Chapter 10
Quantity and Price Competition in Static Oligopoly Models

We saw in the previous chapter that there are two types of oligopoly models, those that assume cooperative behavior and those that assume noncooperative behavior. In Chaps. 10 and 11, we develop the classic models of oligopoly where firms behave noncooperatively. These models represent the most abstract material that is found in the book. Here you will see how some of the great figures in history have thought about the oligopoly problem.

A fundamental question in industrial organization is the extent to which the number of competitors (n) affects price competition. We have seen that price equals marginal cost (MC) in perfect competition and exceeds marginal cost in a monopoly setting. These equilibrium outcomes are illustrated in Fig. 10.1. Point A identifies the monopoly outcome at the monopoly price (p_{M}) and $n = 1$. Point B identifies the perfectly competitive outcome where price (p_{PC}) equals marginal cost and $n =$ many. One of our goals is to determine what happens between these two polar extremes when there are only a few competitors. As you might expect, p_{M} and p_{PC} represent the upper and lower limits on actual prices in most real oligopoly markets. From the previous chapter, we know that a perfect cartel will lead to the monopoly outcome, but what happens if firms behave noncooperatively?

The first formal models of oligopoly were developed by Cournot (1838) and Bertrand (1883). Not only are these models of historical significance, but they also provide the theoretical foundation for more realistic models that will be discussed in applied chapters later in the book. Furthermore, Cournot and Bertrand anticipated the static Nash equilibrium long before game theoretic methods were formally developed. The key difference between the Cournot and Bertrand models is the choice of strategic variable. In Cournot the choice variable is output, and in Bertrand it is price.

These static Cournot and Bertrand models have been extended in two ways. First, in the Cournot–Bertrand model, some firms compete in output (a la Cournot), while others compete in price (a la Bertrand). The second extension allows the

V.J. Tremblay and C.H. Tremblay, *New Perspectives on Industrial Organization*, Springer Texts in Business and Economics, DOI 10.1007/978-1-4614-3241-8_10,

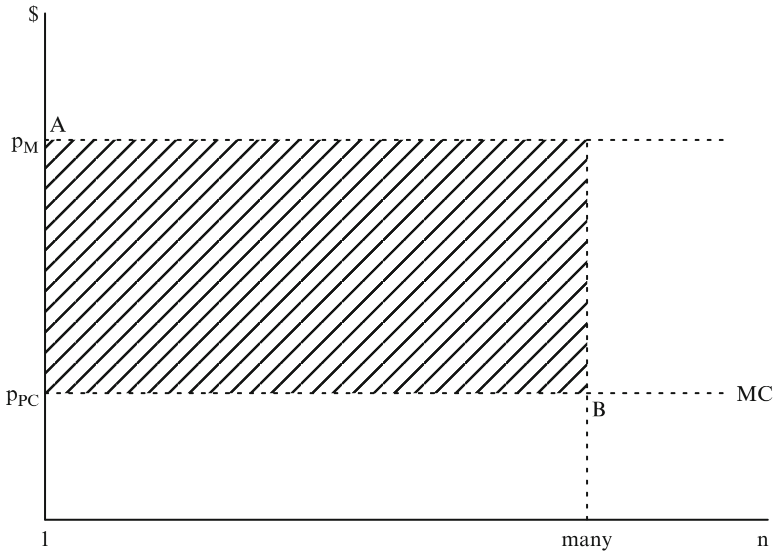

Fig. 10.1 Price competition and the number of firms (n)

choice of strategic variable (output or price) to be endogenously chosen by firms. That is, each firm can choose whether it wants to compete in output or in price. We will see that in contrast to the monopoly case, the choice of strategic variable has a dramatic effect on the equilibrium outcome in an oligopoly setting.

We also consider dynamic versions of these models in Chap. 11. The first is a dynamic version of the Cournot model, where one firm chooses output in the first stage or period and one or more firms choose output in the second stage of the game. This model was first considered by Stackelberg (1934). Other extensions include the dynamic Bertrand model, the dynamic Cournot–Bertrand model, and a model that allows the timing of play to be endogenous. We will see that small changes in the structure of the game concerning the timing of actions and the information possessed by firms, as well as the choice of strategic variable, can profoundly affect market outcomes.

Table 10.1 lists 12 oligopoly models and their key characteristics, labeled M1 through M12. The classic models are Cournot, Bertrand, Dynamic Cournot (or Stackelberg), and Dynamic Bertrand, labeled M1, M2, M5, and M6, respectively. In this chapter, we focus on the static models, M1–M4. In the next chapter, we consider the dynamic models (M5–M8) and cases where the timing of play is endogenous (models M9–M12). We consider the empirical evidence regarding price competition in oligopoly markets in Chap. 12.

Here, discussion begins with a simple market of just two firms that produce homogeneous goods. This minimizes mathematical complexity but still allows us to analyze many of the essential features of firm strategy in the Cournot and Bertrand models. Next, we extend the models to allow for asymmetric costs, more than two firms, and product differentiation. Then, we develop the relatively new

Table 10.1 Twelve Duopoly models: output and price competition in static and dynamic settings

	Timing of actions[b]		
	Static	Dynamic	Endogenous (Early or Late)
Strategic variable[a]			
Output	M1 (Cournot)	M5 (Dynamic-Cournot)[c]	M9
Price	M2 (Bertrand)	M6 (Dynamic-Bertrand)	M10
Output–price	M3 (Cournot–Bertrand)	M7 (Dynamic Cournot–Bertrand)	M11
Endogenous (Output or Price)	M4	M8	M12

[a]Output means that both firms compete in output; Price means that both firms compete in price; Output–price means that one firm competes in output and the other firm competes in price; Endogenous means that firms can choose whether to compete in output or price.
[b]Static means that the game is static (i.e., there is a single stage or period); Dynamic means that the game is dynamic (i.e., there are two stages); Endogenous means that firms choose whether to compete in an early or late period.
[c]The dynamic-Cournot model is also called the Stackelberg model.

Cournot–Bertrand model. Finally, we consider the case where the choice of strategic variable (output versus price) is endogenous and discuss when choice variables are considered strategic substitutes and strategic complements.

10.1 Cournot and Bertrand Models with Homogeneous Products

In this section, we derive the classic models of Cournot (1838) and Bertrand (1883) when products are homogeneous. Because they are prominent in our discussion in later chapters, we formally derive the Nash equilibrium (NE) for each model and describe each result graphically. These are models M1 and M2 in Table 10.1.

10.1.1 The Cournot Model with Two Firms and Symmetric Costs

The first formal model of duopoly was developed by Cournot (1838). He describes a market where there are two springs of water that are owned by different individuals. The owners sell water independently in a given period. Production costs are zero, and demand is negatively sloped. Each owner sets output to maximize its profit at the same moment in time, and the equilibrium price clears the market (p^*).[1] Cournot's goal was to determine the optimal values of firm output, price, and profit. Notice that because the products are homogeneous, $p_1 = p_2 = p^*$ in equilibrium.

[1] This assumes an auctioneer who quotes a market price that just clears the market, which is p^*.

For our purposes, we allow costs to be positive and assume linear demand and cost equations. As in previous chapters, inverse demand is $p = a - bQ$ and firm (owner) i's total cost is $TC_i = cq_i$. Recall that p is price and Q is industry output, where Q is the sum of the output from firm 1 (q_1) and firm 2 (q_2). All parameters are positive: a is the price intercept of demand, $-b$ is the slope of inverse demand, and c is the marginal and average cost of production. To assure firm participation, $a > c$. In terms of notation, subscript i identifies firm 1 or 2, and subscript j represents the other firm.

One goal of this chapter is to learn how to describe this economic problem as a game. Recall that to be a game, we must define the players, their choice variables, their payoffs, the timing of play, and the information set. In this chapter, we only consider static games where players have complete information. That is, decisions are made simultaneously and all of the characteristics of the game are common knowledge. In this case, the relevant characteristics are:

1. Players: Firms (owners) 1 and 2.
2. Strategic variable: Firm i chooses nonnegative values of q_i.
3. Payoffs: Firm i's payoffs are profits; $\pi_i(q_i, q_j) = TR_i - TC_i$, where TR_i is firm i's total revenue $(p \cdot q_i)$. In this model, $\pi_i = p \cdot q_i - cq_i = [a - b(q_i + q_j)] \cdot q_i - cq_i = aq_i - bq_i^2 - bq_iq_j - cq_i$.
4. Information is complete.

Note that linear demand and cost functions produce a profit equation that is quadratic, just as in the monopoly model in Chap. 6.[2] The NE solution to this game turns out to be the same as the Cournot solution and has been called the Cournot equilibrium, the Cournot–Nash equilibrium, or the Nash equilibrium in output to a duopoly game. Here, we call it the Cournot equilibrium.

Recall from Chap. 3 that we derive the NE in two steps. The first step is to find each firm's best-reply function, which identifies firm i's profit maximizing output (q_i^{BR}) for all values of q_j. This is simply firm i's first-order condition of profit maximization, where we take the first derivative of the firm's profit and set it to 0. Second, we must derive the output levels that constitute a mutual best reply, where the best-reply functions for both firms simultaneously hold. This identifies NE output levels. In other words, firm i maximizes its profit with respect to q_i, assuming that firm j chooses its NE output level. The first-order conditions for each firm are[3]

$$\begin{aligned}\frac{\partial \pi_1}{\partial q_1} &= \frac{\partial TR_1}{\partial q_1} - \frac{\partial TC_1}{\partial q_1} \\ &= MR_1 - MC_1 \\ &= (a - 2bq_1 - bq_2) - (c) = 0, \end{aligned} \tag{10.1}$$

[2] In fact, firm i's profit equation would be identical to that of a monopolist if $q_j = 0$.

[3] This produces a maximum because the profit equation for each firm is concave. That is, the second-order condition of profit maximization holds, because the second derivative of the profit equation for each firm is $-2b < 0$. For further discussion of second-order conditions, see the Mathematics and Econometrics Appendix at the end of the book.

$$\begin{aligned}\frac{\partial \pi_2}{\partial q_2} &= \frac{\partial \mathrm{TR}_2}{\partial q_2} - \frac{\partial \mathrm{TC}_2}{\partial q_2} \\ &= \mathrm{MR}_2 - \mathrm{MC}_2 \\ &= (a - 2bq_2 - bq_1) - (c) = 0, \end{aligned} \tag{10.2}$$

where MR_i is firm i's marginal revenue ($\partial \mathrm{TR}_i/\partial q_i$) and MC_i is firm i's marginal cost ($\partial \mathrm{TC}_i/\partial q_i$).[4] Again, these first-order conditions identify each firm's best (profit maximizing) reply to its rival's output level. We will illustrate this graphically momentarily.

At the equilibrium, a NE or a mutual best reply means that (10.1) and (10.2) must both be true. To find the Cournot equilibrium output levels, we solve (10.1) and (10.2) simultaneously for output:

$$q_1^* = q_2^* = \frac{a - c}{3b}. \tag{10.3}$$

Substituting these values into the demand function and firm profit equations gives us NE price and profits:

$$p^* = \frac{(a + 2c)}{3}, \tag{10.4}$$

$$\pi_1^* = \pi_2^* = \frac{(a - c)^2}{9b}. \tag{10.5}$$

Equations (10.3)–(10.5) indicate that the Cournot model gives reasonable comparative static predictions, that is, predictions concerning how the equilibrium will change with demand and cost conditions.[5] Just like the monopoly model, output and profit levels go up with a decrease in marginal cost and an increase in demand (i.e., as a increases and b decreases). Price rises with an increase in marginal cost and demand.

An interesting feature of the model is that it produces a symmetric equilibrium, one where output, price, and profits are the same for both firms. This is evident when we inspect the first-order conditions of both firms. Notice that the conditions are interchangeable when we replace subscript 1 with 2 and subscript 2 with 1. This **interchangeability condition** leads to a symmetric outcome where the NE strategies of each firm can be described by a single equation.[6] Symmetry will typically occur when firms have the same cost functions, produce homogeneous goods, and pursue the same goals. But models may be symmetric under other conditions as well, which we will see later in the chapter.

[4] We derive firm i's marginal revenue as follows. Firm i's total revenue function is $\mathrm{TR}_i = aq_i – bq_i^2 – bq_iq_j$. We obtain the partial derivative of TR_i by taking its derivative and holding rival output (q_j) fixed. Thus, $\partial \pi_i/\partial q_i = a – 2bq_i – bq_j$.

[5] For a discussion of comparative static analysis, see the Mathematics and Econometrics Appendix.

[6] This symmetry condition is sometimes called a level playing field assumption or an exchangeability assumption (Athey and Schmutzler 2001).

