times \sum_{K^+} (-\Lambda_i) & i \in K^- \\ \Lambda_i & i \in K^+, \end{cases}$$

where

$$\Lambda_i = \frac{1}{1-\delta_i}(J_i^n(\mathbf{X}^{\mathrm{d}}) - J_i^n(\mathbf{X}^n)), \;\; K^+ = \{i : \Lambda_i > 0\} \;\; \textit{and} \;\; K^- = \{i : \Lambda_i \le 0\}.$$

Despite its seemingly complex structure, the contract is actually quite simple to implement: at the beginning of their cooperation, the retailers who strictly benefit from the contract compensate the retailers whose profit is reduced (as a result of ordering system-optimal quantities) through the activation bonus B_i. In addition, the retailers agree that in the case of any defection, all benefits should be forfeited and allocated among the retailers who suffer a loss after such an action.[4] Thus, there is no incentive for any retailer to defect from the strategy which prescribes ordering system-optimal quantity, sharing entire residuals, and receiving dual allocations. The transfer payment is zero as long as the retailers follow the contract—it serves as a threat that prevents them from defection.[5] One could, alternatively, develop a contract in which retailers whose profit decreases after ordering system-optimal quantity receive compensations for their losses at the end of every period. This type of contract would not require activation bonuses, but may lead to more complex implementation, as the payments need to be calculated and exchanged at the end of every period (in our contract, this happens only if there was a defection in a given period).

Note that the eviction contract works not only for dual allocations, but also for any other allocation rule that induces full participation and complete residual sharing, but not a first-best inventory decision. This can be easily confirmed by observing that the proof does not depend upon any pre-specified allocation rules.

4.7 Concluding Remarks

In this work, we study a repeated inventory-sharing game with n retailers in which the retailers distribute the profit from transshipments according to the dual allocations. Each retailer faces stochastic demand and salvages all unsold inventory at the end of each period. Using the standard tools from the theory of repeated games, we show that the use of NRS induces complete sharing in an SPNE of an infinitely repeated game (providing that the discount factor of future payoffs is large enough), while the retailers always withhold residuals if the game is repeated a finite number of times. We also show that complete sharing can be an SPNE even if the punishment is not executed over an infinite horizon but instead lasts only for a finite number of periods. Clearly, shorter punishment periods require larger discount factors, and a punishment that lasts only a few periods will induce complete sharing only with the retailers whose discounting of the future periods is very small. In addition, we provide some analytical results for the asymptotic behavior of the retailers' ordering quantities and the lower bounds on discount factors that induce complete sharing for large number of players.

[4]The amount of transfer payments $d_i \leq 0$ (realized when a player benefits from a defection) removes from a retailer all possible gains from that defection. $\Delta_{it} > 0$ (which leads to $d_i > 0$) implies that a retailer observes a loss as a result of someone's defection (and is, therefore, compensated from payments of those who benefit); this retailer receives a fraction of total transfer payments proportional to her loss as compared to the total losses observed by the system.

[5]In the whole contract lifetime, the discretionary transfer payment happens at most $n-1$ times, as the number of inventory-sharing retailers is reduced from n to 1.

While there can be a significant difference in optimal profits generated by decentralized retailers and those generated in a centralized system, a decentralized model will result in a system-optimal outcome if the retailers are symmetric. As this condition may not be satisfied in many cases, we derive another condition, (4.4), that leads to a first-best outcome. When this condition is not satisfied, we develop a contract that induces the retailers to order a first-best quantity whenever the complete sharing condition holds.

We note that our model assumes that all leftover inventory is salvaged at the end of each period. The reason for this is twofold. On one hand, because we were mainly interested in studying the impact of repeated interactions on the retailers' sharing decisions in the second stage, a more complex model in which the retailers are allowed to carry inventory from one period to another would lead to a more complicated model that is beyond the scope of this work. On another hand, such situations do occur in industries where products have short life cycles, long lead times, and unpredictable demands, like apparel, Christmas toys, and high-tech electronic components. Retailers in these industries are often open to inventory-sharing agreements with others.

Our inventory-sharing model may require a neutral third party for its implementation—monitoring of residuals, making effective transshipment decisions, and allocation of profits among the members. While this is easily realized within a trade association or when the retailers belong to a larger organization, it might be more difficult to execute when the retailers are independent. It is, therefore, interesting to observe emergence of companies such as iSuppli Corp., which act as neutral intermediaries among independent entities and, at the same time, improve the market's efficiency.

When dual allocations are used in one-shot setting, the retailers withhold their residuals, and our aim was to study if this property persists when the retailers interact repeatedly. Note, however, that many of our results can be extended to alternative allocation rules (though some extensions may require certain modifications in proofs and results).

Appendix

Proof of Theorem 3. In order to prove this theorem, we first introduce the following notation: let $F^m(y) = P\{\sum_{i=1}^m D_i \le y\}$, $\hat{F}^m(y) = P\{\frac{1}{m}\sum_{i=1}^m D_i \le y\}$, and $\mathsf{E}[D_i] = \mu$. Note that $F^m(y) = \hat{F}^m(\frac{y}{m})$. We will also need the following lemmas.

Lemma 1. *In an inventory-sharing game with symmetric retailers facing strictly increasing and independent distribution functions, a retailer defecting from strategy* $(X^{\mathrm{d}}, \bar{H}_i, \bar{E}_i)$ *maximizes her benefit from defection if she orders* X^{d}.

Proof of Lemma 1. If we have n symmetric retailers, the dual price of retailer i's residual will be either 0 or p, depending on the amount she is sharing with the others.

