Chapter 12
Planning Production on an Unreliable Machine for Multiple Items Subject to Stochastic Demand

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Abstract We develop an extension of the classical newsvendor model that incorporates multiple items, setup times, and an unreliable machine. This model is motivated by applications at metal stamping plants where machine reliability is a key source of uncertainty. Given a fixed production schedule, a finite horizon, and a known demand distribution, we formulate an extension of the newsvendor model, derive important properties of this model, and exploit these properties to provide a solution algorithm that determines the cost minimizing production quantities. Finally, we present three simple extensions to the model: (1) a method for rescheduling within the planning horizon, (2) an extension to evaluate whether or not to purchase the option to run overtime within the planning horizon, and (3) an extension that permits the modeling of a machine that operates at a different speed depending on the part being produced.

Keywords Multiple items • Setup times • Unreliable machines • Cost minimization • Solution algorithm • Rescheduling

12.1 Introduction

In this chapter, we develop extension of the classical newsvendor model. We model a single, unreliable machine that repetitively produces a set of parts in batches subject to shortage and overage (inventory-holding) costs. Our model makes the following assumptions. First, we assume that there is only a single demand point for all parts, and that it occurs at the end of a finite production horizon. Second, the demand for each part is a random variable with a known distribution, where the uncertainty in the demand quantity is not resolved until the demand point.

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Lastly, we assume a fixed production sequence. Under these assumptions, our model determines an optimal production quantity for each part. The development of this model is motivated by applications at metal stamping plants (Kletter 1994). This model could be used as part of a manufacturing control system, embedded in a software tool that would receive data in real time from the shop floor and assist plant management in decision making.

This chapter is structured as follows. Section 2 presents a review of the literature. A model is then formulated in Sect. 3 as an extension of a classical newsvendor problem. In Sect. 4, we derive properties of the objective function that are exploited to develop a solution algorithm, presented in Sect. 5 and that takes advantage of the special structure of the model. In Sect. 6, we show numerical results from exercise of the model. Finally, in Sect. 7, three extensions to the model are presented, including the incorporation of options to run overtime.

12.2 Literature Review

We briefly review the literature that is related to our model. We will divide our literature review into two parts: those that model the problem of planning production quantities on an unreliable machine, and those that use a newsvendor model for problems closely related to the one we study here.

12.2.1 Unreliable Machine

The presence of machine unreliability in a manufacturing system has been studied in a variety of different contexts, including problems of sequencing, scheduling, and lot sizing. We briefly review each of these areas.

We first discuss sequencing of jobs on an unreliable machine. The earliest work is that of Glazebrook (1984) who models the problem as a rather general cost-discounted Markov decision process. He shows the conditions under which the optimal policy is of an index type (i.e., the job to be processed is the one with the smallest *Gittins index*; see Gittins 1979). Pinedo and Rammouz (1988) find the optimal nonpreemptive policies for several objective functions in the case of a Poisson failure process. For a general failure process and a discrete time model, Birge and Glazebrook (1988) find bounds on the error of following the strategy that is optimal when the failure process is memoryless. Birge et al. (1990) study in greater detail the problem of minimizing weighted flow-time and obtain results that are consistent with and complementary to Pinedo and Rammouz. Epstein et al. (2010) analyze optimal sequencing on an unreliable machine where the machine may slow or stop completely. For a detailed and current overview of this research area, see the surveys contained in Lee (2004), Pinedo (2008), Diedrich et al. (2009), and Racke (2009).

There is also a significant body of work on lot sizing on an unreliable machine. Yano and Lee (1995) provide a broad review of the literature in this area.

See Schmidt (2000) for an overview of the scheduling literature in cases where the machine is continuously available for processing, for example, when there is incomplete information about when the machine may change availability. Al-Salamah and Abudari (2011) model a production process with failures: however, failures don't result in the stoppage of the machine but instead produces nonconforming items. El-Ferik (2008) determines optimal production quantities under an assumption that the production facility is subject to random failure, where preventative maintenance schedules need to be balanced with production. Halim et al. (2010) show the effect on optimal lot sizing when an unreliable machine is subject to fuzzy demand and repair time. Giri et al. (2005) model a two-stage production system where the upstream stage is subject to failures but the downstream stage is not. They assume that after a machine failure, production of the affected lot is not resumed.

In the case of an infinite planning horizon, Giri and Dohi (2004) derive optimal lot sizes under a net present value (NPV) approach.

Of particular note is the work of Groenevelt et al. (1992a, 1992b), who extend the basic economic manufacturing quantity (EMQ) model to incorporate the effects of machine breakdowns. The first paper assumes that repairs are instantaneous but bear a fixed cost. The second paper assumes (as we do) that repairs are not instantaneous but instead consume machine time. This model permits any repair time distribution, but assumes that the time between failures is exponentially distributed. Under the assumption of lost sales, the authors seek an optimal lot size and safety stock level to minimize cost subject to a constraint on the service level. They require, however, some awkward assumptions regarding safety stock to achieve separability in the optimization of the lot sizes and safety stock level. The authors do not explore the impacts of multiple parts sharing the same machine.

Other authors, such as Sethi and Zhang (1994) have approached similar problems from a control theoretic perspective. These authors consider the problem of finding an optimal setup schedule (a sequence of parts and the times at which the changeovers will occur) for an unreliable machine. They show that in the limit (as the length of the horizon tends to infinity), the stochastic problem can be reduced to a deterministic problem, and show how to obtain the optimal control policy. The authors also cite many other similar works.

Reiman and Wein (1998) study a two customer class, single server system with setups. The authors use heavy traffic diffusion approximations to analyze a system with a renewal arrival process, general service times, and either setup costs or setup times. They solve a control problem to minimize a linear function of the queue length plus setup costs, if any. Within these heavy traffic diffusion approximations, one could model the unreliability of the machine within the service time distribution.