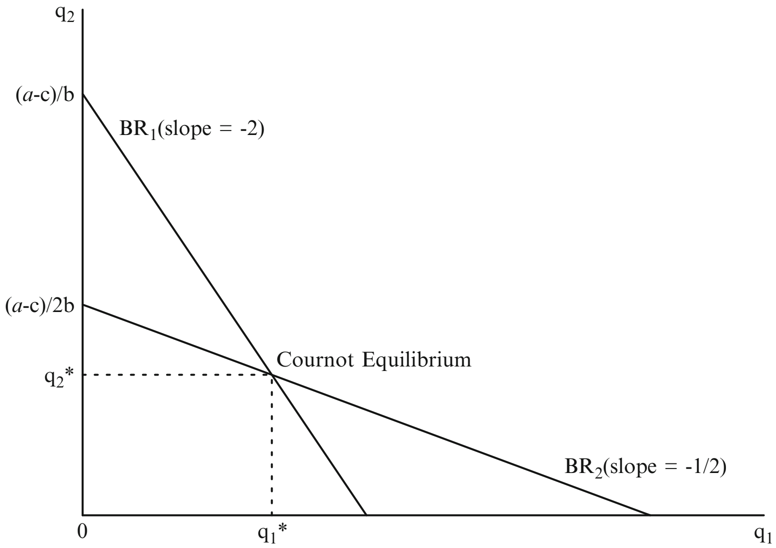

Fig. 10.2 Best-reply functions and the Cournot equilibrium

Now that we have derived the NE for the Cournot model, we want to describe it graphically. One way to do this is to graph the best-reply functions, which are obtained by solving each firm's first-order condition for q_2.[7] From (10.1) and (10.2), the best-reply functions for firm 1 (BR_1) and firm 2 (BR_2) are

$$BR_1 : q_2 = \frac{a-c}{b} - 2q_1, \qquad (10.6)$$

$$BR_2 : q_2 = \frac{a-c}{2b} - \frac{1}{2}q_1. \qquad (10.7)$$

Notice that these functions are linear and are expressed in slope-intercept form. Both have a negative slope, BR_1 is steeper than BR_2, and BR_1 has a higher intercept than BR_2.[8]

The best-reply functions are graphed in Fig. 10.2, with q_2 on the vertical axis and q_1 on the horizontal axis.[9] The best-reply functions hold simultaneously where they intersect, which identifies the Cournot equilibrium. At this point each firm is maximizing profit, given its belief that its rival is doing the same, a belief that is consistent with actual behavior at the equilibrium. That is, this point represents

[7] We solve for q_2 because q_2 will be on the vertical axis and q_1 will be on the horizontal axis in our figures.

[8] That is, the q_2 intercept is $(a–c)/b$ for BR_1 and $(a–c)/(2b)$ for BR_2. The slope is –2 for BR_1 and –½ for BR_2.

[9] As we demonstrate in Appendix 10.A, the equilibrium is stable because BR_1 is steeper than BR_2.

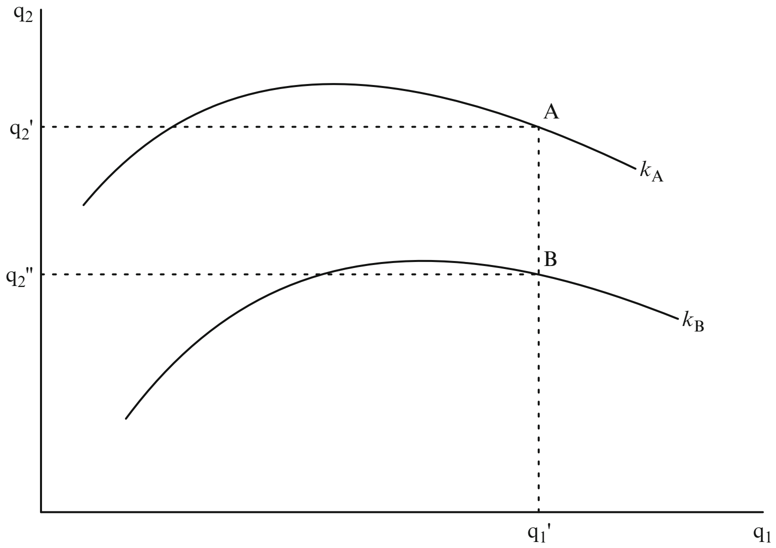

Fig. 10.3 Firm 1's isoprofits

a mutual best reply, and neither firm has an incentive to deviate from it.[10] The diagram can also be used to visualize the comparative static results that output rises with parameter a and falls with parameters b and c.

Another way of depicting the Cournot equilibrium is with isoprofit curves. Recall from the previous chapter that a firm's isoprofit equation maps out all combinations of q_1 and q_2 for a constant level of profit, k. Based on the linear demand and cost functions above, the isoprofit equation for firm i includes all q_1–q_2 pairs of points that satisfy: $\pi_i = k = aq_i - bq_i^2 - bq_iq_j - cq_i$. We obtain firm 1's isoprofit equation by solving for q_2: $q_2 = (aq_1 - cq_1 - bq_1^2 - k)/(bq_1)$, which is a quadratic function. Two isoprofit curves for firm 1 are graphed in Fig. 10.3 for different values of k. Notice that they are concave to the q_1 axis and that firm 1's profits rise as we move to a lower isoprofit curve (i.e., $k_B > k_A$). The reason for this is that for a given value of q_1 (e.g., q_1'), firm 1's profits increase as q_2 falls (from q_2' to q_2''). Parallel results hold for firm 2; the only difference is that its isoprofit curve is concave to the q_2 axis.

Isoprofit curves can be used to identify the cartel outcome and to derive a firm's best-reply function. Consider firm 1's problem when $q_2 = q_2'$, as described in

[10] Recall from Chap. 4 that two players have reached a NE when firm i's best reply to s_j^* is s_i^*, for all $i = 1$ or 2 and $j \neq i$. In other words, firm i chooses s_i^* based on the belief that firm j chooses s_j^*. The NE is reached when this belief is correct for both firms. In the Cournot model, this means that (1) when firm 2 chooses q_2^*, firm 1's best reply is q_1^* and (2) when firm 1 chooses q_1^*, firm 2's best reply is q_2^*. Thus, the q_1^*–q_2^* pair is a mutual best reply and neither firm has an incentive to change its level of output.

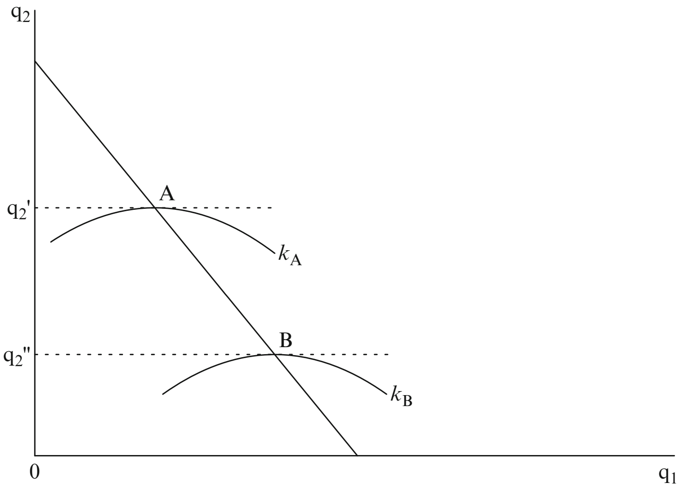

Fig. 10.4 Derivation of firm 1's best-reply function

Fig. 10.4. To obtain firm 1's best reply, firm 1 will choose the level of output that maximizes its profits, given the constraint that $q_2 = q_2'$. This occurs on the lowest possible isoprofit curve, at tangency point A. Notice that this is simply a constrained optimization problem. Similarly, when $q_2 = q_2''$, the tangency point is at B. The locus of these tangency points for all values of q_2 generates firm 1's best-reply function, depicted as the solid line in the figure. The same approach can be used to derive firm 2's best-reply function.

Figure 10.5 describes the Cournot equilibrium with respect to best reply and isoprofit curves. At the equilibrium, it is clear that each firm is maximizing its profit given that its rival is producing at the equilibrium level of output. That is, firm 1's isoprofit curve, π_1^*, is tangent to firm 2's optimal output (dashed) line at q_2^*; similarly, firm 2's isoprofit curve, π_2^*, is tangent to firm 1's output (dashed) line at q_1^*. Thus, this is a NE because it is a mutual best reply and neither firm has an incentive to deviate. However, both firms can earn higher profits if they cut production, which would move them into the shaded, lens-shaped region in Fig. 10.5. As we saw in Chap. 9, the cartel outcome occurs in this region where the isoprofit functions are tangent.

10.1.2 The Cournot Model with Two Firms and Asymmetric Costs

We next consider the Cournot model when there is a dominant firm. A dominant firm has a larger market share than its competitors, which can arise when the firm produces a superior product or produces at lower cost than its competitors.[11]

[11] In addition, this firm typically takes a leadership role in choosing output or price, an issue we take up in the next chapter.

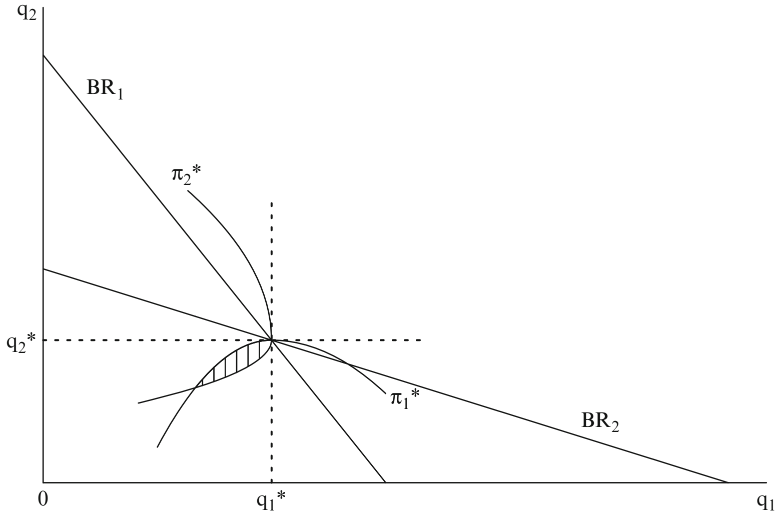

Fig. 10.5 The Cournot equilibrium with best-reply functions and isoprofits

In this section, we consider the case where firm 1 has a cost advantage over firm 2. The only difference from the previous model is that firm i's total cost becomes $\text{TC}_i = c_i q_i$, where $c_1 < c_2$. Thus, the firm's profits become $\pi_i = aq_i - bq_i^2 - bq_i q_j - c_i q_i$.

We obtain the NE using the same method as before. We solve the first-order conditions simultaneously for output and plug these optimal values into the demand and profit equations to obtain the Cournot equilibrium. In this case, the first-order conditions are

$$\begin{aligned}\frac{\partial \pi_1}{\partial q_1} &= \text{MR}_1 - \text{MC}_1 \\ &= (a - 2bq_1 - bq_2) - (c_1) = 0, \end{aligned} \tag{10.8}$$

$$\begin{aligned}\frac{\partial \pi_2}{\partial q_2} &= \text{MR}_2 - \text{MC}_2 \\ &= (a - 2bq_2 - bq_1) - (c_2) = 0. \end{aligned} \tag{10.9}$$

Marginal revenue is unchanged, but firm i's marginal cost is now c_i. Cournot values are

$$q_1^* = \frac{a - 2c_1 + c_2}{3b}, \tag{10.10}$$

$$q_2^* = \frac{a + c_1 - 2c_2}{3b}, \tag{10.11}$$

$$p^* = \frac{(a + c_1 + c_2)}{3}, \tag{10.12}$$

$$\pi_1^* = \frac{(a - 2c_1 + c_2)^2}{9b}, \tag{10.13}$$

$$\pi_2^* = \frac{(a + c_1 - 2c_2)^2}{9b}. \tag{10.14}$$

Although firms face different costs, the model is symmetric because the interchangeability condition holds. In other words, we can write firm i's first-order condition as

$$\frac{\partial \pi_i}{\partial q_i} = a - 2bq_i - bq_j - c_i = 0. \tag{10.15}$$

As a result, the NE can be written more compactly as

$$q_i^* = \frac{a - 2c_i + c_j}{3b}, \tag{10.16}$$

$$p^* = \frac{(a + c_i + c_j)}{3}, \tag{10.17}$$

$$\pi_i^* = \frac{(a - 2c_i + c_j)^2}{9b}. \tag{10.18}$$

Note that as c_i approaches c_j (value c), the solution approaches the Cournot equilibrium with symmetric costs found in (10.3)–(10.5). The key insight from studying the asymmetric cost case is that firm i's output and profit levels rise as rival costs increase (see 10.16 and 10.18). Thus, by having lower costs, firm 1 is the superior firm in that ${q_1}^* > {q_2}^*$ and ${\pi_1}^* > {\pi_2}^*$.

The effect of this cost asymmetry on Cournot output levels can be seen in a graph of best-reply functions. Again, the best-reply functions are derived by solving each firm's first-order conditions for q_2:

$$\text{BR}_1 : q_2 = \frac{a - c_1}{b} - 2q_1, \tag{10.19}$$

$$\text{BR}_2 : q_2 = \frac{a - c_2}{2b} - \frac{1}{2}q_1. \tag{10.20}$$

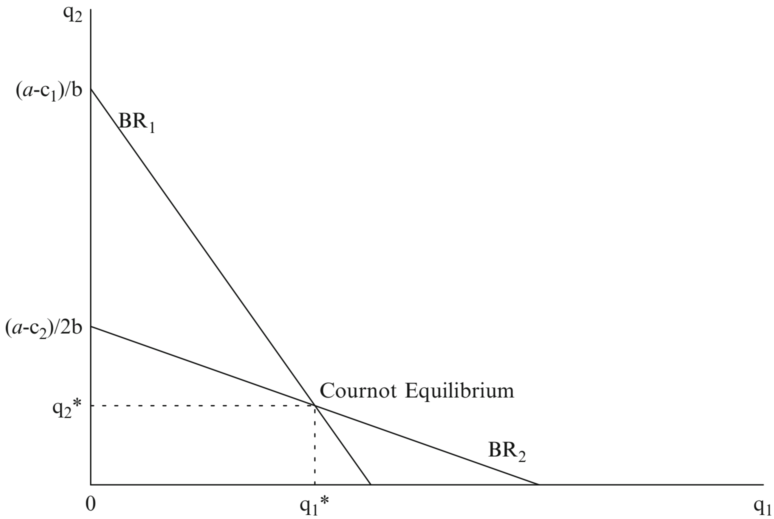

Fig. 10.6 The Cournot equilibrium when firm 1 has lower costs than firm 2

Compared to the symmetric case, the slopes are unchanged but the distance between the q_2 intercepts for the two firms widens. The best replies are plotted in Fig. 10.6 and show that a cost advantage for firm 1 increases the equilibrium value of q_1 and decreases the equilibrium value of q_2.

