Chapter 2
Framework for Examining the Transformation of Mathematics and Mathematics Learning in the Transition from School to University

An Analysis of German Textbooks from Upper Secondary School and the First Semester

Maike Vollstedt, Aiso Heinze, Kristin Gojdka and Stefanie Rach

2.1 Introduction

Many first-year students experience the transition from school to university as a challenging enterprise. This is especially true for mathematics programs. Comparatively high dropout rates of freshman students after the first or second semester indicate that this transition is the main obstacle for students to finish their studies in mathematics. For example, in Germany, universities and mathematics departments are faced with dropout rates of up to 50 % of first-year students in mathematics. According to surveys, students report that this is mainly caused by the enormous pressure to perform and a lack of motivation (Heublein et al. 2009). However, most of the surveys do not use instruments detecting the specific situation for the subject mathematics. We assume that the high dropout rate during the transition from school to university is rooted in the necessity of coping with two discontinuities: the discontinuity of the learning subject and the discontinuity of the way of learning. Accordingly, managing the transition from school to university successfully means individually developing two ways of transformation to overcome these discontinuities. First, a transformation from school mathematics to academic mathematics, so that academic mathematics can be recognized as an extension of school mathematics and the individual mathematical knowledge learned in school can serve as a basis for further learning (Deiser and Reiss 2013, Chap. 3). Second, a

M. Vollstedt (✉) · A. Heinze · K. Gojdka · S. Rach
Institute for Mathematics, Mathematics Education
Free University Berlin, Königin-Luise-Str. 24-26, 14195 Berlin, Germany
e-mail: vollstedt@math.fu-berlin.de

A. Heinze
Department of Mathematics Education, IPN—Leibniz Institute for Science and Mathematics Education, Olshausenstraße 62, 24098 Kiel, Germany
e-mail: heinze@ipn.uni-kiel.de

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transformation from school learning to academic learning is required, so that learning strategies acquired implicitly in school can be adapted for academic learning processes (Pepin 2013, Chap. 4).

In this contribution, we describe an approach to examine and describe the two discontinuities and the related requirements for the corresponding transformations through textbook comparisons. Mathematics textbooks play a decisive role for students' learning processes at school (Rezat 2009) as well as at university (Alsina 2001). Accordingly, we first assume that the way mathematics is presented in the textbooks can be considered as an indicator of the character of school mathematics and academic mathematics. Second we assume that the didactical structure of the textbooks can be considered as an indicator of the requirements of students' learning strategies. Based on these assumptions, we developed a theory-driven framework to compare textbooks using certain criteria related to the character of mathematics and the requirements for students' learning strategies. Using this framework for textbook comparison, we expect empirical results that help to specify which aspects of mathematics and mathematics learning are constitutive elements of the discontinuities in the transition from school to university. Empirical findings of two feasibility studies using two school textbooks, two university-level textbooks, as well as lecture notes handed out by a mathematics professor indicate that the application of this framework yields reliable results.

2.2 Theoretical Background

During the last decade, the previously mentioned discontinuities and the related challenges for first-year mathematics students were investigated from different perspectives. For example, the investigation took the transformation of mathematical contents, knowledge, learning strategies from school to university into account, as well as the students' motivation and self-regulation (e.g., Deiser and Reiss 2013, Chap. 3; Pepin 2013, Chap. 4; Kaiser and Buchholtz 2013, Chap. 5; De Guzmán et al. 1998; Hoyles et al. 2001; Rach and Heinze 2011). In the following, we discuss theoretical observations and empirical results from studies focusing on the transformation of mathematics from school to academic level as well as the transformation of the corresponding learning processes at school and at university. Moreover, we present some results on textbook research, as our framework for examining transformation processes in the transition phase from school to university is based on a textbook approach.

2.2.1 The Character of Mathematics at School and at University

Mathematics as it is taught in high school is not just academic mathematics in a simplified form; mathematics as "school mathematics" has its own character (e.g., Biermann and Jahnke 2013, Chap. 1; Hoyles et al. 2001; Heinze and Reiss 2007).

Mathematics as a school subject must contribute to the aim of general education. This means, in particular, that the character of school mathematics must enable students to learn mathematics in such a way that they can use their mathematical knowledge for solving everyday problems and as a sound basis for their vocational education (Heymann 2003). Accordingly, mathematical content, which is relevant for the application of mathematics as tool (e.g., percentages and algebraic manipulations) but which is hardly interesting from a scientific mathematical perspective, is comparatively strongly emphasized (e.g., Dörfler and McLone 1986). Mathematics at university has a different character, because it is considered a scientific discipline. Here, the mathematical content is organized and presented in an axiomatic and rigorous manner. In the first semesters, applications of mathematics for solving real-world problems hardly play any role.