For example, if $\sum_{j \neq i}(\bar{E}_j - \bar{H}_j) = k > 0$, the retailers other than i need k additional units of products. Then, retailer i will receive p per unit if $0 < \bar{H}_i < k$, while she will get nothing otherwise. More formally, retailer i's total expected profit when she orders X_i and other retailers order $\mathbf{X}^{\mathrm{d}}_{-i}$ is given by

$$\begin{aligned} J_i^{\mathrm{d}}(X_i|\mathbf{X}^{\mathrm{d}}_{-i}) = {} & r\mathsf{E}[\min\{X_i, D_i\}] + v\mathsf{E}[H_i] - cX_i \\ & + p\int_0^{\infty} f^{n-1}\left((n-1)X^{\mathrm{d}} + k\right)\int_{X_i-k}^{X_i}(X_i - u)f(u)\mathrm{d}u\mathrm{d}k \\ & + p\int_0^{\infty} f^{n-1}\left((n-1)X^{\mathrm{d}} - k\right)\int_{X_i}^{X_i+k}(u - X_i)f(u)\mathrm{d}u\mathrm{d}k, \end{aligned}$$

where $f^{n-1}((n-1)X^{\mathrm{d}} + y)$ is the probability density when the residual demand (resp., inventory) for the remaining $(n-1)$ retailers is $y > 0$ (resp., $(-y) > 0$), and its first derivative is given by

$$\begin{aligned} (J_i^{\mathrm{d}})'(X_i|\mathbf{X}^{\mathrm{d}}_{-i}) = {} & r - c - (r-v)F(X_i) \\ & + p\int_0^{\infty}[F(X_i) - F(X_i - k)]f^{n-1}\left((n-1)X^{\mathrm{d}} + k\right)\mathrm{d}k \\ & - p\int_0^{\infty}[F(X_i + k) - F(X_i)]f^{n-1}\left((n-1)X^{\mathrm{d}} - k\right)\mathrm{d}k \\ & - p\int_0^{\infty} k\Big[f(X_i - k)f^{n-1}\left((n-1)X^{\mathrm{d}} + k\right) \\ & - f(X_i + k)f^{n-1}\left((n-1)X^{\mathrm{d}} - k\right)\Big]\mathrm{d}k. \end{aligned} \tag{4.6}$$

Retailer i can increase her profit if she deviates whenever her dual price is zero. In other words, she maximizes her profit if she withholds part of her residual inventory/demand to make it lower than the total residual demand/inventory from other retailers. Under this kind of strategy, her total expected profit will be increased to

$$\begin{aligned} J_i^{\mathrm{def}}(X_i|\mathbf{X}^{\mathrm{d}}_{-i}) = {} & r\mathsf{E}[\min\{X_i, D_i\}] + v\mathsf{E}[H_i] - cX_i \\ & + p\int_0^{\infty} f^{n-1}\left((n-1)X^{\mathrm{d}} + k\right)\int_{X_i-k}^{X_i}(X_i - u)f(u)\mathrm{d}u\mathrm{d}k \\ & + p\int_0^{\infty} f^{n-1}\left((n-1)X^{\mathrm{d}} - k\right)\int_{X_i}^{X_i+k}(u - X_i)f(u)\mathrm{d}u\mathrm{d}k \\ & + p\int_0^{\infty} kf^{n-1}\left((n-1)X^{\mathrm{d}} + k\right)F(X_i - k)\mathrm{d}k \\ & + p\int_0^{\infty} kf^{n-1}\left((n-1)X^{\mathrm{d}} - k\right)[1 - F(X_i + k)]\mathrm{d}k, \end{aligned}$$

and its derivatives are

$$(J_i^{\text{def}})'(X_i|\mathbf{X}_{-i}^{\text{d}}) = r - c - (r-v)F(X_i) + p\int_0^{\infty} [F(X_i) - F(X_i - k)]\, f^{n-1}\left((n-1)X^{\text{d}} + k\right) \mathrm{d}k - p\int_0^{\infty} [F(X_i + k) - F(X_i)]\, f^{n-1}\left((n-1)X^{\text{d}} - k\right) \mathrm{d}k, \quad (4.7)$$

$$(J_i^{\text{def}})''(X_i|\mathbf{X}_{-i}^{\text{d}}) = -tf(X_i) - p\int_0^{\infty} \Big[f(X_i - k) f^{n-1}\left((n-1)X^{\text{d}} + k\right) + f(X_i + k) f^{n-1}\left((n-1)X^{\text{d}} - k\right)\Big] \mathrm{d}k < 0. \quad (4.8)$$

Because all demands follow an identical distribution, it follows from (4.6) and (4.7) that

$$\begin{aligned}
&[(J_i^{\text{def}})' - (J_i^{\text{d}})'](X_i|\mathbf{X}_{-i}^{\text{d}}) \\
&= p\int_0^{\infty} k\Big[f(X_i - k) f^{n-1}\left((n-1)X^{\text{d}} + k\right) - f(X_i + k) f^{n-1}\left((n-1)X^{\text{d}} - k\right)\Big] \mathrm{d}k \\
&= E\left[X_i - D_i | \sum_{m=1}^{n} D_m = (n-1)X^{\text{d}} + X_i\right] \\
&= \frac{n-1}{n}\left(X_i - X^{\text{d}}\right).
\end{aligned}$$

Recall that $X^{\text{d}} = \arg\max J_i^{\text{d}}(X_i|\mathbf{X}_{-i}^{\text{d}})$, and consequently $(J_i^{\text{d}})'(X^{\text{d}}|\mathbf{X}_{-i}^{\text{d}})) = 0$. This implies

$$\begin{aligned}
(J_i^{\text{def}})'(X^{\text{d}}|\mathbf{X}_{-i}^{\text{d}}) &= (J_i^{\text{d}})'(X^{\text{d}}|\mathbf{X}_{-i}^{\text{d}}) + [(J_i^{\text{def}})'(X^{\text{d}}|\mathbf{X}_{-i}^{\text{d}}) - (J_i^{\text{d}})'(X^{\text{d}}|\mathbf{X}_{-i}^{\text{d}})] \\
&= 0 + \frac{n-1}{n}(X^{\text{d}} - X^{\text{d}}) = 0.
\end{aligned}$$