This model also reveals that if firm 2 has an extreme cost disadvantage compared to firm 1, firm 1 will have a monopoly position. For sufficiently high c_2, the best-reply functions intersect at a negative value of q_2 (see Fig. 10.7). Firm 2 will shut down ($q_2^* = 0$), leaving firm 1 as the sole producer. From (10.19), when $q_2^* = 0$ firm 1's best reply is $q_1^* = (a - c_1)/(2b)$, the monopoly level of output. In this case, the Cournot equilibrium is the same as the monopoly solution that we derived in Chap. 6, with $q_2^* = 0$, $q_1^* = (a - c_1)/(2b)$, $p^* = (a + c_1)/2$, and $\pi_1^* = (a - c_1)^2/(4b)$. This demonstrates that the monopoly outcome is a NE.

10.1.3 The Cournot Model with n Firms and Symmetric Costs

Next we consider the Cournot model with symmetric costs and n firms. Our goal is to see how NE values change as n starts at 1 (monopoly) and approaches infinity (perfect competition). The model is general in that it describes the NE for any market structure from monopoly through perfect competition.

We continue to assume that demand and costs functions are linear. The only difference is that with n firms, $Q = q_1 + q_2 + q_3 + \cdots + q_n$. With these assumptions, the model is symmetric and firm i's profit equation can be written

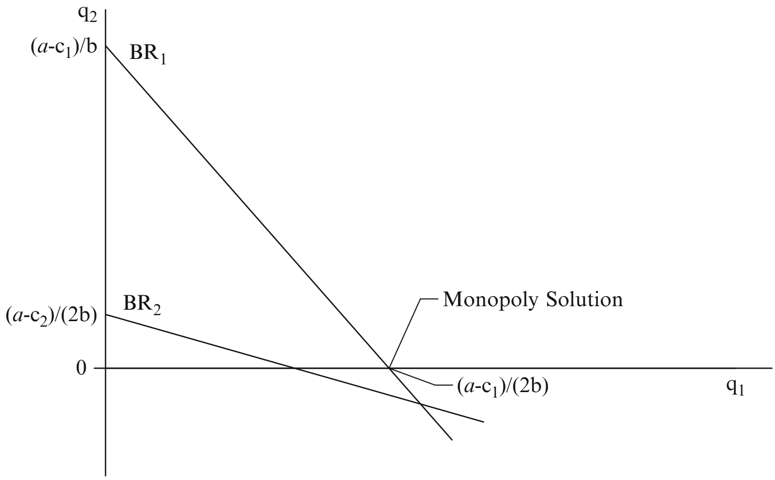

Fig. 10.7 The Cournot equilibrium when firm 2 shuts down, leaving firm 1 in a monopoly position

as $\pi_i = p \cdot q_i - cq_i = [a - b(q_1 + q_2 + q_3 + \cdots + q_n)]q_i - cq_i$. For notational convenience, we can rewrite this as

$$\pi_i = [a - b(q_i + Q_{-i})]q_i - c_i q_i, \tag{10.21}$$

where Q_{-i} is the sum of rival output (i.e., $Q_{-i} = Q - q_i$ or $Q = q_i + Q_{-i}$). The first-order condition for firm i is

$$\begin{aligned}\frac{\partial \pi_i}{\partial q_i} &= \mathrm{MR}_i - \mathrm{MC}_i \\ &= (a - 2bq_i - bQ_{-i}) - c = 0.\end{aligned} \tag{10.22}$$

Given symmetry, output is the same for each firm and $Q_{-i} = (n - 1)q_i$ in equilibrium.[12] Using this fact and the demand and profit equations above, the Cournot equilibrium with n firms is

$$q_i^* = \frac{a - c}{b(n+1)}, \tag{10.23}$$

$$p^* = \frac{a}{n+1} + c\frac{n}{n+1}, \tag{10.24}$$

[12] This is true only in equilibrium. We can set $Q_{-i} = (n-1)q_i$ in the first-order condition because optimal output levels are embedded in it. In other words, it is true that $q_1^* = q_2^* = q_3^* = \ldots = q_n^*$, but it need not be true that $q_1 = q_2 = q_3 = \ldots = q_n$. Thus, we can make this substitution in the first-order condition but not in the profit equation, (10.21).

$$\pi_i^* = \frac{(a-c)^2}{b(n+1)^2}, \tag{10.25}$$

$$Q^* = nq_i^* = \frac{a-c}{b}\frac{n}{n+1}. \tag{10.26}$$

The Cournot model with n firms produces two substantive implications:

- When $n = 1$, the NE is the monopoly outcome, where $q_i^* = Q^* = (a - c)/(2b)$, $p^* = (a + c)/2$, and $\pi_i^* = (a - c)^2/(4b)$. Thus, the monopoly outcome is a NE when $n = 1$, as in the model in the previous section.
- As n approaches infinity, the NE approaches the perfectly competitive outcome, where $p^* = c$, $\pi_i^* = 0$, and $Q^* = (a - c)/b$.

This demonstrates a key principle in oligopoly theory, the **Cournot Limit Theorem**: the Cournot equilibrium equals the monopoly outcome when n equals 1 and approaches the competitive equilibrium as n approaches infinity.[13]

The Cournot Limit Theorem yields predictable implications regarding the effect of n on allocative efficiency. We illustrate this in Fig. 10.8, where the monopoly outcome is represented by price p_1 and quantity Q_1 and the perfectly competitive outcome by p_∞ and Q_∞. As discussed in Chap. 6, total (consumer plus producer) surplus is maximized in perfect competition and equals area $ap_\infty A_\infty$. For a monopolist, total surplus equals area $ap_\infty \text{EA}_1$, implying a deadweight or efficiency loss equal to area $A_1\text{EA}_\infty$. In the Cournot model, the price–quantity pair moves to A_2 with two firms, A_4 with four firms, A_{10} with ten firms, etc. As n approaches infinity, the price–quantity pair approaches A_∞. In other words, competition reduces price and allocative inefficiency: the efficiency loss falls and approaches zero as the number of firms increases from 1 to infinity.

---

[13] We can see this more generally from firm i's first-order condition of profit maximization. Assume that the firm's profit equals $\pi_i = p(Q)q_i – \text{TC}(q_i)$, where $p(Q)q_i$ is total revenue and TC (q_i) is total cost. The first-order condition is

$$\frac{\partial \pi_i}{\partial q_i} = p + \frac{\partial p}{\partial q_i} q_i - \text{MC}_i = 0,$$

where MC_i is firm i's marginal cost. Given symmetry, $q_i = Q/n$, where Q is industry output. Thus,

$$\frac{\partial \pi_i}{\partial q_i} = p + \frac{\partial p}{\partial q_i}\frac{Q}{n} - \text{MC}_i = 0.$$

Notice that if $n = 1$, this is the first-order condition of a monopolist [see Chap. 6, Eq. (6.7)]. Furthermore, as n approaches infinity, Q/n approaches 0 and price approaches marginal cost, the perfectly competitive outcome.

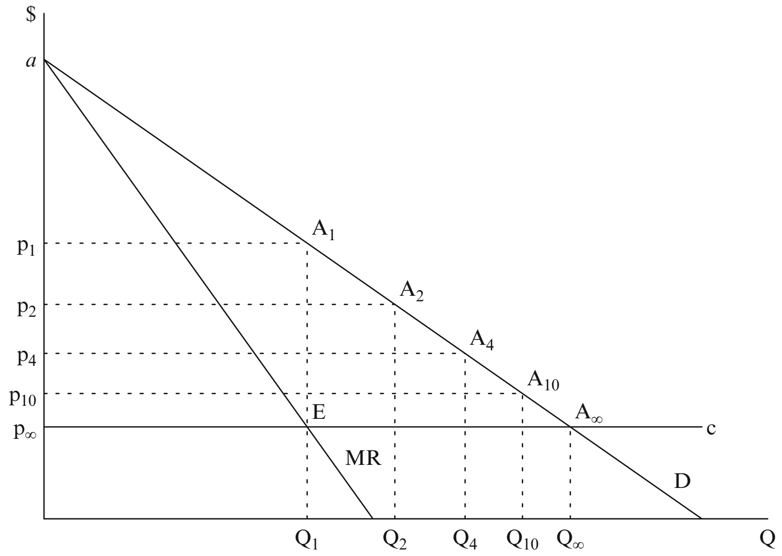

Fig. 10.8 The Cournot equilibrium and the number of competitors

If this implication were always true, there would be little left to say about the effect of market structure on allocative efficiency. Unfortunately, this is not the case, which we will see with the Bertrand model.

10.1.4 The Bertrand Model

The second duopoly model of note was developed by Bertrand (1883) when he reviewed the Cournot model. Bertrand criticized Cournot's assumption that firms compete in output, as Bertrand believed that most real firms set price, not output. In a later review of both Cournot and Bertrand's work, Fisher (1898, 126) reiterated Bertrand's concern, stating that price is a more "natural" choice variable. Recall that for a monopolist, the optimal quantity–price pair is the same whether the firm chooses output or price as the choice variable. We will see that this is not the case in an oligopoly market.

To make it easier to compare and contrast the Cournot and Bertrand models, we use the same demand and cost conditions and begin the discussion by assuming a duopoly setting with symmetric costs and homogeneous goods. Recall that the demand function in the Cournot model is expressed as an inverse demand function, $p = a - bQ$. In the Bertrand model, however, we are interested in the demand function as it has the choice variable on the right-hand side of the demand equation. Solving for output, the demand function is $Q = (a - p)/b$. For this demand function, the quantity intercept is a/b and the slope is $-1/b$.[14] In the Bertrand model, firm i's problem is to maximize $\pi_i(p_i, p_j)$ with respect to p_i instead of q_i. Once firms set prices, consumers determine quantity demanded.

It turns out that the solution to the Bertrand problem is also a NE, which is called a Bertrand equilibrium, a Bertrand–Nash equilibrium, or the NE in prices to a homogeneous goods duopoly game. We simply call it a Bertrand equilibrium. The formal characteristics of this static game are as follows:

1. Players: Firms 1 and 2.
2. Strategic Variable: Firm i chooses a nonnegative value of p_i.
3. Payoffs: Firm i's payoffs are profits: $\pi_i(p_i, p_j) = p_i \cdot q_i - cq_i$.
4. Information is complete.

If the profit equation of each firm were differentiable, we could find the Bertrand equilibrium using the same approach that we used to find the Cournot equilibrium. We would use calculus to identify the first-order conditions with respect to price for each firm and solve them simultaneously to obtain NE prices. Unfortunately, the firm's demand and, therefore, profit equations are discontinuous. Thus, we are unable to differentiate in this case.

Why is there a discontinuity? Consider firm i's demand function. Because the products are homogeneous, consumers will always purchase from the cheapest seller. If prices are the same (i.e., $p \equiv p_i = p_j$), consumers are indifferent between purchasing from firms 1 and 2. In this case, the usual assumption is that half of the consumers purchase from firm 1 and the other half from firm 2. Under these conditions and assuming that prices are less than a, quantity demanded for firm i is

$$q_i = \left\{ \begin{array}{cc} 0 & \text{if } p_i > p_j \\ \dfrac{a-p}{2b} & \text{if } p_i = p_j \\ \dfrac{a-p_i}{b} & \text{if } p_i < p_j \end{array} \right\} \tag{10.27}$$

The discontinuity is easy to see in the graph of firm i's demand, $d_i \equiv q_i$, found in Fig. 10.9 for a given $p_j < a$. Demand is 0 when $p_i > p_j$ and equals the market demand, $(a - p_i)/b$ when $p_i < p_j$. By assumption, demand is half the market demand, $(a - p_i)/(2b)$, when $p_i = p_j$. Thus, the firm's demand consists of the grey line segments and the point $(a - p_j)/(2b)$ when $p_i = p_j$.

The discontinuity in demand creates a discontinuity in profits, as seen in Fig. 10.10. Recall that for a monopolist, profits are quadratic for linear demand and cost functions. In this Bertrand duopoly case, firm i has a monopoly position and faces a profit equation that is quadratic when $p_i < p_j$, where $\pi_i = (a - p_i)(p_i - c)/b$. Firm i's profits are 0 when $p_i > p_j$, because firm j now has the monopoly position. When $p_i = p_j$, profits are split evenly between firms, and $\pi_i = (a - p)(p - c)/(2b)$.

[14] That is, $dQ/dp = -1/b$, while the slope of the inverse demand function (dp/dQ) is $-b$. In addition, the price intercept equals a.