12.2.2 Related Newsvendor Models

The classic "newsvendor" model has been the subject of many extensions that are similar to those we consider here. For many decades, researchers have considered models with multiple items (Evans 1967, Smith et al. 1980). Rose (1992) considers uncertain replenishment, but assumes demand is deterministic. Others have studied multiple time periods for production (Bitran et al. 1986; Matsuo 1990; Ciarallo et al. 1994). Jain and Silver (1995) model uncertainty in supply and permit the option to reserve reliable capacity for a premium charge. Dada et al. (2007) consider a newsvendor that purchases a single item provided by multiple suppliers, some of which are unreliable, and develop a model for optimal supplier selection. Huggins and Olsen (2010) extend the basic newsvendor problem for a single item to permit expediting for unmet demand.

12.3 Formulation

The mathematical structure of our model will closely parallel that of the classic newsboy model, which we now briefly describe. The following table (Table 12.1) lists the notation that we will use throughout this section.

For simplicity we will first state the formulation as a single part formulation, dropping the index i from our notation. The problem is then to choose an order-up-to quantity y to minimize the expected purchase, holding, and shortage costs. Mathematically, we can state the problem as

$$C^*(x) = \min_{y \geq x} c(y-x) + p\int_{y}^{\infty}(t-y)g(t)\mathrm{d}t + h\int_{0}^{y}(y-t)g(t)\mathrm{d}t. \tag{12.1}$$

Table 12.1 Notation for formulation

i	Index which denotes different parts to be produced; $i = 1,\ldots,N$
y_i	Decision variable denoting order-up-to level for part i
x_i	Current inventory level of part i
c_i	Unit purchase price of part i
h_i	Cost per unit of inventory remaining at the end of the period for part i
p_i	Unit shortage cost for part i
s_i	Time required to set up the machine to begin producing part i
$g_i(\cdot)$	PDF of demand for part i
$f_i(t;T)$	PDF that in T units of time, the cumulative output of the machine is t units of part i
$F_i(\cdot)$, $G_i(\cdot)$	CDFs for the PDFs f_i, g_i
$\overline{F}_i(\cdot)$, $\bar{G}_i(\cdot)$	$1-F_i(\cdot)$, $1-G_i(\cdot)$

The problem is solved by finding the value of y such that $\partial C(x)/\partial y$ is zero. To find this partial derivative, we need to employ Leibnitz's theorem for differentiation of an integral:

$$\frac{\partial}{\partial y}\int_{p(y)}^{q(y)} f(x,y)\mathrm{d}x = \int_{p(y)}^{q(y)} \frac{\partial f(x,y)}{\partial y}\mathrm{d}x + \frac{\partial q(y)}{\partial y} f(q(y),y) - \frac{\partial p(y)}{\partial y} f(p(y),y) \quad (12.2)$$

(Beyer 1987). We will use this extensively in our analysis. From this rule, it is easy to see that the optimal solution y^* to the newsboy model occurs at the point where $G(y^*) = (p-c)/(p+h)$, unless this implies $y^* < x$, in which case it is optimal not to order.

We now extend this basic single part model to our multiple part, unreliable production model, for now ignoring overtime opportunities. The problem is to find the optimal order-up-to levels to minimize the sum of purchasing, holding, and shortage costs over all parts. Let y, x, c, p, h, and $g(\cdot)$ retain the same meanings as above, except now we add a subscript i, for each part $i = 1,\ldots,N$. We assume without loss of generality that the parts are indexed in the order in which they will be produced. In practice, the order-up-to strategy would be implemented as follows: produce the first part until the inventory level reaches the optimal y_1, then the production switches to the part 2 until its inventory level reaches the optimal y_2, etc.

Let T denote the amount of time available for production, and the time available after setups as $T_i = T - s_1 - \cdots - s_i$. If we are already setup to produce part 1, then we set $s_1 = 0$. We assume for simplicity that each part is produced at the same rate when the machine is working. We now introduce machine unreliability into the formulation by including the PDF $f(t;T)$. Kletter (1996) provides a variety of formulas for this distribution in the case where the interarrival time of machine failures and repairs are exponentially distributed. However, the results below are independent of the form of this distribution.

We can now write the problem as

$$C^*(x) = \min_{y_1 \geq x_1,\ldots,y_N \geq x_N} C(y,x), \quad (12.3)$$

where

$$C(y,x) = \sum_{i=1}^{N} C_i(y,x), \quad (12.4)$$

and

$$C_i(y,x) = c_i \int_{x_i}^{y_i} (t-x_i) f\left(\sum_{j=1}^{i-1} y_j - x_j + t - x_i; T_i\right)\mathrm{d}t + c_i(y_i - x_i)\bar{F}\left(\sum_{j=1}^{i} y_j - x_j; T_i\right)$$
$$+ p_i \int_{x_i}^{y_i}\int_{u}^{\infty} (t-u) g_i(t)\mathrm{d}t\, f\left(\sum_{j=1}^{i-1} y_j - x_j + u - x_i; T_i\right)\mathrm{d}u$$

$$
\begin{aligned}
&+p_i\bar{F}\left(\sum_{j=1}^{i} y_j - x_j; T_i\right)\int_{y_i}^{\infty}(t-y_i)g_i(t)\mathrm{d}t\\
&+p_iF\left(\sum_{j=1}^{i-1} y_j - x_j; T_i\right)\int_{x_i}^{\infty}(t-x_i)g_i(t)\mathrm{d}t\\
&+h_i\int_{x_i}^{y_i}\int_{0}^{u}(u-t)g_i(t)\mathrm{d}t f\left(\sum_{j=1}^{i-1} y_j - x_j + u - x_i; T_i\right)\mathrm{d}u\\
&+h_i\bar{F}\left(\sum_{j=1}^{i} y_j - x_j; T_i\right)\int_{0}^{y_i}(y_i-t)g_i(t)\mathrm{d}t\\
&+h_iF\left(\sum_{j=1}^{i-1} y_j - x_j; T_i\right)\int_{0}^{x_i}(x_i-t)g_i(t)\mathrm{d}t,
\end{aligned}
\tag{12.5}
$$

where the summations from 1 to $i-1$ are taken to be null at $i = 1$.