Since $J_i^{\text{def}}(X_i|\mathbf{X}_{-i}^{\text{d}})$ is a concave function, the optimal ordering decision when player i defects, X_i^{def}, should satisfy $(J_i^{\text{def}})'(X_i^{\text{def}}|\mathbf{X}_{-i}^{\text{d}})) = 0$. Thus, $X_i^{\text{def}} = X^{\text{d}}$, and a retailer contemplating a defection maximizes her profit if she orders at the decentralized optimal level. □

Lemma 2. *In an inventory-sharing game with n symmetric retailers and strictly increasing demand distribution function, the expected profit for each retailer,* $J^{\text{d}}\left(X^{\text{d}}(n), n\right)$*, is increasing in n, where* $X^{\text{d}}(n)$ *is the NE ordering decision for each retailer in the decentralized system.*

Proof of Lemma 2. Consider a game with $n+1$ symmetric retailers, and let $\mathscr{S}$ be any n-members subset of these retailers. In terms of cooperative game theory, the value of the coalition $\mathscr{S}$ corresponds to the profit generated by its members; because the retailers are symmetric, it can be written as $V^*_{\mathscr{S}} = nJ^{\mathrm{d}}(X,n)$, where $J^{\mathrm{d}}(X,n)$ denotes the expected profit generated by an arbitrary retailer in a game with n symmetric retailers under dual allocations. However, in an $(n+1)$-retailer game with dual allocations, each retailer will receive a payoff $J^{\mathrm{d}}(X,n+1)$. Because dual allocations belong to the core, we must have $nJ^{\mathrm{d}}(X,n+1) > V^*_{\mathscr{S}} = nJ^{\mathrm{d}}(X,n)$. It is then straightforward that $J^{\mathrm{d}}\left(X^{\mathrm{d}}(n+1),n+1\right) \geq J^{\mathrm{d}}\left(X^{\mathrm{d}}(n),n+1\right) \geq J^{\mathrm{d}}\left(X^{\mathrm{d}}(n),n\right)$. □

We can now prove the theorem. Consider the model with n symmetric retailers and suppose that there were no prior defections. That is, each retailer orders X^{d} and shares her entire residuals. Recall that we have shown in Lemma 1 that defecting retailers maximize their profit if they order X^{d} and deviate in the amount they share with others. Under demand realization $\mathbf{D}$, let $\bar{P}_i^{\mathrm{def}}(\mathbf{X}^{\mathrm{d}},\mathbf{D},n)$ denote the highest payoff that retailer i can generate if she defects in a game with n players, while the other retailers cooperate, and recall that $P_i^d(\mathbf{X}^{\mathrm{d}},\mathbf{D},n)$ is her profit in the current period if she shares all of her residuals. After defection, she will receive $J_i(X_1)$ in all subsequent periods. Thus, a possible deviation by player i is deterred if her discount factor satisfies

$$\bar{P}_i^{\mathrm{def}}(\mathbf{X}^{\mathrm{d}},\mathbf{D},n) + \frac{\delta}{1-\delta}J_i(X_1) < \frac{\delta}{1-\delta}J_i^{\mathrm{d}}(\mathbf{X}^{\mathrm{d}},n) + P_i^d(\mathbf{X}^{\mathrm{d}},\mathbf{D},n), \forall \mathbf{D}, \tag{4.9}$$

where $J_i^{\mathrm{d}}(\mathbf{X}^{\mathrm{d}},n)$ denotes the payoff that retailer i receives when n retailers use dual allocations, order $\mathbf{X}^{\mathrm{d}}$, and share their entire residuals. It is easy to verify that (4.9) holds whenever

$$\delta > \delta_{i,n} = \frac{1}{1+\frac{J_i^{\mathrm{d}}(\mathbf{X}^{\mathrm{d}},n)-J_i(X_1)}{\sup_{\mathbf{D}}\{\bar{P}_i^{\mathrm{def}}(\mathbf{X}^{\mathrm{d}},\mathbf{D},n)-P_i^d(\mathbf{X}^{\mathrm{d}},\mathbf{D},n)\}}}. \tag{4.10}$$

Note that the upper bound of the extra profit that one can get out of deviation, $\sup_{\mathbf{D}}\{\bar{P}_i^{\mathrm{def}}(\mathbf{X}^{\mathrm{d}},\mathbf{D},n) - P_i^d(\mathbf{X}^{\mathrm{d}},\mathbf{D},n)\}$, can be obtained by comparing two cases: (1) the extra profit generated when $D_i = 0$ and the total residual demand of the remaining retailers is slightly below X^{d}; and (2) the extra profit generated when $D_{-i} = 0$ and D_i is slightly above nX^{d}. In the first case, this profit is pX^{d}; in the second case, this profit would be $p(n-1)X^{\mathrm{d}}$, assuming that demand can achieve values above nX^{d}. However, note that in most real-life situations there is an $M > 0$ such that $P(D_i > M)$ is negligible (if demand distribution has a finite support with upper bound $\bar{D}$, then $M = \bar{D}$), and the maximum benefit from defection is $p(M - X^{\mathrm{d}})$. Let us denote $\hat{n} = \min\{n : nX^{\mathrm{d}} \geq M\}$. Then, whenever $n \geq \hat{n}$, it implies that $\sup_{\mathbf{D}}\{\bar{P}_i^{\mathrm{def}}(\mathbf{X}^{\mathrm{d}},\mathbf{D},n) - P_i^d(\mathbf{X}^{\mathrm{d}},\mathbf{D},n)\} = \max\{pX^{\mathrm{d}}, p(M-X^{\mathrm{d}})\}$, and (4.10) corresponds to

$$\delta > \delta_{i,n} = \frac{p\max\{X^{\mathrm{d}}, M-X^{\mathrm{d}}\}}{p\max\{X^{\mathrm{d}}, M-X^{\mathrm{d}}\} + J_i^{\mathrm{d}}(\mathbf{X}^{\mathrm{d}},n) - J_i(X_1)}.$$