Mathematics as a tool and mathematics as a scientific discipline can be considered as two sides of the same domain: "It is a tool in the study of the sciences, and it is an object of study in its own right" (Hoyles et al. 2001, p. 841). The fact that these two sides of mathematics are reflected at school and at university in quite a different way has serious consequences for the role of important characteristics of mathematics like proving, rigor, or formalism. For example, most of the mathematical concepts in school are introduced and used informally (Engelbrecht 2010). Accordingly, students mainly work with a "concept image" of a concept (in the sense of Tall and Vinner 1981), whereas the "concept definition" of most of the concepts does not play a prominent role. In mathematic courses at university, concepts are mostly introduced by a formal definition, i.e., as concept definition (Deiser and Reiss 2013, Chap. 3). This is necessary to meet the standards of rigor. A similar situation can be observed for the role of mathematical proofs. If mathematics is considered as a scientific theory, then there is the need for scientific evidence of statements and for explanations why these statements are true (e.g., Hanna and Jahnke 1996). In contrast, if mathematics is considered as a tool, proofs play a minor role. In this case, proofs are often omitted because it is enough to know that a statement is true (e.g., that the tool works well).

2.2.2 Learning Mathematics at School and at University

Though learners at both school and university learn mathematics, there are remarkable differences in students' learning activities. These differences constitute a discontinuity in the transition phase from school to university which, in consequence, requires a transformation of individual learning strategies. The two most important differences between school and university in this respect are the formal organization of learning opportunities and the individual learning strategies necessary for an effective use of these learning opportunities (Pepin 2013, Chap.4).

In most German universities, teaching and learning mathematics for first-year students is structured in three complementary activities. Each week, there are one or two 90-min lectures given by a mathematics professor, a self-study phase where 3–5 challenging tasks (mainly proof tasks) are solved as obligatory homework, and a 90-minute tutorial where a senior mathematics student discusses the solutions of the homework with a group of 20–30 students. The self-study phase is organized by the students in their private time. Mostly, students work in small groups (2–6 students) on their homework where they are individually and cooperatively involved in problem-solving activities. Moreover, they are supposed to recapitulate their lecture notes and use additional literature. In summary, these learning opportunities are quite different from the learning opportunities in school. In school, German students attend 3–5 mathematics lessons per week (each 45 min) which are prepared and structured by teachers with respect to the cognitive and affective learning prerequisites of the students. Mathematics instruction encompasses phases of teacher talk and student's private work (single, partner, or group work) and class work (in a questioning–answering format). Homework mainly serves as a supplement to the content of the previous lesson and offers opportunities for practicing. In most of the phases, students receive precise instructions about what they should do and what they should achieve. Therefore, it is a kind of guided learning with specific learning tasks which aims at the acquisition of different aspects of competencies (conceptual knowledge, procedural knowledge, etc.; Kaiser 1999; Kawanaka et al. 1999).

The differences between school and university in the formal organization of learning opportunities and in the character of mathematics imply different requirements for students' learning strategies. Because mathematic lectures are both rigorous and formal in university lectures, university students need to apply specific elaboration strategies to understand the mathematical content. New mathematical concepts cannot be grasped through formal concept definitions, so it is necessary that students connect the presented concept definition to an already existing concept image from an intuitive use of this concept in school or that they individually develop a new concept image (cf. Engelbrecht 2010). In addition to learning concepts, students have to acquire problem-solving competencies so that they are able to solve the weekly challenging proof problems as homework. As mathematics is frequently taught using completed theories or as elegant solutions in lectures and tutorials, problem-solving strategies are mainly dealt with implicitly and students are not offered direct accessible models for the trial-and-error process of creating new knowledge (e.g., Dreyfus 1991). Accordingly, they have to elaborate on the proofs and to reflect on proving processes. The use of self-explanations can be considered as an effective learning strategy in this respect (e.g., Chi et al. 1989; Reiss et al. 2006).

In summary, at university, mathematics as a scientific product is presented to the students who, in turn, have to find and apply learning strategies on their own to make this product accessible for their individual learning processes. At school, however, mathematics is presented in the framework of a didactical structure. The teachers prepare mathematics in such a way that it is accessible to the students. Students' learning is guided by sequences of specific chosen tasks which implicitly induces the application of adequate learning strategies (Pepin 2013, Chap. 4).

2.2.3 Approaching the Discontinuities by Textbook Comparisons

There are different possibilities to approach the two discontinuities between school and university. For example, taking the student perspective, you can ask students about their perception of mathematics and mathematics learning at school and at university or you can compare mathematical competencies between freshmen before they commence their studies and again after one semester. Similarly, you can take the teacher's perspective and ask teachers at school and university about their view on mathematics and mathematics learning. A third possibility is to take the observer perspective and to observe and analyze mathematics and mathematics learning in both institutions.

In this contribution, we choose the third possibility by taking the observer perspective with a specific focus. We analyzed school and university textbooks and compared our findings to the previously mentioned discontinuities. We are well aware that (1) textbooks obviously only represent a small section of the learning opportunities students are offered at schools and universities. Furthermore, we are also aware that (2) the impact of textbooks strongly depends on the individual use of textbooks which again is influenced by cultural traditions (e.g., Pepin and Haggarty 2001). Nevertheless, research on mathematics textbooks shows that textbooks have a close connection to the curriculum and, therefore, they reflect the differences between the mathematics curriculum at school and at university. Geoffrey Howson (1995) describes textbooks as a mediator between intended and implemented curriculum. They are designed as a means to transfer the intended contents to the lesson or function as a device for self-study phases. However, a textbook cannot be identified with either the intended or the implemented curriculum as publishers as well as teachers choose which contents to include in the book or to impart in the lesson, respectively (cf. Howson 1995). Hence, Schmidt et al. (2001) introduced the notion of a "potentially implemented curriculum" which is represented by a textbook.