Each $C_i(y,x)$ represents the expected purchasing, holding, and shortage costs incurred for part i given a set of order-up-to levels y_i. We have written $C_i(y,x)$ as the sum of eight terms. The first two terms express the expected purchasing cost, where the first term is the expected purchasing cost if the realized uptime of the machine is such that the available supply of the ith part is between the values of 0 and $y_i - x_i$ and the second term is the expected purchasing cost if the realized uptime of the machine is such that the available supply of the ith part is the desired value $y_i - x_i$. There is no purchasing cost if the available supply of the ith part is not greater than zero. The next three terms represent the expected shortage costs. The first of these terms is the expected shortage cost if the available supply is between 0 and $y_i - x_i$, the second term is the expected shortage cost if the available supply is $y_i - x_i$, and the third is the expected shortage cost if the available supply is 0. Similarly, the last three terms represent the expected holding costs, where the first of these terms is the expected holding cost if the available supply of the ith part is between 0 and $y_i - x_i$, the second term is the expected holding cost if the available supply is $y_i - x_i$, and the third is the expected holding cost if the available supply is 0.

12.4 Properties of the Objective Function

To obtain the optimal order quantities, we wish to show that the total cost function is convex with respect to the order quantities. If this is so, we can find minimizing order quantities by finding where the partial derivative of the total cost function is zero.

12.4.1 First Order Optimality Condition

We begin by finding the partial derivative of the total cost function with respect to y_N. Using Leibnitz's rule, we obtain

$$\frac{\partial}{\partial y_N} C(y,x) = (c_N - p_N \bar{G}_N(y_N) + h_N \bar{G}_N(y_N)) \bar{F} \left(\sum_{j=1}^{N} y_j - x_j; T_N \right). \tag{12.6}$$

When written as the product of two terms as we have done, this derivative has a nice interpretation. The first term is the derivative of the cost function for the classical newsboy problem. This term is multiplied by the probability that we can complete our production plan in the time available.

Because of this structure, the first order optimality condition is reduced to $G_N(y_N) = (p_N - c_N)/(p_N + h_N)$, the solution to the classical newsboy problem. As before, it is easy to show that if this implies $y_N < x_N$, then the optimal y_N is x_N. The optimal y_N should not be dependent on the other y_i, because once we have produced parts $1,\ldots,N-1$, all we can do is try to minimize the costs for part N. The optimal y_N should not be dependent on the machine's reliability, because the best thing to do is attempt to achieve the optimal order-up-to quantity exactly.

We now turn to the more difficult task of taking the partial derivative of the total cost function $C(y,x)$ with respect to y_i for $i < N$. After simplification, the result is

$$\begin{aligned} \frac{\partial}{\partial y_i} C(y,x) = \bar{F} \left(\sum_{j=1}^{i} y_j - x_j; T_i \right) & (c_i - p_i \bar{G}_i(y_i) + h_i G_i(y_i)) \\ & + \sum_{k=i+1}^{N} (p_k - c_k) \left[F \left(\sum_{j=1}^{k} y_j - x_j; T_k \right) - F \left(\sum_{j=1}^{k-1} y_j - x_j; T_k \right) \right] \\ & - \sum_{k=i+1}^{N} (p_k + h_k) \int_{x_k}^{y_k} G_k(u) f \left(\sum_{j=1}^{k-1} y_j - x_j + u - x_k; T_k \right) \mathrm{d}u, \end{aligned} \tag{12.7}$$

where the summations from $i+1$ to N are taken to be null at $i = N$. This expression is easier to interpret if we rewrite it as

$$\begin{aligned} \frac{\partial}{\partial y_i} C(y,x) = \bar{F} \left(\sum_{j=1}^{i} y_j - x_j; T_i \right) & (c_i - p_i \bar{G}_i(y_i) + h_i G_i(y_i)) \\ & + \sum_{k=i+1}^{N} (p_k - c_k) \left[F \left(\sum_{j=1}^{k} y_j - x_j; T_k \right) - F \left(\sum_{j=1}^{k-1} y_j - x_j; T_k \right) \right] \end{aligned}$$

$$+\sum_{k=i+1}^{N}(p_k+h_k)\int_{x_k}^{y_k}\bar{G}_k(u)f\left(\sum_{j=1}^{k-1}y_j-x_j+u-x_k;T_k\right)\mathrm{d}u$$

$$-\sum_{(k)=i+1}^{N}(p_k+h_k)\int_{x_k}^{y_k}f\left(\sum_{j=1}^{k-1}y_j-x_j+u-x_k;T_k\right)\mathrm{d}u, \quad (12.8)$$

and then simplify to obtain

$$\frac{\partial}{\partial y_i}C(y,x)=\bar{F}\left(\sum_{j=1}^{i}y_j-x_j;T_i\right)(c_i-p_i\bar{G}_i(y_i)+h_iG_i(y_i))$$

$$-\sum_{k=i+1}^{N}(c_k+h_k)\left[F\left(\sum_{j=1}^{k}y_j-x_j;T_k\right)-F\left(\sum_{j=1}^{k-1}y_j-x_j;T_k\right)\right]$$

$$+\sum_{k=i+1}^{N}(p_k+h_k)\int_{x_k}^{y_k}\bar{G}_k(u)f\left(\sum_{j=1}^{k-1}y_j-x_j+u-x_k;T_k\right)\mathrm{d}u. \quad (12.9)$$

The first term is analogous to $\partial C(y,x)/\partial y_N$ discussed above. The second two terms give the impact of the choice of y_i on the parts $k=i+1,\ldots,N$. The first of these terms represents the marginal cost of machine time. The expression in square brackets is the probability that machine output is insufficient to produce up to y_k but sufficient to start production of part k. As this probability increases, total cost decreases at rate c_k+h_k, assuming that the units built are not sold. The final term is the marginal cost of lost sales. The integral represents the expected sales given that machine output is greater than zero but less than y_k. As this increases, shortage costs are accrued at a rate p_k and holding costs, which have already been charged in the second term, are avoided at a rate h_k.

It can be seen from the first order condition that as T tends to infinity, the optimal y_i each approach their "newsboy point" y_i^N, that is, the point where $G(y_i)=(p_i-c_i)/(p_i+h_i)$. However, we can prove a stronger result, as stated by the following:

Theorem 1. *The optimal y_i are never greater than y_i^N, their respective newsboy points.*

Proof. We have already shown that the optimal y_N is y_N^N, the newsboy point for part N. Suppose that we have shown that the optimal y_k are not greater than y_k^N for $k=i+1,\ldots,N$. We will now show that the optimal y_i is also less than or equal to y_i^N. We first require the following.