Because the players are symmetric, let $\delta_n = \delta_{i,n}$. Since $J_i(X_1)$ does not depend on n and we showed in Lemma 2 that $J_i^{\mathrm{d}}(\mathbf{X}^{\mathrm{d}},n)$ increases with n, δ_n is decreasing in n. Finally, let $\delta_n^* = \delta_n$. □

Proof of Proposition 2. When each retailer orders X^{d}, the total expected profit for each of them can be determined by

$$\begin{aligned}
J(\mathbf{X}^{\mathrm{d}}) &= r\mathsf{E}[\min\{X^{\mathrm{d}}, D\}] + v\mathsf{E}[H] - cX^{\mathrm{d}} \\
&\quad + p\int_0^\infty kf(X^{\mathrm{d}}-k)\left[1-\hat{F}^{n-1}\left(X^{\mathrm{d}}+\frac{k}{n-1}\right)\right]\mathrm{d}k \\
&\quad + p\int_0^\infty kf(X^{\mathrm{d}}+k)\hat{F}^{n-1}\left(X^{\mathrm{d}}-\frac{k}{n-1}\right)\mathrm{d}k \\
&= (r-c)X^{\mathrm{d}} - (r-v)\left[X^{\mathrm{d}}F(X^{\mathrm{d}}) - \int_0^{X^{\mathrm{d}}} yf(y)\mathrm{d}y\right] \\
&\quad + p\int_0^\infty kf(X^{\mathrm{d}}-k)\left[1-\hat{F}^{n-1}\left(X^{\mathrm{d}}+\frac{k}{n-1}\right)\right]\mathrm{d}k \\
&\quad + p\int_0^\infty kf(X^{\mathrm{d}}+k)\hat{F}^{n-1}\left(X^{\mathrm{d}}-\frac{k}{n-1}\right)\mathrm{d}k.
\end{aligned}$$

If we let $\sigma^2 = Var[D_i]$, then by the central limit theorem (CLT) we have

$$\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^{m} D_i \sim N\left(\mu, \frac{\sigma^2}{m}\right).$$

Suppose first that $X^{\mathrm{d}} > \mu$. Then, we have $\lim_{n\to\infty}[1-\hat{F}^{n-1}(X^{\mathrm{d}}+\frac{k}{n-1})] = 0$ and $\lim_{n\to\infty}\hat{F}^{n-1}(X^{\mathrm{d}}-\frac{k}{n-1}) = 1$; hence, the derivative of $J(\cdot|\mathbf{X}_{-i}^{\mathrm{d}})$ evaluated at X^{d} becomes

$$J'(X^{\mathrm{d}}|\mathbf{X}_{-i}^{\mathrm{d}}) = r-c-(r-v)F(X^{\mathrm{d}}) - p + pF(X^{\mathrm{d}}) = -(c-v) + t[1-F(X^{\mathrm{d}})],$$

which is a decreasing function of X^{d}. Thus, if $t=0$ or $F(\mu) \geq 1-\frac{c-v}{t} = \frac{r-c-p}{t}$, then $J'(X^{\mathrm{d}}|\mathbf{X}_{-i}^{\mathrm{d}}) \leq 0$ for any $X^{\mathrm{d}} \in (\mu,\infty)$, and the retailer maximizes her profit by choosing $X^{\mathrm{d}} \to \mu^+$. Otherwise, $X^{\mathrm{d}} = \inf\{x : F(x) > \frac{r-c-p}{t}\}$ is an optimal solution within (μ,∞).

If $X^{\mathrm{d}} < \mu$, $\lim_{n\to\infty}[1-\hat{F}^{n-1}(X^{\mathrm{d}}+\frac{k}{n-1})] = 1$ and $\lim_{n\to\infty}\hat{F}^{n-1}(X^{\mathrm{d}}-\frac{k}{n-1}) = 0$. The derivative of $J(\cdot|\mathbf{X}^{\mathrm{d}}_{-i})$ evaluated at X^{d} becomes

$$J^{'}(X^{\mathrm{d}}|\mathbf{X}^{\mathrm{d}}_{-i}) = r-c-(r-v)F(X^{\mathrm{d}})+pF(X^{\mathrm{d}}) = (r-c)-tF(X^{\mathrm{d}}),$$

which is again a decreasing function of X^{d}. In this case, if $F(\mu) \leq \frac{r-c}{t}$ or $t=0$, then $J^{'}(X^{\mathrm{d}}|\mathbf{X}^{\mathrm{d}}_{-i}) \geq 0$ for any $X^{\mathrm{d}} \in (\infty,\mu)$, and the retailer maximizes her profit by choosing $X^{\mathrm{d}} \to \mu^{-}$. Otherwise, $X^{\mathrm{d}} = \sup\{x : F(x) < \frac{r-c}{t}\}$ is an optimal solution within $(-\infty,\mu)$.

From the above, we can conclude that whenever $F(\mu) \in [\frac{r-c-p}{t}, \frac{r-c}{t}]$ or $t=0$, the retailer should select $X^{\mathrm{d}} \to \mu$. Otherwise, because $\frac{r-c-p}{t} \leq \frac{r-c}{t}$, any local optimum is also a global optimum whenever $F(\mu) \notin [\frac{r-c-p}{t}, \frac{r-c}{t}]$. □

Proof of Corollary 1. Suppose first that $t>0$. If $F(\mu) > \frac{r-c}{t}$, it follows from Proposition 2 that $\lim_{n\to\infty} X^{\mathrm{d}}(n) = \sup\{x : F(x) < \frac{r-c}{t}\}$. This implies that $F(X^{\mathrm{d}}) \leq \frac{r-c}{t} < F(\mu)$, hence $X^{\mathrm{d}} < \mu$. On the other hand, when there is no cooperation among the retailers, the optimal ordering level X^1 can be determined by the newsvendor model, $F(X^1) = \frac{r-c}{r-v}$. Recall that we assume $p = r-v-t \geq 0$, which implies $r-v \geq t$, therefore $F(X^1) \leq F(X^{\mathrm{d}})$, and $X^1 \leq X^{\mathrm{d}}$.