There is already sufficient research on textbooks: Some studies take a comparative cultural perspective by investigating the structure and use of textbooks in different countries (Howson 1995; Pepin and Haggarty 2001; Valverde 2002). Other studies look at textbooks from a sociocultural perspective when investigating their structure (Rezat 2006), the students' use of textbooks (Rezat 2009), the difficulty of tasks (Brändström 2005), or the role of textbooks for the establishment of misconceptions (Kajander and Lovric 2009). Most of these studies deal with school textbooks, whereas textbooks at university level are not as well researched.

Regarding our investigation of the two discontinuities during the transition from school to universities, we assume that both the discontinuity in the character of mathematics and the discontinuity in mathematics learning are reflected in the textbooks. According to Pepin and Haggarty (2001), textbooks show mathematical intentions that can be divided into three areas: "What mathematics is represented in textbooks; beliefs about the nature of mathematics that are implicit in textbooks; and the presentation of mathematical knowledge" (Pepin and Haggarty 2001,

p. 160). Therefore, we expect that differences in the character of school mathematics and academic mathematics can be found in mathematics textbooks. In addition, we expect that differences in mathematics learning at school and at university are also reflected in the textbooks. A mathematics textbook in school is a focal point for the interaction between the teacher and mathematics, between the student and the teacher, as well as between the student and mathematics. Rezat (2009, p. 66) therefore suggests enhancing the didactic triangle into a didactic tetrahedron incorporating the textbook as fourth element.

In summary, mathematics textbooks play an important role for students' learning processes at school as well as at university. As textbooks represent potentially implemented curricula, we assume that the way mathematics is presented in the textbooks represents the character of school mathematics and academic mathematics. Moreover, the didactical structure of the textbooks indicates the requirements for students learning strategies because textbooks influence the interaction between teachers, learners, and mathematics.

2.3 Research Objectives

Our research aims at examining and describing the two discontinuities in the transition from school to university regarding the character of mathematics and the way of learning mathematics. To achieve this goal, we use different approaches. In the following, we will present an approach that is based on a comparison of textbooks at school and textbooks at university. Hence, the specific goals of this contribution are as follows:

1. The elaboration of a theory-based framework for analyzing and comparing mathematics textbooks at the upper secondary level and the first semester at universities.
2. The presentation of results of feasibility studies to show that this framework allows a reliable data collection for textbook comparisons.

The feasibility studies were conducted with a small number of textbooks. This means particularly that we cannot yet report clear results concerning differences between textbooks at school and at university. Nevertheless, there are some tendencies that we will address in the discussion section.

2.4 A Framework for Textbook Comparison

For the analysis of mathematics textbooks at school and university levels, we apply a framework that is derived from a psychological and a didactical perspective. It consists of six criteria that can be divided into general and content-specific ones (see Fig. 2.1). General criteria are not bound to mathematical contents but could be

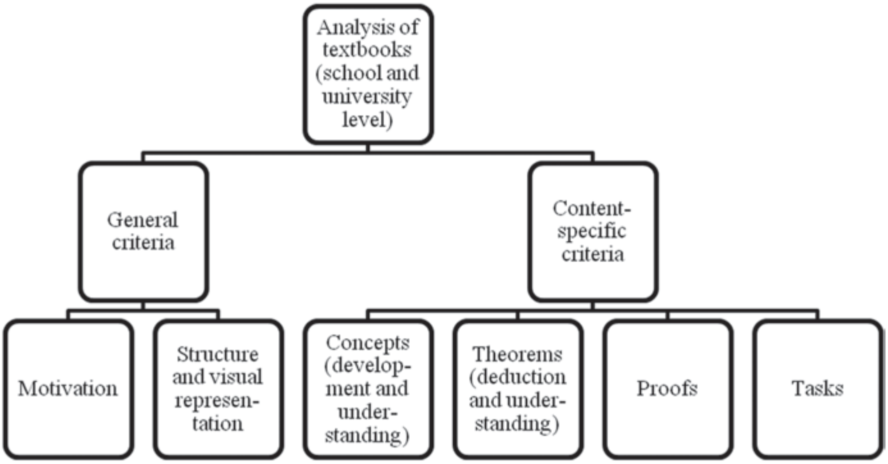

Fig. 2.1 Framework for the analysis of textbooks used at school and university levels

applied for the analysis of textbooks from any other subject. We restrict ourselves to motivation and the structure and visual representation of the contents. Content-specific criteria investigate aspects that are particular for mathematics such as the development and understanding of concepts and the deduction and understanding of theorems, proofs, and tasks.

In the following, we elaborate on these six criteria. In Sect. 2.5, we show examples of the operationalization for some of the criteria that were used as rating schemes for data collection.