Lemma 1.

$$\frac{\partial}{\partial y_i}C(y,x)\geq\bar{F}\left(\sum_{j=1}^{i}y_j-x_j;T_i\right)(c_i-p_i\bar{G}_i(y_i)+h_iG_i(y_i))$$

Proof.

$$\frac{\partial}{\partial y_i}C(y,x) = \bar{F}\left(\sum_{j=1}^{i} y_j - x_j; T_i\right)(c_i - p_i\bar{G}_i(y_i) + h_iG_i(y_i))$$
$$+ \sum_{k=i+1}^{N}(p_k - c_k)\left[F\left(\sum_{j=1}^{k} y_j - x_j; T_k\right) - F\left(\sum_{j=1}^{k-1} y_j - x_j; T_k\right)\right]$$
$$- \sum_{k=i+1}^{N}(p_k + h_k)\int_{x_k}^{y_k} G_k(u) f\left(\sum_{j=1}^{k-1} y_j - x_j + u - x_k; T_k\right) \mathrm{d}u$$

(from equation (12.7))

$$\geq \bar{F}\left(\sum_{j=1}^{i} y_j - x_j; T_i\right)(c_i - p_i\bar{G}_i(y_i) + h_iG_i(y_i))$$
$$+ \sum_{k=i+1}^{N}(p_k - c_k)\left[F\left(\sum_{j=1}^{k} y_j - x_j; T_k\right) - F\left(\sum_{j=1}^{k-1} y_j - x_j; T_k\right)\right]$$
$$- \sum_{k=i+1}^{N}(p_k + h_k)G_k(y_k)\int_{x_k}^{y_k} f\left(\sum_{j=1}^{k-1} y_j - x_j + u - x_k; T_k\right) \mathrm{d}u$$

(because $G_k(\cdot)$ is nondecreasing)

$$= \bar{F}\left(\sum_{j=1}^{i} y_j - x_j; T_i\right)(c_i - p_i\bar{G}_i(y_i) + h_iG_i(y_i))$$
$$- \sum_{k=i+1}^{N}(c_k - p_k\bar{G}_k(y_k) + h_kG_k(y_k))$$
$$\times \left[F\left(\sum_{j=1}^{k} y_j - x_j; T_k\right) - F\left(\sum_{j=1}^{k-1} y_j - x_j; T_k\right)\right]$$
$$\geq \bar{F}\left(\sum_{j=1}^{i} y_j - x_j; T_i\right)(c_i - p_i\bar{G}_i(y_i) + h_iG_i(y_i)). \tag{12.10}$$

(because $G_k(y_k) \leq (p_k - c_k)/(p_k + h_k)$ for $k = i+1,\ldots,N$, by the induction hypothesis). □

Using this result, it immediately follows that for any $y_i > y_i^N, \partial C(y,x)/\partial y_i$ is positive. Therefore, if $x_i < y_i^N$, the optimal y_i lies between x_i and y_i^N. If $x_i \geq y_i^N$, then it is optimal not to produce (the optimal y_i equals x_i).

12.4.2 Convexity of the Total Cost Function

In this section, we prove the following:

Theorem 2. *If* $x_k \leq y_k \leq y_k^N$ *for* $k = i, i+1, \ldots, N$, *then* $\partial^{\mathrm{y}} C(y,x)/\partial y_i^{\mathrm{y}}$ *is nonnegative.*

Proof. We once again use Leibnitz's rule to take the second partial derivative with respect to y_i to obtain

$$\begin{aligned}\frac{\partial^2}{\partial y_i^2} C(y,x) = &\left[\bar{F}\left(\sum_{j=1}^{i} y_j - x_j; T_i\right)(p_i g_i(y_i) + h_i g_i(y_i))\right] \\ &+ \left[-f\left(\sum_{j=1}^{i} y_j - x_j; T_i\right)(c_i - p_i \bar{G}_i(y_i) + h_i G_i(y_i))\right] \\ &+ \left[\sum_{k=i+1}^{N}\left\{(p_k - c_k)\left(f\left(\sum_{j=1}^{k} y_j - x_j; T_k\right) - f\left(\sum_{j=1}^{k-1} y_j - x_j; T_k\right)\right)\right.\right. \\ &\left.\left. -(p_k + h_k)\int_{x_k}^{y_k} G_k(u)\frac{\partial}{\partial y_i}\left\{f\left(\sum_{j=1}^{k-1} y_j - x_j + u - x_k; T_k\right)\right\} \mathrm{d}u\right\}\right]. \end{aligned} \tag{12.11}$$

We now show that this second partial derivative is nonnegative. We have written the second partial derivative as the sum of three (square bracketed) terms. The first term can be seen to be nonnegative by inspection. The second square bracketed term is nonnegative if $-c_i + p_i\bar{G}_i(y_i) - h_i G_i(y_i)$ is nonnegative, which is true if $G_i(y_i) \leq (p_i - c_i)/(p_i + h_i)$, which is always true for $y_i < y_i^N$. Showing that the third bracketed term is nonnegative is slightly more difficult. We note that for each k,

$$\begin{aligned}&(p_k - c_k)\left(f\left(\sum_{j=1}^{k} y_j - x_j; T_k\right) - f\left(\sum_{j=1}^{k-1} y_j - x_j; T_k\right)\right) \\ &\quad -(p_k + h_k)\int_{x_k}^{y_k} G_k(u)\frac{\partial}{\partial y_i}\left\{f\left(\sum_{j=1}^{k-1} y_j - x_j + u - x_k; T_k\right)\right\} \mathrm{d}u \\ &\geq (p_k - c_k)\left(f\left(\sum_{j=1}^{k} y_j - x_j; T_k\right) - f\left(\sum_{j=1}^{k-1} y_j - x_j; T_k\right)\right) \\ &\quad -(p_k + h_k)G_k(y_k)\int_{x_k}^{y_k} \frac{\partial}{\partial y_i}\left\{f\left(\sum_{j=1}^{k-1} y_j - x_j + u - x_k; T_k\right)\right\} \mathrm{d}u\end{aligned}$$