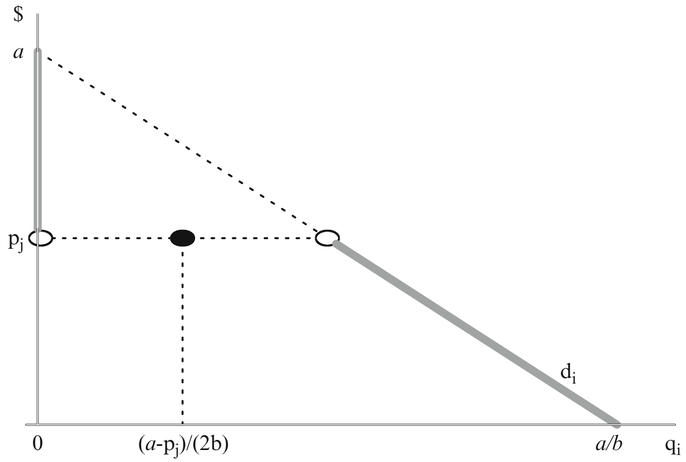

Fig. 10.9 Firm *i*'s demand function in a Bertrand game

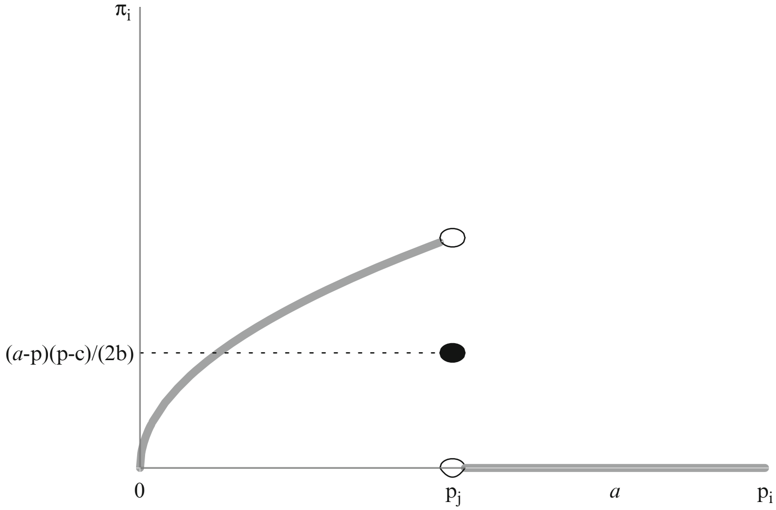

Fig. 10.10 Firm *i*'s profits in a Bertrand game

Even though we cannot differentiate the profit equations, we can use the characteristics of a NE to identify the Bertrand solution. Recall that players will have no incentive to deviate at the NE. In this game, there is a unique NE where $p_i = p_j = c$ for $c < a$. The proof is rather intuitive, and we provide it below.[15]

[15] The proof assumes that prices are infinitely divisible.

We need to show that neither player has an incentive to deviate when price equals marginal cost. For this to be a unique equilibrium, we also need to show that there are no other equilibrium outcomes.

Proof Consider all relevant strategic possibilities where p_i and p_j are positive but less than $a > c$.

- $p_i > p_j > c$: This is not a NE because firm i can increase its profits by setting its price between p_j and c.
- $p_i > p_j < c$: This is not a NE because $\pi_j < 0$ and firm j can increase its profits by shutting down.
- $p_i = p_j > c$: This is not a NE because each firm can increase its profit by cutting price below its rival's price and above c.
- $p_i = p_j < c$: This is not a NE because $\pi_i < 0$ and $\pi_j < 0$. Both firms can increase profit by shutting down.
- $p_i = p_j = c$: This is a NE because neither firm can increase profit by raising or lowering price or by shutting down.

The only outcome where neither firm has an incentive to deviate occurs where $p_i = p_j = c$, the unique Bertrand equilibrium to this game. The intuition behind this result is that each firm has an incentive to undercut the price of its rival until price equals marginal cost. This is called **price undercutting** and is normally associated with a price war. Notice that the model produces a perfectly competitive outcome: $p = c$, $\pi_i = 0$, and $Q = (a - c)/b$. Comparative static results are the same as in perfect competition. That is, the equilibrium price increases with marginal cost, and industry production increases with demand and decreases with marginal cost.

The Bertrand solution shows how different the outcome can be in an oligopoly market when we change the strategic variable from output to price. Recall that in the monopoly case the solution is the same whether the firm maximizes profit with respect to output or price, but this is not the case with oligopoly. Although the assumptions of the Bertrand model are identical to the Cournot model except that price is the choice variable instead of output, the outcome is dramatically different. This demonstrates that a firm's strategic choice, as well as its demand and cost conditions, affects the NE in an oligopoly setting.

We next consider the case when $n > 2$. It is easy to verify that the Bertrand model with symmetric costs produces the perfectly competitive result as long as $n > 1$. That is, price undercutting will lead to price competition that is so fierce that only 2 or more firms are necessary to generate a perfectly competitive outcome. This result sharply contrasts with the Cournot outcome where infinitely many competitors are required for a competitive outcome. Because this Bertrand result is so extreme and generally inconsistent with reality, it is called the **Bertrand paradox**.

The analysis so far suggests that neither the Cournot nor the Bertrand model is totally satisfactory. The Cournot model produces the more realistic outcome that price falls with the number of competitors, but the Bertrand model assumes more realistically that firms compete in price rather than output. Nevertheless, the Cournot model may be more realistic than it appears. In the next chapter, we will see that when firms compete in a dynamic game, where the quantities of output are chosen in the first period and prices are chosen in the second period, the NE is Cournot. In defense of the Bertrand model, there are various ways in which firms can avoid the Bertrand paradox.

The Bertrand paradox vanishes when one firm has a competitive cost advantage over its rivals. Returning to the duopoly case, let $c_1 < c_2$. With this cost asymmetry, undercutting produces an outcome where firm 1 charges the highest possible price that is just below c_2.[16] Thus, there will be only one seller in the market, but its price may be below its simple monopoly price. Note that this is different from the Cournot model, where both the high and low cost firms may coexist. An important implication of the Bertrand model with cost asymmetries is that it shows how the presence of a potential entrant can reduce the price charged by a monopolist. Later we will see that product differentiation can also be used to overcome the Bertrand paradox.

10.2 Cournot and Bertrand Models with Differentiated Products

We begin our discussion of differentiated oligopoly with a model that assumes multicharacteristic product differentiation. Recall from Chap. 7 that this occurs when consumers value variety and products differ on a number of characteristics. Later in the chapter we consider models with different types of product differentiation. Our goal is to understand how product differentiation affects equilibrium prices, production, profits, and allocative efficiency. In this chapter, we assume that firms have already chosen product characteristics. Thus, the degree of product differentiation is predetermined. In a later chapter, we will analyze how firms make product design decisions.

To keep things simple, we assume a duopoly market where firms face the same variable costs, although fixed or quasi-fixed costs may differ by firm. Thus, any cost difference between brands is due to a difference in set-up costs, not marginal cost.

10.2.1 The Cournot Model with Multicharacteristic Differentiation

Consider a Cournot duopoly with multiproduct differentiation. From Chap. 7 we saw that the inverse demand functions for each firm are

$$p_1 = a - q_1 - dq_2, \tag{10.28}$$

$$p_2 = a - q_2 - dq_1. \tag{10.29}$$

[16] This also assumes that c_2 is less than firm 1's simple monopoly price (p_m).

Recall that parameter d is an index of product differentiation. Products 1 and 2 are homogeneous when $d = 1$; when $d = 0$, the products are unrelated and each firm is a monopolist. Thus, with product differentiation d ranges from 0 to 1, and the degree of product differentiation increases as d gets closer to 0. Firm i's total cost equation is $\text{TC}_i = cq_i - F_i$, where F_i is the firm's fixed (or quasi-fixed) cost. Given these demand and cost conditions, firm i's profits are $\pi_i(q_i, q_j) = \text{TR}_i - \text{TC}_i = (a - q_i - dq_j)q_i - cq_i - F_i = aq_i - q_i^2 - dq_iq_j - cq_i - F_i$.

The profit equation is differentiable, so the Cournot equilibrium can be derived in the same way as in the homogeneous goods case. That is, we obtain the first-order conditions and solve them simultaneously for output. The first-order conditions, which are similar to (10.1) and (10.2), are[17]

$$\frac{\partial \pi_1}{\partial q_1} = \text{MR}_1 - \text{MC}_1 = (a - 2q_1 - dq_2) - (c) = 0, \tag{10.30}$$

$$\frac{\partial \pi_2}{\partial q_2} = \text{MR}_2 - \text{MC}_2 = (a - 2q_2 - dq_1) - (c) = 0. \tag{10.31}$$

Solving these equations simultaneously for p_1 and p_2 yields the NE output levels. Substituting them into the demand and profit equations above gives their NE values. Notice that the interchangeability condition holds, making for a symmetric Cournot equilibrium:

$$q_i^* = \frac{a - c}{2 + d}, \tag{10.32}$$

$$p_i^* = \frac{(a + c + cd)}{2 + d}, \tag{10.33}$$

$$\pi_i^* = \frac{(a - c)^2}{(2 + d)^2} - F_i. \tag{10.34}$$

We graph the best-reply functions and the Cournot equilibrium in Fig. 10.11, which we will use to compare with the equilibrium in the differentiated Bertrand model.

The main reason for studying the differentiated Cournot model is to determine how product differentiation affects the equilibrium.[18] The key results are:

- The equilibrium converges to the homogeneous Cournot equilibrium as d approaches 1 (i.e., Figs. 10.2 and 10.10 become the same).

[17] Notice that the second-order conditions of profit maximization hold, because the second derivative of the profit equation for each firm is $-2 < 0$.

[18] The effects of a change in marginal cost and a change in the demand intercept are the same as in the case with homogeneous goods.

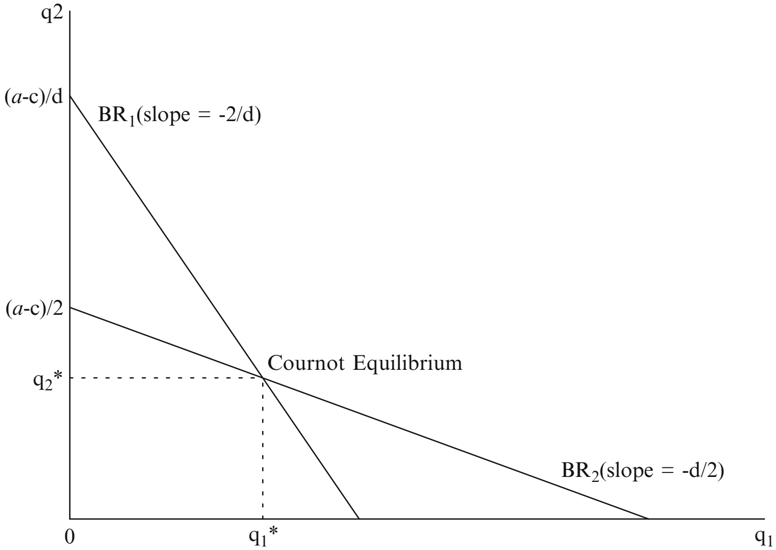

Fig. 10.11 The Cournot equilibrium with product differentiation

- The equilibrium converges to the monopoly equilibrium as d approaches 0. Recall from Chap. 6 that when there is a monopoly firm where $b = 1$, then $Q^* = q_i^* = (a - c)/2$, $p^* = (a + c)/2$ and $\pi_i = (a - c)^2/4 - F_i$.
- Greater product differentiation (i.e., lower d) leads to higher prices and profits.

The effect of product differentiation on price, which is described in (10.33), is exhibited in Fig. 10.12. It illustrates that when the two products are unrelated (i.e., $d = 0$), the equilibrium price (p_i^*) equals the monopoly price, $(a + c)/2$. At the other extreme when the two products are perfect substitutes (i.e., $d = 1$), the equilibrium price equals the homogeneous Cournot price, $(a + 2c)/3$. In between, the price falls with d. The result that greater product differentiation leads to less price competition is called the **principle of product differentiation**.

10.2.2 The Bertrand Model with Multicharacteristic Differentiation

We now want to analyze the Bertrand model with multicharacteristic differentiation. Firms face the same demand structure as in the differentiated Cournot model, (10.28) and (10.29). In the Bertrand model, the demand function is used in place of the inverse demand function. All choice variables (prices) appear on the right-hand side of each equation. Solving the system of inverse demand functions

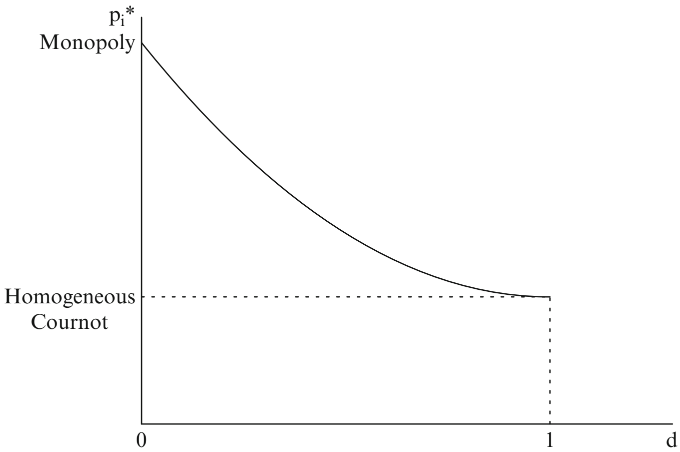

Fig. 10.12 The Cournot equilibrium price for different levels of product differentiation (d)

from the Cournot model simultaneously for q_1 and q_2 yields the following demand system:

$$q_1 = \alpha - \beta p_1 + \delta p_2, \tag{10.35}$$

$$q_2 = \alpha - \beta p_2 + \delta p_1, \tag{10.36}$$

where $\alpha \equiv a(1 - d)/x$, $\beta \equiv 1/x$, $\delta \equiv d/x$, and $x \equiv (1 - d^2)$.[19] Note that when there is product differentiation, d ranges from 0 to 1 and β exceeds δ. With this demand system, firm i's profits are $\pi_i(p_i, p_j) = \text{TR}_i - \text{TC}_i = p_i(\alpha - \beta p_i + \delta p_j) - c(\alpha - \beta p_i + \delta p_j) - F_i$.