2.4.1 General Criteria

The general criteria considered in our framework relate to self-determination theory of motivation and the structure and visual representation.

2.4.1.1 Self-Determination Theory of Motivation

According to self-determination theory (SDT; Deci and Ryan 1985; Ryan and Deci 2002), motivated actions can be distinguished by their degree of self-determination and regulation. Actions can be amotivated, extrinsically motivated, or intrinsically motivated. Forming one end of a self-determination continuum, amotivation is characterized by non-regulation. The other end is marked by intrinsic motivation that is assigned by intrinsic regulation. The different degrees of extrinsic motivation lying in between distinguish four different types of regulations: external, introjected, identified, and integrated regulations (see Ryan and Deci 2002; see also

Pepin 2013, Chap. 4 for the role of self-regulated learning at the transition from school to university).

SDT postulates three basic psychological needs to explain the relation of motivation and goals to health and well-being. The need for competence refers to “feeling effective in one’s ongoing interactions with the social environment and experiencing opportunities to exercise and express one’s capacities” (Ryan and Deci 2002, p. 7) and the need for social relatedness corresponds to “feeling connected to others, to caring for and being cared for by those others, to having a sense of belongingness both with other individuals and with one’s community” (Ryan and Deci 2002, p. 7), while the need for autonomy focuses on “being the perceived origin or source of one’s own behavior” (Ryan and Deci 2002, p. 8). These needs are assumed to be innate, culturally universal, and equally relevant for extrinsic and intrinsic motivation (Ryan and Deci 2002).

Different studies have analyzed mathematics lessons with respect to the implementation of the three basic needs (Rakoczy 2008; Kunter 2005; Daniels 2008). In these studies, the following aspects turned out to be important for motivated learning.

Implementation of Perceived Autonomy On the one hand, students should have the possibility to make deliberate choices in their learning process so as to give room for their own demands. In the context of textbooks, they should be able to have the choice as to which explanations, examples, and tasks they want to deal with to organize their own learning process. One way to offer this possibility is to provide different ways of introducing new contents and offering different examples and tasks. It is important that these different ways of approaching a certain concept do not offer different degrees of complexity or contents. On the contrary, they follow the same aim by offering different approaches to the same learning goals and demands disguised in different representations and methods.

On the other hand, the topics dealt with in the lessons should be personally relevant to the students (see also Vollstedt 2011) in order to help them realize the value of their actions. They experience them as leading to their goals concerning their own values. Hence, they have a higher feeling of autonomy which is related to their motivation to learn (see Rakoczy 2008, p. 41). Textbooks therefore implement autonomy when they allow contexts which relate to the students’ lives and which are personally relevant to the students. Through this, mathematics can become more important for the students.

Implementation of Perceived Competence This aspect complements the suggestion to give room for own decisions mentioned above. The students perceive themselves as competent when they can come to the right conclusions. Therefore, different levels of difficulty are needed for the tasks and introductory parts of the sections (Rakoczy 2008). Depending on their own level of achievement, students can then choose which task to deal with subsequently. To enable this choice, the task or introduction should be marked, for instance, according to its degree of difficulty.

The second way to foster the students’ perception of competence is to give them guidance through the book by following a certain structure. The fragmentation of major contents into subchapters as well as a general guidance through each chapter offer the students a sense of security (Rakoczy 2008; Kunter 2005). Another element of structure is the occurrence of advance organizers at the beginning or the end of the text (Ausubel 1960). Without this structure they might have the feeling of getting lost and not being able to cope with the demands made on them.

Implementation of Perceived Social Relatedness Most of the aspects of the experience of social relatedness in mathematics lessons refer to the relation between the students and the teacher or between the students themselves. These are difficult to transfer to textbooks. It can, however, be evaluated to what extent books stimulate or cultivate cooperative learning so that the students' need for social relatedness is met by the conceptual design of the book.

2.4.1.2 Structure and Visual Representation

This criterion is divided into the following subsections: comprehension of the text, formalism, and visual representation. The variables for the first one result from studies carried out by Langer et al. (1973, 1974, 2006). They were complemented by aspects of formalism (Kettler 1998) as this is one of the characteristic elements of mathematical texts. The last element concerns the role and quality of graphical representations, which is based on the work of Mayer et al. (Mayer and Gallini 1990; Mayer and Moreno 1998; Mayer and Johnson 2008).

Comprehension of the Text and Formalism According to Langer et al., four elements are important to understand texts: simplicity, coherence/organization, conciseness, and motivational additives. Simplicity refers to the diction and the syntax of the text. No matter what level of difficulty characterizes the content, familiar words are combined to short sentences with easy structure and difficult words (foreign words or technical terms) are explained. Coherence describes the inner logical structure of the text in which sentences combine to form a stringent idea, whereas organization refers to the outer structure of the text (sections related to each other are in close distance, sections are divided by headlines, and important aspects are highlighted). The level of conciseness relates to the length of the text in comparison to its informational content, i.e., whether the phrasing is scant or wordy. Motivational additives then embrace elements which the author uses to raise the reader's interest. The complementation of the element formalism adjusts the theory to mathematical texts insofar as it judges the frequency of the occurrence of mathematical elements. According to Kettler (1998), the amount of mathematical symbols can have an impact on the reaction of the reader as the readers' sympathy decreases when the degree of symbolism increases.