(because $G_k(\cdot)$ is nondecreasing)

$$\geq (p_k - c_k)\left(f\left(\sum_{j=1}^{k} y_j - x_j; T_k\right) - f\left(\sum_{j=1}^{k-1} y_j - x_j; T_k\right)\right)$$
$$-(p_k + h_k)\frac{(p_k - c_k)}{(p_k + h_k)}\int_{x_k}^{y_k} \frac{\partial}{\partial y_i}\left\{f\left(\sum_{j=1}^{k-1} y_j - x_j + u - x_k; T_k\right)\right\}\mathrm{d}u$$

(because $G_k(y_k) \leq (p_k - c_k)/(p_k + h_k)$)

$$= (p_k - c_k)\left(f\left(\sum_{j=1}^{k} y_j - x_j; T_k\right) - f\left(\sum_{j=1}^{k-1} y_j - x_j; T_k\right)\right)$$
$$-(p_k - c_k)\left(f\left(\sum_{j=1}^{k} y_j - x_j; T_k\right) - f\left(\sum_{j=1}^{k-1} y_j - x_j; T_k\right)\right) = 0. \quad (12.12)$$

□

Given the other $y_j, j \neq i$, this result allows us to find the optimal y_i by determining if $\exists\ y_i \in [x_i, y_i^N]$ such that $\partial C(y,x)/\partial y_i = 0$. If such a y_i exists then it is optimal, otherwise, the optimal policy is not to order. Since $\partial C(y,x)/\partial y_i$ is a nondecreasing function of y_i over the range $[x_i, y_i^N]$ when $y_k \leq y_k^N$ for $k = i+1,\ldots,N$, the optimal y_i can be found by simple binary search.

Given the above results, after we have found y_N we can find the other y_i by solving the above problem as a $N-1$ dimensional unconstrained minimization problem on the interval $x_i \leq y_i \leq y_i^N, i = 1,\ldots,N-1$. For a discussion of general algorithms to solve such problems, see Bazaraa et al. (1993). Below we present a solution algorithm that exploits the special structure of the model.

12.5 Solution Algorithm

The difficulty in finding the optimal production quantities is that the first order condition tells us that $N-1$ of the y_i are mutually dependent. We now describe a solution procedure that exploits the special structure of these dependencies. In particular, consider the difference

$$\hat{C}_{i+1} = \frac{\partial C(y,x)}{\partial y_{i+1}} - \frac{\partial C(y,x)}{\partial y_i}$$
$$= \bar{F}\left(\sum_{j=1}^{i+1} y_j - x_j; T_{i+1}\right)(c_{i+1} - p_{i+1}\bar{G}_{i+1}(y_{i+1}) + h_{i+1}G_{i+1}(y_{i+1}))$$

$$
\begin{aligned}
&-\bar{F}\left(\sum_{j=1}^{i} y_j - x_j; T_i\right)(c_i - p_i\bar{G}_i(y_i) + h_i G_i(y_i)) \\
&-(p_{i+1} - c_{i+1})\left[F\left(\sum_{j=1}^{i+1} y_j - x_j; T_{i+1}\right) - F\left(\sum_{j=1}^{i} y_j - x_j; T_{i+1}\right)\right] \\
&+(p_{i+1} + h_{i+1})\int_{x_{i+1}}^{y_{i+1}} G_{i+1}(u) f\left(\sum_{j=1}^{i} y_j - x_j + u - x_{i+1}; T_{i+1}\right) \mathrm{d}u.
\end{aligned}
\tag{12.13}
$$

Note that if y_i is optimal, $\partial C(y,x)/\partial y_i$ is zero, so that $\hat{C}_{i+1} = C(y,x)/y_{i+1}$. The reason that this is significant is because $\hat{C}_{i+1}$ is a function only of $y_1,\ldots,y_i$. Therefore if the optimal y_1 is known then $\hat{C}_2$ can be used to find the optimal y_2, and then $\hat{C}_3$ can be used to find the optimal y_3, and so forth.

Since the optimal y_1 is not known, we must use a search technique to find it. We now prove three important properties that will be helpful in this regard.

Let the production quantities that result from the above procedure be denoted by $\hat{y}_i$. We first show that $\hat{y}_N = y_N^N$ iff $\partial C(y,x)/\partial y_1 = 0$. Observe that $\hat{C}_N$ is exactly equal to $\partial C(y,x)/\partial y_N - \partial C(y,x)/\partial y_{N-1}$, and thus $\hat{y}_N = y_N^N$ iff $\partial C(y,x)/\partial y_{N-1} = 0$. Further, for any i, $\hat{C}_{i+1} = C(y,x)/y_{i+1}$ iff $\partial C(y,x)/\partial y_i = 0$. Therefore, $\hat{y}_N = y_N^N$ iff $\partial C(y,x)/\partial y_1 = 0$.

The second property is that if the guess for the optimal value of y_1 is too large, $\hat{y}_N > y_N^N$. We have shown above that if $x_k \le y_k \le y_i^N$ for $k = i, i+1,\ldots,N$, then $\partial^2 C(y,x)/\partial y_i^2 \ge 0$. Accordingly, if the guess for the optimal value of y_1 is too large, $\partial C(y,x)/\partial y_1 > 0$, so that in order for $\hat{C}_2 = 0, \hat{y}_2$ must be chosen such that $\partial C(y,x)/\partial y_2 > 0$, so that $\hat{y}_2$ will be greater than the optimal y_2. Repeating this argument, we see that each $\hat{y}_i$ will be greater than the optimal y_i, and thus $\hat{y}_N > y_N^N$. By analogous reasoning, we can conclude that if the guess for the optimal value of y_1 is too small, $\hat{y}_N < y_N^N$.