If, on the other hand, $F(\mu) < \frac{r-c-p}{t}$, then $\lim_{n\to\infty} X^{\mathrm{d}}(n) = \inf\{x : F(x) > \frac{r-c-p}{t}\}$. This implies that $F(\mu) < \frac{r-c-p}{t} \leq F(X^{\mathrm{d}})$, hence $\mu < X^{\mathrm{d}}$. Consequently, $F(X^1) = \frac{r-c}{r-v} \geq \frac{r-c-p}{r-v-p} = \frac{r-c-p}{t} = F(X^{\mathrm{d}})$, so $X^1 \geq X^{\mathrm{d}}$.

When $t=0$, each retailer orders the expected demand value, and the result is straightforward. □

Proof of Theorem 4. Recall that the lower bound of δ_n satisfies

$$\begin{aligned}\delta_n^* &= \frac{p\max\{X^{\mathrm{d}},M-X^{\mathrm{d}}\}}{p\max\{X^{\mathrm{d}},M-X^{\mathrm{d}}\}+J_i^{\mathrm{d}}(\mathbf{X}^{\mathrm{d}},n)-J_i(X_1)} \\ &= \frac{\rho(X^{\mathrm{d}})}{\rho(X^{\mathrm{d}})+J_i^{\mathrm{d}}(\mathbf{X}^{\mathrm{d}},n)-J_i(X_1)}\forall i. \end{aligned} \tag{4.11}$$

In addition, in the model without cooperation, each retailer's profit is maximized at $X^1 = F^{-1}\left(\frac{r-c}{r-v}\right)$, and equals

$$J^1(X^1) = (r-v)\int_0^{X^1} yf(y)\mathrm{d}y = (r-v)\xi(X^1). \tag{4.12}$$

If $X^{\mathrm{d}} = \mu$, it follows from the CLT that

$$\lim_{n\to\infty} 1-\hat{F}^{n-1}\left(X^{\mathrm{d}}+\frac{k}{n-1}\right) = \lim_{n\to\infty}\hat{F}^{n-1}\left(X^{\mathrm{d}}-\frac{k}{n-1}\right) = \frac{1}{2},$$

which implies

$$\begin{aligned} J_i^{\mathrm{d}}(\mathbf{X}^{\mathrm{d}},n) &= (r-c)X^{\mathrm{d}} - (r-v)\left[X^{\mathrm{d}}F(X^{\mathrm{d}}) - \int_0^{X^{\mathrm{d}}} yf(y)\mathrm{d}y\right] \\ &\quad + p\int_0^{\infty} kf(X^{\mathrm{d}}-k)\left[1-\hat{F}^{n-1}\left(X^{\mathrm{d}}+\frac{k}{n-1}\right)\right]\mathrm{d}k \\ &\quad + p\int_0^{\infty} kf(X^{\mathrm{d}}+k)\hat{F}^{n-1}\left(X^{\mathrm{d}}-\frac{k}{n-1}\right)\mathrm{d}k \\ &= (r-c)\mu - (r-v)\left[\mu F(\mu) - \int_0^{\mu} yf(y)\mathrm{d}y\right] \\ &\quad + \frac{p}{2}\left[\int_0^{\infty} kf(\mu-k)\mathrm{d}k + \int_0^{\infty} kf(\mu+k)\mathrm{d}k\right] \\ &= [r-c-tF(\mu)]\mu + t\int_0^{\mu} yf(y)\mathrm{d}y \\ &= [r-c-tF(\mu)]\mu + t\xi(\mu). \end{aligned} \tag{4.13}$$

By substituting (4.12) and (4.13) into (4.11), we obtain

$$\delta_\infty^* = \frac{\rho(\mu)}{\rho(\mu) + [r-c-tF(\mu)]\mu + t\xi(\mu) - (r-v)\xi(X^1)}.$$

If $X^{\mathrm{d}} = \sup\{x : F(x) < \frac{r-c}{t}\} < \mu$, we have $\lim_{n\to\infty} 1 - \hat{F}^{n-1}(X^{\mathrm{d}} + \frac{k}{n-1}) = 1$ and $\lim_{n\to\infty} \hat{F}^{n-1}(X^{\mathrm{d}} - \frac{k}{n-1}) = 0$, hence

$$\begin{aligned} J_i^{\mathrm{d}}(\mathbf{X}^{\mathrm{d}},n) &= (r-c)X^{\mathrm{d}} - (r-v)\left[X^{\mathrm{d}}F(X^{\mathrm{d}}) - \int_0^{X^{\mathrm{d}}} yf(y)\mathrm{d}y\right] \\ &\quad + p\int_0^{\infty} kf(X^{\mathrm{d}}-k)\left[1-\hat{F}^{n-1}\left(X^{\mathrm{d}}+\frac{k}{n-1}\right)\right]\mathrm{d}k \\ &\quad + p\int_0^{\infty} kf(X^{\mathrm{d}}+k)\hat{F}^{n-1}\left(X^{\mathrm{d}}-\frac{k}{n-1}\right)\mathrm{d}k \\ &= (r-c)X^{\mathrm{d}} - (r-v)\left[X^{\mathrm{d}}F(X^{\mathrm{d}}) - \int_0^{X^{\mathrm{d}}} yf(y)\mathrm{d}y\right] + p\int_0^{\infty} kf(X^{\mathrm{d}}-k)\mathrm{d}k \\ &= t\int_0^{X^{\mathrm{d}}} yf(y)\mathrm{d}y \\ &= t\xi(X^{\mathrm{d}}). \end{aligned} \tag{4.14}$$