In this case, firm i's profit equation is differentiable in p_i, and we can find the NE using the same method as in the Cournot model. First we differentiate each firm's profit equation with respect to its own price to derive the first-order conditions. The first-order conditions are[20]

$$\begin{aligned}\frac{\partial \pi_1}{\partial p_1} &= \frac{\partial \text{TR}_1}{\partial p_1} - \frac{\partial \text{TC}_1}{\partial p_1} \\ &= \text{MR}p_1 - \text{MC}p_1 \\ &= (\alpha - 2\beta p_1 + \delta p_2) - (-\beta c) = 0,\end{aligned} \tag{10.37}$$

[19] Detailed derivations can be found in Shy (1995, 162–163).

[20] Notice that the second-order conditions of profit maximization hold, because the second derivative of the profit equation for each firm is $-2\beta < 0$, as $\beta > 0$.

$$\begin{aligned}\frac{\partial \pi_2}{\partial p_2} &= \frac{\partial \mathrm{TR}_2}{\partial p_2} - \frac{\partial \mathrm{TC}_2}{\partial p_2} \\ &= \mathrm{MR}p_2 - \mathrm{MC}p_2 \\ &= (\alpha - 2\beta p_2 + \delta p_1) - (-\beta c) = 0,\end{aligned} \tag{10.38}$$

where $\mathrm{MR}p_i$ is firm i's marginal revenue with respect to a change in p_i and $\mathrm{MC}p_i$ is firm i's marginal cost with respect to a change p_i. Second, solving these equations simultaneously for p_1 and p_2 yields the NE prices. Substituting the optimal prices into the demand and profit equations above gives their equilibrium values. Given that the interchangeability condition is met, the Bertrand equilibrium is

$$p_i^* = \frac{\alpha + \beta c}{2\beta - \delta}, \tag{10.39}$$

$$q_i^* = \frac{\beta[\alpha - c(\beta - \delta)]}{2\beta - \delta}, \tag{10.40}$$

$$\pi_i^* = \frac{\beta[\alpha - c(\beta - \delta)]^2}{(2\beta - \delta)^2} - F_i. \tag{10.41}$$

Because this model produces a different outcome from previous models, we describe its best-reply and isoprofit functions. Solving each firm's first-order condition for p_2 gives the best replies

$$\mathrm{BR}_1 : p_2 = -\frac{\alpha + \beta c}{\delta} + \frac{2\beta}{\delta} p_1, \tag{10.42}$$

$$\mathrm{BR}_2 : p_2 = \frac{\alpha + \beta c}{2\beta} + \frac{\delta}{2\beta} p_1. \tag{10.43}$$

The best-reply functions are linear, but unlike the Cournot model they have a positive slope.[21] These functions and their corresponding isoprofit curves are graphed in Fig. 10.13. Bertrand equilibrium prices occur where the best-reply functions intersect. Notice that both firms are better off if they move into the shaded region by raising prices above the NE prices. Thus, the cartel outcome is in this region.

Previously we saw that one way for a firm to avoid the Bertrand paradox is to gain a cost advantage over its competitors. Another way is for firms to differentiate

[21] For BR_1, the slope is $2\beta/\delta$ and the p_2 intercept is $-(\alpha + \beta c)/\delta$. For BR_2, the slope is $\delta/2\beta$ and the p_2 intercept is $(\alpha + \beta c)/2\beta$. For the equilibrium to be stable, an issue that we discuss in the Appendix 10.A, BR_1 must be steeper than BR_2 (i.e., $\beta > \delta/2$).

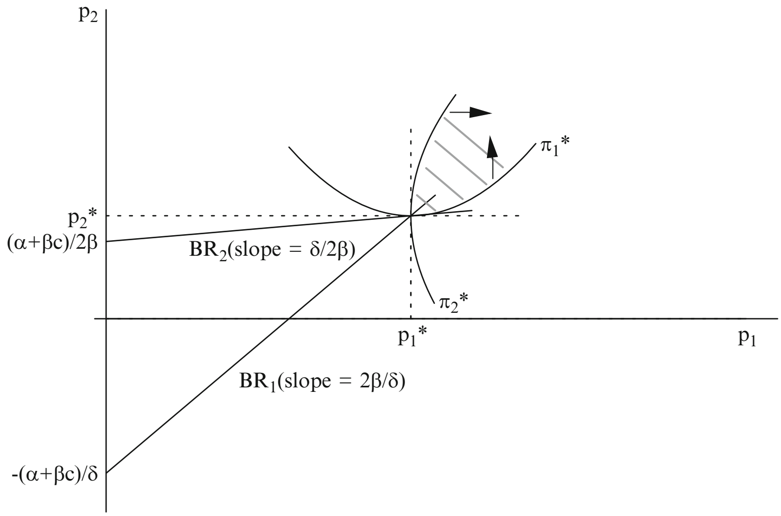

Fig. 10.13 The Bertrand equilibrium with product differentiation

their products. If we rewrite the equilibrium (10.39)–(10.41) in the original parameters a, c, and d,

$$p_i^* = \frac{a + c - ad}{2 - d}, \tag{10.44}$$

$$q_{\mathrm{i}}^* = \frac{a - c}{2 + d + d^2}, \tag{10.45}$$

$$\pi_i^* = \frac{(a - c)^2(1 - d)}{(2 - d)^2(1 + d)}. \tag{10.46}$$

Recall that the degree of product differentiation increases as d approaches 0. When products are homogeneous ($d = 1$), the model produces the simple Bertrand outcome with price equal to marginal cost (c) and profits equal to zero. It produces the monopoly outcome when the products are unrelated ($d = 0$), again verifying that the monopoly outcome is a NE. Finally, equilibrium prices and profits increase as products become more differentiated (i.e., as $d \to 0$). Thus, this analysis provides further verification of the principle of product differentiation and demonstrates that another way for firms to avoid the Bertrand paradox is to differentiate their products.

10.2.3 The Bertrand Model with Horizontal and Vertical Differentiation

In this section, we consider Bertrand models with horizontal and vertical differentiation. We saw in Chap. 7 that differentiation is horizontal when consumers disagree over their preference ordering of a product's characteristic, as is the case with a red versus a blue VW Jetta. Some consumers prefer the red and others the blue Jetta, *ceteris paribus*. When consumers agree over the preference ordering of a characteristic, we have vertical differentiation. Product quality is an example of a vertical characteristic. These models as well as previous models with product differentiation provide a theoretical framework for later analysis when firms compete in other dimensions, such as product characteristics and advertising.

10.2.3.1 Price Competition in the Linear City Model

Recall that in the Hotelling model discussed in Chap. 7 brands differ in terms of a single characteristic (θ). Consumers have different tastes, with some consumers preferring brands with high levels of θ and others preferring brands with low levels of θ. The Hotelling model is represented by a main street of unit length that starts at 0 and ends at 1. Location is indexed by parameter θ. Consumers live on main street and are uniformly distributed.

In this example, two supermarkets (1 and 2) compete for consumer business, with store 1 located at θ_1 and store 2 located at θ_2, with $0 \leq \theta_1 \leq ½ \leq \theta_2 \leq 1$. Stores 1 and 2 are homogeneous when they both locate at ½ but become increasingly differentiated as they move further and further apart. Suppose store 1 is located at position 0, and store 2 is located at position 1.[22] With positive transportation costs (t), consumers will prefer the store closest to home. As we saw in Chap. 7, these assumptions produce the following linear demand functions:

$$q_1 = \frac{N[t(\theta_2 - \theta_1) - p_1 + p_2]}{2t}, \tag{10.47}$$

$$q_2 = \frac{N[t(\theta_2 - \theta_1) + p_1 - p_2]}{2t}, \tag{10.48}$$

[22] To simplify the analysis, we also assume that the market is covered (i.e., no consumer refrains from purchase) and that consumers have unit demands (i.e., each consumer buys just one unit of brand 1 from store 1 or one unit of brand 2 from store 2). To review these concepts, see Chap. 7.

where N is the number of consumers. Note that the model shows that demand increases as stores move further apart. For now, we assume that store location is fixed or predetermined. Firm i's total cost is $\text{TC}_i = \text{cq}_i - \text{F}_i$, and its profit equation is $\pi_i(p_i, \quad p_j) = \text{TR}_i - \text{TC}_i = p_i\{N[t(\theta_2 - \theta_1) - p_i + p_j]\}/(2t) - c\{N[t(\theta_2 - \theta_1) - p_i + p_j]\}/(2t) - F_i$.

As in the previous model, store i's profit equation is differentiable in p_i, enabling us to derive the NE by differentiation. The first-order conditions are[23]

$$\begin{aligned}\frac{\partial \pi_1}{\partial p_1} &= \text{MR}p_1 - \text{MC}p_1 \\ &= \frac{N[t(\theta_2 - \theta_1) - 2p_1 + p_2]}{2t} - \frac{(-cN)}{2t} \\ &= \frac{N[t(\theta_2 - \theta_1) - 2p_1 + p_2 + c]}{2t} = 0,\end{aligned} \tag{10.49}$$

$$\begin{aligned}\frac{\partial \pi_2}{\partial p_2} &= \text{MR}p_2 - \text{MC}p_2 \\ &= \frac{N[t(\theta_2 - \theta_1) - 2p_2 + p_1]}{2t} - \frac{(-cN)}{2t} \\ &= \frac{N[t(\theta_2 - \theta_1) - 2p_2 + p_1 + c]}{2t} = 0.\end{aligned} \tag{10.50}$$

Notice that the interchangeability condition holds. Solving the first-order conditions for prices and substituting them into the demand and profit equations yields the Bertrand equilibrium:

$$p_i^* = c + t(\theta_2 - \theta_1), \tag{10.51}$$

$$q_i^* = N(\theta_2 - \theta_1)/2, \tag{10.52}$$

$$\pi_i^* = \frac{Nt(\theta_2 - \theta_1)^2}{2} - F_i. \tag{10.53}$$

Consistent with the principle of product differentiation, price competition falls and profits rise as stores 1 and 2 move further apart [i.e., as the distance $(\theta_2 - \theta_1)$ increases]. As in the previous model of multicharacteristic differentiation, the linear city model generates positively sloped best-reply functions. Deriving and graphing the best-reply functions is left as an exercise at the end of the chapter.

---

[23] Notice that the second-order conditions of profit maximization hold, because the second derivative of the profit equation for each firm is $-N/t < 0$.

10.2.3.2 Price Competition in the Circular City Model

Next we consider the circular city model of horizontal differentiation where main street is bent around to form a circle (see Chap. 7). The advantage of this model is that it allows us to investigate the NE in a differentiated market with n firms. The model is symmetric, and firm i's demand function is

$$q_i = \frac{t/n - p_i + p}{t}, \tag{10.54}$$

where p represents the price charged by rivals.[24] Firm i's profits are $\pi_i(p_i, p_j) = \text{TR}_i - \text{TC}_i = p_i[(t/n - p_i + p)/t] - c[(t/n - p_i + p)/t] - F_i$.

The profit equation is differentiable in p_i, enabling us to derive the NE by differentiation. The interchangeability condition holds, and the first-order condition for firm i is

$$\begin{aligned} \frac{\partial \pi_i}{\partial p_i} &= \text{MR}p_i - \text{MC}p_i \\ &= \frac{t/n - 2p_i + p}{t} - \frac{-c}{t} \\ &= \frac{t/n - 2p_i + p + c}{t} = 0 \end{aligned} \tag{10.55}$$

The firm's best-reply function is $p_i^{\text{BR}} = (t/n + c + p)/2$, which has a positive slope like the other Bertrand models with product differentiation. Because the problem is symmetric, $p_i = p$ in equilibrium. Thus, the Bertrand equilibrium is

$$p_i^* = c + t/n. \tag{10.56}$$

$$q_i^* = 1/n. \tag{10.57}$$

$$\pi_i^* = t/n^2 - F_i. \tag{10.58}$$

In this model, price approaches marginal cost and profits decline as the number of competitors increases. This is the same result as in the Cournot limit theorem. Even though the price equals marginal cost in the homogeneous goods Bertrand model when there are 2 or more firms, the differentiated Bertrand model has similar implications as Cournot regarding the effect of market structure on price competition.

[24] In this model, the number of consumers (N) is normalized to 1 for simplicity.