The third and final property that we wish to show is that $\hat{C}_{i+1}$ is an increasing function of y_{i+1}. This property is particularly important, as it allows us to find $\hat{y}_{i+1}$ by simple binary search. To prove this, we take the partial derivative of $\hat{C}_{i+1}$ with respect to y_{i+1} and simplify to obtain

$$
\frac{\partial}{\partial y_{i+1}}\hat{C}_{i+1} = \bar{F}\left(\sum_{j=1}^{i+1} y_j - x_j; T_{i+1}\right)(p_{i+1} g_{i+1}(y_{i+1}) + h_{i+1} g_{i+1}(y_{i+1})), \tag{12.14}
$$

which is clearly nonnegative since each term is nonnegative, and thus the result is proven.

Using these properties, we are now ready to state the following:

Algorithm:

1. Preprocessing. Compute the y_i^N. If any $x_i \geq y_i^N$, then the optimal $\hat{y}_i = x_i$ and it is optimal not to produce this part. Remove all such parts from the list of parts to be produced over the horizon.
2. Initialization. Set $\hat{y}_1 = y_1^N$. Set $U = y_1^N$ and $L = x_1$.
3. Main loop. For each $i = 2, \ldots, N$, find the $\hat{y}_i$ such that $\hat{C}_i = 0$. If any $\hat{y}_i > y_i^N$, then $\hat{y}_1$ is too large. Set $U = \hat{y}_1$, $\hat{y}_1 = (U + L)/2$, and repeat Step 3.
4. Optimality test. If $|\hat{\mathrm{y}}_\mathrm{N} - \mathrm{y}_\mathrm{N}^\mathrm{N}| < \varepsilon$, then the $\hat{y}_i$ are optimal. Stop.
5. Adjustment step. If $\hat{y}_N > y_N^N$, then $\hat{y}_1$ is too large. Set $U = \hat{y}_1$, $\hat{y}_1 = (U + L)/2$, and go to Step 3. If $\hat{y}_N < y_N^N$, then $\hat{y}_1$ is too small. Set $L = \hat{y}_1$, $\hat{y}_1 = (U + L)/2$, and go to Step 3.

The algorithm essentially performs a binary search on the guess for the optimal y_1 by maintaining an upper and lower bound (U and L) on the optimal value. The algorithm terminates when the current value of $\hat{y}_N$ is within some small positive ε of y_N^N.

Because the properties that we have proven above are valid only if $x_i \leq y_i \leq y_i^N$ for $i = 1, \ldots, N$, we must take care to ensure that this remains true throughout the algorithm. We perform the test in Step 2 to ensure that we do not proceed if any $y_i > y_i^N$. We set $L = x_1$ so that $\hat{y}_1 \geq x_1$. Lastly, in a preprocessing step we remove a part i from consideration if $x_i > y_i^N$. We can do this because, for any such part, the optimal $\hat{y}_i$ is x_i, and it is thus optimal not to produce that part. Since the part would not be produced, it has no effect on the other parts.

12.6 Numerical Results

In this section, we present numerical results from an implementation of the solution algorithm described in the previous section.

For simplicity we describe a two part ($N = 2$) system, with identical parameters for the two parts. The base case parameters used are summarized in Table 12.2. We assume that the demand distribution g is normally distributed with mean equal to 100 and standard deviation equal to 10. We also assume that f, the distribution of output of the machine over a horizon of length T, has a mean of T a standard deviation of $0.1T$, which equates to a coefficient of variation of 0.1.

In this two part example, we know that y_2^* will equal its newsvendor point, which in this case is equal to 97.47. To find y_1^*, we developed a simple spreadsheet model that computes $\hat{C}_i$. The solution procedure then simply searches over values of y_1 until $\hat{C}_2 = 0$. The solution procedure converges to a value equal to the optimal solution within four digits of precision in just 13 iterations. In this case, the value is $y_1^* = 85.31$.

Table 12.2 Base case parameters for experiments

Parameter	Value
T	160
x_i	0
c_i	2
h_i	1
p_i	4
s_i	0

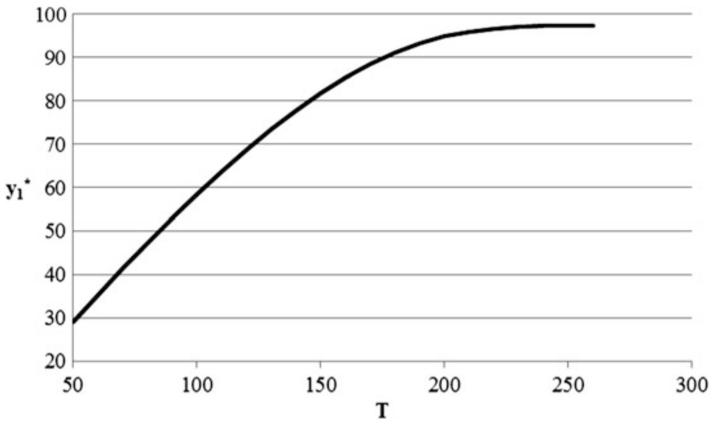

Fig. 12.1 Optimal production quantity of part 1 as a function of T

Next, we wish to show the effect of the time constraint on the optimal value for y_1. In Fig. 12.1 we show the results of varying the value of T, which also impacts our production distribution f. As expected, the value of T has a major effect on the production schedule, until T becomes sufficiently large, at which point y_1^* approaches its newsvendor point.

Finally, in Fig. 12.2 we show the effect of varying the coefficient of variation in the production schedule. This was set to 0.1 in the base case. We see that increased variability has the effect of causing greater levels of planned production as a hedge against this uncertainty.

12.7 Extensions to the Model

In this section, we present three simple extensions to the model: (1) a method for rescheduling within the planning horizon, (2) an extension to evaluate whether or not to purchase the option to run overtime within the planning horizon, and (3) an extension that permits the modeling of a machine that operates at a different speed depending on the part being produced.