By substituting (4.12) and (4.14) into (4.11), we obtain

$$\delta_\infty^* = \frac{\rho(X^{\mathrm{d}})}{\rho(X^{\mathrm{d}}) + t\xi(X^{\mathrm{d}}) - (r-v)\xi(X^1)}.$$

Finally, if $X^{\mathrm{d}} = \inf\{x : F(x) > \frac{r-c-p}{t}\} > \mu$, we have $\lim_{n\to\infty} 1 - \hat{F}^{n-1}(X^{\mathrm{d}} + \frac{k}{n-1}) = 0$ and $\lim_{n\to\infty} \hat{F}^{n-1}(X^{\mathrm{d}} - \frac{k}{n-1}) = 1$, hence

$$\begin{aligned}
J_i^{\mathrm{d}}(\mathbf{X}^{\mathrm{d}}, n) &= (r-c)X^{\mathrm{d}} - (r-v)\left[X^{\mathrm{d}}F(X^{\mathrm{d}}) - \int_0^{X^{\mathrm{d}}} yf(y)\mathrm{d}y\right] \\
&\quad + p\int_0^\infty kf(X^{\mathrm{d}} - k)\left[1 - \hat{F}^{n-1}\left(X^{\mathrm{d}} + \frac{k}{N-1}\right)\right]\mathrm{d}k \\
&\quad + p\int_0^\infty kf(X^{\mathrm{d}} + k)\hat{F}^{n-1}\left(X^{\mathrm{d}} - \frac{k}{n-1}\right)\mathrm{d}k \\
&= (r-c)X^{\mathrm{d}} - (r-v)\left[X^{\mathrm{d}}F(X^{\mathrm{d}}) - \int_0^{X^{\mathrm{d}}} yf(y)dy\right] + p\int_0^\infty kf(X^{\mathrm{d}} + k)\mathrm{d}k \\
&= p\mu + t\int_0^{X^{\mathrm{d}}} yf(y)\mathrm{d}y \\
&= p\mu + t\xi(X^{\mathrm{d}}). \qquad (4.15)
\end{aligned}$$

By substituting (4.12) and (4.15) into (4.11), we obtain

$$\delta_\infty^* = \frac{\rho(X^{\mathrm{d}})}{\rho(X^{\mathrm{d}}) + p\mu + t\xi(X^{\mathrm{d}}) - (r-v)\xi(X^1)}.$$

□

Proof of Proposition 5. Retailers have the same demand distribution $F(\cdot)$, price, r, salvage value, v, transshipping cost, t, and unit profit from transshipment, $p = r - v - t$. Denote $X = \sum_j X_j$, $X_{-i} = \sum_{j\neq i} X_j$ and let f^m the *p.d.f* of mD_i. It can be verified that

$$\begin{aligned}
\frac{\partial J_i^{\mathrm{d}}}{\partial X_i} - \frac{\partial J^n}{\partial X_i} &= p\int_0^\infty kf(X_i - k)f^{n-1}(X_{-i} + k)\mathrm{d}k - p\int_0^\infty kf(X_i + k)f^{n-1}(X_{-i} - k)\mathrm{d}k \\
&= p\,\mathsf{E}[X_i - D_i \,|\, X = D]\, f^n(X).
\end{aligned}$$

Denote $O_i = \left(\frac{\partial J_i^{\mathrm{d}}}{\partial X_i} - \frac{\partial J^n}{\partial X_i}\right)|_{\mathbf{X}^n}$. Achieving first best requires $O_i = 0$ for all i. However, for any $i \neq j$,

$$\begin{aligned}
O_i - O_j &= p f^n(X)\,\mathsf{E}[X_i^n - X_j^n + D_j - D_i \,|\, D = X] \\
&= p f^n(X)\,\left[X_i^n - X_j^n + \mathsf{E}[D_j - D_i \,|\, D = X]\right] \\
&= p f^n(X)\,(X_i^n - X_j^n).
\end{aligned}$$

It therefore requires $X_i^n = X_j^n$, $\forall i, j$. This is obviously not true given that each X_i^n has to satisfy its FOC with a different c_i:

$$\frac{\partial J^n}{\partial X_i^n} = r - c_i + (r-v)F(X_i^n) + pPr\{D_i \le X_i^n, D > X^n\} - pPr\{D_i \ge X_i^n, D < X^n\}$$
$$= 0. \qquad \square$$

Proof of Theorem 5. The eviction contract described in Theorem 5 will be an optimal contract if it satisfies the following constraints:

1. *Participation constraint*—each retailer is better off if she adopts the contract.
2. *Early adoption constraint*—each retailer prefers to adopt the contract in the current period than in the later period.
3. *Continuation constraints*—each retailer is better off if she does not deviate in any period.

We now show that the eviction contract satisfies all three constraints.

PARTICIPATION CONSTRAINT: If retailer i adopts the contract in period 1, her infinite horizon discounted payoff is given by

$$B_i + \sum_{t=1}^{\infty} \delta_i^{t-1} J_i^n(\mathbf{X}^n) = B_i + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n).$$

If the contract is not adopted and each retailer orders the individually optimal quantity (under the dual allocation rule), her payoff is

$$\sum_{t=1}^{\infty} \delta_i^{t-1} J_i^n(\mathbf{X}^{\mathrm{d}}) = \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^{\mathrm{d}}).$$

The participation constraint is satisfied if

$$B_i + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) \ge \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^{\mathrm{d}}).$$

First, suppose that $\Lambda_i > 0$, which implies $B_i = \frac{1}{1-\delta_i}[J_i^n(\mathbf{X}^{\mathrm{d}}) - J_i^n(\mathbf{X}^n)]$. In other words, retailer i's profit is larger if the retailers order $\mathbf{X}^{\mathrm{d}}$, and she receives a positive bonus to compensate for ordering $\mathbf{X}^n$. Then,

$$B_i + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) = \frac{1}{1-\delta_i}[J_i^n(\mathbf{X}^{\mathrm{d}}) - J_i^n(\mathbf{X}^n)] + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) = \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^{\mathrm{d}}),$$

and hence i is not better off if she does not adopt the contract.