10.2.3.3 The Bertrand Model with Vertical Differentiation

Next, we consider a Bertrand model developed by Choi and Shin (1992), where differentiation is vertical. Firm 1 produces the brand of higher quality or reliability. Recall from Chap. 7 that z_i indexes the quality of brand i, where $z_1 > z_2 > 0$, and the degree of vertical differentiation is $z \equiv z_1 - z_2$. Product quality is assumed to be predetermined. Consumers all prefer the high quality brand but some have a stronger preference for quality than others. A consumer's preference for quality is represented by ϕ, and the diversity of consumer tastes ranges from ϕ_L to ϕ_H, with $\phi_H > \phi_L > 0$ and $\phi_H - \phi_L = 1$.[25]

This model assumes the Mussa and Rosen specification of vertical differentiation, which produces the following linear demand functions:

$$q_1 = \frac{N(z\phi_H - p_1 + p_2)}{z}, \tag{10.59}$$

$$q_2 = \frac{N(-z\phi_L + p_1 - p_2)}{z}. \tag{10.60}$$

Demand for firm i's brand goes up with an increase in the number of consumers, a drop in the firm's own price, and an increase in its rival's price. Costs are assumed to be the same as before. Because $\phi_H \neq -\phi_L$, the problem is not symmetric, and the profit equations for each firm are: $\pi_1(p_1, p_2) = TR_1 - TC_1 = p_1[(z\phi_H - p_1 + p_2)/z] - c[(z\phi_H - p_1 + p_2)/z] - F_1$; $\pi_2(p_1, p_2) = TR_2 - TC_2 = p_2[(-z\phi_L + p_1 - p_2)/z] - c[(-z\phi_L + p_1 - p_2)/z] - F_2$. Profit equations are differentiable, and the first-order conditions are[26]

$$\begin{aligned}\frac{\partial \pi_1}{\partial p_1} &= \mathrm{MR}p_1 - \mathrm{MC}p_1\\ &= \frac{N(z\phi_H - 2p_1 + p_2)}{z} - \frac{(-c)}{z}\\ &= \frac{N(z\phi_H - 2p_1 + p_2 + c)}{z} = 0,\end{aligned} \tag{10.61}$$

$$\begin{aligned}\frac{\partial \pi_2}{\partial p_2} &= \mathrm{MR}p_2 - \mathrm{MC}p_2\\ &= \frac{N(-z\phi_L - 2p_2 + p_1)}{z} - \frac{(-c)}{z}\\ &= \frac{N(-z\phi_L - 2p_2 + p_1 + c)}{z} = 0.\end{aligned} \tag{10.62}$$

[25] Later we will see that another constraint will be important, that is $\phi_H > 2\phi_L > 0$.

[26] The second-order conditions of profit maximization hold, because the second derivative of the profit equation for each firm is $-2/z < 0$.

Solving the first-order conditions for prices and substituting them into the demand and profit equations yields the Bertrand equilibrium when differentiation is vertical:

$$p_1^* = c + \frac{z(2\phi_H - \phi_L)}{3} > p_2^* = c + \frac{z(\phi_H - 2\phi_L)}{3}, \tag{10.63}$$

$$q_1^* = \frac{N(2\phi_H - \phi_L)}{3} > q_2^* = \frac{N(\phi_H - 2\phi_L)}{3}, \tag{10.64}$$

$$\pi_1^* = \frac{Nz(2\phi_H - \phi_L)^2}{9} - F_1; \pi_2^* = \frac{Nz(\phi_H - 2\phi_L)^2}{9} - F_2. \tag{10.65}$$

For firm 2 to produce a positive level of output, $\phi_H > 2\phi_L$. Thus, this condition must hold for there to be two firms in the market.

This model of vertical differentiation produces several interesting results. First, the high quality firm sells more output and at a higher price. The high quality firm will also earn greater profit (i.e., have a competitive or a strategic advantage) as long as the difference in fixed costs is not too great. Finally, the principle of differentiation is verified: prices and profits increase as the degree of production differentiation rises (i.e., as z increases).

We also derive and graph the best-reply functions for this model. Recall that we can obtain the best-reply functions by solving each firm's first-order condition with respect to p_2[27]:

$$\text{BR}_1 : p_2 = -(c + z\phi_H) + 2p_1, \tag{10.66}$$

$$\text{BR}_2 : p_2 = \frac{c - z\phi_L}{2} + \frac{1}{2}p_1. \tag{10.67}$$

The best-reply functions are linear and are illustrated in Fig. 10.14. The figure verifies that the high quality producer will charge a higher price than the low quality producer.

10.3 The Cournot–Bertrand Model

One concern with the Cournot and Bertrand models is that they take the strategic variable as given. That is, both firms either compete in output (Cournot) or in price (Bertrand). But why is it not possible for one firm to compete in output and the other in price? After all, Kreps and Scheinkman (1983) argue that it is "witless" to criticize the choice of strategic variable, as it is an empirical question whether or not firms compete in output or in price.

[27] For BR_1, the slope is 2 and the p_2 intercept is $-(c + z\phi_H)$. For BR_2, the slope is 1/2 and the p_2 intercept is $(c - z\phi_L)/2$.

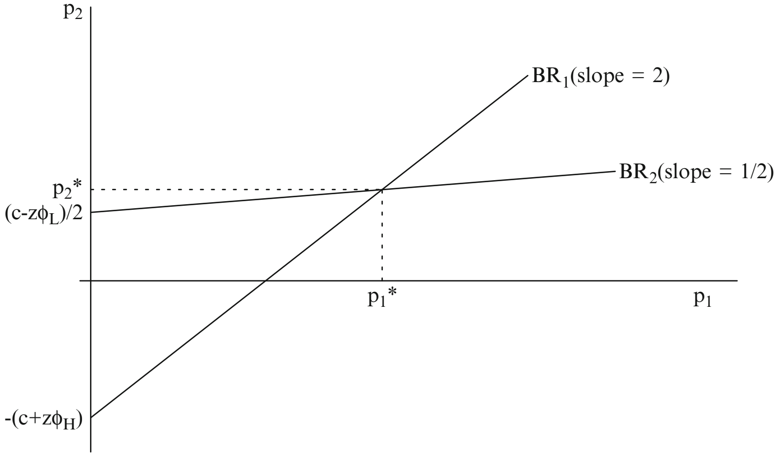

Fig. 10.14 The Bertrand equilibrium with vertical differentiation

Each case is witnessed in the real world. At a farmer's market, Cournot competition is common. Farmers compete in output, choosing how much to bring to market and then allowing price to adjust once there. In contrast, fast food restaurants typically compete in price, as in Bertrand. In addition, a mixture of Cournot and Bertrand behavior is observed in the market for small cars. In each period, Honda and Subaru dealers set quantities and let price adjust to clear the market. On the other hand, Saturn and Scion dealers fill consumer orders at a fixed price.[28] This type of behavior can be described by a Cournot–Bertrand model where one firm competes in output and the other firm competes in price. This corresponds to model M3 in Table 10.1, which was developed by Singh and Vives (1984), C. Tremblay and V. Tremblay (2011a), and V. Tremblay et al. (forthcoming-a).

The assumptions of the Cournot–Bertrand are the same as those of the Cournot and Bertrand models, except that firm 1 competes in output and firm 2 competes in price. This requires that the demand system have the two choice variables (q_1 and p_2) on the right-hand side of each demand equation. We use the system of inverse demand functions for the Cournot model found in (10.28) and (10.29), which assumes multicharacteristic differentiation. Solving that system simultaneously for p_1 and q_2 yields the following demand equations:

$$p_1 = (a - ad) - (1 - d^2)q_1 + dp_2, \tag{10.68}$$

[28] Historically, the market for personal computers provides another example of Cournot–Bertrand type behavior. That is, Dell set price and built computers to order, while IBM shipped completed computers to dealers who let price adjust to clear the market. Cournot–Bertrand behavior can also be found in the aerospace connector industry where leading distributors compete in price and smaller distributors compete in output.

$$q_2 = a - p_2 - dq_1. \tag{10.69}$$

Recall that each firm is a monopolist when $d = 0$ and that products are perfectly homogeneous when $d = 1$. In order to simplify the calculations, we set marginal cost equal to zero.[29]

The first thing to note is that the model is naturally asymmetric because firms have different choice variables. This is clear from the firms' profit maximization problems:

$$\max_{q_1} \ \pi_1 = \mathrm{TR}_1 - \mathrm{TC}_1 = [(a - ad) - (1 - d^2)q_1 + dp_2]q_1 - F_1, \tag{10.70}$$

$$\max_{p_2} \ \pi_2 = \mathrm{TR}_2 - \mathrm{TC}_2 = p_2(a - p_2 - dq_1) - F_2. \tag{10.71}$$

In this model, firm 1 maximizes profit with respect to output, and firm 2 maximizes profit with respect to price. One can see from the first-order conditions that the interchangeability condition does not hold[30]:

$$\begin{aligned} \frac{\partial \pi_1}{\partial q_1} &= \frac{\partial \mathrm{TR}_1}{\partial q_1} - \frac{\partial \mathrm{TC}_1}{\partial q_1} \\ &= \mathrm{MR}_1 - \mathrm{MC}_1 \\ &= [(a - ad) - 2(1 - d^2)q_1 + dp_2] - (0) = 0, \end{aligned} \tag{10.72}$$

$$\begin{aligned} \frac{\partial \pi_2}{\partial p_2} &= \frac{\partial \mathrm{TR}_2}{\partial p_2} - \frac{\partial \mathrm{TC}_2}{\partial p_2} \\ &= \mathrm{MR}p_2 - \mathrm{MC}p_2 \\ &= [a - 2p_2 - dq_1] - (0) = 0. \end{aligned} \tag{10.73}$$

Solving this system of first-order conditions simultaneously gives the NE values of choice variables, q_1 and p_2. This produces what is called the Cournot–Bertrand equilibrium:

$$p_1^* = \frac{a(2 - d - 2d^2 + d^3)}{4 - 3d^2} > p_2^* = \frac{a(2 - d - d^2)}{4 - 3d^2}, \tag{10.74}$$

---

[29] With this assumption, p_i can be thought of as the difference between the price and marginal cost.

[30] Notice that the second-order condition holds for each firm. That is $\partial^2\pi_1/\partial {q_1}^2 = -2(1-d^2) < 0$, and $\partial^2\pi_2/\partial {p_2}^2 = -2$.

$$q_1^* = \frac{a(2-d)}{4-3d^2} > q_2^* = \frac{a(2-d-d^2)}{4-3d^2}, \tag{10.75}$$

$$\pi_1^* = \frac{a^2(2-d)^2(1-d^2)}{(4-3d^2)^2} - F_1; \quad \pi_2^* = \frac{a^2(2-d-d^2)^2}{(4-3d^2)^2} - F_2. \tag{10.76}$$

The NE in the Cournot–Bertrand model has several interesting properties. First, firm 1 charges a higher price and produces more output. Second, firm 1 earns greater profit as long as the difference in fixed costs is not too great. Third, the degree of product differentiation has a dramatic effect on the equilibrium. As expected, when $d = 0$, firms are not direct competitors and each firm behaves as a monopolist. When products are perfect substitutes ($d = 1$), however, firm 1 produces the competitive output level, price equals marginal cost (which is 0), and firm 2 exits the market.[31] The mere threat of a Bertrand-type competitor that produces a perfectly homogeneous good is enough to assure a competitive equilibrium even when there is only one Cournot-type firm left in the market. In this model, the Bertrand paradox applies even in the monopoly case. This provides a dramatic example where a potential entrant reduces market power. The Cournot–Bertrand equilibrium is also consistent with the principle of product differentiation.

To further analyze the Cournot–Bertrand model, we describe the NE in terms of best-reply and isoprofit diagrams. We obtain the best-reply functions by solving each firm's first-order condition for p_2:

$$\text{BR}_1 : p_2 = \frac{ad-a}{d} + \frac{2(1-d^2)}{d} q_1, \tag{10.77}$$

$$\text{BR}_2 : p_2 = \frac{a}{2} - \frac{d}{2} q_1. \tag{10.78}$$

The best-reply and isoprofit curves are illustrated in Fig. 10.15. The natural asymmetry of the model is evident from the fact that firm 1's best reply has a positive slope and firm 2's best reply has a negative slope.[32] Furthermore, firm 1's profits increase in p_2, and its isoprofit curve is convex to the q_1 axis. In contrast, firm 2's profits decrease in q_1, and its isoprofit curve is concave to the p_2 axis. Finally, notice that both firms are better off if they move into the lens-shaped region where firm 1 reduces production and firm 2 raises price. The cartel outcome would occur in this region. These unique features of best-reply and isoprofit curves occur because the model mixes Cournot and Bertrand strategic choices. Again, even though firms face the same demand and cost conditions, the choice of different strategic variables leads to dramatically different results.

[31] This is similar to the outcome of a "contestable market", as discussed in Chap. 5. For further discussion, see C. Tremblay and V. Tremblay (2011a) and C. Tremblay, M. Tremblay, and V. Tremblay (2011).

[32] For BR_1, the slope is $2(1–d^2)/d$ and the p_2 intercept is $(ad–a)/d$. For BR_2, the slope is $–d/2$ and the p_2 intercept is $a/2$.