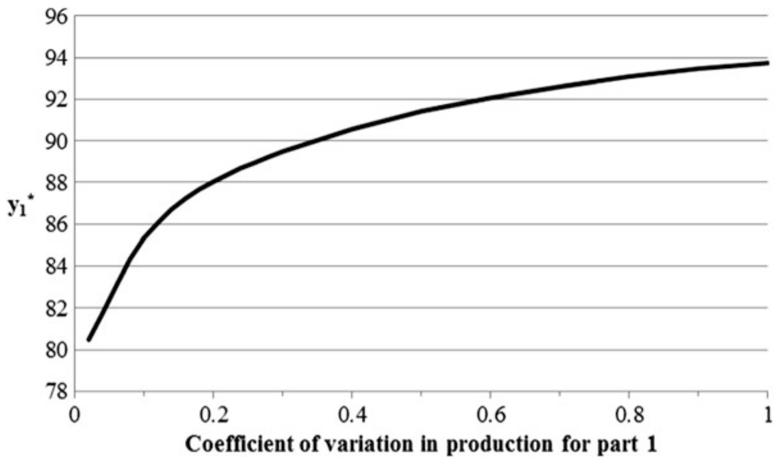

Fig. 12.2 Optimal production quantity of part 1 as a function of the variability in production

12.7.1 Dynamic Rescheduling

In the development above, we have discussed how to determine a set of production quantities to minimize expected total cost. We have assumed up to this point that this plan, once established, is fixed. That is, the plan is implemented by producing the predetermined optimal quantity of each part, and then switching production to the next part. Of course, as this plan is implemented, the reliability of the machine may be much higher or much lower than expected. As a result, if we were given the opportunity to do so, we might adjust the production plan based on the actual realized reliability of the machine.

Suppose we are now permitted to modify our choice of production quantity for the current part. We propose a simple method that allows someone on the shop floor to determine when to stop production of the current part based on the part's inventory level. We denote this production level as the *critical inventory level*. The method described below could produce a chart with two axes: the horizontal axis will be time, and the vertical axis will be the critical inventory level. In this sense, this method could produce a visual aide for a production manager on the factory floor.

Suppose for a particular future point in time t_1 we would like to determine the amount of inventory at or above which it is optimal to stop producing the current part and switch to the next part. We denote this production level as the *critical inventory level*, and determine it as follows. The first step is to update the horizon length in the equations above by replacing T with $T - t_1$. This reflects the amount of time that would be remaining for production at time t_1. Next, search over values for x_1, at each iteration finding the optimal y_i, until we identify the *lowest* x_1 such that at the optimum, $y_1 = x_1$. This is the critical inventory level, since this is the inventory level at which is it optimal not to produce.

We now prove the existence of such a critical inventory level. Recall that we are only interested in the *lowest* x_1 such that at the optimum $y_1 - x_1 = 0$, so the only question we must answer is whether or not such an x_1 exists. But this is clearly so, since if we set $x_1 = G_1{}^{-1}((p_1 - c_1)/(p_1 + h_1))$, we know $y_1 \leq G_1{}^{-1}((p_1 - c_1)/(p_1 + h_1))$ and since we must constrain y_1 to be at least $x_1, y_1 = x_1$.

We can vary the value of t_1 and find the critical inventory levels at each point over the planning horizon. The optimal dynamic operating policy is therefore implemented on the shop floor by producing until the inventory level crosses this curve. Once this happens and production is switched to the next part, the model should be solved again to find the critical inventory level as a function of time for the next part.

12.7.2 Including Options to Run Overtime

In the development above, we purposely omitted any discussion of how to make optimal overtime decisions. Suppose now that there are $m = 1, \ldots, N_{\mathrm{OT}}$ opportunities over the horizon to run overtime, and for simplicity assume that they are each of duration OT at cost w_{m}. Without loss of generality, we will assume that the opportunities are indexed in the order of increasing cost.

In the development above, we computed optimal production quantities ignoring overtime opportunities. This is equivalent to assuming that we choose not to run overtime, and the resulting expected cost is the expected cost of this strategy. Suppose instead that we decide that we are going to run overtime once. To evaluate the expected cost of this strategy we simply replace T by $T + \mathrm{OT}$ and find the optimal production quantities to compute the minimum expected cost, and then add w_{m}. Note that it does not matter where within the planning horizon that we run overtime, since all overtime opportunities occur before the demand point. As a result, we can find the optimal policy by simply running the solution algorithm above $N_{\mathrm{OT}} + 1$ times, with T taking on the values $T, T + \mathrm{OT}, T + 2\mathrm{OT}, \ldots, T + N_{\mathrm{OT}}$ OT, and choosing the strategy with lowest expected cost. Intuitively, increasing the length of the horizon will have a non-increasing benefit. If this is true, which we leave as a conjecture, the evaluation of policies can be stopped when the total cost increases from the previous iteration.

12.7.3 Extension to Different Machine Speeds

For notational convenience, up to this point we have assumed that the machine operates at the same speed when producing different parts. If the speeds are different, then the requirements on the machine need to be expressed in common units, such as time, instead of parts. This can be accommodated easily, replacing all expressions such as $F\left(\sum_{j=1}^{i} y_j - x_j; T_i\right)$ with $F\left(\sum_{j=1}^{i} \frac{y_j - x_j}{P_j}; T_i\right)$, where P_j is the speed at which the machine produces part j when it is working. Our solution procedure for finding the optimal y_i is unchanged by this modification.

12.8 Conclusion and Future Research

In this chapter, we have extended the basic newsvendor model to an unreliable machine that must produce multiple parts in a given period of time. We have seen that with an infinite production horizon, the problem simply decomposes into a single item newsvendor problem for each part. However, under a time constraint, the optimal production quantities are reduced from those in the infinite horizon case. We showed that the optimal production quantities are mutually dependent on one another, but have discovered a special structure in these relationships, and also proven other important properties of the objective function and decision variables. These results allowed us to construct a simple algorithm that performs a binary search for the optimal value of the production quantity for the first part. In each iteration, we exploit the special structure of the problem to easily determine the production quantities for the other parts, and easily test if the overall solution is optimal.

Our formulation has assumed that the production sequence is fixed. This could be the result of sequence-dependent setup costs or setup times, or a function of the timing of arrivals of materials from upstream suppliers. A future research topic could include relax this assumption and allow the decision maker to change the sequence, possibly with penalty costs associated with changes. We have also assumed that demand is satisfied for all parts at the end of the horizon. Another future research topic could be an extension to multiple time periods, or allowing different parts to have demand pull from inventory at different points in time.