Visual Representation Concerning the characteristic visual representation, two major types are distinguished: the role and quality of graphical representations. Mayer and Gallini (1990) differentiate five roles of illustrations:

1. Decoration: The graphical representation has no direct relation to the text but serves as motivational element.

2. Representation: The contents are represented in another way without adding information (e.g., diagrams).
3. Transformation: The graphical representation serves to ease the memory of easily understandable information; additional information may be included.
4. Organization: The graphical representation is supposed to structure the text and to organize its elements.
5. Interpretation: Graphical representations are to help the reader to understand difficult relationships.

The quality of a graphical representation can be judged by the occurrence of the split-attention effect (Mayer and Moreno 1998) and the redundancy effect (Mayer and Johnson 2008). Figures are more difficult to understand when it is necessary to split the reader's attention between more than one source which can only be understood in relation with each other. The effect can be minimized when the sources can be integrated into one main source, for instance, by incorporating the values of angles directly into the figure instead of placing them next to it. The redundancy effect occurs when the same information is given in the text as well as in the figure. The different kinds of representation do not show any relations or help in another way toward a better understanding. The effect can be minimized when only keywords are integrated in the representations, whereas it can be maximized by giving the whole text again in the figure (Sweller 2005). Mayer et al. were able to show that graphical representations can enhance understanding and remembrance of information, whereas improper use hinders them.

2.4.2 Content-Specific Criteria

In contrast to general criteria, which can also be applied to other subjects, content-specific criteria investigate aspects which are specific for mathematics. The following sections give more details about the development and understanding of concepts and theorems, the role of proofs, and tasks.

2.4.2.1 Development and Understanding of Concepts

The way how mathematical concepts are developed influences their fundamental understanding (Vollrath 1984): Can students give a definition of the concept and can they decide whether an example fits the category of the respective concept? Can students give examples or counterexamples and do they know characteristic properties of the concept? Can the concept be applied when solving problems and can the students integrate the concept into a network of subconcepts and generic terms? All but the very first aspect are necessary to develop a deep understanding of concepts.

The background theory we apply to the development and understanding of concepts relates to instructional psychology as well as to the theory of mental models (*Grundvorstellungen*, vom Hofe 1995). Klauer and Leutner (2007) name different possible functions of teaching which are necessary to reach a teaching goal. In our model, we primarily focus on three of them which are important for the development of concepts: First, the *transformation of information* characterized with regard to the way the concept is introduced as well as the precision and formalism of this introduction. The second function concerns the *processing of information* and focuses on the possibilities of understanding, recalling, expanding, and reviewing the concept as well as on the way how concepts are distinguished from others with the help of examples and counterexamples. The third under consideration is *transfer*. This entails looking at the number of adequate mental models and the number of equivalent definitions given in the book.

2.4.2.2 Development and Understanding of Theorems

The development and understanding of theorems is partly analogous to the development and understanding of concepts as well as proofs (see below). The first considered aspect is the way the theorem is introduced with the help of an example or a problem which motivates the theorem. The second aspect then deals with the mathematical development of the theorem. The formulation of the theorem then takes into consideration the precision and formalism of the formulations used to state the theorem. Finally, the last aspect concerns the differentiation of the respective theorem from others with the help of illustrating examples and/or counterexamples for the application of the theorem.

2.4.2.3 Presentation of the Proving Process and Proofs

Proving something is an essential mathematical activity (e.g., Heinze and Reiss 2007). To prove that a mathematical theorem is true, it is crucial to detect connections between mathematical structures and to show that the correctness of these connections can be universally argued. By doing so, learners have the possibility of experiencing mathematics as a process and not as a set science. Boero (1999) distinguishes six phases of a proving process:

1. Production of a conjecture;
2. Formulation of the statement according to shared textual conventions;
3. Exploration of the content (and limits of validity) of the conjecture;
4. Selection and enchaining of coherent, theoretical arguments into a deductive chain;
5. Organisation of the enchained arguments into a proof that is acceptable according to current mathematical standards; and
6. Approaching a formal proof.

From this theoretical basis, we distinguish between elements of the proving process (the role of advanced organizers and the generation of a proof idea) and the formulation of proofs, i.e., preciseness and formalism of the proofs as well as the number of different methods which were presented.

2.4.2.4 Tasks

Tasks are a central element in mathematics textbooks and fundamental for the students' learning process (Rezat 2009). The characteristics taken in our framework to judge the tasks in school and university textbooks are based on the educational standards passed by The Standing Conference of the Ministers of Education and Cultural Affairs of the Länder in the Federal Republic of Germany (KMK 2004)[1]. They distinguish between five key content areas, six general mathematical competences (cognitive processes), and three levels of demand. As our study compares textbook sections with similar content only, the different key contents of educational standards can be neglected. The tasks from the different textbooks are therefore analyzed concerning their main mathematical competencies and levels of demand only. Each task has to be judged with respect to the competence:

1. Argue mathematically;
2. Solve problems mathematically;
3. Model mathematically;
4. Use mathematical representations;
5. Deal with symbolic, formal, and technical elements of mathematics; and
6. Communicate.