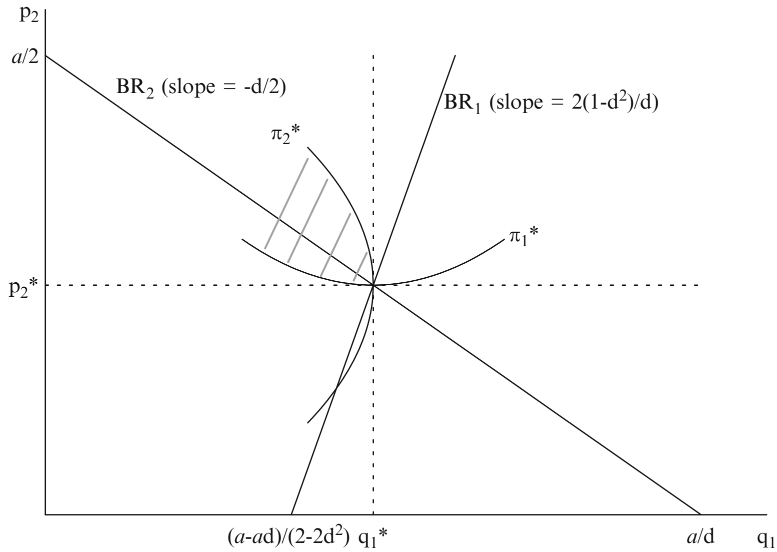

Fig. 10.15 The Cournot–Bertrand equilibrium

10.4 The Choice of Output or Price Competition

We have examined the classic Cournot and Bertrand models in homogeneous and differentiated goods markets and the more recent Cournot–Bertrand model in a differentiated goods market. These models produce several important conclusions:

1. In a market with homogeneous goods, prices and profits are substantially higher in the Cournot model than in the Bertrand model.
2. Although the equilibrium in a monopoly setting is invariant to the choice of strategic variable, output or price, this is not true in an oligopoly setting. The perfectly competitive solution is reached in the Bertrand model when products are homogeneous and there are two or more competitors. In contrast, the Cournot solution approaches the perfectly competitive equilibrium only as n approaches infinity.
3. Competition diminishes with product differentiation in both the Cournot and Bertrand models, and the two models are much more alike in differentiated goods markets.
4. A duopoly model becomes naturally asymmetric when firms compete in different choice variables. In the Cournot–Bertrand model, the firm that chooses to compete in output has a strategic advantage over the firm that chooses to compete in price as long as the Bertrand-type firm does not have a significant cost advantage.

These results raise the following question: If given the option, why would a firm choose to compete in price instead of output? Clearly, when products are homogeneous, output competition is a more profitable strategic choice. Yet, some firms compete in price.

One explanation, provided by Kreps and Scheinkman (1983), involves the nature of technology and the ease with which a firm can adjust output relative to price. They argue that when it is time consuming and costly to change production capacity or output, firms will compete in output and let price adjust to clear the market (as in Cournot). This would be true at a farmers' market, for example, where each farmer brings a fixed supply of produce to the market at a given point in time. Other examples include many heavy manufacturing industries, where it takes a considerable amount of time to produce a product from start to finish. Under these conditions, firms compete in output rather than price.

When price adjustments are relatively more costly than output adjustments, firms set prices and let production adjust to meet demand (as in Bertrand). Most inexpensive restaurants face this situation. Once menus are printed, it is costly to change price in response to short-term demand fluctuations, and a good chef can easily adjust to an increase in demand for pancakes relative to scrambled eggs. Other examples where output can adjust quickly and firms compete in price are the software and banking service industries.[33]

The optimal choice of strategic variable, output or price, can also be influenced by product differentiation and cost asymmetries. These issues are addressed by Singh and Vives (1984), Häckner (2000), and V. Tremblay et al. (forthcoming-a). Singh and Vives (1984) and V. Tremblay et al. (forthcoming-a) consider a duopoly model with multicharacteristic differentiation. A common feature of their work is that the decision to compete in output or price is endogenous. This leads to four possible outcomes:

1. Cournot (C): Both firms choose to compete in output.
2. Bertrand (B): Both firms choose to compete in price.
3. Cournot–Bertrand (CB): Firm 1 chooses to compete in output, and firm 2 chooses to compete in price.
4. Bertrand–Cournot (BC): Firm 1 chooses to compete in price, and firm 2 chooses to compete in output.

This possibility corresponds to model M4 in Table 10.1.

To illustrate their findings, we use a numerical example based on the demand system found in (10.28) and (10.29). Note that this is the demand system for the Cournot model, and it translates to demand system (10.35) and (10.36) in the Bertrand model and demand system (10.68) and (10.69) in the Cournot–Bertrand model. To compare profits in each case, we set $c = 0$, $a = 25$, and $d = 1/2$.

[33] Kreps and Scheinkman actually proposed a two-stage game, where each firm makes its decision on the sticky (long run) variable in the first stage and the flexible (short-run) variable adjusts to equilibrium in the second stage. This leads to the same result, however: (1) When output is sticky, firms compete in output, and price adjusts to meet demand, as in Cournot; (2) When price is sticky, firms compete in price, and output adjusts to meet demand, as in Bertrand.

		Firm 2	
		q_2	p_2
Firm 1	q_1	$\pi_1^C = 100 - F_1^C$, $\pi_2^C = 100 - F_2^C$	$\pi_1^{CB} = 99.9 - F_1^C$, $\pi_2^{CB} = 92.5 - F_2^B$
	p_1	$\pi_1^{BC} = 92.5 - F_1^B$, $\pi_2^{BC} = 99.9 - F_2^C$	$\pi_1^B = 92.6 - F_1^B$, $\pi_2^B = 92.6 - F_2^B$

Fig. 10.16 Payoff matrix for the Cournot (C), Bertrand (B), Cournot-Bertrand (CB), and Bertrand-Cournot (BC) outcomes

Figure 10.16 displays the results. Notationally, π_i^C is firm i's profit when both firms compete in output (Cournot), π_i^B is firm i's profits when both firms compete in price (Bertrand), π_i^{CB} is firm i's profits when firm 1 competes in output and firm 2 competes in price (Cournot–Bertrand), and π_i^{BC} is firm i's profits when firm 1 competes in price and firm 2 competes in output (Bertrand–Cournot). When fixed or set-up costs are positive, F_i^C equals fixed costs when firm i competes in output, and F_i^B equals fixed costs when firm i competes in price. As Fig. 10.16 shows, optimal play depends upon our assumptions about fixed costs. If fixed costs are sufficiently low and are the same regardless of the strategic choice (i.e., $F_1^C = F_1^B$ and $F_2^C = F_2^B$), as in the Singh and Vives (1984) model, then both firms are better off competing in output. That is, if firms had the choice, they would always prefer to compete in output because this is the dominant strategy.

The intuition behind this result relates to how the choice of strategic variable affects a firm's price elasticity of demand. When firm j's output is fixed, the slope of firm i's demand function is close to the slope of the market demand function.[34] When firm j's price is fixed, firm i's demand function is relatively more elastic, because firm i can steal sales by undercutting j's price. Thus, firm demand functions are more elastic and equilibrium prices are lower under price competition than under output competition.

Nevertheless, there are three conditions under which price competition is more profitable than output competition. First, as discussed above, it may be prohibitively costly to change price relative to output, causing price competition to be more profitable than output competition. This can occur when fixed costs associated with output competition (F_i^C) are substantially higher than the fixed cost associated with price competition (F_i^B). With output competition, a firm must bring a substantial quantity of output to market, but sales take time and the firm must have a storage facility to hold inventory. A firm that competes in price, however, may fill customer orders only after an order is placed. In the example in Fig. 10.16, if $F_i^C = 10$ and $F_i^B = 0$, then the dominant strategy for both firms is to compete in price (see Fig. 10.17).

[34] The slopes of firm and market demand functions converge as the degree of product differentiation diminishes, and the slopes are the same when products are homogeneous.

Firm 1 \ Firm 2	q_2	p_2
q_1	$\pi_1^C = 100$, $\pi_2^C = 90$	$\pi_1^{CB} = 99.9$, $\pi_2^{CB} = 92.5$
p_1	$\pi_1^{BC} = 92.5$, $\pi_2^{BC} = 89.9$	$\pi_1^B = 92.6$, $\pi_2^B = 92.6$

Fig. 10.17 Payoff matrix when there are asymmetric costs

Second, Häckner (2000) showed that price dominates output competition when brands are differentiated vertically and this differentiation is sufficiently great. Finally, V. Tremblay et al. (forthcoming-a) showed that in a dynamic setting the follower is just as likely to compete in price as in output, regardless of whether the leader competes in output or price. We take up this issue in the next chapter.

V. Tremblay et al. (forthcoming-b) find that cost asymmetries explain the Cournot–Bertrand behavior in the small car market. They show that Scion dealers compete in price because it has a relatively high cost of competing in output, while Honda dealers compete in output because it has a cost advantage in output competition. If we let Honda be firm 1 and Scion be firm 2 in the example in Fig. 10.16, then this can occur if $F_1^C = F_1^B = F_2^B = 0$ and $F_2^C = 10$. In this case, Firm 1 will compete in output and firm 2 will compete in price. In all, the choice of strategic variable depends on demand and cost conditions and the degree of asymmetry in the model

10.5 Strategic Substitutes and Strategic Complements

An interesting feature of these simple parametric models is that the best-reply functions exhibit a consistent pattern. When firms compete in output, the best reply functions have a negative slope, and when they compete in price with product differentiation, they have a positive slope.[35] Bulow et al. (1985) discovered these patterns and gave them the following names:

- The strategies of two players are **strategic substitutes** when the best-reply functions have a negative slope.

[35] Although there are exceptions when demand and cost functions are nonlinear, Amir and Grilo (1999) call this the "typical geometry" for the Cournot and Bertrand models. Throughout the book, we assume this typical geometry.

- The strategies of two players are **strategic complements** when the best-reply functions have a positive slope.

So far, we have investigated only best-reply functions for price and output, but these definitions apply to other strategic variables as well (e.g., advertising).

In general, whether a strategic variable between two firms is a strategic substitute or complement hinges on how a change in firm *j*'s strategic variable (s_j) affects the marginal returns of firm *i*'s strategic variable (s_i). More formally, given firm *i*'s profit equation, $\pi_i(s_i, s_j)$, which is assumed to be strictly concave in s_i and twice continuously differentiable, marginal returns are defined as $\partial\pi_i/\partial s_i$.[36] The effect of s_j on firm *i*'s marginal returns is $\partial(\partial\pi_i/\partial s_i)/\partial s_j = \partial^2\pi_i/\partial s_i\partial s_j \equiv \pi_{ij}$.

It turns out that s_i and s_j are strategic complements when $\pi_{ij} > 0$ and are strategic substitutes when $\pi_{ij} < 0$. A proof is provided in Appendix 10.B.

10.6 Summary

1. An **oligopoly** is characterized by a market with a few firms that compete in a strategic setting. Each firm's profit and best course of action depend on its own action and the actions of its competitors. A **duopoly** is an oligopoly market with two firms.
2. In the **Cournot model**, each firm simultaneously chooses a level of output that maximizes its own profit. The Cournot outcome is a Nash equilibrium (NE) where each firm correctly assumes that its competitors behave optimally. According to the **Cournot Limit Theorem**, as the number of firms in a market changes from 1 to infinity, the Cournot equilibrium changes from monopoly to perfect competition.
3. The **interchangeability condition** means that the first-order conditions of every firm in a model are interchangeable (by reversing firm subscripts). When this condition holds, the model is symmetric.
4. In the **Bertrand model**, each firm simultaneously chooses its price to maximize its own profit. The Bertrand equilibrium is a NE. When products are homogeneous and firms face the same costs, the Bertrand equilibrium price equals marginal cost as long as there are two or more firms in the market. This occurs because of **price undercutting**, where each firm undercuts the price of its rivals until the competitive price is reached. The implication that prices are competitive as long as there are two or more competitors is called the **Bertrand Paradox**. A firm can avoid the Bertrand Paradox if it has a cost advantage over its competitors.

[36] This concept is discussed more fully in the Mathematics and Econometrics Appendix.

5. The Bertrand model with homogeneous goods and symmetric costs makes it clear that economic theory cannot prove that market prices will fall as the number of competitors increases beyond two firms.
6. According to the **principle of product differentiation**, price competition diminishes as product differentiation increases. Thus, the Bertrand paradox does not arise when products are differentiated.
7. In the **Cournot–Bertrand model**, firm 1 competes in output and firm 2 competes in price, actions that are made simultaneously. This model produces a naturally asymmetric outcome and gives firm 1 a strategic advantage (i.e., it has higher profits) as long as any difference in costs between firms is sufficiently small.
8. When the choice of strategic variable is endogenous, firms will choose to compete in output as long as there are not substantial cost savings associated with price competition and as long as the degree of vertical product differentiation is not too great.
9. When best-reply functions have a negative slope, as in the Cournot model, the strategic variables between firms are **strategic substitutes**. When best replies have a positive slope, as in the Bertrand model, the strategic variables are **strategic complements**.

10.7 Review Questions

1. (Advanced) Consider a market with two firms (1 and 2) that face a linear inverse demand function $p = a - bQ$, where Q is industry output, q_i is the output of firm i (1 or 2), and $Q = q_1 + q_2$. Costs are also linear, with firm i's total cost equaling $\text{TC}_i = cq_i$. In addition, $a > c > 0$, $b > 0$. Find the Cournot equilibrium output for each firm. How will your answer change if $\text{TC}_i = cq_i^2$?
2. Consider a market with two firms (1 and 2) that face a linear demand function $Q = 24 - p$ and a total cost function $\text{TC}_i = cq_i$, $c > 0$. Find the Bertrand equilibrium price. How will your answer change if $c_1 = 10$ and $c_2 = 12$?
3. Explain how an increase in the number of firms affects the equilibrium price and allocative inefficiency in the homogeneous goods Cournot and Bertrand models.
4. Consider the oligopoly problem with n firms in Sect. 10.1.3. Assume that $a = 12$, $b = 1$, and $c = 0$. Use a graph similar to Fig. 10.8 to identify the NE when n equals 1, 2, 3, and infinity. How does total (consumer plus producer) surplus change as n increases?
5. Explain how a cost advantage or product differentiation can allow firms to avoid the Bertrand paradox.
6. In the Bertrand model with horizontal differentiation, explain how the equilibrium changes as t approaches 0. What does this say about the relationship between t and product differentiation?