Now, suppose that $\Lambda_i \leq 0$—that is, retailer i's profit is larger if the retailers order $\mathbf{X}^n$ and she gives a side payment to other retailers to induce their acceptance of the contract. Observe that $J^n(\mathbf{X}^n) \geq J^n(\mathbf{X}^{\mathrm{d}})$, which implies $\sum_i \Lambda_i \leq 0$. This further means that $0 \leq \sum_{K^+} \Lambda_j \leq \sum_{K^-} (-\Lambda_j)$ and

$$0 \leq \frac{\sum_{K^+} (-\Lambda_j)}{\sum_{K^-} \Lambda_j} \leq 1. \tag{4.16}$$

Now,

$$\begin{aligned} B_i + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) &= \frac{1}{1-\delta_i}[J_i^n(\mathbf{X}^{\mathrm{d}}) - J_i^n(\mathbf{X}^n)] \times \frac{\sum_{K^+}(-\Lambda_j)}{\sum_{K^-}\Lambda_j} + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) \\ &\geq \frac{1}{1-\delta_i}[J_i^n(\mathbf{X}^{\mathrm{d}}) - J_i^n(\mathbf{X}^n)] + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) = \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^{\mathrm{d}}), \end{aligned}$$

where the inequality follows from (4.16). Thus, the participation constraint is satisfied for all i.

EARLY ADOPTION CONSTRAINT: If the contract is adopted in period $t = 2$ instead of in period $t = 1$, the retailers order $\mathbf{X}^{\mathrm{d}}$ in period 1, and retailer i realizes the payoff

$$J_i^n(\mathbf{X}^{\mathrm{d}}) + \delta_i B_i + \sum_{t=2}^{\infty} \delta_i^{t-1} J_i^n(\mathbf{X}^n) = J_i^n(\mathbf{X}) + \delta_i B_i + \frac{\delta_i}{1-\delta_i} J_i^n(\mathbf{X}^n).$$

The early adoption constraint holds if

$$B_i + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) \geq J_i^n(\mathbf{X}) + \delta_i B_i + \frac{\delta_i}{1-\delta_i} J_i^n(\mathbf{X}^n).$$

First, suppose that $\Lambda_i > 0$, which implies $B_i = \frac{1}{1-\delta_i}[J_i^n(\mathbf{X}^{\mathrm{d}}) - J_i^n(\mathbf{X}^n)]$. Then,

$$\begin{aligned} J_i^n(\mathbf{X}^{\mathrm{d}}) + \delta_i B_i + \frac{\delta_i}{1-\delta_i} J_i^n(\mathbf{X}^n) &= J_i^n(\mathbf{X}^{\mathrm{d}}) + \frac{\delta_i}{1-\delta_i}[J_i^n(\mathbf{X}^{\mathrm{d}}) - J_i^n(\mathbf{X}^n)] + \frac{\delta_i}{1-\delta_i} J_i^n(\mathbf{X}^n) \\ &= \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^{\mathrm{d}}), \end{aligned}$$

and

$$B_i + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) = \frac{1}{1-\delta_i}[J_i^n(\mathbf{X}^{\mathrm{d}}) - J_i^n(\mathbf{X}^n)] + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) = \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^{\mathrm{d}}).$$

Hence, retailer i does not benefit from late adoption of the contract.

Next, when $\Lambda_i \le 0$, then $J_i^n(\mathbf{X}^{\mathrm{d}}) - J_i^n(\mathbf{X}^n) \le 0$, and (4.16) implies

$$B_i + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) - \left(J_i^n(\mathbf{X}^{\mathrm{d}}) + \delta_i B_i + \frac{\delta_i}{1-\delta_i} J_i^n(\mathbf{X}^n) \right)$$
$$= [J_i^n(\mathbf{X}^{\mathrm{d}}) - J_i^n(\mathbf{X}^n)] \times \frac{\sum_{K^+}(-\Lambda_j)}{\sum_{K^-}\Lambda_j} + J_i^n(\mathbf{X}^n) - J_i^n(\mathbf{X}^{\mathrm{d}})$$
$$\ge J_i^n(\mathbf{X}^{\mathrm{d}}) - J_i^n(\mathbf{X}^n) + J_i^n(\mathbf{X}^n) - J_i^n(\mathbf{X}^{\mathrm{d}}) = 0.$$

Thus, retailer i prefers to adopt the contract in the first period.

CONTINUATION CONSTRAINT: We now want to show that a retailer never benefits from defecting. Recall that Z_t denotes the coalition structure in period t, and suppose that retailer i orders a quantity different from $X_{it}^{Z_t}$ and/or withholds some of her residuals. As a result, she pays a penalty, d_{it}, in period t, and is excluded from inventory sharing in all subsequent periods. We denote, with slight abuse of notation, $\mathbf{X}_t(\hat{\mathbf{h}}_{t-1}) = \mathbf{X}_t$, $\mathbf{H}_t(\hat{\mathbf{h}}_{t-1}) = \mathbf{H}_t$, $\mathbf{E}_t(\hat{\mathbf{h}}_{t-1}) = \mathbf{E}_t$, $d_{it}(\hat{\mathbf{h}}_t) = d_{it}$, and $\Delta_{it}(\hat{\mathbf{h}}_t) = \Delta_{it}$. Then, retailer i's discounted payoff starting from period t is given by

$$J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + d_{it}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + \frac{\delta_i}{1-\delta_i} J_i^1(X_i^1).$$

The continuation constraint holds if

$$J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + d_{it}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + \frac{\delta_i}{1-\delta_i} J_i^1(X_i^1) \le \frac{1}{1-\delta_i} J_i^{Z_t}(\mathbf{X}^{Z_t}).$$

If $i \in I_t^-$, then $\Delta_{it} \le 0$, and $d_{it} = \frac{1}{1-\delta_i}\left[J_i^{Z_t}(\mathbf{X}^{Z_t}) - \delta J_i^1(X_i^1)\right] - J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t)$. Thus, i receives a payoff

$$J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + \frac{1}{1-\delta_i}\left[J_i^{Z_t}(\mathbf{X}^{Z_t}) - \delta_i J_i^1(X_i^1)\right] - J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + \frac{\delta_i}{1-\delta_i} J_i^1(X_i^1)$$
$$= \frac{1}{1-\delta_i} J_i^{Z_t}(\mathbf{X}^{Z_t}),$$

and i does not benefit from defection.