Moreover, each task has to be judged concerning its level of demand, i.e., whether it is necessary to reproduce, to make connections, or to generalize and reflect.

In addition to the task analysis based on the educational standards, the numbers of different solutions and solution approaches are evaluated. Finally, the tasks are analyzed with respect to their relation to mental models, i.e., whether new mental models are developed or whether known mental models are used.

2.4.3 Summary

In the previous subsections, we present general and content-specific criteria for a textbook analysis. All criteria are based on psychological or didactical theories or models. Their significance for mathematics learning is based on evidence from empirical studies (e.g., in case of learning activities) or on theoretical analyses (in the case of the learning content). Accordingly, we assume that these criteria cover important aspects for a comparison of school mathematics with academic mathematics and for a comparison of the requirements of individual mathematics learn-

[1] The underlying competence model coincides in many respects with the competence model of the PISA 2012 study (see OECD 2010).

ing at school and at university. By identifying differences and commonalities, these aspects help to track down the transformation of mathematical contents and the learning of mathematics at the transition between school and university.

2.5 Feasibility Studies

The model presented above was developed in the context of two feasibility studies. The first one was used to check the validity of the model. Both a school and a university book were rated by two field experts. The focus was on the consistency of the two experts' judgment. Based on the results from this first study, the model was refined and then applied in a second study. The aim of this second study was again to test the model for validity as well as to detect differences concerning the methodological and didactical organization of the textbooks. These results form the basis for statements with respect to the transformation of contents or learning strategies at the transition from school to university level.

The studies reported on in this article are part of an ongoing bigger study that compares textbooks from school and university in different countries. In this article, we restrict ourselves to the first two feasibility studies comparing textbooks at school and university levels which are very frequently used in Germany. For the first study, one book from each level was taken: *Lambacher Schweizer Gesamtband Oberstufe* (Brandt and Reinelt 2009) is one of the most commonly used textbooks at school level. Its section about vector spaces was compared to the respective section in the 'Beutelspacher', a very popular linear algebra textbook at university level using a very explanatory approach (Beutelspacher 2010). For *Lambacher Schweizer Gesamtband Oberstufe*, the experts reach a consensus on 16 out of 32 criteria. For Beutelspacher's textbook, this was the case for 24 out of 34 criteria. The differing number of criteria results from the fact that not all criteria could be applied to both books: *Lambacher Schweizer Gesamtband Oberstufe* does not contain proofs and Beutelspacher's book does not contain pictures. Although the consistency of the rating is higher than the anticipated value, it is obvious that the results could be improved.

A closer look at the results shows that, due to a misunderstanding of the coding scheme, some subitems from proof were accessed for the *Lambacher Schweizer Gesamtband Oberstufe* although this textbook does not contain any proofs. Similarly, the criterion of vividness (one item in structure and visual representation/motivational additives dealing with the way contents are presented) was not judged. Our hypothesis is that it did not become totally clear to what extent the items belong to their main categories. Descriptions were therefore refined to make this clearer.

After refining the framework, the second study was conducted in the field of calculus using the standard school textbooks *Lambacher Schweizer* (Drüke-Noe et al. 2008) and *Elemente der Mathematik* (Griesel and Postel 2001; Griesel et al. 2007, 2008) together with the university-level textbooks *Königsberger* (Königsberger 2004) and *Forster* (Forster 2008), which from experience are often used by undergraduate students. In addition, the lecture notes handed out by one of the professors from the mathematics department at our university were coded. In all cases, the sections dealing with real numbers, continuity, and differentiability were rated.

Attention has to be drawn to some specifics concerning the textbooks used. Forster's book does not contain a chapter dealing with real numbers in detail. They are only treated in a very dense compression on one page at the end of the book. Also, there are no solutions for the tasks posed to students. The lecture notes do not contain any tasks or solutions. Therefore, these sections were not rated for these textbooks in our study. Then, the school textbooks only briefly deals with continuity so that the explanatory power of the comparison in this realm is lowered. We added propositions about typical characteristics of $\mathbb{R}$ like uncountability and the embeddedness of $\mathbb{Q}$ to the topic of real numbers where possible. For *continuity*, we looked at the intermediate value theorem, and for *differentiability*, we observed derivation rules, the calculation of turning points, inflection points, as well as convexity and monotony.

Two master mathematics students were responsible for the rating. The categories were quantified with respect to whether the criterion is a conceptual element of the book, i.e., that it occurs in every chapter considered or whether the criterion just occurs sporadically, i.e., there is at least one chapter in which it does not occur. Criteria were considered as consistent if both raters agreed totally with each other in their judgment. Only those characteristics were interpreted that were rated by both raters.