7. Derive and graph the best-reply functions in the Bertrand model with horizontal differentiation discussed in Sect. 10.2.3.1. Show how a change in parameters c, t, N, and $(\theta_2 - \theta_1)$ will affect NE prices.
8. Wal-Mart stores typically locate on the edge of a city, even though potential demand may be greatest at the city's center. Assuming a linear city, is this a good location strategy? Explain.
9. In many markets, high quality brands coexist with low quality brands. If all consumers prefer high to low quality goods, *ceteris paribus*, why do some firms choose to supply low quality goods?
10. (Advanced) Consider a market with two firms (1 and 2) where firm 1 competes in output and firm 2 competes in price. Firm 1's inverse demand is $p_1 = 12 - q_1 + p_2$, firm 2's demand is $q_2 = 24 - p_2 - q_1$, $\text{TC}_i = cq_i$, $c > 0$. Find the Cournot–Bertrand equilibrium price, output, and profit levels for each firm. How does a change in c affect the equilibrium price, output, and profit?
11. Assume a duopoly market where firms can choose to compete in output or in price. Provide a simple numerical example where it is optimal for both firms to compete in price instead of output.
12. (Advanced) Consider the Cournot and Bertrand models of multicharacteristic differentiation that are discussed in Sect. 10.1. Show that choice variables are strategic substitutes in the Cournot model and are strategic complements in the Bertrand model by evaluating the slope of the best-reply functions or the signs of π_{ij} for each firm in each model.
13. In the Bertrand model in Sect. 10.2.2, discuss what happens to Nash prices when $\beta = ½$ and $\delta = 1$. Will the model be stable, as described in Appendix 10.A, if $\beta = ½$ and $\delta = 2$?
14. Assume a duopoly market with two firms, 1 and 2. In case I, firms behave as Cournot competitors, as described in Fig. 10.2. In case II, firms behave as differentiated Bertrand competitors, as described in Fig. 10.13. In case III, firms behave as Cournot–Bertrand competitors, as described in Fig. 10.15. Suppose that the management team of firm 1 is overconfident; they overestimate the demand intercept (a or α in Figs. 10.2, 10.13, and 10.15). Explain how this overconfidence will affect the Nash equilibrium.

Appendix A: Stability of the Cournot and Bertrand Models

Here, we are interested in the stability of the Nash equilibrium (NE) in a Cournot, Bertrand, or Cournot–Bertrand model. According to Mas-Colell (1995: 414), an equilibrium in a static model is stable when the "adjustment process in which the firms take turns myopically playing a best response to each others' current strategies converges to the Nash equilibrium from any strategy pair in a neighborhood of the equilibrium." For a stable NE, the best-reply functions must meet certain regularity conditions.

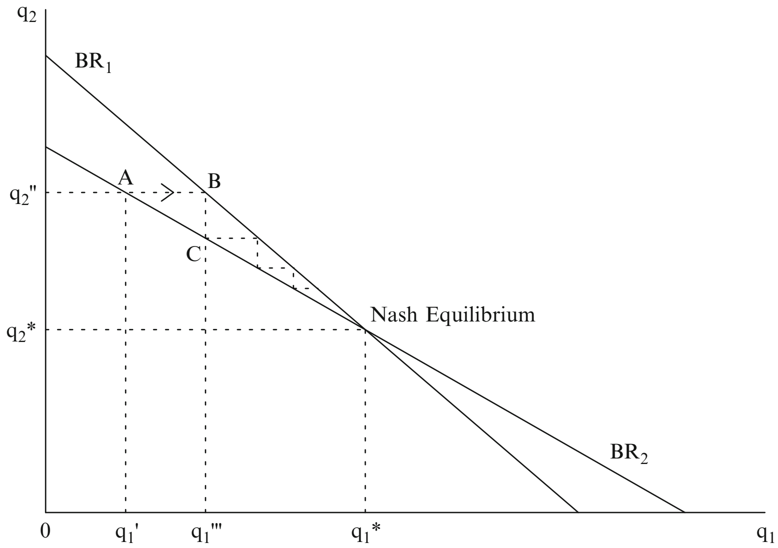

Fig. 10.18 A stable Cournot model

First, we consider the Cournot model developed in Sect. 10.2.1. The Cournot equilibrium is stable when BR_1 is steeper than BR_2, as in Fig. 10.18. To see this, assume that firm 1 chooses a disequilibrium level of output, q_1'. Firm 2's best reply to q_1' is q_2''. When firm 2 chooses q_2'', firm 1's best response is q_1'''. Thus, the adjustment process moves from point A, to B, to C in the graph, a process that continues until the NE is reached. At equilibrium, Firm 1's best reply to q_2^* is q_1^*, and firm 2's best reply to q_1^* is q_2^* (i.e., they are a mutual best reply). The equilibrium is unstable when BR_1 is flatter than BR_2, as illustrated in Fig. 10.19. In this case, when starting at q_1' the adjustment process moves away from the NE.

We now investigate stability of the Cournot equilibrium more generally. In Appendix 10.B, we prove that the slopes of the best-reply functions are $\partial q_1^{BR}/\partial q_2 = -\pi_{12}/\pi_{11}$ for firm 1 and $\partial q_2^{BR}/\partial q_1 = -\pi_{21}/\pi_{22}$ for firm 2. In the graph with q_2 on the vertical axis, the slope of firm 1's best reply is $-\pi_{11}/\pi_{12}$. Thus, stability of the equilibrium in the Cournot model requires that $|-\pi_{11}/\pi_{12}| > |-\pi_{21}/\pi_{22}|$. Because $\pi_{ii} < 0$ and $\pi_{ij} < 0$, we can rewrite the stability condition as $\pi_{11}\pi_{22} - \pi_{12}\pi_{21} > 0$. In the example from Sect. 10.2.1, $\pi_{11} = \pi_{22} = -2$ and $\pi_{12} = \pi_{21} = -d$. Thus, the slope of firm 1's best reply is $-2/d$, the slope of firm 2's best reply is $-d/2$, and the stability condition is $\pi_{11}\pi_{22} - \pi_{12}\pi_{21} = 4 - d^2 > 0$. Thus, the equilibrium is stable when $d < 2$.

Next, we consider the differentiated Bertrand model developed in Sect. 10.2.2. The Bertrand equilibrium is stable when BR_1 is steeper than BR_2, as in Fig. 10.20. If we begin at a disequilibrium point, p_1', firm 2's best reply is p_2''. When firm

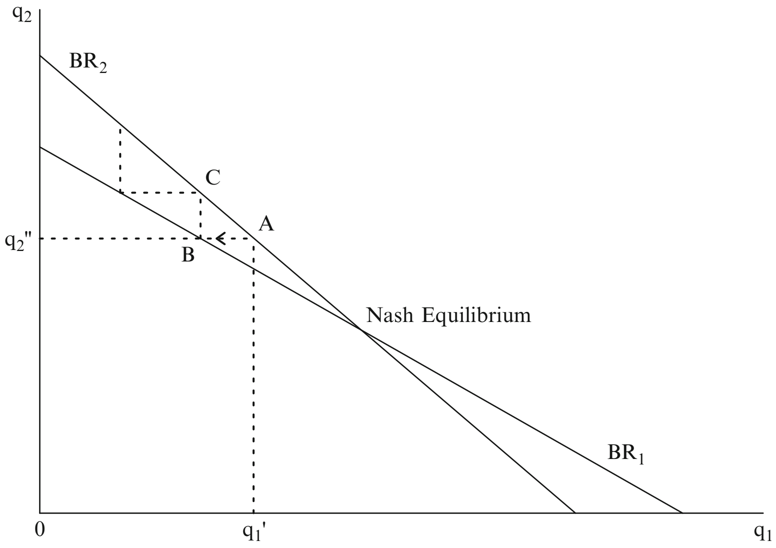

Fig. 10.19 An unstable Cournot model

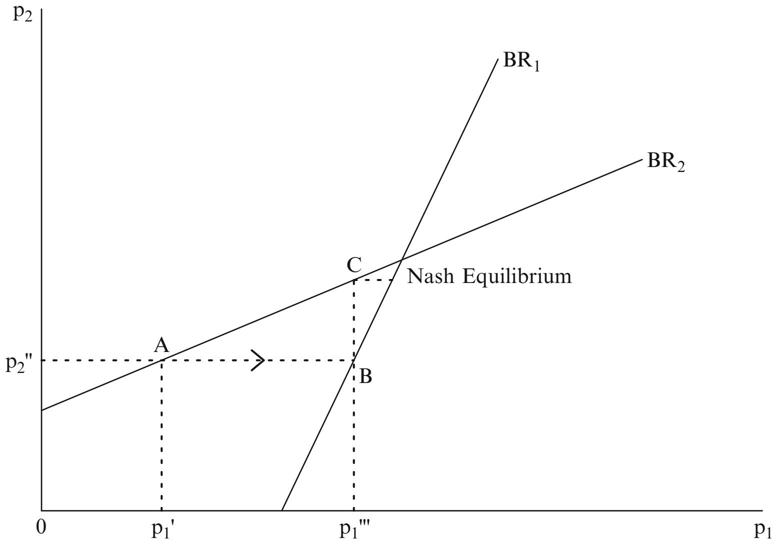

Fig. 10.20 A stable Bertrand model

2 chooses p_2'', firm 1's best response is p_1''', etc. Thus, the adjustment process moves from point A, to B, to C and converges to the NE. This equilibrium is unstable, however, when BR_1 is flatter than BR_2, as in Fig. 10.21. In this case, the adjustment process moves away from the NE.

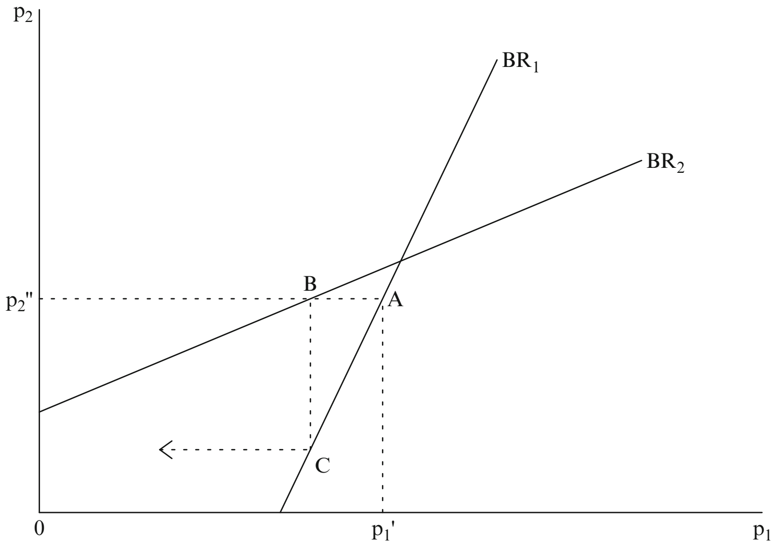

Fig. 10.21 An unstable Bertrand model

In the example from Sect. 10.2.2, $\pi_{11} = \pi_{22} = -2\beta$ and $\pi_{12} = \pi_{21} = \delta$. Thus, the slope of firm 1's best reply is $2\beta/\delta$, the slope of firm 2's best reply is $\delta/2\beta$, and the stability condition is $\pi_{11}\pi_{22} - \pi_{12}\pi_{21} = 4\beta^2 - \delta^2 > 0$. Therefore, Bertrand equilibrium is stable when $\beta > \delta/2$.

Analysis of stability conditions for the Cournot–Bertrand model can be found in V. Tremblay et al. (forthcoming-a).

Appendix B: Strategic Substitutes and Complements and the Slope of the Best-Reply Functions

As discussed in the text, the two strategic variables of firms i and j, s_i and s_j, are strategic complements when $\pi_{ij} > 0$ and are strategic substitutes when $\pi_{ij} < 0$. The proof follows from the first- and second-order conditions of profit maximization and the application of the implicit-function theorem, which is discussed in the Mathematics and Econometrics Appendix at the end of the book. Recall that firm i's best-reply function is derived by solving the firm's first-order condition for s_i, s_i^{BR}, which is the optimal value of s_i given s_j. Even though we are using a general function, embedded in the first-order condition is s_i^{BR}. Thus, we can use the implicit-function theorem to obtain the slope of firm i's best-reply function:

$$\frac{\partial s_i^{\mathrm{BR}}}{\partial s_j} = \frac{-\pi_{ij}}{\pi_{ii}}, \tag{10.79}$$

where $\pi_{ii} \equiv \partial^2\pi_i/\partial s_i^2$, which is negative from our concavity assumption (ensuring that the second-order condition of profit maximization is met). Thus, the sign of $\partial s_i^{BR}/\partial s_j$ equals the sign of π_{ij}. To summarize:

- When $\pi_{ij} < 0$, the best-reply functions have a negative slope and s_i and s_j are strategic substitutes, as in the Cournot model.
- When $\pi_{ij} > 0$, the best-reply functions have a positive slope and s_i and s_j are strategic complements, as in the differentiated Bertrand model.

In the mixed Cournot and Bertrand model developed in Sect. 10.3, $\pi_{12} = d > 0$ and $\pi_{21} = -d < 0$. This verifies that firm 1's best-reply function has a positive slope, and firm 2's best-reply function has a negative slope (Fig. 10.15). It also implies that q_1 and p_2 are strategic complements for the Cournot-type firm and are strategic substitutes for the Bertrand-type firm.