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References

Al-Salamah, M., & Abudari, A. (2012). Production lot sizing & process targeting under process deterioration & machine breakdown conditions. *Modeling and Simulation in Engineering*, Hindawi Publishing Corporation, 7 pages doi:10.1155/2012/393495

Bazaraa, M. S., Sherali, H. D., & Shetty, C. M. (1993). *Nonlinear programming: Theory and algorithms* (2nd ed.). New York: Wiley.

Beyer, W. H. (1987). *CRC handbook of mathematical sciences* (6th ed.). Boca Raton: CRC Press.

Birge, J., Frenk, J. B. G., Mittenthal, J., & Rinnooy Kan, A. H. G. (1990). Single-machine scheduling subject to stochastic breakdowns. *Naval Research Logistics, 37*, 661–677.

Birge, J. R., & Glazebrook, K. D. (1988). Assessing the effects of machine breakdowns in stochastic scheduling. *Operations Research Letters, 7*(6), 267–271.

Bitran, G. R., Haas, E. A., & Matsuo, H. (1986). Production planning of style goods with high setup costs and forecast revisions. *Operations Research, 34*(2), 226–236.

Ciarallo, F. W., Akella, R., & Morton, T. E. (1994). A periodic review, production planning model with uncertain capacity and uncertain demand. *Management Science, 40*(3), 320–332.

Dada, M., Petruzzi, N. C., & Schwarz, L. B. (2007). A newsvendor's procurement problem when suppliers are unreliable. *Manufacturing & Service Operations Management, 9*(1), 9–32.

Diedrich, F., Jansen, K., Schwarz, U. M., & Trystram, D. (2009) A survey on approximation algorithms for scheduling with machine unavailability. In Algorithmics of large and complex networks: design, analysis, and simulation (pp. 50–64). Springer, Berlin.

El-Ferik, S. (2008). Economic production lot-sizing for an unreliable machine under imperfect age-based maintenance policy. *European Journal of Operational Research, 186*(1), 150–163.

Epstein, L., Levin, A., Marchetti-Spaccamela, A., Megow, N., Mestre, J., Skutella, M., et al. (2010). *Universal sequencing on an unreliable machine. Proceedings of the 14th international conference on integer programming and combinatorial optimization* (IPCO XIV). Springer, Berlin.

Evans, R. V. (1967). Inventory control of a multiproduct system with a limited production resource. *Naval Research Logistics Quarterly, 14*(2), 173–184.

Giri, B. C., Yun, W. Y., & Dohi, T. (2005). Optimal lot sizing in an unreliable two-stage serial production-inventory system. *International Transactions in Operational Research, 12*(1), 63–82.

Giri, B. C., & Dohi, T. (2004). Optimal lot sizing for an unreliable production system based on net present value approach. *International Journal of Production Economics, 92*(2), 157–167.

Gittins, J. C. (1979). Bandit processes and dynamic allocation indices. *Journal of the Royal Statistical Society, Series B, 41*, 148–177.

Glazebrook, K. D. (1984). Scheduling stochastic jobs on a single machine subject to breakdowns. *Naval Research Logistics Quarterly, 31*, 251–264.

Groenevelt, H., Pintelon, L., & Seidmann, A. (1992a). Production batching with machine breakdowns and safety stocks. *Operations Research, 40*(5), 959–971.

Groenevelt, H., Pintelon, L., & Seidmann, A. (1992b). Production lot sizing with machine breakdowns. *Management Science, 38*(1), 104–123.

Halim, K. A., Giri, B. C., & Chaudhuri, K. S. (2010). Lot sizing in an unreliable manufacturing system with fuzzy demand and repair time. *International Journal of Industrial and Systems Engineering, 5*(4), 485–500.

Huggins, E. L., & Olsen T. L. (2010). Inventory control with generalized expediting. *Operations Research, 58*(5), 1414–1426.

Jain, K., & Silver, E. A. (1995). The single period procurement problem where dedicated supplier capacity can be reserved. *Naval Research Logistics Quarterly, 42*, 915–934.

Kletter, D. B. (1994). *Determining Production Lot Sizes and Safety Stocks for an Automobile Stamping Plant*. S.M. Thesis, MIT.

Kletter, D. B. (1996) *Planning and Control of an Unreliable Machine In a Multi-item Production-Inventory System*. Ph.D. Thesis, MIT.

Lee, C. Y. (2004). Machine scheduling with availability constraints. In J. Y.-T. Leung (Ed.), *Handbook of scheduling*. Boca Raton: CRC Press.

Matsuo, H. (1990). A stochastic sequencing problem for style goods with forecast revisions and hierarchical structure. *Management Science, 36*(3), 332–347.

Pinedo, M., & Rammouz, E. (1988). A note on stochastic scheduling on a single machine subject to breakdown and repair. *Probability in the Engineering and Informational Sciences, 2*, 41–49.

Pinedo, M. (2008). *Scheduling: Theory, Algorithms, and Systems* (3rd ed.). Englewood Cliffs: Prentice-Hall.

Racke, H. (2009). Survey on oblivious routing strategies. In K. Ambos-Spies, B. Lowe, & W. Merkle (Eds.), *Mathematical theory and computational practice, Proceedings of 5th Conference on Computability in Europe (CiE)* (pp. 419–429). Springer, Berlin.

Reiman, M. I., & Wein, L. M. (1998). Dynamic scheduling of a two-class queue with setups. *Operations Research, 46*(4), 532–547.

Rose, J. S. (1992). The newsvendor with known demand and uncertain replenishment: applications to quality control and container fill. *Operations Research Letters, 11*(2), 111–117.

Schmidt, G. (2000). Scheduling with limited machine availability. *European Journal of Operational Research, 121*, pp. 1–15.

Sethi, S. P., & Zhang, Q. (1994). *Hierarchical production and setup scheduling in a single machine system, Chapter 8 in hierarchical decision making in stochastic manufacturing systems*. Boston: Birkhäuser.

Smith, S. A., Chambers, J. C., & Shlifer, E. (1980). Optimal inventories based on job completion rate for repairs requiring multiple items. *Management Science, 26*(8), 849–852.

Yano, C. A., & Lee, H. L. (1995). Lot sizing with random yields: a review. *Operations Research, 43*(2), 311–334.