2.6 Exemplary Results of the Second Feasibility Study

Section 2.4 gave a general introduction to the framework used in the studies described in Sect. 2.5. In the following subsections, exemplary operationalizations are given to illustrate how we transferred the model into ratable characteristic features. Cohen's kappa is reported to indicate the strength of the interrater agreement as a reliability measure. According to the Landis and Koch (1977, p. 165) interpretation scale, the strength of agreement is fair if $0.2<\kappa<0.4$, moderate if $0.4<\kappa<0.6$, substantial if $0.6<\kappa<0.8$, and almost perfect if $0.8<\kappa<1$. In general, a reliability of $\kappa>0.6$ is considered as an acceptable agreement, so that the value of the corresponding criterion can be interpreted.

2.6.1 Motivation

One of the aspects of motivation according to SDT (see above) is the experience of social relatedness. When analyzing textbooks, you therefore have to judge to what extent the book supports group work. The following feature characteristics were developed:

1. The book explicitly invites the students to work on the tasks in groups. This method is part of the book's conceptual design.
2. Students are sporadically invited by the book to work in groups.
3. There are no tasks which are supposed to be worked on in groups.

Interrater agreement on social relatedness in the second feasibility study was substantial ($\kappa = 0.632$).

2.6.2 Structure and Visual Representation

The structure of the textbooks comprises motivational additives. One aspect considered in this realm is vividness:

1. It is part of the book's conceptual design that descriptions are padded with anecdotes and that stories are used to convey facts.
2. Some information is always presented in the same dreary and unvaried way.
3. The text deals with the contents in a very prosaic way, i.e., facts are conveyed by using factual language. There is no supplementary information in terms of anecdotes or stories.

Interrater agreement on vividness was substantial ($\kappa = 0.650$).

The role of graphical representations is operationalized as follows:

1. The graphical representation contains more information than can be found in the text. These mostly comprise tasks that are introduced by a text in which information has to be taken from the corresponding graphical representation.
2. The graphical representation contains the same information as the text but eventually offers another way of access. Graphical representations that present the text in a modified display format belong to this group. They can, for instance, be restructured to be learned or understood more easily.
3. The graphical representation does not have any information content. Pictures with motivating character belong to this group.

Interrater agreement on the role of graphical representation is only moderate ($\kappa = 0.451$).

2.6.3 Development and Understanding of Concepts

To develop a sound concept definition and concept image of a mathematical concept, it has to be linked to inner-mathematical as well as extra-mathematical, i.e., applied contexts and examples. The following characterizations were developed to operationalize the introduction of a new concept:

1. It is part of the book's conceptual design that a new concept is introduced by using an applied or inner-mathematical example or problem.
2. Sporadically, an applied or inner-mathematical example or problem is used to introduce a new concept.
3. There is no introduction.

The characterization of the introduction of a new concept showed substantial interrater agreement ($\kappa = 0.745$).

To understand a mathematical concept properly, the corresponding information has to be processed in different steps. The conceptualization of reviewing is as follows:

1. The book requests the reader (after some time) to be able to actively name and use already known concepts as well as their characteristic properties. Occasionally, contents that have already been learned are referred to, or they are necessary to solve tasks, respectively (active).
2. The book reminds the reader of learned contents and of characteristics of learned concepts (passive).
3. The book proceeds in the contents without testing concepts which have already been learned or including characteristics of learned concepts in the contents. The particular chapters are strictly delimited from each other.

The interrater agreement on reviewing shows moderate strength ($\kappa = 0.548$).

2.6.4 Development and Understanding of Theorems

The operationalization of the development and understanding of theorems is divided into three subsections dealing with the introduction of the theorem, its formulation, and its demarcation from other theorems. The development of the theorem is an example of the first section.

1. It is part of the book's conceptual design that the development of the theorem is described.
2. It is sporadically shown how the theorem can be developed.
3. There is only a formal formulation of the theorem.

Interrater agreement on the development of a theorem showed only moderate strength ($\kappa = 0.417$).

The next operationalization presented is the one of the degree of formalism. It belongs to the formulation of the theorem.

1. The formulation/notation of the theorem equally consists of mathematical symbols and (German) language.
2. The formulation/notation of the theorem consists mainly of mathematical symbols.
3. The formulation/notation of the theorem consists mainly of (German) language.

The interrater agreement on the degree of formalism of this operationalization was substantial (κ = 0.714).

The demarcation of a mathematical theorem can, for instance, be characterized by using explicating examples and counterexamples as this marks the theorem's applicability. The operationalization of this characteristic shows substantial strength in interrater agreement (κ = 0.696).

1. There are examples and counterexamples for the theorem given.
2. There are either examples or counterexamples to mark the applicability of the theorem.
3. No examples of applicability of the theorem are used.

2.6.5 Presentation of the Proving Process and Proofs

To learn how to prove a mathematical proposition, it is necessary to understand how to come to the idea of the proof. Therefore, students have to understand how a proof is developed and how to write it down properly. The operationalization of the generation of a proof idea is given below:

1. It is part of the book's conceptual design to show the derivation of the proof ideas.
2. Proof ideas are sporadically derived.
3. It is never shown how a proof idea can develop.

Substantial interrater agreement (κ = 0.632) could be reached for the generation of a proof idea.