Now, suppose $i \in I_t^+$, and consequently $\Delta_{it} > 0$. This implies

$$\frac{1}{1-\delta_i}\left[J_i^{Z_t}(\mathbf{X}^{Z_t}) - \delta_i J_i^1(X_i^1)\right] - J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) \ge 0. \tag{4.17}$$

Notice that

$$\sum_i \Delta_{it} = \sum_i \left\{ \frac{1}{1-\delta_i}\left[J_i^{Z_t}(\mathbf{X}^{Z_t}) - \delta_i J_i^1(X_i^1)\right] - J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) \right\}$$
$$= \frac{\delta_i}{1-\delta_i}\left\{J^{Z_t}(\mathbf{X}^{Z_t}) - J^1(\mathbf{X}_1)\right\} + J^{Z_t}(\mathbf{X}^{Z_t}) - J^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) \ge 0,$$

where the inequality holds because $\mathbf{X}^{Z_t}$ with complete residual sharing maximizes the system profit when the state is Z_t and systems with inventory-sharing retailers generate higher profit than systems without inventory sharing. As a result, $\sum_{I_t^+} \Delta_{jt} \geq \sum_{I_t^-} (-\Delta_{jt})$, and

$$0 \leq \frac{\sum_{I_t^-} (-\Delta_{jt})}{\sum_{I_t^+} \Delta_{jt}} \leq 1. \tag{4.18}$$

Thus, retailer i receives a payoff

$$J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + \left\{ \frac{1}{1-\delta_i} \left[J_i^{Z_t}(\mathbf{X}^{Z_t}) - \delta_i J_i^1(X_i^1) \right] - J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) \right\} \times \frac{\sum_{I_t^-} (-\Delta_{jt})}{\sum_{I_t^+} \Delta_{jt}}$$

$$+ \frac{\delta_i}{1-\delta_i} J_i^1(X_i^1) \leq J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + \frac{1}{1-\delta_i} \left[J_i^{Z_t}(\mathbf{X}^{Z_t}) - \delta_i J_i^1(X_i^1) \right] - J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t)$$

$$+ \frac{\delta_i}{1-\delta} J_i^1(X_i^1) = \frac{1}{1-\delta_i} J_i^{Z_t}(\mathbf{X}^{Z_t}),$$

where the inequality follows from (4.17) and (4.18). As a result, i prefers not to defect in any period. □

References

Anupindi, R., Bassok, Y., & Zemel, E. (2001). A general framework for the study of decentralized distribution system. *M&SOM*, *3*(4) 349–368.

Bagwell, K., & Staiger, R. W. (1997). Collusion over the business cycle. *The RAND Journal of Economics*, *28*(1) 82–106.

Dong, L., & Rudi, N. (2004). Who benefits from transshipment? Exogenous vs. endogenous wholesale prices. *Management Science*, *50*(5) 645–657.

Granot, D., & Sošić, G. (2003). A three stage model for a decentralized distribution system of retailers. *Operations Research*, *51*(5) 771–784.

Haltiwanger, J., & Harrington, J. E., Jr (1991). The impact of cyclical demand movements on collusive behavior. *The RAND Journal of Economics*, *22*(1) 89–106.

Hu, X., Duenyas, I., & Kapuscinski, R. (2007). Existence of coordinating transshipment prices in a two-location inventory problem. *Management Science*, *53*(8) 1289–1302.

Huang, X., & Sošić, G. (2010a). Transshipment of inventories: dual allocation vs. transshipment prices. *M&SOM*, *12*(2) 299–312.

Lippman, S. A., & McCardle, K. F. (1997). The competitive newsboy. *Operations Research*, *45*(1) 54–65.

Parlar, M. (1988). Game theoretic analysis of the substitutable product inventory problem with random demands. *Naval Research Logistics*, *35* 397–409.

Rotemberg, J. J., & Saloner, G. (1986). A supergame-theoretic model of price wars during booms. *The American Economic Review*, *76*(3) 390–407.

Rudi, N., Kapur, S., & Pyke, D. F. (2001). A two-location inventory model with transshipment and local decision making. *Management Science*, *47*(12) 1668–1680.

Shao, J., Krishnan, H., & McCormick, S. T. (2011). Incentives for transshipment in a supply chain with decentralized retailers. *MSOM*, *13*(3):361–372.

Sošić, G. (2004). Transshipment of inventories among retailers: myopic vs. farsighted stability. *Management Science*, *52*(10) 1493–1508.

Wang, Q., & Parlar, M. (1994). A three-person game theory model of the substitutable product inventory problem with random demands. *European Journal of Operational Research*, *76*(1) 83–97.

Wee, K. E., & Dada, M. (2005). Optimal policies for transshipping inventory in a retail network. *Management Science*, *51*(10) 1519–1533.

Yang, H., & Schrage, L. (2009). Conditions that cause risk pooling to increase inventory. *European Journal of Operational Research*, *192*(3) 837–851.

Zhang, J. (2005). Transshipment and its impact on supply chain members' performance. *Management Science*, *51*(10) 1534–1539.

Zhao, H., Deshpande, V., & Ryan, J. K. (2005). Inventory sharing and rationing in decentralized dealer networks. *Management Science*, *51*(4) 531–547.