As there are several ways as to how to come to a proof idea, it is necessary to illustrate different approaches or methods on how to reach a proof:

1. It is part of the book's conceptual design that the assertion is proven in different ways or that the proof idea is sketched, respectively.
2. It is sporadically shown how an assertion can be proven in another way.
3. The assertion is proven in at most one way.

The strength of the interrater agreement concerning the number of methods to prove is substantial with κ = 0.632.

2.6.6 Tasks

To work on mathematical tasks actively is a fundamental part in the process of learning mathematics. Therefore, our model distinguishes between different dimensions of the tasks referring to the national educational standards, contents, and solutions. In the realm of the contents, it was rated to what extent different mental models are part of the book's conceptual design.

1. The tasks only make use of mental models that have been addressed beforehand.
2. New mental models are introduced by means of tasks. They are just briefly presented; there is no sufficient implementation.
3. The readers must develop new mental models on their own while working on the tasks.

The interrater agreement on the use of mental models was perfect ($\kappa = 1$).

One aspect that was rated concerning the solutions of the tasks was the explication of the approach to the solution:

1. The idea of the approach to the solution is described and the approach to the solution is explained.
2. Only the approach to the solution is indicated. It is, however, not stated how it has arisen.
3. Only the solution is given without elaborating on the approach to the solution.

The interrater agreement of this operationalization was moderate ($\kappa = 0.591$).

2.7 Discussion

The goal of the research presented here is to develop a theory-based framework for a mathematics textbook analysis. The aim is to allow a reliable rating of different criteria to compare textbooks for schools and for universities. As presented in Sect. 2.4, the criteria are derived from theories and models concerning mathematical learning activities and the character of mathematics. Two feasibility studies were conducted to evaluate the framework: one to validate and complete the framework and a second to check whether a reliable rating of the criteria is possible. In Sect. 2.6, exemplary results on the reliability values of different rating criteria are presented. The results indicate that the development of operationalizations which allow reliable ratings for a mathematics textbook analysis is possible for many criteria. However, in several cases, the interrater agreement cannot be considered as acceptable, and hence, a further improvement in the feature characteristic descriptions is necessary.

On the basis of the reliable rating criteria presented in Sect. 2.6, some tendencies about commonalities and differences between mathematics textbooks for school and for university can be described. However, as the “sample” of textbooks included in this feasibility studies is quite small (two textbooks for schools, two for universities, and one lecture note for a university course), the results should not be over-interpreted. In our study, we did not find differences between school and university textbooks for:

- The motivation criterion “social relatedness” because there were hardly tasks requiring collaborative activities.
- The criterion relating to the understanding of theorems which addresses the explication of examples and counterexamples. This is because only examples were presented for both types of textbooks.

- The proof criteria "developing a proof idea" and "different proofs for a theorem" because in both types of textbooks the idea was developed for a minimum of proof and, in general, only one proof was presented.
- The task criterion "use of different mental models in tasks," because, in general, only the mental models introduced before were addressed in the tasks.

In contrast to these commonalities, the textbooks for schools and for universities in our sample also revealed some differences, for example:

- For the structure criterion "motivational additives" the textbooks for schools contain some additional information about the mathematical facts in terms of stories and anecdotes raising the readers' interests, whereas we did not find such motivational additives in textbooks for universities.
- The introduction of new concepts is in textbooks for schools almost always developed on the basis of inner-mathematical or extra-mathematical examples, whereas in textbooks for universities such an introduction is rarely given.
- The degree of formalism for the formulation of theorems in textbooks for school almost always consists of continuous written language, whereas in textbooks for university a mixture of continuous text and symbols is used.

From analyzing these commonalities and differences, some anticipated findings have become evident which already give indications about the transformation problems in students' learning during the transition stage from school to universities. For example, mathematical proofs are treated inadequately in both types of textbooks. However, proofs are underemphasized at school so that students do not experience negative consequences. In contrast, at university, proofs are one of the main aspects in mathematics courses; however, university-level textbooks do not give didactical support to learn how to prove a task. Another example is the introduction of new concepts. In school textbooks, there are frequently inner-mathematical or extra-mathematical examples to motivate the new concepts. At university, such motivation is rarely given. This means that students have to elaborate on that question by themselves which requires specific learning strategies.

Already these first ideas from our feasibility studies indicate that the two hypothesized transformations from school to university are not independent but interwoven. The transformation of the character of mathematics with a stronger emphasis on concepts and proofs requires an increasing learning effort. However, the change from school-based to academic learning opportunities requires a transformation of individual learning strategies to grasp the academic mathematics.

To make sound statements about these tendencies revealed from our study, more substantial studies with more textbooks have to be carried out. The findings gained from this study, however, are an initial starting point and can be seen as basis for a future refinement of the model. This refined framework then is supposed to be used in further studies comparing international textbooks.

School and university textbooks should be revised in such a way that their contents and methodologies are better adapted to each other. This would help diminish the transformation challenges experienced during transition from school to university. The transition from school mathematics to university mathematics is supposed to be easier for students if mathematical methods and university standards were applied in school textbooks. The same can be said for improving the didactical quality of university textbooks.

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